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a functorial description with applications

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On Gorenstein projective, injective and flat dimensions—A functorial description with applications [☆]

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Abstract

Gorenstein homological dimensions are refinements of the classical homological dimensions, and finiteness singles out modules with amenable properties reflecting those of modules over Gorenstein rings.

As opposed to their classical counterparts, these dimensions do not immediately come with practical and robust criteria for finiteness, not even over commutative noetherian local rings. In this paper we enlarge the class of rings known to admit good criteria for finiteness of Gorenstein dimensions:

It now includes, for instance, the rings encountered in commutative algebraic geometry and, in the noncommutative realm, k-algebras with a dualizing complex.

0. Introduction

An important motivation for studying homological dimensions goes back to 1956 when Auslander, Buchsbaum and Serre proved the following theorem: A commutative noetherian local ring R is regular if the residue field k has finite projective dimension and only if all R-modules have finite projective dimension. This introduced the theme that finiteness of a homological dimension for all modules singles out rings with special properties.

Subsequent work showed that over any commutative noetherian ring, modules of finite projective or injective dimension have special properties resembling those of modules over regular rings. This is one reason for studying homological dimensions of individual modules.

This paper is concerned with homological dimensions for modules over associative rings. In the introduction we restrict to a commutative noetherian local ring R with residue field k.

Pursuing the themes described above, Auslander and Bridger [2,3] introduced a homological dimension designed to single out modules with properties similar to those of modules over Gorenstein rings. They called it the G-dimension, and it has strong parallels to the projective dimension: R is Gorenstein if the residue field k has finite G-dimension and only if all finitely generated R-modules have finite G-dimension. A finitely generated R-module M of finite G-dimension satisfies an analogue of the Auslander–Buchsbaum formula:

$$\text{G-dim}_R M = \text{depth}\, R - \text{depth}_R M.$$

Like other homological dimensions, the G-dimension is introduced by first defining the modules of dimension 0, and then using these to resolve arbitrary modules. Let us recall the definition: A finitely generated R-module M has G-dimension 0 if

$$\text{Ext}_R^m(M, R) = 0 = \text{Ext}_R^m\big(\text{Hom}_R(M, R), R\big) \quad \text{for } m > 0$$

and M is reflexive, that is, the canonical map

$$M \to \text{Hom}_R\big(\text{Hom}_R(M, R), R\big)$$

is an isomorphism. Here we encounter a first major difference between G-dimension and projective dimension. A projective R-module M is described by vanishing of the cohomology functor $\text{Ext}_R^1(M, -)$, and projectivity of a finitely generated module can even be

verified by computing a single cohomology module, $\text{Ext}_R^1(M, k)$. However, verification of G-dimension 0 requires, a priori, the computation of infinitely many cohomology modules. Indeed, recent work of Jorgensen and Şega [34] shows that for a reflexive module M the vanishing of $\text{Ext}_R^{>0}(M, R)$ and $\text{Ext}_R^{>0}(\text{Hom}_R(M, R), R)$ cannot be inferred from vanishing of any finite number of these cohomology modules.

Since the modules of G-dimension 0 are not described by vanishing of a (co)homology functor, the standard computational techniques of homological algebra, like dimension shift, do not effectively apply to deal with modules of finite G-dimension. This has always been the Achilles' heel of the theory and explains the interest in finding alternative criteria for finiteness of this dimension.

G-dimension also differs from projective dimension in that it is defined only for finitely generated modules. To circumvent this shortcoming, Enochs and Jenda [14] proposed to study a homological dimension based on a larger class of modules: An R-module M is called Gorenstein projective, if there exists an exact complex

$$P = \cdots \to P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\partial_0^P} P_{-1} \to \cdots$$

of projective modules, such that $M \cong \text{Coker}\,\partial_1^P$ and $\text{Hom}_R(P, Q)$ is exact for any projective R-module Q. This definition allows for nonfinitely generated modules and is, indeed, satisfied by all projective modules. It was already known from [2] that a finitely generated R-module M is of G-dimension 0 precisely when there exists an exact complex $L \cdots \to L_1 \to L_0 \to \cdots$ of finitely generated free modules, such that $M \cong \text{Coker}\,\partial_1^L$ and $\text{Hom}_R(L, R)$ is exact. Avramov, Buchweitz, Martsinkovsky, and Reiten (see [8]) proved that for finitely generated modules, the Gorenstein projective dimension agrees with the G-dimension.

Gorenstein flat and injective modules were introduced along the same lines [14,16]. Just as the G-dimension has strong similarities with the projective dimension, the Gorenstein injective and flat dimensions are, in many respects, similar to the classical flat and injective dimensions. However, these new dimensions share the problem encountered already for G-dimension: It is seldom practical to study modules of finite dimension via modules of dimension 0.

The goal of this paper is to remedy this situation. We do so by establishing a conjectured characterization of modules of finite Gorenstein dimensions in terms of vanishing of homology and invertibility of certain canonical maps. It extends an idea of Foxby, who found an alternative criterion for finite G-dimension of finitely generated modules; see [48]. Before describing the criteria for finiteness of Gorenstein projective, injective and flat dimensions, we present a few applications.

The study of Gorenstein dimensions takes cues from the classical situation; an example: It is a trivial fact that projective modules are flat but a deep result, due to Gruson–Raynaud [41] and Jensen [33], that flat R-modules have finite projective dimension. For Gorenstein dimensions the situation is more complicated. It is true but not trivial that Gorenstein projective R-modules are Gorenstein flat; in fact, the proof relies on the very result by Gruson, Raynaud and Jensen mentioned above. However, very little is known about Gorenstein

projective dimension of Gorenstein flat R-modules. As a first example of what can be gained from our characterization of modules of finite Gorenstein dimensions, we address this question; see 4.2:

Theorem I. *If R has a dualizing complex, then the following are equivalent for an R-module M:*

(i) *M has finite Gorenstein projective dimension, $\operatorname{Gpd}_R M < \infty$.*
(ii) *M has finite Gorenstein flat dimension, $\operatorname{Gfd}_R M < \infty$.*

As the hypothesis of Theorem I indicates, our characterization of finite Gorenstein dimensions requires the underlying ring to have a dualizing complex. By Kawasaki's proof of Sharp's conjecture [36], this is equivalent to assuming that R is a homomorphic image of a Gorenstein local ring.

While the Gorenstein analogues of the Auslander–Buchsbaum–Serre theorem and the Auslander–Buchsbaum formula were among the original motives for studying G-dimension, the Gorenstein equivalent of another classic, the Bass formula, has proved more elusive. It was first established over Gorenstein rings [15], later over Cohen–Macaulay local rings with dualizing module [8]. The tools invented in this paper enable us to remove the Cohen–Macaulay hypothesis; see 6.3:

Theorem II. *If R has a dualizing complex, and N is a nonzero finitely generated R-module of finite Gorenstein injective dimension, then*

$$\operatorname{Gid}_R N = \operatorname{depth} R.$$

As a third application we record the following result, proved in 5.7 and 6.9:

Theorem III. *If R has a dualizing complex, then any direct product of Gorenstein flat R-modules is Gorenstein flat, and any direct sum of Gorenstein injective modules is Gorenstein injective.*

Over any noetherian ring, a product of flat modules is flat and a sum of injectives is injective; this is straightforward. The situation for Gorenstein dimensions is, again, more complicated and, hitherto, Theorem III was only known for some special rings.

The proofs of Theorems I–III above rely crucially on a description of finite Gorenstein homological dimensions in terms of two full subcategories of the derived category of R-modules. They are the so-called Auslander categories, $\mathsf{A}(R)$ and $\mathsf{B}(R)$, associated to the dualizing complex; they were first studied in [17,22].

We prove that the modules in $\mathsf{A}(R)$ are precisely those of finite Gorenstein projective dimension, see Theorem 4.1, and the modules in $\mathsf{B}(R)$ are those of finite Gorenstein injective dimension, see Theorem 4.4. For many applications it is important that these two categories are related through an equivalence that resembles Morita theory:

$$\mathsf{A}(R) \underset{\mathbf{R}\operatorname{Hom}_R(D,-)}{\overset{D \otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathsf{B}(R);$$

where D is the dualizing complex. This may be viewed as an extension of well-known facts: If R is Cohen–Macaulay, the equivalences above restrict to the subcategories of modules of finite flat and finite injective dimension. If R is Gorenstein, that is, $D = R$, the subcategories of modules of finite flat and finite injective dimension even coincide, and so do $\mathsf{A}(R)$ and $\mathsf{B}(R)$.

For a Cohen–Macaulay local ring with a dualizing module, this description of finite Gorenstein homological dimensions in terms of $\mathsf{A}(R)$ and $\mathsf{B}(R)$ was established in [17]. The present version extends it in several directions: The underlying ring is not assumed to be either commutative, or Cohen–Macaulay, or local.

In general, we work over an associative ring with unit. For the main results, the ring is further assumed to admit a dualizing complex. Also, we work consistently with complexes of modules. Most proofs, and even the definition of Auslander categories, require complexes, and it is natural to state the results in the same generality.

The characterization of finite Gorenstein homological dimensions in terms of Auslander categories is proved in Section 4; Sections 5 and 6 are devoted to applications. The main theorems are proved through new technical results on preservation of quasi-isomorphisms; these are treated in Section 2. The first section fixes notation and prerequisites, and in the third we establish the basic properties of Gorenstein dimensions in the generality required for this paper.

1. Notation and prerequisites

In this paper, all rings are assumed to be associative with unit, and modules are, unless otherwise explicitly stated, left modules. For a ring R we denote by R^{opp} the opposite ring, and identify right R-modules with left R^{opp}-modules in the natural way. Only when a module has bistructure, do we include the rings in the symbol; e.g., $_S M_R$ means that M is an (S, R^{opp})-bimodule.

We consistently use the notation from the appendix of [8]. In particular, the category of R-complexes is denoted $\mathsf{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$, and $\square$ to denote boundedness conditions. For example, $\mathsf{C}_{\sqsupset}(R)$ is the full subcategory of $\mathsf{C}(R)$ of right-bounded complexes.

The derived category is written $\mathsf{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$, and $\square$ to denote homological boundedness conditions. Superscript "f" signifies that the homology is degreewise finitely generated. Thus, $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ denotes the full subcategory of $\mathsf{D}(R)$ of homologically right-bounded complexes with finitely generated homology modules. The symbol "$\simeq$" is used to designate isomorphisms in $\mathsf{D}(R)$ and quasi-isomorphisms in $\mathsf{C}(R)$. For the derived category and derived functors, the reader is referred to the original texts, Verdier's thesis [45] and Hartshorne's notes [29], and further to a modern account: the book by Gelfand and Manin [26].

Next, we review a few technical notions for later use.

1.1. Definition. Let S and R be rings. If S is left noetherian and R is right noetherian, we refer to the ordered pair $\langle S, R \rangle$ as a *noetherian pair* of rings.

A *dualizing complex* for a noetherian pair of rings $\langle S, R \rangle$ is a complex ${}_S D_R$ of bimodules meeting the requirements:

(1) The homology of D is bounded and degreewise finitely generated over S and over R^{opp}.
(2) There exists a quasi-isomorphism of complexes of bimodules, ${}_S P_R \xrightarrow{\simeq} {}_S D_R$, where ${}_S P_R$ is right-bounded and consists of modules projective over both S and R^{opp}.
(3) There exists a quasi-isomorphism of complexes of bimodules, ${}_S D_R \xrightarrow{\simeq} {}_S I_R$, where ${}_S I_R$ is bounded and consists of modules injective over both S and R^{opp}.
(4) The homothety morphisms

$$\check{\chi}_D^{\langle S,R \rangle} : {}_S S_S \to \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}({}_S D_R, {}_S D_R)$$

and

$$\hat{\chi}_D^{\langle S,R \rangle} : {}_R R_R \to \mathbf{R}\mathrm{Hom}_S({}_S D_R, {}_S D_R),$$

are bijective in homology. That is to say,

- $\check{\chi}_D^{\langle S,R \rangle}$ is invertible in $\mathsf{D}(S)$ (equivalently, invertible in $\mathsf{D}(S^{\mathrm{opp}})$), and
- $\hat{\chi}_D^{\langle S,R \rangle}$ is invertible in $\mathsf{D}(R)$ (equivalently, invertible in $\mathsf{D}(R^{\mathrm{opp}})$).

If R is both left and right noetherian (e.g., commutative and noetherian), then a dualizing complex for R means a dualizing complex for the pair $\langle R, R \rangle$ (in the commutative case the two copies of R are tacitly assumed to have the same action on the modules).

For remarks on this definition and comparison to other notions of dualizing complexes in noncommutative algebra, we refer to Appendix A. At this point we just want to mention that 1.1 is a natural extension of existing definitions: When $\langle S, R \rangle$ is a noetherian pair of algebras over a field, Definition 1.1 agrees with the one given by Yekutieli–Zhang [51]. If R is commutative and noetherian, then 1.1 is clearly the same as Grothendieck's definition [29, V.§2].

The next result is proved in Appendix A.

1.2. Proposition. *Let $\langle S, R \rangle$ be a noetherian pair. A complex ${}_S D_R$ is dualizing for $\langle S, R \rangle$ if and only if it is dualizing for the pair $\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle$.*

1.3. Equivalence. If ${}_S D_R$ is a dualizing complex for the noetherian pair $\langle S, R \rangle$, we consider the adjoint pairs of functors,

$$\mathsf{D}(R) \xrightleftharpoons[\mathbf{R}\mathrm{Hom}_S({}_S D_R, -)]{{}_S D_R \otimes^{\mathbf{L}}_R -} \mathsf{D}(S).$$

These functors are represented by ${}_S P_R \otimes_R -$ and $\mathrm{Hom}_S({}_S P_R, -)$, where ${}_S P_R$ is as in 1.1(2); see also Appendix A.

The *Auslander categories* $\mathsf{A}(R)$ and $\mathsf{B}(S)$ with respect to the dualizing complex $_SD_R$ are defined in terms of natural transformations being isomorphisms:

$$\mathsf{A}(R) = \left\{ X \in \mathsf{D}_\square(R) \;\middle|\; \begin{array}{l} \eta_X : X \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(_SD_R, {}_SD_R \otimes_R^{\mathbf{L}} X) \text{ is an iso-} \\ \text{morphism in } \mathsf{D}(R), \text{ and } _SD_R \otimes_R^{\mathbf{L}} X \text{ is bounded} \end{array} \right\},$$

and

$$\mathsf{B}(S) = \left\{ Y \in \mathsf{D}_\square(S) \;\middle|\; \begin{array}{l} \varepsilon_Y : {}_SD_R \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S(_SD_R, Y) \xrightarrow{\simeq} Y \text{ is an isomor-} \\ \text{phism in } \mathsf{D}(S), \text{ and } \mathbf{R}\mathrm{Hom}_S(_SD_R, Y) \text{ is bounded} \end{array} \right\}.$$

All R-complexes of finite flat dimension belong to $\mathsf{A}(R)$, while S-complexes of finite injective dimension belong to $\mathsf{B}(S)$, cf. [5, Theorem 3.2].

The Auslander categories $\mathsf{A}(R)$ and $\mathsf{B}(S)$ are clearly triangulated subcategories of $\mathsf{D}(R)$ and $\mathsf{D}(S)$, respectively, and the adjoint pair $(_SD_R \otimes_R^{\mathbf{L}} -, \mathbf{R}\mathrm{Hom}_S(_SD_R, -))$ restricts to an equivalence:

$$\mathsf{A}(R) \underset{\mathbf{R}\mathrm{Hom}_S(_SD_R,-)}{\overset{_SD_R \otimes_R^{\mathbf{L}} -}{\rightleftarrows}} \mathsf{B}(S).$$

In the commutative setting, this equivalence, introduced in [5], is sometimes called *Foxby equivalence*.

1.4. Finitistic dimensions. We write $\mathrm{FPD}(R)$ for the (left) finitistic projective dimension of R, i.e.,

$$\mathrm{FPD}(R) = \sup \left\{ \mathrm{pd}_R M \;\middle|\; \begin{array}{l} M \text{ is an } R\text{-module of} \\ \text{finite projective dimension} \end{array} \right\}.$$

Similarly, we write $\mathrm{FID}(R)$ and $\mathrm{FFD}(R)$ for the (left) finitistic injective and (left) finitistic flat dimension of R.

When R is commutative and noetherian, it is well known from [6, Corollary 5.5] and [41, II. Theorem 3.2.6] that

$$\mathrm{FID}(R) = \mathrm{FFD}(R) \leqslant \mathrm{FPD}(R) = \dim R.$$

If, in addition, R has a dualizing complex, then $\dim R$ is finite by [29, Corollary V.7.2], and hence so are the finitistic dimensions.

Jensen [33, Proposition 6] proved that if $\mathrm{FPD}(R)$ is finite, then any R-module of finite flat dimension has finite projective dimension as well. For noncommutative rings, finiteness of $\mathrm{FPD}(R)$ has proved difficult to establish; indeed, even for finite-dimensional algebras over a field it remains a conjecture. Therefore, the following result is of interest:

1.5. Proposition. *Assume that the noetherian pair $\langle S, R \rangle$ has a dualizing complex $_S D_R$. If $X \in \mathsf{D}(R)$ has finite* $\mathrm{fd}_R X$, *then there is an inequality,*

$$\mathrm{pd}_R X \leqslant \max\big\{\mathrm{id}_S(_S D_R) + \sup(_S D_R \otimes^{\mathbf{L}}_R X),\ \sup X\big\} < \infty.$$

Moreover, $\mathrm{FPD}(R)$ *is finite if and only if* $\mathrm{FFD}(R)$ *is finite.*

Proof. When $R = S$ is commutative, this was proved by Foxby [20, Corollary 3.4]. Recently, Jørgensen [35] has generalized the proof to the situation where R and S are k-algebras. The further generalization stated above is proved in A.1.

Dualizing complexes have excellent duality properties; we shall only need that for commutative rings:

1.6. Duality. Let R be commutative and noetherian with a dualizing complex D. Grothendieck [29, V.§2] considered the functor $-^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-, D)$. As noted ibid. $-^{\dagger}$ sends $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ to itself and, in fact, gives a duality on that category. That is, there is an isomorphism

$$X \xrightarrow{\simeq} X^{\dagger\dagger} = \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(X, D), D\big) \tag{1.6.1}$$

for $X \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$, see [29, Proposition V.2.1].

We close this section by recalling the definitions of Gorenstein homological dimensions; they go back to [8,13,14,16].

1.7. Gorenstein projective dimension. An R-module A is *Gorenstein projective* if there exists an exact complex P of projective modules, such that A is isomorphic to a cokernel of P, and $\mathrm{H}(\mathrm{Hom}_R(P, Q)) = 0$ for all projective R-modules Q. Such a complex P is called a *complete projective resolution* of A.

The *Gorenstein projective dimension*, $\mathrm{Gpd}_R X$, of $X \in \mathsf{D}_{\sqsupset}(R)$ is defined as

$$\mathrm{Gpd}_R X = \inf\left\{\sup\{\ell \in \mathbb{Z} \mid A_\ell \neq 0\}\ \middle|\ \begin{matrix} A \in \mathsf{C}_{\sqsupset}(R) \text{ is isomorphic to } X \text{ in } \mathsf{D}(R) \\ \text{and every } A_\ell \text{ is Gorenstein projective} \end{matrix}\right\}.$$

1.8. Gorenstein injective dimension. The definitions of *Gorenstein injective modules* and *complete injective resolutions* are dual to the ones given in 1.7, see also [8, (6.1.1) and (6.2.2)]. The *Gorenstein injective dimension*, $\mathrm{Gid}_R Y$, of $Y \in \mathsf{D}_{\sqsubset}(R)$ is defined as

$$\mathrm{Gid}_R Y = \inf\left\{\sup\{\ell \in \mathbb{Z} \mid B_{-\ell} \neq 0\}\ \middle|\ \begin{matrix} B \in \mathsf{C}_{\sqsubset}(R) \text{ is isomorphic to } Y \text{ in } \mathsf{D}(R) \\ \text{and every } B_\ell \text{ is Gorenstein injective} \end{matrix}\right\}.$$

1.9. Gorenstein flat dimension. An R-module A is *Gorenstein flat* if there exists an exact complex F of flat modules, such that A is isomorphic to a cokernel of F, and $\mathrm{H}(J \otimes_R F) = 0$ for all injective R^{opp}-modules J. Such a complex F is called a *complete flat resolution* of A.

The definition of the *Gorenstein flat dimension*, $\operatorname{Gfd}_R X$, of $X \in \mathsf{D}_{\sqsupset}(R)$ is similar to that of the Gorenstein projective dimension given in 1.7, see also [8, (5.2.3)].

2. Ubiquity of quasi-isomorphisms

In this section we establish some important, technical results on preservation of quasi-isomorphisms. It is, e.g., a crucial ingredient in the proof of the main Theorem 4.1 that the functor $- \otimes_R A$ preserves certain quasi-isomorphisms, when A is a Gorenstein flat module. This is established in Theorem 2.15 below. An immediate consequence of this result is that Gorenstein flat modules may sometimes replace real flat modules in representations of derived tensor products. This corollary, 2.16, plays an important part in the proof of Theorem 3.5.

Similar results on representations of the derived Hom functor are used in the proofs of Theorems 3.1 and 3.3. These are also established below.

The section closes with three approximation results for modules of finite Gorenstein homological dimension. We recommend that this section is consulted as needed rather than read linearly.

The first lemmas are easy consequences of the definitions of Gorenstein projective, injective, and flat modules.

2.1. Lemma. *If M is a Gorenstein projective R-module, then $\operatorname{Ext}_R^m(M, T) = 0$ for all $m > 0$ and all R-modules T of finite projective or finite injective dimension.*

Proof. For a module T of finite projective dimension, the vanishing of $\operatorname{Ext}_R^m(M, T)$ is an immediate consequence of the definition of Gorenstein projective modules.

Assume that $\operatorname{id}_R T = n < \infty$. Since M is Gorenstein projective, we have an exact sequence,

$$0 \to M \to P_0 \to P_{-1} \to \cdots \to P_{1-n} \to C \to 0,$$

where the P's are projective modules. Breaking this sequence into short exact ones, we see that $\operatorname{Ext}_R^m(M, T) = \operatorname{Ext}_R^{m+n}(C, T)$ for $m > 0$, so the Exts vanish as desired since $\operatorname{Ext}_R^w(-, T) = 0$ for $w > n$. $\square$

Similarly one establishes the next two lemmas.

2.2. Lemma. *If N is a Gorenstein injective R-module, then $\operatorname{Ext}_R^m(T, N) = 0$ for all $m > 0$ and all R-modules T of finite projective or finite injective dimension.*

2.3. Lemma. *If M is a Gorenstein flat R-module, then $\operatorname{Tor}_m^R(T, M) = 0$ for all $m > 0$ and all R^{opp}-modules T of finite flat or finite injective dimension.*

From Lemma 2.1 it is now a three step process to arrive at the desired results on preservation of quasi-isomorphisms by the Hom functor. We give proofs for the results regarding the covariant Hom functor; those on the contravariant functor have similar proofs.

2.4. Lemma. *Assume that $X, Y \in \mathsf{C}(R)$ with either $X \in \mathsf{C}_\sqsupset(R)$ or $Y \in \mathsf{C}_\sqsupset(R)$. If $\mathrm{H}(\mathrm{Hom}_R(X_\ell, Y)) = 0$ for all $\ell \in \mathbb{Z}$, then $\mathrm{H}(\mathrm{Hom}_R(X, Y)) = 0$.*

This result can be found in [19, Lemma 6.7]. However, since this reference is not easily accessible, we provide an argument here:

Proof. Fix an $n \in \mathbb{Z}$; we shall prove that $\mathrm{H}_n(\mathrm{Hom}_R(X, Y)) = 0$. Since

$$\mathrm{H}_n\big(\mathrm{Hom}_R(X, Y)\big) = \mathrm{H}_0\big(\mathrm{Hom}_R\big(X, \Sigma^{-n}Y\big)\big)$$

we need to show that every morphism $\alpha : X \to \Sigma^{-n}Y$ is null-homotopic. Thus for a given morphism α we must construct a family $(\gamma_m)_{m \in \mathbb{Z}}$ of degree 1 maps, $\gamma_m : X_m \to (\Sigma^{-n}Y)_{m+1} = Y_{n+m+1}$, such that

$$\alpha_m = \gamma_{m-1}\partial_m^X + \partial_{m+1}^{\Sigma^{-n}Y}\gamma_m \qquad (*)_m$$

for all $m \in \mathbb{Z}$. We do so by induction on m: Since X or $\Sigma^{-n}Y$ is in $\mathsf{C}_\sqsupset(R)$ we must have $\gamma_m = 0$ for $m \ll 0$. For the inductive step, assume that γ_m has been constructed for all $m < \tilde{m}$. By assumption $\mathrm{H}(\mathrm{Hom}_R(X_{\tilde{m}}, \Sigma^{-n}Y)) = 0$, so applying $\mathrm{Hom}_R(X_{\tilde{m}}, -)$ to

$$\Sigma^{-n}Y = \cdots \to Y_{n+\tilde{m}+1} \xrightarrow{\partial_{\tilde{m}+1}^{\Sigma^{-n}Y}} Y_{n+\tilde{m}} \xrightarrow{\partial_{\tilde{m}}^{\Sigma^{-n}Y}} Y_{n+\tilde{m}-1} \to \cdots$$

yields an exact complex. Using that α is a morphism and that $(*)_{\tilde{m}-1}$ holds, we see that $\alpha_{\tilde{m}} - \gamma_{\tilde{m}-1}\partial_{\tilde{m}}^X$ is in the kernel of $\mathrm{Hom}_R(X_{\tilde{m}}, \partial_{\tilde{m}}^{\Sigma^{-n}Y})$:

$$\begin{aligned}
\partial_{\tilde{m}}^{\Sigma^{-n}Y}\big(\alpha_{\tilde{m}} - \gamma_{\tilde{m}-1}\partial_{\tilde{m}}^X\big) &= \big(\alpha_{\tilde{m}-1} - \partial_{\tilde{m}}^{\Sigma^{-n}Y}\gamma_{\tilde{m}-1}\big)\partial_{\tilde{m}}^X \\
&= \big(\gamma_{\tilde{m}-2}\partial_{\tilde{m}-1}^X\big)\partial_{\tilde{m}}^X \\
&= 0.
\end{aligned}$$

By exactness, $\alpha_{\tilde{m}} - \gamma_{\tilde{m}-1}\partial_{\tilde{m}}^X$ is also in the image of $\mathrm{Hom}_R(X_{\tilde{m}}, \partial_{\tilde{m}+1}^{\Sigma^{-n}Y})$, which means that there exists $\gamma_{\tilde{m}} : X_{\tilde{m}} \to Y_{n+\tilde{m}+1}$ such that $\partial_{\tilde{m}+1}^{\Sigma^{-n}Y}\gamma_{\tilde{m}} = \alpha_{\tilde{m}} - \gamma_{\tilde{m}-1}\partial_{\tilde{m}}^X$. $\quad\square$

2.5. Lemma. *Assume that $X, Y \in \mathsf{C}(R)$ with either $X \in \mathsf{C}_\sqsubset(R)$ or $Y \in \mathsf{C}_\sqsubset(R)$. If $\mathrm{H}(\mathrm{Hom}_R(X, Y_\ell)) = 0$ for all $\ell \in \mathbb{Z}$, then $\mathrm{H}(\mathrm{Hom}_R(X, Y)) = 0$.*

2.6. Proposition. *Consider a class $\mathfrak{U}$ of R-modules, and let $\alpha : X \to Y$ be a morphism in $\mathsf{C}(R)$, such that*

$$\mathrm{Hom}_R(U, \alpha) : \mathrm{Hom}_R(U, X) \xrightarrow{\simeq} \mathrm{Hom}_R(U, Y)$$

is a quasi-isomorphism for every module $U \in \mathfrak{U}$. Let $\tilde{U} \in \mathsf{C}(R)$ be a complex consisting of modules from $\mathfrak{U}$. The induced morphism,

$$\mathrm{Hom}_R(\tilde{U}, \alpha) : \mathrm{Hom}_R(\tilde{U}, X) \to \mathrm{Hom}_R(\tilde{U}, Y),$$

is then a quasi-isomorphism, provided that either

(a) $\tilde{U} \in \mathsf{C}_\sqsupset(R)$*, or*
(b) $X, Y \in \mathsf{C}_\sqsupset(R)$*.*

Proof. Under either hypothesis, (a) or (b), we must verify exactness of

$$\mathrm{Cone}\big(\mathrm{Hom}_R(\tilde{U}, \alpha)\big) \cong \mathrm{Hom}_R\big(\tilde{U}, \mathrm{Cone}(\alpha)\big).$$

Condition (b) implies that $\mathrm{Cone}(\alpha) \in \mathsf{C}_\sqsupset(R)$. In any event, Lemma 2.4 informs us that it suffices to show that the complex $\mathrm{Hom}_R(\tilde{U}_\ell, \mathrm{Cone}(\alpha))$ is exact for all $\ell \in \mathbb{Z}$, and this follows as all

$$\mathrm{Hom}_R(\tilde{U}_\ell, \alpha) : \mathrm{Hom}_R(\tilde{U}_\ell, X) \xrightarrow{\simeq} \mathrm{Hom}_R(\tilde{U}_\ell, Y)$$

are assumed to be quasi-isomorphisms in $\mathsf{C}(R)$. $\quad\square$

2.7. Proposition. *Consider a class $\mathfrak{V}$ of R-modules, and let $\alpha : X \to Y$ be a morphism in* $\mathsf{C}(R)$*, such that*

$$\mathrm{Hom}_R(\alpha, V) : \mathrm{Hom}_R(Y, V) \xrightarrow{\simeq} \mathrm{Hom}_R(X, V)$$

is a quasi-isomorphism for every module $V \in \mathfrak{V}$. Let $\tilde{V} \in \mathsf{C}(R)$ be a complex consisting of modules from $\mathfrak{V}$. The induced morphism,

$$\mathrm{Hom}_R(\alpha, \tilde{V}) : \mathrm{Hom}_R(Y, \tilde{V}) \to \mathrm{Hom}_R(X, \tilde{V}),$$

is then a quasi-isomorphism, provided that either

(a) $\tilde{V} \in \mathsf{C}_\sqsubset(R)$*, or*
(b) $X, Y \in \mathsf{C}_\sqsubset(R)$*.*

2.8. Theorem. *Let $V \xrightarrow{\simeq} W$ be a quasi-isomorphism between R-complexes, where each module in V and W has finite projective dimension or finite injective dimension. If $A \in$* $\mathsf{C}_\sqsupset(R)$ *is a complex of Gorenstein projective modules, then the induced morphism*

$$\mathrm{Hom}_R(A, V) \to \mathrm{Hom}_R(A, W)$$

is a quasi-isomorphism under each of the next two conditions:

(a) $V, W \in \mathsf{C}_\sqsubset(R)$*, or*
(b) $V, W \in \mathsf{C}_\sqsupset(R)$*.*

Proof. By Proposition 2.6(a) we may immediately reduce to the case, where A is a Gorenstein projective module. In this case we have quasi-isomorphisms $\mu : P \xrightarrow{\simeq} A$ and $\nu : A \xrightarrow{\simeq} \tilde{P}$ in $\mathsf{C}(R)$, where $P \in \mathsf{C}_\sqsupset(R)$ and $\tilde{P} \in \mathsf{C}_\sqsubset(R)$ are, respectively, the "left half" and "right half" of a complete projective resolution of A.

Let T be any R-module of finite projective or finite injective dimension. Lemma 2.1 implies that a complete projective resolution stays exact when the functor $\mathrm{Hom}_R(-, T)$ is applied to it. In particular, the induced morphisms

$$\mathrm{Hom}_R(\mu, T) : \mathrm{Hom}_R(A, T) \xrightarrow{\simeq} \mathrm{Hom}_R(P, T), \qquad (*)$$

and

$$\mathrm{Hom}_R(\nu, T) : \mathrm{Hom}_R(\tilde{P}, T) \xrightarrow{\simeq} \mathrm{Hom}_R(A, T) \qquad (\sharp)$$

are quasi-isomorphisms. From $(*)$ and Proposition 2.7(a) it follows that under assumption (a) both $\mathrm{Hom}_R(\mu, V)$ and $\mathrm{Hom}_R(\mu, W)$ are quasi-isomorphisms. In the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_R(A, V) & \longrightarrow & \mathrm{Hom}_R(A, W) \\
\mathrm{Hom}(\mu,V) \downarrow \simeq & & \simeq \downarrow \mathrm{Hom}(\mu,W) \\
\mathrm{Hom}_R(P, V) & \xrightarrow{\simeq} & \mathrm{Hom}_R(P, W)
\end{array}
$$

the lower horizontal morphism is obviously a quasi-isomorphism, and this makes the induced morphism $\mathrm{Hom}_R(A, V) \to \mathrm{Hom}_R(A, W)$ a quasi-isomorphism as well.

Under assumption (b), the induced morphism $\mathrm{Hom}_R(\tilde{P}, V) \to \mathrm{Hom}_R(\tilde{P}, W)$ is a quasi-isomorphism by Proposition 2.6(b). As the induced morphisms $(\sharp)$ are quasi-isomorphisms, it follows by Proposition 2.7(b) that so are $\mathrm{Hom}_R(\nu, V)$ and $\mathrm{Hom}_R(\nu, W)$. From the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_R(A, V) & \longrightarrow & \mathrm{Hom}_R(A, W) \\
\mathrm{Hom}(\nu,V) \uparrow \simeq & & \simeq \uparrow \mathrm{Hom}(\nu,W) \\
\mathrm{Hom}_R(\tilde{P}, V) & \xrightarrow{\simeq} & \mathrm{Hom}_R(\tilde{P}, W)
\end{array}
$$

we conclude that also its top vertical morphism is a quasi-isomorphism. $\quad\square$

2.9. Theorem. *Let $V \xrightarrow{\simeq} W$ be a quasi-isomorphism between R-complexes, where each module in V and W has finite projective dimension or finite injective dimension. If $B \in \mathsf{C}_\sqsubset(R)$ is a complex of Gorenstein injective modules, then the induced morphism*

$$\mathrm{Hom}_R(W, B) \to \mathrm{Hom}_R(V, B)$$

is a quasi-isomorphism under each of the next two conditions:

(a) $V, W \in \mathsf{C}_{\sqsupset}(R)$, *or*
(b) $V, W \in \mathsf{C}_{\sqsubset}(R)$.

2.10. Corollary. *Assume that $X \simeq A$, where $A \in \mathsf{C}_{\sqsupset}(R)$ is a complex of Gorenstein projective modules. If $U \simeq V$, where $V \in \mathsf{C}_{\sqsubset}(R)$ is a complex in which each module has finite projective dimension or finite injective dimension, then*

$$\mathbf{R}\mathrm{Hom}_R(X, U) \simeq \mathrm{Hom}_R(A, V).$$

Proof. We represent $\mathbf{R}\mathrm{Hom}_R(X, U) \simeq \mathbf{R}\mathrm{Hom}_R(A, V)$ by the complex $\mathrm{Hom}_R(A, I)$, where $V \xrightarrow{\simeq} I \in \mathsf{C}_{\sqsubset}(R)$ is an injective resolution of V. From 2.8(a) we get a quasi-isomorphism $\mathrm{Hom}_R(A, V) \xrightarrow{\simeq} \mathrm{Hom}_R(A, I)$, and the result follows. $\square$

2.11. Remark. There is a variant of Theorem 2.8 and Corollary 2.10. If R is commutative and noetherian, and A is a complex of *finitely generated* Gorenstein projective R-modules, then we may relax the requirements on the modules in V and W without changing the conclusions of 2.8 and 2.10: It is sufficient that each module in V and W has finite *flat* or finite injective dimension. This follows immediately from the proofs of 2.8 and 2.10, when one takes [8, Proposition 4.1.3] into account.

2.12. Corollary. *Assume that $Y \simeq B$, where $B \in \mathsf{C}_{\sqsubset}(R)$ is a complex of Gorenstein injective modules. If $U \simeq V$, where $V \in \mathsf{C}_{\sqsupset}(R)$ is a complex in which each module has finite projective dimension or finite injective dimension, then*

$$\mathbf{R}\mathrm{Hom}_R(U, Y) \simeq \mathrm{Hom}_R(V, B). \qquad \square$$

Next, we turn to tensor products and Gorenstein flat modules. The first lemma follows by applying Pontryagin duality to Lemma 2.4 for R^{opp}.

2.13. Lemma. *Assume that $X \in \mathsf{C}(R^{\mathrm{opp}})$ and $Y \in \mathsf{C}(R)$ with either $X \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$ or $Y \in \mathsf{C}_{\sqsubset}(R)$. If $\mathrm{H}(X_\ell \otimes_R Y) = 0$ for all $\ell \in \mathbb{Z}$, then $\mathrm{H}(X \otimes_R Y) = 0$.*

2.14. Proposition. *Consider a class $\mathfrak{W}$ of R^{opp}-modules, and let $\alpha: X \to Y$ be a morphism in $\mathsf{C}(R)$, such that*

$$W \otimes_R \alpha: W \otimes_R X \xrightarrow{\simeq} W \otimes_R Y$$

is a quasi-isomorphism for every module $W \in \mathfrak{W}$. Let $\tilde{W} \in \mathsf{C}(R^{\mathrm{opp}})$ be a complex consisting of modules from $\mathfrak{W}$. The induced morphism,

$$\tilde{W} \otimes_R \alpha: \tilde{W} \otimes_R X \to \tilde{W} \otimes_R Y,$$

is then a quasi-isomorphism, provided that either

(a) $\tilde{W} \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$, *or*
(b) $X, Y \in \mathsf{C}_{\sqsubset}(R)$.

Proof. It follows by Lemma 2.13 that $\mathsf{Cone}\,(\tilde{W} \otimes_R \alpha) \simeq \tilde{W} \otimes_R \mathsf{Cone}\,(\alpha)$ is exact under either assumption, (a) or (b). $\square$

2.15. Theorem. *Let* $V \xrightarrow{\simeq} W$ *be a quasi-isomorphism between complexes of* R^{opp}*-modules, where each module in* V *and* W *has finite injective dimension or finite flat dimension. If* $A \in \mathsf{C}_{\sqsupset}(R)$ *is a complex of Gorenstein flat modules, then the induced morphism*

$$V \otimes_R A \to W \otimes_R A$$

is a quasi-isomorphism under each of the next two conditions:

(a) $V, W \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$, *or*
(b) $V, W \in \mathsf{C}_{\sqsubset}(R^{\mathrm{opp}})$.

Proof. Using Proposition 2.14(a), applied to R^{opp}, we immediately reduce to the case, where A is a Gorenstein flat module. In this case we have quasi-isomorphisms $\mu : F \xrightarrow{\simeq} A$ and $\nu : A \xrightarrow{\simeq} \tilde{F}$ in $\mathsf{C}(R)$, where $F \in \mathsf{C}_{\sqsupset}(R)$ and $\tilde{F} \in \mathsf{C}_{\sqsubset}(R)$ are complexes of flat modules. To be precise, F and $\tilde{F}$ are, respectively, the "left half" and "right half" of a complete flat resolution of A. The proof now continues as the proof of Theorem 2.8; only using Proposition 2.14 instead of 2.6 and 2.7, and Lemma 2.3 instead of 2.1. $\square$

2.16. Corollary. *Assume that* $X \simeq A$, *where* $A \in \mathsf{C}_{\sqsupset}(R)$ *is a complex of Gorenstein flat modules. If* $U \simeq V$, *where* $V \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$ *is a complex in which each module has finite flat dimension or finite injective dimension, then*

$$U \otimes_R^{\mathbf{L}} X \simeq V \otimes_R A.$$

Proof. We represent $U \otimes_R^{\mathbf{L}} X \simeq V \otimes_R^{\mathbf{L}} A$ by the complex $P \otimes_R A$, where $\mathsf{C}_{\sqsupset}(R^{\mathrm{opp}}) \ni P \xrightarrow{\simeq} V$ is a projective resolution of V. By Theorem 2.15(a) we get a quasi-isomorphism $P \otimes_R A \xrightarrow{\simeq} V \otimes_R A$, and the desired result follows. $\square$

The Gorenstein dimensions refine the classical homological dimensions. On the other hand, the next three lemmas show that a module of finite Gorenstein projective/injective/flat dimension can be approximated by a module, for which the corresponding classical homological dimension is finite.

2.17. Lemma. *Let* M *be an* R*-module of finite Gorenstein projective dimension. There is then an exact sequence of* R*-modules,*

$$0 \to M \to H \to A \to 0,$$

where A *is Gorenstein projective and* $\mathrm{pd}_R H = \mathrm{Gpd}_R M$.

Proof. If M is Gorenstein projective, we take $0 \to M \to H \to A \to 0$ to be the first short exact sequence in the "right half" of a complete projective resolution of M.

We may now assume that $\mathrm{Gpd}_R\, M = n > 0$. By [30, Theorem 2.10] there exists an exact sequence,

$$0 \to K \to A' \to M \to 0,$$

where A' is Gorenstein projective, and $\mathrm{pd}_R\, K = n - 1$. Since A' is Gorenstein projective, there exists (as above) a short exact sequence,

$$0 \to A' \to Q \to A \to 0,$$

where Q is projective, and A is Gorenstein projective. Consider the push-out:

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
A & \!\!=\!\!=\!\! & A \\
\uparrow & & \uparrow \\
0 \to K \to Q \to & H & \to 0 \\
\| & \uparrow & \uparrow \\
0 \to K \to A' \to & M & \to 0 \\
& \uparrow & \uparrow \\
& 0 & 0
\end{array}
$$

The second column of this diagram is the desired sequence. To see this we must argue that $\mathrm{pd}_R\, H = n$: The class of Gorenstein projective modules is projectively resolving by [30, Theorem 2.5], so if H were projective, exactness of the second column would imply that $\mathrm{Gpd}_R\, M = 0$, which is a contradiction. Consequently $\mathrm{pd}_R\, H > 0$. Applying, e.g., [46, Example 4.1.2(1)] to the first row above it follows that $\mathrm{pd}_R\, H = \mathrm{pd}_R\, K + 1 = n$. □

The next two lemmas have proofs similar to that of 2.17.

2.18. Lemma. *Let N be an R-module of finite Gorenstein injective dimension. There is then an exact sequence of R-modules,*

$$0 \to B \to H \to N \to 0,$$

where B is Gorenstein injective and $\mathrm{id}_R\, H = \mathrm{Gid}_R\, N$.

2.19. Lemma. *Assume that R is right coherent, and let M be an R-module of finite Gorenstein flat dimension. There is then an exact sequence of R-modules,*

$$0 \to M \to H \to A \to 0,$$

where A is Gorenstein flat and $\mathrm{fd}_R\, H = \mathrm{Gfd}_R\, M$.

3. Measuring Gorenstein dimensions

Gorenstein dimensions are defined in terms of resolutions, and when a finite resolution is known to exist, the minimal length of such can be determined by vanishing of certain derived functors. We collect these descriptions in three theorems, which mimic the style of Cartan and Eilenberg. Such results have previously in [8,13,14,16,30] been established in more restrictive settings, and the purpose of this section is to present them in the more general setting of complexes over associative rings.

We start by investigating the Gorenstein projective dimension.

3.1. Theorem. *Let $X \in \mathsf{D}_{\sqsupset}(R)$ be a complex of finite Gorenstein projective dimension. For $n \in \mathbb{Z}$ the following are equivalent*:

  (i) $\mathrm{Gpd}_R X \leqslant n$.
 (ii) $n \geqslant \inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U)$ *for all* $U \in \mathsf{D}_{\square}(R)$ *of finite projective or finite injective dimension with* $\mathrm{H}(U) \neq 0$.
(iii) $n \geqslant -\inf \mathbf{R}\mathrm{Hom}_R(X, Q)$ *for all projective R-modules Q.*
(iv) $n \geqslant \sup X$ *and, for any right-bounded complex $A \simeq X$ of Gorenstein projective modules, the cokernel $\mathrm{C}_n^A = \mathrm{Coker}(A_{n+1} \to A_n)$ is a Gorenstein projective module.*

Moreover, the following hold:

$$
\mathrm{Gpd}_R X = \sup\{\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U) \mid \mathrm{pd}_R U < \infty \text{ and } \mathrm{H}(U) \neq 0\}
$$
$$
= \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(X, Q) \mid Q \text{ is projective}\}
$$
$$
\leqslant \mathrm{FPD}(R) + \sup X.
$$

Proof. The proof of the equivalence of (i)–(iv) is cyclic. Clearly, (ii) is stronger than (iii), and this leaves us three implications to prove.

(i) $\Rightarrow$ (ii): Choose a complex $A \in \mathsf{C}_{\square}(R)$ consisting of Gorenstein projective modules, such that $A \simeq X$ and $A_\ell = 0$ for $\ell > n$. First, let U be a complex of finite projective dimension with $\mathrm{H}(U) \neq 0$. Set $i = \inf U$ and note that $i \in \mathbb{Z}$ as $U \in \mathsf{D}_{\square}(R)$ with $\mathrm{H}(U) \neq 0$. Choose a bounded complex $P \simeq U$ of projective modules with $P_\ell = 0$ for $\ell < i$. By Corollary 2.10 the complex $\mathrm{Hom}_R(A, P)$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(X, U)$ in $\mathsf{D}(\mathbb{Z})$; in particular, $\inf \mathbf{R}\mathrm{Hom}_R(X, U) = \inf \mathrm{Hom}_R(A, P)$. For $\ell < i - n$ and $q \in \mathbb{Z}$, either $q > n$ or $q + \ell \leqslant n + \ell < i$, so the module

$$
\mathrm{Hom}_R(A, P)_\ell = \prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(A_q, P_{q+\ell})
$$

vanishes. Hence, $\mathrm{H}_\ell(\mathrm{Hom}_R(A, P)) = 0$ for $\ell < i - n$, and $\inf \mathbf{R}\mathrm{Hom}_R(X, U) \geqslant i - n = \inf U - n$ as desired.

Next, let U be a complex of finite injective dimension and choose a bounded complex $I \simeq U$ of injective modules. Set $i = \inf U$ and consider the soft truncation $V = I_i \supset$.

The modules in V have finite injective dimension and $U \simeq V$, whence $\operatorname{Hom}_R(A, V) \simeq$ $\mathbf{R}\operatorname{Hom}_R(X, U)$ by Corollary 2.10, and the proof continues as above.

(iii) $\Rightarrow$ (iv): This part evolves in three steps. First we establish the inequality $n \geqslant \sup X$, next we prove that the nth cokernel in a bounded complex $A \simeq X$ of Gorenstein projectives is again Gorenstein projective, and finally we give an argument that allows us to conclude the same for $A \in \mathsf{C}_{\sqsupset}(R)$.

To see that $n \geqslant \sup X$, it is sufficient to show that

$$\sup\left\{-\inf \mathbf{R}\operatorname{Hom}_R(X, Q) \mid Q \text{ is projective}\right\} \geqslant \sup X. \tag{$*$}$$

By assumption, $g = \operatorname{Gpd}_R X$ is finite; i.e., $X \simeq A$ for some complex

$$A = 0 \to A_g \xrightarrow{\partial_g^A} A_{g-1} \to \cdots \to A_i \to 0,$$

and it is clear $g \geqslant \sup X$ since $X \simeq A$. For any projective module Q, the complex $\operatorname{Hom}_R(A, Q)$ is concentrated in degrees $-i$ to $-g$,

$$0 \to \operatorname{Hom}_R(A_i, Q) \to \cdots \to \operatorname{Hom}_R(A_{g-1}, Q) \xrightarrow{\operatorname{Hom}_R(\partial_g^A, Q)} \operatorname{Hom}_R(A_g, Q) \to 0,$$

and isomorphic to $\mathbf{R}\operatorname{Hom}_R(X, Q)$ in $\mathsf{D}(\mathbb{Z})$, cf. Corollary 2.10. First, consider the case $g = \sup X$: The differential $\partial_g^A : A_g \to A_{g-1}$ is not injective, as A has homology in degree $g = \sup X = \sup A$. By the definition of Gorenstein projective modules, there exists a projective module Q and an injective homomorphism $\varphi : A_g \to Q$. Because ∂_g^A is not injective, $\varphi \in \operatorname{Hom}_R(A_g, Q)$ cannot have the form $\operatorname{Hom}_R(\partial_g^A, Q)(\psi) = \psi \partial_g^A$ for some $\psi \in \operatorname{Hom}_R(A_{g-1}, Q)$. That is, the differential $\operatorname{Hom}_R(\partial_g^A, Q)$ is not surjective; hence $\operatorname{Hom}_R(A, Q)$ has nonzero homology in degree $-g = -\sup X$, and $(*)$ follows.

Next, assume that $g > \sup X = s$ and consider the exact sequence

$$0 \to A_g \to \cdots \to A_{s+1} \to A_s \to \mathsf{C}_s^A \to 0. \tag{$\star$}$$

It shows that $\operatorname{Gpd}_R \mathsf{C}_s^A \leqslant g - s$, and it is easy to check that equality must hold; otherwise, we would have $\operatorname{Gpd}_R X < g$. A straightforward computation based on Corollary 2.10, cf. [8, Lemma 4.3.9], shows that for all $m > 0$, all $n \geqslant \sup X$, and all projective modules Q one has

$$\operatorname{Ext}_R^m\left(\mathsf{C}_n^A, Q\right) = \operatorname{H}_{-(m+n)}\left(\mathbf{R}\operatorname{Hom}_R(X, Q)\right). \tag{$\sharp$}$$

By [30, Theorem 2.20] we have $\operatorname{Ext}_R^{g-s}(\mathsf{C}_s^A, Q) \neq 0$ for some projective Q, whence $\operatorname{H}_{-g}(\mathbf{R}\operatorname{Hom}_R(X, Q)) \neq 0$ by $(\sharp)$, and $(*)$ follows. We conclude that $n \geqslant \sup X$.

It remains to prove that C_n^A is Gorenstein projective for any right-bounded complex $A \simeq X$ of Gorenstein projective modules. By assumption, $\operatorname{Gpd}_R X$ is finite, so a bounded complex $\tilde{A} \simeq X$ of Gorenstein projective modules does exist. Consider the cokernel $\mathsf{C}_n^{\tilde{A}}$. Since $n \geqslant \sup X = \sup \tilde{A}$, it fits in an exact sequence $0 \to \tilde{A}_t \to \cdots \to \tilde{A}_{n+1} \to \tilde{A}_n \to$

$C_n^{\tilde{A}} \to 0$, where all the $\tilde{A}_\ell$'s are Gorenstein projective. By ($\sharp$) and [30, Theorem 2.20] it now follows that also $C_n^{\tilde{A}}$ is Gorenstein projective. With this, it is sufficient to prove the following:

> If $P, A \in C_\sqsubset(R)$ are complexes of, respectively, projective and Gorenstein projective modules, and $P \simeq X \simeq A$, then the cokernel C_n^P is Gorenstein projective if and only if C_n^A is so.

Let A and P be two such complexes. As P consists of projectives, there is a quasi-isomorphism $\pi : P \xrightarrow{\simeq} A$, cf. [4, 1.4.P], which induces a quasi-isomorphism between the truncated complexes, $\subset_n \pi : \subset_n P \xrightarrow{\simeq} \subset_n A$. The mapping cone

$$\mathsf{Cone}\,(\subset_n \pi) = 0 \to C_n^P \to P_{n-1} \oplus C_n^A \to P_{n-2} \oplus A_{n-1} \to \cdots$$

is a bounded exact complex, in which all modules but the two left-most ones are known to be Gorenstein projective modules. It follows by the resolving properties of Gorenstein projective modules, cf. [30, Theorem 2.5], that C_n^P is Gorenstein projective if and only if $P_{n-1} \oplus C_n^A$ is so, which is tantamount to C_n^A being Gorenstein projective.

(iv) $\Rightarrow$ (i): Choose a projective resolution P of X; by (iv) the truncation $\subset_n P$ is a complex of the desired type.

To show the last claim, we still assume that $\mathrm{Gpd}_R X$ is finite. The two equalities are immediate consequences of the equivalence of (i), (ii), and (iii). Moreover, it is easy to see how a complex $A \in C_\square(R)$ of Gorenstein projective modules, which is isomorphic to X in $D(R)$, may be truncated to form a Gorenstein projective resolution of the top homology module of X, cf. ($\star$) above. Thus, by the definition we automatically obtain the inequality $\mathrm{Gpd}_R X \leqslant \mathrm{FGPD}(R) + \sup X$, where

$$\mathrm{FGPD}(R) = \sup \left\{ \mathrm{Gpd}_R\, M \,\middle|\, \begin{array}{l} M \text{ is an } R\text{-module with finite} \\ \text{Gorenstein projective dimension} \end{array} \right\}$$

is the (left) finitistic Gorenstein projective dimension, cf. 1.4. Finally, we have $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$ by [30, Theorem 2.28]. $\quad\square$

3.2. Corollary. *Assume that R is left coherent, and let $X \in D_\sqsubset(R)$ be a complex with finitely presented homology modules. If X has finite Gorenstein projective dimension, then*

$$\mathrm{Gpd}_R X = -\inf \mathbf{R}\mathrm{Hom}_R(X, R).$$

Proof. Under the assumptions, X admits a resolution by finitely generated projective modules, say P; and thus, $\mathrm{Hom}_R(P, -)$ commutes with arbitrary sums. The proof is now a straightforward computation. $\quad\square$

Next, we turn to the Gorenstein injective dimension. The proof of Theorem 3.3 below relies on Corollary 2.12 instead of 2.10 but is otherwise similar to the proof of Theorem 3.1; hence it is omitted.

3.3. Theorem. *Let $Y \in \mathsf{D}_{\sqsubset}(R)$ be a complex of finite Gorenstein injective dimension. For $n \in \mathbb{Z}$ the following are equivalent*:

(i) $\operatorname{Gid}_R Y \leqslant n$.
(ii) $n \geqslant -\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U, Y)$ *for all $U \in \mathsf{D}_{\square}(R)$ of finite injective or finite projective dimension with $\mathrm{H}(U) \neq 0$.*
(iii) $n \geqslant -\inf \mathbf{R}\mathrm{Hom}_R(J, Y)$ *for all injective R-modules J.*
(iv) $n \geqslant -\inf Y$ *and, for any left-bounded complex $B \simeq Y$ of Gorenstein injective modules, the kernel $\mathrm{Z}_{-n}^B = \mathrm{Ker}(B_{-n} \to B_{-(n+1)})$ is a Gorenstein injective module.*

Moreover, the following hold:

$$\operatorname{Gid}_R Y = \sup\{-\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U, Y) \mid \mathrm{id}_R U < \infty \text{ and } \mathrm{H}(U) \neq 0\}$$
$$= \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(J, Y) \mid J \text{ is injective}\}$$
$$\leqslant \mathrm{FID}(R) - \inf Y.$$

The next result is a straightforward application of Matlis' structure theorem for injective modules to the equality in 3.3.

3.4. Corollary. *Assume that R is commutative and noetherian. If $Y \in \mathsf{D}_{\sqsubset}(R)$ is a complex of finite Gorenstein injective dimension, then*

$$\operatorname{Gid}_R Y = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), Y) \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Finally, we treat the Gorenstein flat dimension.

3.5. Theorem. *Assume that R is right coherent, and let $X \in \mathsf{D}_{\sqsupset}(R)$ be a complex of finite Gorenstein flat dimension. For $n \in \mathbb{Z}$ the following are equivalent*:

(i) $\operatorname{Gfd}_R X \leqslant n$.
(ii) $n \geqslant \sup(U \otimes_R^{\mathbf{L}} X) - \sup U$ *for all $U \in \mathsf{D}_{\square}(R^{\mathrm{opp}})$ of finite injective or finite flat dimension with $\mathrm{H}(U) \neq 0$.*
(iii) $n \geqslant \sup(J \otimes_R^{\mathbf{L}} X)$ *for all injective R^{opp}-modules J.*
(iv) $n \geqslant \sup X$ *and, for any right-bounded complex $A \simeq X$ of Gorenstein flat modules, the cokernel $\mathrm{C}_n^A = \mathrm{Coker}(A_{n+1} \to A_n)$ is Gorenstein flat.*

Moreover, the following hold:

$$\operatorname{Gfd}_R X = \sup\{\sup(U \otimes_R^{\mathbf{L}} X) - \sup U \mid \mathrm{id}_{R^{\mathrm{opp}}} U < \infty \text{ and } \mathrm{H}(U) \neq 0\}$$
$$= \sup\{\sup(J \otimes_R^{\mathbf{L}} X) \mid J \text{ is injective}\}$$
$$\leqslant \mathrm{FFD}(R) + \sup X.$$

Proof. The proof of the equivalence of (i)–(iv) is cyclic. The implication (ii) $\Rightarrow$ (iii) is immediate, and this leaves us three implications to prove.

(i) $\Rightarrow$ (ii): Choose a complex $A \in \mathsf{C}_\square(R)$ of Gorenstein flat modules, such that $A \simeq X$ and $A_\ell = 0$ for $\ell > n$. First, let $U \in \mathsf{D}(R^{\mathrm{opp}})$ be a complex of finite injective dimension with $\mathrm{H}(U) \neq 0$. Set $s = \sup U$ and pick a bounded complex $I \simeq U$ of injective modules with $I_\ell = 0$ for $\ell > s$. By Corollary 2.16 the complex $I \otimes_R A$ is isomorphic to $U \otimes_R^{\mathbf{L}} X$ in $\mathsf{D}(\mathbb{Z})$; in particular, $\sup(U \otimes_R^{\mathbf{L}} X) = \sup(I \otimes_R A)$. For $\ell > n + s$ and $q \in \mathbb{Z}$ either $q > s$ or $\ell - q \geqslant \ell - s > n$, so the module

$$(I \otimes_R A)_\ell = \coprod_{q \in \mathbb{Z}} I_q \otimes_R A_{\ell - q}$$

vanishes. Hence, $\mathrm{H}_\ell(I \otimes_R A) = 0$ for $\ell > n + s$, forcing $\sup(U \otimes_R^{\mathbf{L}} X) \leqslant n + s = n + \sup U$ as desired.

Next, let $U \in \mathsf{D}(R^{\mathrm{opp}})$ be a complex of finite flat dimension and choose a bounded complex $F \simeq U$ of flat modules. Set $s = \sup U$ and consider the soft truncation $V = \subset_s F$. The modules in V have finite flat dimension and $U \simeq V$, hence $V \otimes_R A \simeq U \otimes_R^{\mathbf{L}} X$ by Corollary 2.16, and the proof continues as above.

(iii) $\Rightarrow$ (iv): By assumption, $\mathrm{Gfd}_R X$ is finite, so a bounded complex $A \simeq X$ of Gorenstein flat modules does exist. For any injective R^{opp}-module J, we have $J \otimes_R^{\mathbf{L}} X \simeq J \otimes_R A$ in $\mathsf{D}(\mathbb{Z})$ by Corollary 2.16, so

$$\begin{aligned}
\sup(J \otimes_R^{\mathbf{L}} X) &= \sup(J \otimes_R A) \\
&= -\inf \mathrm{Hom}_{\mathbb{Z}}(J \otimes_R A, \mathbb{Q}/\mathbb{Z}) \\
&= -\inf \mathrm{Hom}_{R^{\mathrm{opp}}}\big(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})\big) \\
&= -\inf \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}\big(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})\big),
\end{aligned}$$

where the last equality follows from Corollary 2.12, as $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is a complex of Gorenstein injective modules by [30, Theorem 3.6]. As desired, we now have:

$$\begin{aligned}
n &\geqslant \sup\big\{\sup(J \otimes_R^{\mathbf{L}} X) \mid J \text{ is injective}\big\} \\
&= \sup\big\{-\inf \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}\big(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})\big) \mid J \text{ is injective}\big\} \\
&\geqslant -\inf \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \\
&= \sup A = \sup X,
\end{aligned}$$

where the inequality follows from 3.3 (applied to R^{opp}). The rest of the argument is similar to the one given in the proof of Theorem 3.1. It uses that the class of Gorenstein flat modules is resolving, and here we need the assumption that R is right-coherent, cf. [30, Theorem 3.7].

(iv) $\Rightarrow$ (i): Choose a projective resolution P of X; by (iv) the truncation $\subset_n P$ is a complex of the desired type.

For the second part, we can argue, as we did in the proof of Theorem 3.1, to see that $\mathrm{Gfd}_R X \leqslant \mathrm{FGFD}(R) + \sup X$, where

$$\mathrm{FGFD}(R) = \sup \left\{ \mathrm{Gfd}_R M \,\middle|\, \begin{array}{l} M \text{ is an } R\text{-module with finite} \\ \text{Gorenstein flat dimension} \end{array} \right\}$$

is the (left) finitistic Gorenstein flat dimension, cf. 1.4. By [30, Theorem 3.24] we have $\mathrm{FGFD}(R) = \mathrm{FFD}(R)$, and this concludes the proof. $\square$

The next corollary is immediate by Matlis' structure theorem for injective modules.

3.6. Corollary. *Assume that R is commutative and noetherian. If $X \in \mathsf{D}_{\sqsupset}(R)$ is a complex of finite Gorenstein flat dimension, then*

$$\mathrm{Gfd}_R X = \sup\{\sup(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R^{\mathbf{L}} X) \mid \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

The next two results deal with relations between the Gorenstein projective and flat dimensions. Both are Gorenstein versions of well-established properties of the classical homological dimensions.

3.7. Proposition. *Assume that R is right coherent and any flat R-module has finite projective dimension. For every $X \in \mathsf{D}_{\sqsupset}(R)$ the next inequality holds*

$$\mathrm{Gfd}_R X \leqslant \mathrm{Gpd}_R X.$$

Proof. Under the assumptions, it follows by [30, proof of Proposition 3.4] that every Gorenstein projective R-module also is Gorenstein flat. $\square$

We now compare the Gorenstein projective and Gorenstein flat dimension to Auslander and Bridger's G-dimension. In [3] Auslander and Bridger introduce the G-dimension, $\mathrm{G}\text{-}\dim_R(-)$, for finitely generated modules over a left and right noetherian ring.

The G-dimension is defined via resolutions consisting of modules from the so-called G-class, $\mathrm{G}(R)$. The G-class consists exactly of the finite R-modules M with $\mathrm{G}\text{-}\dim_R M = 0$ (together with the zero-module). The basic properties are catalogued in [3, Proposition 3.8(c)].

When R is commutative and noetherian, [8, Section 2.3] introduces a G-dimension, also denoted $\mathrm{G}\text{-}\dim_R(-)$, for complexes in $\mathsf{D}_{\sqsupset}^{\mathrm{f}}(R)$. For modules it agrees with Auslander and Bridger's G-dimension. However, the definition given in [8, Section 2.3] makes perfect sense over any two-sided noetherian ring.

3.8. Proposition. *Assume that R is left and right coherent. For a complex $X \in \mathsf{D}_{\sqsupset}(R)$ with finitely presented homology modules, the following hold*:

(a) *If every flat R-module has finite projective dimension, then*

$$\mathrm{Gpd}_R X = \mathrm{Gfd}_R X.$$

(b) *If R is left and right noetherian, then*

$$\mathrm{Gpd}_R X = \text{G-dim}_R X.$$

Proof. Since R is right coherent and flat R-modules have finite projective dimension, Proposition 3.7 implies that $\mathrm{Gfd}_R X \leqslant \mathrm{Gpd}_R X$. To prove the opposite inequality in (a), we may assume that $n = \mathrm{Gfd}_R X$ is finite. Since R is left coherent and the homology modules of X are finitely presented, we can pick a projective resolution P of X, where each P_ℓ is finitely generated. The cokernel C_n^P is finitely presented, and by Theorem 3.5 it is Gorenstein flat.

Following the proof of [8, Theorem 5.1.11], which deals with commutative, noetherian rings and is propelled by Lazard's [37, Lemma 1.1], it is easy, but tedious, to check that over a left coherent ring, any finitely presented Gorenstein flat module is also Gorenstein projective. Therefore, C_n^P is actually Gorenstein projective, which shows that $\mathrm{Gpd}_R X \leqslant n$ as desired.

Next, we turn to (b). By the "if" part of [8, Theorem 4.2.6], every module in the G-class is Gorenstein projective in the sense of Definition 1.7. (Actually, [8, Theorem 4.2.6] is formulated under the assumption that R is commutative and noetherian, but the proof carries over to two-sided noetherian rings as well.) It follows immediately that $\mathrm{Gpd}_R X \leqslant$ G-dim$_R X$.

For the opposite inequality, we may assume that $n = \mathrm{Gpd}_R X$ is finite. Let P be a projective resolution of X by finitely generated modules, and consider cokernel the C_n^P. Of course, C_n^P is finitely generated, and by Theorem 3.1 it is also Gorenstein projective. Now the "only if" part of (the already mentioned "associative version" of) [8, Theorem 4.2.6] gives that C_n^P belongs to the G-class. Hence, $\subset_n P$ is a resolution of X by modules from the G-class and, thus, G-dim$_R X \leqslant n$. $\square$

3.9. Remark. It is natural to ask if finiteness of Gorenstein dimensions is "closed under distinguished triangles." That is, in a distinguished triangle,

$$X \to Y \to Z \to \Sigma X,$$

where two of the three complexes X, Y and Z have finite, say, Gorenstein projective dimension, is then also the third complex of finite Gorenstein projective dimension?

Of course, once we have established the main theorems, 4.1 and 4.4, it follows that over a ring with a dualizing complex, finiteness of each of the three Gorenstein dimensions is closed under distinguished triangles. This conclusion is immediate, as the Auslander categories are triangulated subcategories of $\mathsf{D}(R)$. However, from the definitions and results of this section, it is not immediately clear that the Gorenstein dimensions possess this property in general. However, in [44] Veliche introduces a Gorenstein projective dimension for unbounded complexes. By [44, Theorem 3.2.8(1)], finiteness of this dimension is closed under distinguished triangles; by [44, Theorem 3.3.6] it coincides, for right-bounded complexes, with the Gorenstein projective dimension studied in this paper.

4. Auslander categories

In this section, we prove two theorems linking finiteness of Gorenstein homological dimensions to Auslander categories:

4.1. Theorem. *Let $\langle S, R \rangle$ be a noetherian pair with a dualizing complex $_S D_R$. For $X \in \mathsf{D}_{\sqsupset}(R)$ the following conditions are equivalent*:

(i) $X \in \mathsf{A}(R)$.
(ii) $\mathrm{Gpd}_R X$ *is finite.*
(iii) $\mathrm{Gfd}_R X$ *is finite.*

4.2. Corollary. *Let R be commutative noetherian with a dualizing complex, and let $X \in \mathsf{D}_{\sqsupset}(R)$. Then $\mathrm{Gfd}_R X$ is finite if and only if $\mathrm{Gpd}_R X$ is finite.*

4.3. Corollary. *Let R and S be commutative noetherian local rings and $\varphi : R \to S$ be a local homomorphism. If R has a dualizing complex, then $\mathrm{G\text{-}dim}\, \varphi$ is finite if and only if $\mathrm{Gfd}_R S$ is finite.*

Proof. As R admits a dualizing complex, then [5, Theorem 4.3] yields that $\mathrm{G\text{-}dim}\, \varphi$ is finite precisely when $S \in \mathsf{A}(R)$. It remains to invoke Theorem 4.1. $\quad\square$

4.4. Theorem. *Let $\langle S, R \rangle$ be a noetherian pair with a dualizing complex $_S D_R$. For $Y \in \mathsf{D}_{\sqsubset}(S)$ the following conditions are equivalent*:

(i) $Y \in \mathsf{B}(S)$.
(ii) $\mathrm{Gid}_S Y$ *is finite.*

At least in the case $R = S$, this connection between Auslander categories and Gorenstein dimensions has been conjectured/expected. One immediate consequence of the theorems above is that the full subcategory, of $\mathsf{D}(R)$, of complexes of finite Gorenstein projective/flat dimension is equivalent, cf. 1.3, to the full subcategory, of $\mathsf{D}(S)$, of complexes of finite Gorenstein injective dimension.

The main ingredients of the proofs of Theorems 4.1 and 4.4 are Lemmas 4.6 and 4.7, respectively. However, we begin with the following:

4.5. Lemma. *Let $\langle S, R \rangle$ be a noetherian pair with a dualizing complex $_S D_R$. For $X \in \mathsf{A}(R)$ and $Y \in \mathsf{B}(S)$ the following hold*:

(a) *For all R-modules M with finite $\mathrm{fd}_R M$ there is an inequality,*

$$-\inf \mathbf{R}\mathrm{Hom}_R(X, M) \leqslant \mathrm{id}_S(_S D_R) + \sup\left(_S D_R \otimes_R^{\mathbf{L}} X\right).$$

(b) *For all R^{opp}-modules M with finite $\mathrm{id}_{R^{\mathrm{opp}}} M$ there is an inequality,*

$$\sup\left(M \otimes_R^{\mathbf{L}} X\right) \leqslant \mathrm{id}_S(_S D_R) + \sup\left(_S D_R \otimes_R^{\mathbf{L}} X\right).$$

(c) *For all S-modules N with finite $\mathrm{id}_S N$ there is an inequality*

$$- \inf \mathbf{R}\mathrm{Hom}_S(N, Y) \leqslant \mathrm{id}_S({}_S D_R) - \inf \mathbf{R}\mathrm{Hom}_S({}_S D_R, Y).$$

Proof. (a) If $\mathrm{H}(X) = 0$ or $M = 0$ there is nothing to prove. Otherwise, we compute as follows:

$$\begin{aligned}
- \inf \mathbf{R}\mathrm{Hom}_R(X, M) &= - \inf \mathbf{R}\mathrm{Hom}_R\big(X, \mathbf{R}\mathrm{Hom}_S({}_S D_R, {}_S D_R \otimes_R^{\mathbf{L}} M)\big) \\
&= - \inf \mathbf{R}\mathrm{Hom}_S\big({}_S D_R \otimes_R^{\mathbf{L}} X, {}_S D_R \otimes_R^{\mathbf{L}} M\big) \\
&\leqslant \mathrm{id}_S\big({}_S D_R \otimes_R^{\mathbf{L}} M\big) + \sup\big({}_S D_R \otimes_R^{\mathbf{L}} X\big) \\
&\leqslant \mathrm{id}_S({}_S D_R) - \inf M + \sup\big({}_S D_R \otimes_R^{\mathbf{L}} X\big).
\end{aligned}$$

The first equality follows as $M \in \mathsf{A}(R)$ and the second by adjointness. The first inequality is by [4, Theorem 2.4.I]; the second is by [4, Theorem 4.5(F)], as S is left noetherian and $\mathrm{fd}_R M$ is finite.

(b) Because $\mathrm{id}_{R^{\mathrm{opp}}} M$ is finite and R is right noetherian, [4, Theorem 4.5(I)] implies that the R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ has finite flat dimension. Now the desired result follows directly from (a), as

$$\begin{aligned}
\sup\big(M \otimes_R^{\mathbf{L}} X\big) &= - \inf \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}\big(M \otimes_R^{\mathbf{L}} X, \mathbb{Q}/\mathbb{Z}\big) \\
&= - \inf \mathbf{R}\mathrm{Hom}_R\big(X, \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})\big).
\end{aligned}$$

(c) Again we may assume that $\mathrm{H}(Y) \neq 0$ and $N \neq 0$, and hence:

$$\begin{aligned}
- \inf \mathbf{R}\mathrm{Hom}_S(N, Y) &= - \inf \mathbf{R}\mathrm{Hom}_S\big({}_S D_R \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S({}_S D_R, N), Y\big) \\
&= - \inf \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_S({}_S D_R, N), \mathbf{R}\mathrm{Hom}_S({}_S D_R, Y)\big) \\
&\leqslant \mathrm{pd}_R \mathbf{R}\mathrm{Hom}_S({}_S D_R, N) - \inf \mathbf{R}\mathrm{Hom}_S({}_S D_R, Y).
\end{aligned}$$

The first equality follows as $N \in \mathsf{B}(S)$ and the second by adjointness. The inequality follows from [4, Theorem 2.4.P]. Now, since R is right noetherian and $\mathrm{id}_R N$ is finite, [4, Theorem 4.5(I)] implies that:

$$\mathrm{fd}_R \mathbf{R}\mathrm{Hom}_S({}_S D_R, N) \leqslant \mathrm{id}_{R^{\mathrm{opp}}}({}_S D_R) + \sup N = \mathrm{id}_{R^{\mathrm{opp}}}({}_S D_R) < \infty.$$

Therefore Proposition 1.5 gives the first inequality in:

$$\begin{aligned}
\mathrm{pd}_R \mathbf{R}\mathrm{Hom}_S({}_S D_R, N) &\leqslant \max \left\{ \begin{array}{l} \mathrm{id}_S({}_S D_R) + \sup({}_S D_R \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S({}_S D_R, N)), \\ \sup \mathbf{R}\mathrm{Hom}_S({}_S D_R, N) \end{array} \right\} \\
&\leqslant \max\big\{\mathrm{id}_S({}_S D_R) + \sup N, \ \sup N - \inf({}_S D_R)\big\} \\
&= \mathrm{id}_S({}_S D_R). \qquad \square
\end{aligned}$$

4.6. Lemma. *Let* $\langle S, R \rangle$ *be a noetherian pair with a dualizing complex* $_S D_R$. *If* M *is an* R-*module satisfying*:

(a) $M \in \mathsf{A}(R)$, *and*
(b) $\mathrm{Ext}_R^m(M, Q) = 0$ *for all integers* $m > 0$ *and all projective* R-*modules* Q,

then M *is Gorenstein projective.*

Proof. We are required to construct a complete projective resolution of M. For the left half of this resolution, any ordinary projective resolution of M will do, because of (b). In order to construct the right half, it suffices to construct a short exact sequence of R-modules,

$$0 \to M \to P' \to M' \to 0, \tag{$*$}$$

where P' is projective and M' satisfies (a) and (b). The construction of ($*$) is done in three steps.

(**1**) First, we show that M can be embedded in an R-module of finite flat dimension. Consider resolutions of $_S D_R$, cf. 1.1(2, 3),

$$_S P_R \xrightarrow{\simeq} {}_S D_R \xrightarrow{\simeq} {}_S I_R,$$

where $_S I_R$ is bounded, and let $\lambda : {}_S P_R \xrightarrow{\simeq} {}_S I_R$ be the composite of these two quasi-isomorphisms. Since $M \in \mathsf{A}(R)$, the complex $_S D_R \otimes_R^{\mathbf{L}} M \simeq {}_S P_R \otimes_R M$ belongs to $\mathsf{D}_{\square}(S)$; in particular, $_S P_R \otimes_R M$ admits an S-injective resolution,

$$_S P_R \otimes_R M \xrightarrow{\simeq} J \in \mathsf{C}_{\sqsubset}(S).$$

Concordantly, we get quasi-isomorphisms of R-complexes,

$$M \xrightarrow{\simeq} \mathrm{Hom}_S({}_S P_R, {}_S P_R \otimes_R M) \xrightarrow{\simeq} \mathrm{Hom}_S({}_S P_R, J) \xleftarrow{\simeq} \mathrm{Hom}_S({}_S I_R, J). \tag{$\flat$}$$

Note that since R is right noetherian, and $_S I_R$ is a complex of bimodules consisting of injective R^{opp}-modules, while J is a complex of injective S-modules, the modules in the R-complex

$$F = \mathrm{Hom}_S({}_S I_R, J) \in \mathsf{C}_{\sqsubset}(R)$$

are flat. From ($\flat$) it follows that the modules M and $\mathrm{H}_0(F)$ are isomorphic, and that $\mathrm{H}_\ell(F) = 0$ for all $\ell \neq 0$. Now, $\mathrm{H}_0(F)$ is a submodule of the zeroth cokernel $\mathrm{C}_0^F = \mathrm{Coker}(F_1 \to F_0)$, and C_0^F has finite flat dimension over R since

$$\cdots \to F_1 \to F_0 \to \mathrm{C}_0^F \to 0$$

is exact and $F \in \mathsf{C}_{\sqsubset}(R)$. This proves the first claim.

(**2**) Next, we show that M can be embedded in a flat (actually free) R-module. Note that, by induction on $\mathrm{pd}_R K$, condition (b) is equivalent to

(b$'$) $\mathrm{Ext}_R^m(M, K) = 0$ for all $m > 0$ and all R-modules K with $\mathrm{pd}_R K < \infty$.

By the already established (**1**) there exists an embedding $M \hookrightarrow C$, where C is an R-module of finite flat dimension. Pick a short exact sequence of R-modules,

$$0 \to K \to L \to C \to 0, \tag{$\sharp$}$$

where L is free and, consequently, $\mathrm{fd}_R K < \infty$. Proposition 1.5 implies that also $\mathrm{pd}_R K$ is finite, and hence $\mathrm{Ext}_R^1(M, K) = 0$ by (b$'$). Applying $\mathrm{Hom}_R(M, -)$ to ($\sharp$), we get an exact sequence of abelian groups,

$$\mathrm{Hom}_R(M, L) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}_R^1(M, K) = 0,$$

which yields a factorization,

$$
\begin{array}{ccc}
M & \longrightarrow & C \\
 & \searrow & \uparrow \\
 & & L
\end{array}
$$

As $M \hookrightarrow C$ is a monomorphism, so is the map from M into the free R-module L.

(**3**) Finally, we are able to construct ($*$). Since R is right noetherian there exists by [12, Proposition 5.1] a flat preenvelope $\varphi : M \to F$ of the R-module M. By (**2**), M can be embedded into a flat R-module, and this forces φ to be a monomorphism. Now choose a projective R-module P' surjecting onto F, that is,

$$0 \to Z \to P' \xrightarrow{\pi} F \to 0$$

is exact. Repeating the argument above, we get a factorization

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi} & F \\
 & \underset{\partial}{\searrow} & \uparrow \pi \\
 & & P'
\end{array}
$$

and because φ is injective so is ∂. Thus, we have a short exact sequence

$$0 \to M \xrightarrow{\partial} P' \to M' \to 0. \tag{$\star$}$$

What remains to be proved is that M' has the same properties as M. The projective R-module P' belongs to the $\mathsf{A}(R)$, and by assumption so does M. Since $\mathsf{A}(R)$ is a triangulated subcategory of $\mathsf{D}(R)$, also $M' \in \mathsf{A}(R)$. Let Q be projective; for $m > 0$, we have $\operatorname{Ext}^m_R(M, Q) = 0 = \operatorname{Ext}^m_R(P', Q)$, so it follows from the long exact sequence of Ext modules associated to $(\star)$ that $\operatorname{Ext}^m_R(M', Q) = 0$ for $m > 1$. To see that also $\operatorname{Ext}^1_R(M', Q) = 0$, we consider the exact sequence of abelian groups,

$$\operatorname{Hom}_R(P', Q) \xrightarrow{\operatorname{Hom}_R(\partial, Q)} \operatorname{Hom}_R(M, Q) \to \operatorname{Ext}^1_R(M', Q) \to 0.$$

Since Q is flat and $\varphi : M \to F$ is a flat preenvelope, there exists, for each $\tau \in \operatorname{Hom}_R(M, Q)$, a homomorphism $\tau' : F \to Q$ such that $\tau = \tau' \varphi$; that is, $\tau = \tau' \pi \partial = \operatorname{Hom}_R(\partial, Q)(\tau' \pi)$. Thus, the induced map $\operatorname{Hom}_R(\partial, Q)$ is surjective and, therefore, $\operatorname{Ext}^1_R(M', Q) = 0$. $\quad\square$

Proof of Theorem 4.1. (ii) $\Rightarrow$ (iii): By Proposition 1.5, every flat R-module has finite projective dimension. Furthermore, R is right noetherian, and thus $\operatorname{Gfd}_R X \leqslant \operatorname{Gpd}_R X$ by Proposition 3.7.

(iii) $\Rightarrow$ (i): If $\operatorname{Gfd}_R X$ is finite, then, by definition, X is isomorphic in $\mathsf{D}(R)$ to a bounded complex A of Gorenstein flat modules. Consider resolutions of the dualizing complex, cf. 1.1(2, 3),

$$_SP_R \xrightarrow{\simeq} {_SD_R} \xrightarrow{\simeq} {_SI_R},$$

where $_SI_R$ is bounded, and let $\lambda : {_SP_R} \xrightarrow{\simeq} {_SI_R}$ be the composite quasi-isomorphism. As $\operatorname{id}_{R^{\mathrm{opp}}}({_SD_R})$ is finite, Theorem 3.5 implies that $_SD_R \otimes^{\mathbf{L}}_R X$ is bounded. Whence, to prove that $X \in \mathsf{A}(R)$, we only need to show that

$$\eta_A : A \to \operatorname{Hom}_S({_SP_R}, {_SP_R} \otimes_R A)$$

is a quasi-isomorphism. Even though the modules in $_SP$ are not necessarily finitely generated, we do have $_SP \simeq {_SD} \in \mathsf{D}^{\mathrm{f}}_\square(S)$ by assumption. Since S is left noetherian, there exists a resolution,

$$\mathsf{C}_\square(S) \ni L \xrightarrow[\simeq]{\sigma} {_SP}$$

by finitely generated free S-modules. There is a commutative diagram, in $\mathsf{C}(\mathbb{Z})$,

$$
\begin{array}{ccc}
\operatorname{Hom}_S({_SP}, {_SI_R}) \otimes_R A & \xrightarrow{\text{tensor-eval.}} & \operatorname{Hom}_S({_SP}, {_SI_R} \otimes_R A) \\
{\scriptstyle \operatorname{Hom}_S(\sigma, {_SI_R}) \otimes_R A} \Big\downarrow {\simeq} & & {\simeq} \Big\downarrow {\scriptstyle \operatorname{Hom}_S(\sigma, {_SI_R} \otimes_R A)} \\
\operatorname{Hom}_S(L, {_SI_R}) \otimes_R A & \xrightarrow[\text{tensor-eval.}]{\simeq} & \operatorname{Hom}_S(L, {_SI_R} \otimes_R A)
\end{array}
$$

Since both L and $_SP$ are right bounded complexes of projective modules, the quasi-isomorphism σ is preserved by the functor $\operatorname{Hom}_S(-, U)$ for any S-complex U. This

explains why the right vertical map in the diagram above is a quasi-isomorphism. Since $L \in \mathsf{C}_{\sqsupset}(S)$ consists of finitely generated free S-modules and $_SI_R$ and A are bounded, it follows by, e.g., [4, Lemma 4.4.(F)] that the lower horizontal tensor-evaluation morphism is an isomorphism in the category of $\mathbb{Z}$-complexes. Finally, $\mathrm{Hom}_S(\sigma, {}_SI_R)$ is a quasi-isomorphism between complexes in $\mathsf{C}_{\sqsubset}(R^{\mathrm{opp}})$ consisting of injective modules. This can be seen by using the so-called swap-isomorphism:

$$\mathrm{Hom}_{R^{\mathrm{opp}}}\big(-_R, \mathrm{Hom}_S({}_SP, {}_SI_R)\big) \cong \mathrm{Hom}_S\big({}_SP, \mathrm{Hom}_{R^{\mathrm{opp}}}(-_R, {}_SI_R)\big).$$

Now Theorem 2.15(b) implies that also $\mathrm{Hom}_S(\sigma, {}_SI_R) \otimes_R A$ is a quasi-isomorphism. This argument proves that the lower horizontal tensor-evaluation map in the next commutative diagram of R-complexes is a quasi-isomorphism:

$$
\begin{array}{ccc}
{}_RR_R \otimes_R A \cong A & \xrightarrow{\ \eta_A\ } & \mathrm{Hom}_S({}_SP_R, {}_SP_R \otimes_R A) \\
{\scriptstyle (\mathrm{Hom}_S(\lambda, {}_SI_R) \circ \acute\chi_I^{\langle S,R\rangle}) \otimes_R A}\Big\downarrow {\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow {\scriptstyle \mathrm{Hom}_S({}_SP_R, \lambda \otimes_R A)} \\
\mathrm{Hom}_S({}_SP_R, {}_SI_R) \otimes_R A & \xrightarrow[\text{tensor-eval.}]{\ \simeq\ } & \mathrm{Hom}_S({}_SP_R, {}_SI_R \otimes_R A)
\end{array}
$$

It remains to see that the vertical morphisms in the above diagram are invertible:

- Consider the composite γ of the following two quasi-isomorphisms of complexes of (R, R^{opp})-bimodules, cf. Appendix A,

$${}_RR_R \xrightarrow[\simeq]{\ \acute\chi_I^{\langle S,R\rangle}\ } \mathrm{Hom}_S({}_SI_R, {}_SI_R) \xrightarrow[\simeq]{\ \mathrm{Hom}_S(\lambda, {}_SI_R)\ } \mathrm{Hom}_S({}_SP_R, {}_SI_R).$$

  First note that R and $\mathrm{Hom}_S({}_SP_R, {}_SI_R)$ belong to $\mathsf{C}_{\sqsubset}(R^{\mathrm{opp}})$. Clearly, R is a flat R^{opp}-module; and we have already seen that $\mathrm{Hom}_S({}_SP_R, {}_SI_R)$ consists of injective R^{opp}-modules. Therefore, Theorem 2.15(b) implies that $\gamma \otimes_R A$ is a quasi-isomorphism.[4]
- Since $_SP_R, {}_SI_R \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$, and $_SP_R$ consists of projective R^{opp}-modules, while $_SI_R$ consists of injective R^{opp}-modules, it follows by Theorem 2.15(a) that the induced morphism, $\lambda \otimes_R A : {}_SP_R \otimes_R A \xrightarrow{\simeq} {}_SI_R \otimes_R A$, is a quasi-isomorphism. Hence also $\mathrm{Hom}_S({}_SP_R, \lambda \otimes_R A)$ is a quasi-isomorphism.

(i) $\Rightarrow$ (ii): Let $X \in \mathsf{A}(R)$; we can assume that $\mathrm{H}(X) \neq 0$. By Lemma 4.5,

$$-\inf \mathbf{R}\mathrm{Hom}_R(X, M) \leqslant s = \mathrm{id}_S({}_SD_R) + \sup\big({}_SD_R \otimes_R^{\mathbf{L}} X\big)$$

for all R-modules M with $\mathrm{fd}_R M < \infty$. Set $n = \max\{s, \sup X\}$. Take a projective resolution $\mathsf{C}_{\sqsupset}(R) \ni P \xrightarrow{\simeq} X$. Since $n \geqslant \sup X = \sup P$, we have $\subset_n P \simeq P \simeq X$, and hence

---

[4] Note that the results in Section 2 do not allow us to conclude that the individual morphisms $\acute\chi_I^{\langle S,R\rangle} \otimes_R A$ and $\mathrm{Hom}_S(\lambda, {}_SI_R) \otimes_R A$ are quasi-isomorphisms.

it suffices to show that the cokernel $C_n^P = \mathrm{Coker}(P_{n+1} \to P_n)$ is a Gorenstein projective R-module. By Lemma 4.6 it is enough to prove that

(a) $C_n^P \in \mathsf{A}(R)$, and
(b) $\mathrm{Ext}_R^m(C_n^P, Q) = 0$ for all integers $m > 0$ and all projective R-modules Q.

Consider the exact sequence of complexes

$$0 \to {\sqsubset}_{n-1} P \to {\subset}_n P \to \Sigma^n C_n^P \to 0.$$

Obviously, ${\sqsubset}_{n-1} P$ belongs to $\mathsf{A}(R)$, cf. 1.3, and also ${\subset}_n P \simeq P \simeq X \in \mathsf{A}(R)$. Because $\mathsf{A}(R)$ is a triangulated subcategory of $\mathsf{D}(R)$, we conclude that $\Sigma^n C_n^P$, and hence C_n^P, belongs to $\mathsf{A}(R)$. This establishes (a).

To verify (b), we let $m > 0$ be an integer, and Q be any projective R-module. Since $n \geqslant \sup X$, it is a straightforward computation, cf. [8, Lemma 4.3.9], to see that

$$\mathrm{Ext}_R^m\big(C_n^P, Q\big) \cong \mathrm{H}_{-(m+n)}\big(\mathbf{R}\mathrm{Hom}_R(X, Q)\big),$$

for $m > 0$. Since $-\inf \mathbf{R}\mathrm{Hom}_R(X, Q) \leqslant s \leqslant n$, we see that $\mathrm{Ext}_R^m(C_n^P, Q) = 0$. $\square$

Proof of Theorem 4.4. Using Lemma 4.7 below, the proof is similar to that of Theorem 4.1. Just as the proof of Lemma 4.6 uses R-flat preenvelopes, the proof of Lemma 4.7 below uses S-injective precovers. The existence of such precovers is guaranteed by [43], cf. [12, Proposition 2.2], as S is left noetherian. $\square$

4.7. Lemma. *Let $\langle S, R \rangle$ be a noetherian pair with a dualizing complex ${}_S D_R$. If N is an S-module satisfying*:

(a) $N \in \mathsf{B}(S)$, *and*
(b) $\mathrm{Ext}_S^m(J, N) = 0$ *for all integers $m > 0$ and all injective S-modules J,*

then N is Gorenstein injective.

5. Stability results

We now apply the characterization from the previous section to show that finiteness of Gorenstein dimensions is preserved under a series of standard operations. In this section, all rings are commutative and noetherian.

It is known from [8] that $\mathrm{Gid}_R \mathrm{Hom}_R(X, E) \leqslant \mathrm{Gfd}_R X$ for $X \in \mathsf{D}_\sqsupset(R)$ and injective modules E. Here is a dual result, albeit in a more restrictive setting:

5.1. Proposition. *Let R be commutative and noetherian with a dualizing complex, and let E be an injective R-module. For $Y \in \mathsf{D}_\sqsubset(R)$ there is an inequality,*

$$\mathrm{Gfd}_R \mathrm{Hom}_R(Y, E) \leqslant \mathrm{Gid}_R Y,$$

and equality holds if E is faithful.

Proof. It is sufficient to prove that if N is a Gorenstein injective module, then $\operatorname{Hom}_R(N, E)$ is Gorenstein flat, and that the converse holds, when E is faithful.

Write $-^\vee = \operatorname{Hom}_R(-, E)$ for short, and set $d = \operatorname{FFD}(R)$, which is finite as R has a dualizing complex, cf. 1.4. From Theorem 3.5 we know that if C is any module with $\operatorname{Gfd}_R C < \infty$ then, in fact, $\operatorname{Gfd}_R C \leqslant d$.

Now assume that N is Gorenstein injective, and consider part of the left half of a complete injective resolution of N,

$$0 \to C_d \to I_{d-1} \to \cdots \to I_0 \to N \to 0. \tag{$*$}$$

The I_ℓ's are injective R-modules and C_d is Gorenstein injective. In particular, $C_d \in \mathsf{B}(R)$ by Theorem 4.4, and $C_d^\vee \in \mathsf{A}(R)$ by [8, Lemma 3.2.9(b)], so $\operatorname{Gfd}_R C_d^\vee \leqslant d$. Applying the functor $-^\vee$ to $(*)$ we obtain an exact sequence:

$$0 \to N^\vee \to I_0^\vee \to \cdots \to I_{d-1}^\vee \to C_d^\vee \to 0,$$

where the $I_\ell^\vee$'s are flat R-modules. From Theorem 3.5 we conclude that $N^\vee$ is Gorenstein flat.

Finally, we assume that E is faithfully injective and that $N^\vee$ is Gorenstein flat, in particular, $N^\vee \in \mathsf{A}(R)$. This forces $N \in \mathsf{B}(R)$, again by [8, Lemma 3.2.9(b)], that is, $\operatorname{Gid}_R N$ is finite. By Lemma 2.18 there exists an exact sequence

$$0 \to B \to H \to N \to 0,$$

where B is Gorenstein injective and $\operatorname{id}_R H = \operatorname{Gid}_R N$. By the first part of the proof, $B^\vee$ is Gorenstein flat, and by assumption, so is $N^\vee$. Therefore, exactness of

$$0 \to N^\vee \to H^\vee \to B^\vee \to 0$$

forces $H^\vee$ to be Gorenstein flat by the resolving property of Gorenstein flat modules, cf. [30, Theorem 3.7]. In particular, $\operatorname{Gid}_R N = \operatorname{id}_R H = \operatorname{fd}_R H^\vee = \operatorname{Gfd}_R H^\vee = 0$. Here the second equality follows from [31, Theorem 1.5]. $\quad\square$

The next result is an immediate corollary of [8, Theorem 6.4.2 and 6.4.3] and 5.1.

5.2. Corollary. *Let R be commutative and noetherian with a dualizing complex, and let F be a flat R-module. For $Y \in \mathsf{D}_\sqsubset(R)$ there is an inequality,*

$$\operatorname{Gid}_R(Y \otimes_R F) \leqslant \operatorname{Gid}_R Y,$$

and equality holds if F is faithful.

5.3. Theorem. *Let $\varphi : R \to S$ be a homomorphism of commutative noetherian rings with* $\operatorname{fd} \varphi$ *finite. Assume that R has a dualizing complex D and $E = D \otimes_R^{\mathbf{L}} S$ is dualizing for S. For $Y \in \mathsf{D}_\square(R)$ the following hold:*

$$\operatorname{Gfd}_R Y < \infty \quad \Rightarrow \quad \operatorname{Gfd}_S\bigl(Y \otimes_R^{\mathbf{L}} S\bigr) < \infty, \tag{a}$$

$$\mathrm{Gid}_R Y < \infty \quad \Rightarrow \quad \mathrm{Gid}_S\big(Y \otimes_R^{\mathbf{L}} S\big) < \infty. \tag{b}$$

Either implication may be reversed under each of the next two extra conditions:

- *φ is faithfully flat,*
- *φ is local and the complex Y belongs to $\mathsf{D}_\square^{\mathrm{f}}(R)$.*

When $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism, the assumption that the base-changed complex $D \otimes_R^{\mathbf{L}} S$ is dualizing for S is tantamount to φ being Gorenstein (at the maximal ideal $\mathfrak{n}$ of S). For details see [5, Theorem 7.8].

Proof. We only prove the statements for the Gorenstein injective dimension; the proof for the Gorenstein flat dimension is similar.

In view of Theorem 4.4, we need to see that the base changed complex $Y \otimes_R^{\mathbf{L}} S$ belongs to $\mathsf{B}(S)$ when $Y \in \mathsf{B}(R)$. This is a special case of [9, Proposition 5.9], from where it also follows that the implication may be reversed when φ is faithfully flat.

Next, let φ be local, Y be in $\mathsf{D}_\square^{\mathrm{f}}(R)$, and assume that $Y \otimes_R^{\mathbf{L}} S \in \mathsf{B}(S)$. The aim is to show that $Y \in \mathsf{B}(R)$. First, we verify that $\mathbf{R}\mathrm{Hom}_R(D, Y)$ has bounded homology. As $E = D \otimes_R^{\mathbf{L}} S$ is a dualizing complex for S, we may compute as follows:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_S\big(E, Y \otimes_R^{\mathbf{L}} S\big) &\simeq \mathbf{R}\mathrm{Hom}_S\big(D \otimes_R^{\mathbf{L}} S, Y \otimes_R^{\mathbf{L}} S\big) \\
&\simeq \mathbf{R}\mathrm{Hom}_R\big(D, Y \otimes_R^{\mathbf{L}} S\big) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(D, Y) \otimes_R^{\mathbf{L}} S. \tag{$*$}
\end{aligned}$$

Here the first isomorphism is trivial, the second is adjointness, and the third follows from [8, (A.4.23)].

The remainder of the proof is built up around two applications of Iversen's amplitude inequality, which is now available for unbounded complexes [23, Theorem 3.1]. The amplitude inequality yields

$$\mathrm{amp}\big(\mathbf{R}\mathrm{Hom}_R(D, Y)\big) \leqslant \mathrm{amp}\big(\mathbf{R}\mathrm{Hom}_R(D, Y) \otimes_R^{\mathbf{L}} S\big), \tag{$\sharp$}$$

as φ is assumed to be of finite flat dimension. Here the amplitude of a complex X is defined as $\mathrm{amp}\, X = \sup X - \inf X$. From $(*)$ we read off that the homology of $\mathbf{R}\mathrm{Hom}_R(D, Y) \otimes_R^{\mathbf{L}} S$ is bounded, and by $(\sharp)$ this shows that the homology of $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded as well.

Finally, consider the commutative diagram

$$
\begin{array}{ccc}
E \otimes_S^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S\big(E, Y \otimes_R^{\mathbf{L}} S\big) & \xleftarrow[\simeq]{\gamma_{Y \otimes^{\mathbf{L}} S}} & D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R\big(D, Y \otimes_R^{\mathbf{L}} S\big) \\
{\scriptstyle \varepsilon_{Y \otimes^{\mathbf{L}} S}} \downarrow {\scriptstyle \simeq} & & {\scriptstyle \simeq} \uparrow {\scriptstyle D \otimes^{\mathbf{L}} \omega_{DYS}} \\
Y \otimes_R^{\mathbf{L}} S & \xleftarrow[\varepsilon_Y \otimes^{\mathbf{L}} S]{} & \big(D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)\big) \otimes_R^{\mathbf{L}} S
\end{array}
$$

where $\gamma_{Y \otimes_R^L S}$ is a natural isomorphism induced by adjointness and commutativity of the derived tensor product. The diagram shows that $\varepsilon_Y \otimes_R^L S$ is an isomorphism. As $D \otimes_R^L \mathbf{R}\mathrm{Hom}_R(D, Y)$ has degreewise finitely generated homology, we may apply [32, Proposition 2.10] to conclude that ε_Y is an isomorphism as well. □

5.4. Localization. Working directly with the definition of Gorenstein flat modules (see [8, Lemma 5.1.3]), it is easily verified that the inequality

$$\mathrm{Gfd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant \mathrm{Gfd}_R X$$

holds for all complexes $X \in \mathsf{D}_\sqsupset(R)$ and all prime ideals $\mathfrak{p}$ of R.

Turning to the Gorenstein projective and injective dimensions, it is natural to ask if they do not grow under localization. When R is local and Cohen–Macaulay with a dualizing module, Foxby settled the question affirmatively in [22, Corollary 3.5]. More recently, Foxby extended the result for Gorenstein projective dimension to commutative noetherian rings of finite Krull dimension; see [11, 5.5(b)]. Unfortunately, it is not clear how to apply the ideas of that proof to the Gorenstein injective dimension, but there is a partial result:

5.5. Proposition. *Let R be commutative and noetherian with a dualizing complex. For any complex $Y \in \mathsf{D}_\sqsubset(R)$ and any prime ideal $\mathfrak{p}$ of R, there is an inequality*

$$\mathrm{Gid}_{R_\mathfrak{p}} Y_\mathfrak{p} \leqslant \mathrm{Gid}_R Y.$$

Proof. It suffices to show that if N is a Gorenstein injective R-module, then $N_\mathfrak{p}$ is Gorenstein injective over $R_\mathfrak{p}$. This is proved in the exact same manner as in [22, Corollary 3.5] using Theorem 4.4. □

It is immediate from Definition 1.9 that a direct sum of Gorenstein flat modules is Gorenstein flat. It has also been proved [18] that, over a right coherent ring, a colimit of Gorenstein flat modules indexed by a filtered set is Gorenstein flat.

One may suspect that also a product of Gorenstein flat modules is Gorenstein flat. In the sequel this is proved for commutative noetherian rings with a dualizing complex. To this end, the next lemma records an important observation.

5.6. Lemma. *Let R be a commutative noetherian ring with a dualizing complex. The Auslander categories $\mathsf{A}(R)$ and $\mathsf{B}(R)$ are closed under direct products of modules and under colimits of modules indexed by a filtered set.*

Proof. There are four parts to the lemma; they have similar proofs, so we shall only prove the first claim, that $\mathsf{A}(R)$ is closed under set indexed products of modules.

Let D be a dualizing complex for R, and let $L \xrightarrow{\simeq} D$ be a resolution of D by finitely generated free modules. Consider a family of modules $\{M_i\}_{i \in I}$ from $\mathsf{A}(R)$ and set $M = \prod_{i \in I} M_i$. The canonical chain map

$$L \otimes_R M = L \otimes_R \left(\prod_{i \in I} M_i \right) \xrightarrow{\alpha} \prod_{i \in I} (L \otimes_R M_i)$$

is an isomorphism. This is a straightforward verification; it hinges on the fact that the module functors $L_n \otimes_R -$ commute with arbitrary products, as the modules L_n are finitely generated and free.

For each $i \in I$, the complex $L \otimes_R M_i$ represents $D \otimes_R^{\mathbf{L}} M_i$, and by [9, Proposition 4.8(a)] there are inequalities $\sup(L \otimes_R M_i) \leqslant \sup D$. Thus, the isomorphism α and the fact that homology commutes with products yields

$$\sup\big(D \otimes_R^{\mathbf{L}} M\big) = \sup(L \otimes_R M) \leqslant \sup D.$$

In particular, $D \otimes_R^{\mathbf{L}} M$ is in $\mathsf{D}_\square(R)$. Next, consider the commutative diagram,

$$
\begin{array}{ccc}
\prod_{i \in I} M_i = M & \xrightarrow{\ \eta_M\ } & \mathrm{Hom}_R(L, L \otimes_R M) \\
{\scriptstyle \prod_{i \in I} \eta_{M_i}} \Big\downarrow {\scriptstyle \simeq} & & {\scriptstyle \cong} \Big\downarrow {\scriptstyle \mathrm{Hom}_R(L,\alpha)} \\
\prod_{i \in I} \mathrm{Hom}_R(L, L \otimes_R M_i) & \xrightarrow[\ \beta\ ]{\cong} & \mathrm{Hom}_R\big(L, \prod_{i \in I}(L \otimes_R M_i)\big)
\end{array}
$$

The canonical map β is an isomorphism of complexes, as $\mathrm{Hom}_R(L, -)$ commutes with products. The map $\prod_{i \in I} \eta_{M_i}$ is a quasi-isomorphism, because each η_{M_i} is one. The upshot is that η_M is a quasi-isomorphism, and M belongs to $\mathsf{A}(R)$. $\square$

5.7. Theorem. *Let R be commutative and noetherian with a dualizing complex. A direct product of Gorenstein flat modules is Gorenstein flat.*

Proof. Let $A^{(i)}$ be a family of Gorenstein flat modules. By Lemma 5.6 the product $\prod_i A^{(i)}$ is in $\mathsf{A}(R)$ and, therefore, $\mathrm{Gfd}_R \prod_i A^{(i)}$ is finite, in fact, at most $d = \mathrm{FFD}(R)$, cf. Theorem 3.5. For each $A^{(i)}$ take a piece of a complete flat resolution:

$$0 \to A^{(i)} \to F_0^{(i)} \to \cdots \to F_{1-d}^{(i)} \to Z_{-d}^{(i)} \to 0,$$

where the F's are flat and the Z's are Gorenstein flat. Taking products we get an exact sequence:

$$0 \to \prod_i A^{(i)} \to \prod_i F_0^{(i)} \to \cdots \to \prod_i F_{1-d}^{(i)} \to \prod_i Z_{-d}^{(i)} \to 0.$$

Since R is noetherian, the modules $\prod_i F_\ell^{(i)}$ are flat. As noted above $\prod_i Z_{-d}^{(i)}$ has Gorenstein flat dimension at most d, which forces the product $\prod_i A^{(i)}$ to be Gorenstein flat, cf. Theorem 3.5. $\square$

On a parallel note, it is immediate from Definition 1.8 that a product of Gorenstein injective modules is Gorenstein injective. We remark that via Theorem 4.1 this gives a different proof that $\mathsf{B}(R)$ is closed under direct products of modules. This shows that information flows in both directions between Auslander categories and Gorenstein dimensions.

Over a ring with a dualizing complex, the proof above is easily modified to show that a direct sum of Gorenstein injectives is, again, Gorenstein injective. In view of Lemma 5.6 it is natural to expect that even a colimit of Gorenstein injective modules will be Gorenstein injective. This is proved in the next section; see Theorem 6.9.

5.8. Local (co)homology. Let R be commutative and noetherian, and let $\mathfrak{a}$ be an ideal of R. The right derived local cohomology functor with support in $\mathfrak{a}$ is denoted $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Its right adjoint, $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$, is the left derived local homology functor with support in $\mathfrak{a}$. Derived local (co)homology is represented on $\mathsf{D}(R)$ as

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}R \otimes_R^{\mathbf{L}} - \simeq \mathrm{C}(\mathfrak{a}) \otimes_R^{\mathbf{L}} -$$

and

$$\mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}R, -) \simeq \mathbf{R}\mathrm{Hom}_R\big(\mathrm{C}(\mathfrak{a}), -\big),$$

where $\mathrm{C}(\mathfrak{a})$ is the so-called Čech, or stable Koszul, complex on $\mathfrak{a}$; it is defined as follows: Let $a \in R$; the complex concentrated in homological degrees 0 and -1:

$$\mathrm{C}(a) = 0 \longrightarrow R \xrightarrow{\rho_a} R_a \longrightarrow 0,$$

where R_a is the localization of R with respect to $\{a^n\}_{n \geqslant 0}$ and ρ_a is the natural homomorphism $r \mapsto r/1$, is the Čech complex on a. When the ideal $\mathfrak{a}$ is generated by $a_1, \ldots, a_n$, the Čech complex on $\mathfrak{a}$ is the tensor product $\bigotimes_{i=1}^n \mathrm{C}(a_i)$. Observe that the flat dimension of $\mathrm{C}(\mathfrak{a})$ is finite.

The above representations of derived local (co)homology will be used without mention in the proofs of Theorems 5.9 and 6.5. For local cohomology this representation goes back to Grothendieck [28, Proposition 1.4.1]; see also [1, Lemma 3.1.1] and the corrections in [42, Section 1]. Local homology was introduced by Matlis [38, §4], when $\mathfrak{a}$ is generated by a regular sequence, and for modules over local Cohen–Macaulay rings the representation above is implicit in [38, Theorem 5.7]. The general version above is due to Greenlees and May [27, Section 2]; see also [42, Section 1] for corrections.

Since $\mathrm{C}(\mathfrak{a})$ has finite flat dimension, it is immediate that $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ preserves homological boundedness as well as finite flat and finite injective dimension, see also [21, Theorem 6.5]. However, $\mathrm{C}(\mathfrak{a})$ has even finite projective dimension. This calls for an argument: Let $a \in R$ and consider the short exact sequence

$$0 \longrightarrow R[X] \xrightarrow{aX-1} R[X] \xrightarrow{\alpha} R_a \longrightarrow 0,$$

where α is the homomorphism $f(X) \mapsto f(1/a)$. This short exact sequence is a bounded free resolution of R_a, whence the projective dimension of R_a is at most one. Let L_a be the complex

$$L_a \;=\; 0 \xrightarrow{\qquad} \begin{matrix} R[X] \\ \oplus \\ R \end{matrix} \xrightarrow{\;(aX-1\ \ \iota)\;} R[X] \xrightarrow{\qquad} 0$$

concentrated in homological degrees 0 and -1, where ι denotes the natural embedding of R into $R[X]$. It is straightforward to verify that L_a is a bounded free resolution of the Čech complex $\mathrm{C}(a)$. Thus, if the ideal $\mathfrak{a}$ is generated by $a_1, \dots, a_n$, then $L = \bigotimes_{i=1}^{n} L_{a_i}$ is a bounded free resolution of $\mathrm{C}(\mathfrak{a})$. This shows that the projective dimension of $\mathrm{C}(\mathfrak{a})$ is finite.

The last two results investigate preservation of finite Gorenstein dimensions by local (co)homology functors.

5.9. Theorem. *Let R be a commutative noetherian ring, and let $\mathfrak{a}$ be an ideal of R. For $Y \in \mathsf{D}_{\square}(R)$ the following hold*:

$$\mathrm{Gfd}_R Y < \infty \quad \Rightarrow \quad \mathrm{Gfd}_R(\mathbf{R}\Gamma_{\mathfrak{a}} Y) < \infty. \tag{a}$$

If, in addition, R admits a dualizing complex, then

$$\mathrm{Gid}_R Y < \infty \quad \Rightarrow \quad \mathrm{Gid}_R(\mathbf{R}\Gamma_{\mathfrak{a}} Y) < \infty. \tag{b}$$

Moreover, if R has a dualizing complex, both implications may be reversed if $\mathfrak{a}$ is in the Jacobson radical of R and $Y \in \mathsf{D}_{\square}^{\mathrm{f}}(R)$.

Note that in Theorem 5.9 we use the existence of a dualizing complex to establish preservation of finite Gorenstein injective dimension. In 5.10 below the dualizing complex is used to establish preservation of finite Gorenstein flat dimension.

5.10. Theorem. *Let R be a commutative noetherian ring, and let $\mathfrak{a}$ be an ideal of R. For $X \in \mathsf{D}_{\square}(R)$ the following hold*:

$$\mathrm{Gid}_R X < \infty \quad \Rightarrow \quad \mathrm{Gid}_R\big(\mathbf{L}\Lambda^{\mathfrak{a}} X\big) < \infty. \tag{a}$$

If, in addition, R admits a dualizing complex, then

$$\mathrm{Gfd}_R X < \infty \quad \Rightarrow \quad \mathrm{Gfd}_R\big(\mathbf{L}\Lambda^{\mathfrak{a}} X\big) < \infty. \tag{b}$$

Moreover, both implications may be reversed if $(R, \mathfrak{m})$ is local and complete in its $\mathfrak{m}$-adic topology, $\mathfrak{a}$ is in the Jacobson radical of R, and $X \in \mathsf{D}_{\square}^{\mathrm{art}}(R)$; that is, X has bounded homology and its individual homology modules are artinian.

Proof of Theorem 5.9. Since $\mathbf{R}\Gamma_{\mathfrak{a}} Y \simeq \mathrm{C}(\mathfrak{a}) \otimes_R^{\mathbf{L}} Y$ the implication (a) follows from [8, Theorem 6.4.5], and (b) from a routine application of Corollary 5.2.

Now assume that R has a dualizing complex D and $\mathfrak{a}$ is in the Jacobson radical. The arguments showing that the implications in (a) and (b) can be reversed are similar; we only write out the details for the latter.

Assume that $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ and $\mathbf{R}\Gamma_{\mathfrak{a}} Y \in \mathsf{B}(R)$. First, we show that $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded. By 5.8 the projective dimension of $\mathrm{C}(\mathfrak{a})$ is finite and, therefore, $\mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\mathfrak{a}), -)$ preserves homological boundedness. In particular, $\mathbf{L}\Lambda^{\mathfrak{a}} \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$ is bounded since, already, $\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$ is. Observe that

$$\mathbf{L}\Lambda^{\mathfrak{a}} \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y) \simeq \mathbf{R}\mathrm{Hom}_R\big(D, \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} Y\big)$$
$$\simeq \mathbf{R}\mathrm{Hom}_R\big(D, \mathbf{L}\Lambda^{\mathfrak{a}} Y\big)$$
$$\simeq \mathbf{R}\mathrm{Hom}_R\big(D, Y \otimes_R^{\mathbf{L}} \hat{R}_{\mathfrak{a}}\big)$$
$$\simeq \mathbf{R}\mathrm{Hom}_R(D, Y) \otimes_R^{\mathbf{L}} \hat{R}_{\mathfrak{a}}.$$

Here the first isomorphism is swap, the second is by [1, Corollary 5.1.1(i)], the third follows from [24, Proposition 2.7], and the last is by [4, Lemma 4.4(F)]. As $\mathfrak{a}$ is in the Jacobson radical of R, the completion $\hat{R}_{\mathfrak{a}}$ is a faithful flat R-module by [39, Theorem 8.14] and, therefore, $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded.

To show that $\varepsilon_Y : D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y) \to Y$ is invertible, we consider the commutative diagram

$$
\begin{array}{ccc}
\mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} Y & \xleftarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_{\mathbf{R}\Gamma_{\mathfrak{a}} Y}} & \mathbf{L}\Lambda^{\mathfrak{a}}\big(D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)\big) \\
& & \\
\beth_Y \Big\downarrow \simeq & & \simeq \Big\uparrow \\
& & \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}}\big(D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)\big) \\
& & \simeq \Big\downarrow \beth_{D\otimes^{\mathbf{L}}\mathbf{R}\mathrm{Hom}(D,Y)} \\
\mathbf{L}\Lambda^{\mathfrak{a}} Y & \xleftarrow[\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y]{} & \mathbf{L}\Lambda^{\mathfrak{a}}\big(D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)\big)
\end{array}
$$

The top horizontal morphism is invertible as $\mathbf{R}\Gamma_{\mathfrak{a}} Y \in \mathsf{B}(R)$. The vertical morphisms $\beth$ are invertible, again by [1, Corollary 5.1.1(i)], and the third vertical morphism is induced by tensor evaluation, cf. [4, Lemma 4.4(F)]. The diagram shows that $\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y$ is an isomorphism. Now, since $D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)$ belongs to $\mathsf{D}^{\mathrm{f}}_{\square}(R)$, cf. [5, (1.2.1) and (1.2.2)], it follows by [24, Proposition 2.7] that we may identify $\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y$ with $\varepsilon_Y \otimes_R \hat{R}_{\mathfrak{a}}$. Whence, ε_Y is an isomorphism by faithful flatness of $\hat{R}_{\mathfrak{a}}$. $\square$

Proof of Theorem 5.10. Assume that the Gorenstein injective dimension of X is finite. Let L be the bounded free resolution of $\mathrm{C}(\mathfrak{a})$ described in 5.8. By assumption there exists a bounded complex, say B, consisting of Gorenstein injective modules and quasi-isomorphic to X. We may represent $\mathbf{L}\Lambda^{\mathfrak{a}} X$ by the bounded complex $\mathrm{Hom}_R(L, B)$. It is readily seen that the individual modules in the latter complex consist of products of Gorenstein injective modules. Consequently, they are Gorenstein injective themselves; see [30, Theorem 2.6]. In particular, $\mathbf{L}\Lambda^{\mathfrak{a}} X$ has finite Gorenstein injective dimension.

In the presence of a dualizing complex, a similar argument applies when the Gorenstein flat dimension of X is finite. This time we use that a product of Gorenstein flat modules is Gorenstein flat by 5.7.

As in the proof of Theorem 5.9 we only argue why the implication in (b) can be reversed; the arguments for reversing the implication in (a) are similar.

When $(R, \mathfrak{m}, k)$ is complete in its $\mathfrak{m}$-adic topology it admits a dualizing complex [29, V.10.4]. Moreover, Matlis duality [39, Theorem 18.6(v)] and the assumption that X has bounded artinian homology yields $X \simeq X^{\vee\vee}$ where $-^{\vee} = \operatorname{Hom}_R(-, \mathrm{E}_R(k))$ is the Matlis duality functor. Here, $\mathrm{E}_R(k)$ is the injective hull of the residue field k. By [24, (2.10)] we have

$$\mathbf{L}\Lambda^{\mathfrak{a}} X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}\big(X^{\vee}\big)^{\vee}. \tag{$*$}$$

As the complex $X^{\vee}$ has finite homology, see [39, Theorem 18.6(v)], and the functor $-^{\vee}$ is faithful and exact, we have the following string of biimplications:

$$
\begin{aligned}
\mathbf{L}\Lambda^{\mathfrak{a}} X \in \mathsf{B}(R) \quad &\Leftrightarrow \quad \mathbf{R}\Gamma_{\mathfrak{a}}\big(X^{\vee}\big) \in \mathsf{A}(R) \\
&\Leftrightarrow \quad X^{\vee} \in \mathsf{A}(R) \\
&\Leftrightarrow \quad X \in \mathsf{B}(R).
\end{aligned}
$$

Here the first biimplication follows from $(*)$ in conjunction with [8, Lemma 3.2.9(a)]; the second follows from Theorem 5.9 and the third from [8, Lemma 3.2.9(a)]. $\quad\square$

5.11. Observation. Let R be a commutative noetherian ring with a dualizing complex, and let $X \in \mathsf{D}_{\square}(R)$. We will demonstrate that $\mathbf{L}\Lambda^{\mathfrak{a}} X$ has finite, say, Gorenstein injective dimension, when and only when $\mathbf{R}\Gamma_{\mathfrak{a}} X$ has finite Gorenstein injective dimension. The argument is propelled by the isomorphisms

$$\mathbf{R}\Gamma_{\mathfrak{a}} X \xrightarrow{\;\simeq\;} \mathbf{R}\Gamma_{\mathfrak{a}}\mathbf{L}\Lambda^{\mathfrak{a}} X \quad \text{and} \quad \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} X \xrightarrow{\;\simeq\;} \mathbf{L}\Lambda^{\mathfrak{a}} X, \tag{$\sharp$}$$

which are valid for any $X \in \mathsf{D}(R)$; for details consult [1, Corollary 5.1.1]. The next string of implications

$$\mathbf{L}\Lambda^{\mathfrak{a}} X \in \mathsf{B}(R) \quad \Rightarrow \quad \mathbf{R}\Gamma_{\mathfrak{a}}\mathbf{L}\Lambda^{\mathfrak{a}} X \in \mathsf{B}(R) \quad \Rightarrow \quad \mathbf{R}\Gamma_{\mathfrak{a}} X \in \mathsf{B}(R),$$

where the first follows from 5.9 and the second from $(\sharp)$, together with

$$\mathbf{R}\Gamma_{\mathfrak{a}} X \in \mathsf{B}(R) \quad \Rightarrow \quad \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} X \in \mathsf{B}(R) \quad \Rightarrow \quad \mathbf{L}\Lambda^{\mathfrak{a}} X \in \mathsf{B}(R),$$

where the first follows from 5.10 and the second from $(\sharp)$, prove the claim. A similar argument is available for Gorenstein flat dimension.

6. Bass and Chouinard formulas

The theorems in Section 3 give formulas for measuring Gorenstein dimensions. We close the paper by establishing alternative formulas that allow us to measure or even compute Gorenstein injective dimension. In this section R is a commutative noetherian ring.

6.1. Width. Recall that when $(R, \mathfrak{m}, k)$ is a local ring, the *width* of an R-complex $X \in D_{\sqsupset}(R)$ is defined as $\mathrm{width}_R X = \inf(k \otimes^{\mathbf{L}}_R X)$. There is always an inequality,

$$\mathrm{width}_R X \geqslant \inf X, \tag{6.1.1}$$

and by Nakayama's lemma, equality holds for $X \in D^{\mathrm{f}}_{\sqsupset}(R)$.

6.2. Observation. Let R be a commutative and noetherian ring with a dualizing complex. It is easy to see that the functor $-^{\dagger}$, cf. 1.6, maps $\mathsf{A}(R)$ to $\mathsf{B}(R)$ and vice versa, cf. [8, Lemma 3.2.9]. Consequently,

$$\mathrm{Gid}_R Y < \infty \quad \Leftrightarrow \quad \mathrm{Gfd}_R Y^{\dagger} < \infty \quad \Leftrightarrow \quad \mathrm{Gpd}_R Y^{\dagger} < \infty$$

for $Y \in D^{\mathrm{f}}_{\square}(R)$ by Theorems 4.1 and 4.4 and duality (1.6.1).

6.3. Theorem. *Let R be a commutative noetherian local ring. For $Y \in D_{\square}(R)$ there is an inequality,*

$$\mathrm{Gid}_R Y \geqslant \mathrm{depth}\, R - \mathrm{width}_R Y.$$

If, in addition, R admits a dualizing complex, and $Y \in D^{\mathrm{f}}_{\square}(R)$ is a complex of finite Gorenstein injective dimension, then the next equality holds,

$$\mathrm{Gid}_R Y = \mathrm{depth}\, R - \inf Y.$$

In particular,

$$\mathrm{Gid}_R N = \mathrm{depth}\, R$$

for any finitely generated R-module $N \neq 0$ of finite Gorenstein injective dimension.

Proof. Set $d = \mathrm{depth}\, R$ and pick an R-regular sequence $\boldsymbol{x} = x_1, \ldots, x_d$. Note that the module $T = R/(\boldsymbol{x})$ has $\mathrm{pd}_R T = d$. We may assume that $\mathrm{Gid}_R Y < \infty$, and the desired inequality now follows from the computation:

$$\begin{aligned}
\mathrm{Gid}_R Y &\geqslant -\inf \mathbf{R}\mathrm{Hom}_R(T, Y) \\
&\geqslant -\mathrm{width}_R \mathbf{R}\mathrm{Hom}_R(T, Y) \\
&= \mathrm{pd}_R T - \mathrm{width}_R Y \\
&= \mathrm{depth}\, R - \mathrm{width}_R Y.
\end{aligned}$$

The first inequality follows from Theorem 3.3 and the second from (6.1.1). The first equality is by [10, Theorem 4.14(b)] and the last by definition of T.

Next, assume that R admits a dualizing complex, and that $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ with $\mathrm{Gid}_R Y$ finite. It suffices to prove the inequality

$$\mathrm{Gid}_R Y \leqslant \mathrm{depth}\, R - \inf Y.$$

Duality (1.6.1) yields $\mathrm{Gid}_R Y = \mathrm{Gid}_R Y^{\dagger\dagger}$, and by Theorem 3.3 there exists an injective R-module J, such that $\mathrm{Gid}_R Y^{\dagger\dagger} = -\inf \mathbf{R}\mathrm{Hom}_R(J, Y^{\dagger\dagger})$. In the computation,

$$\mathrm{Gid}_R Y = -\inf \mathbf{R}\mathrm{Hom}_R\left(Y^{\dagger}, J^{\dagger}\right)$$
$$\leqslant \mathrm{Gpd}_R Y^{\dagger} - \inf J^{\dagger}$$
$$\leqslant \mathrm{Gpd}_R Y^{\dagger} + \mathrm{id}_R D,$$

the first (in)equality is by swap and the second by Theorem 3.1, as $J^{\dagger}$ is a complex of finite flat dimension and hence of finite projective dimension. The final inequality is by [4, Theorem 2.4.I]. Recall from 6.2 that $\mathrm{Gpd}_R Y^{\dagger}$ is finite; since $Y^{\dagger}$ has bounded and degreewise finitely generated homology, it follows from Proposition 3.8(b) and [8, Theorem 2.3.13] that $\mathrm{Gpd}_R Y^{\dagger} = \mathrm{depth}\, R - \mathrm{depth}_R Y^{\dagger}$. Thus, we may continue as follows:

$$\mathrm{Gid}_R Y \leqslant \mathrm{depth}\, R - \mathrm{depth}_R Y^{\dagger} + \mathrm{id}_R D$$
$$= \mathrm{depth}\, R - \inf Y - \mathrm{depth}_R D + \mathrm{id}_R D$$
$$= \mathrm{depth}\, R - \inf Y.$$

Both equalities stem from well-known properties of dualizing complexes, see [9, (3.1)(a) and (3.5)]. □

A dualizing complex D for a commutative noetherian local ring R is said to be *normalized* if $\inf D = \mathrm{depth}\, R$, see [5, 2.5]. There is an equality $\mathrm{depth}_R Y = \inf Y^{\dagger}$ for all $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$, when the dual $Y^{\dagger}$ is taken with respect to a normalized dualizing complex, see [9, 3.1(a) and 3.2(a)]. Any dualizing complex can be normalized by an appropriate suspension.

6.4. Corollary. *Let R be a commutative noetherian and local ring, and let D be a normalized dualizing complex for R. The next equalities hold for $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$,*

$$\mathrm{Gid}_R Y = \mathrm{Gpd}_R Y^{\dagger} = \mathrm{Gfd}_R Y^{\dagger},$$

where $Y^{\dagger} = \mathbf{R}\mathrm{Hom}_R(Y, D)$.

Proof. By Observation 6.2 the three dimensions $\mathrm{Gid}_R Y$, $\mathrm{Gpd}_R Y^{\dagger}$, and $\mathrm{Gfd}_R Y^{\dagger}$ are simultaneously finite, and in this case (1.6.1) and Theorem 6.3 give:

$$\mathrm{Gid}_R\, Y = \mathrm{Gid}_R\, Y^{\dagger\dagger} = \mathrm{depth}\, R - \inf Y^{\dagger\dagger} = \mathrm{depth}\, R - \mathrm{depth}_R\, Y^{\dagger},$$

where the last equality uses that the dualizing complex is normalized. By [8, Theorem 2.3.13] we have $\mathrm{G\text{-}dim}_R\, Y^{\dagger} = \mathrm{depth}\, R - \mathrm{depth}_R\, Y^{\dagger}$, and $\mathrm{G\text{-}dim}_R\, Y^{\dagger} = \mathrm{Gpd}_R\, Y^{\dagger} = \mathrm{Gfd}_R\, Y^{\dagger}$ by Proposition 3.8. □

Theorem 6.3 is a Gorenstein version of Bass' formula for injective dimension of finitely generated modules. In [7] Chouinard proved a related formula:

$$\mathrm{id}_R\, N = \sup\{\mathrm{depth}\, R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}}\, N_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, R\}$$

for any module N of finite injective dimension over a commutative noetherian ring. In the following, we extend this formula to Gorenstein injective dimension over a ring with dualizing complex. The first result in this direction is inspired by [32, Theorem 8.6].

6.5. Theorem. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring with a dualizing complex. Denote by $\mathrm{E}_R(k)$ the injective hull of the residue field. For a complex $Y \in \mathsf{D}_{\square}(R)$ of finite Gorenstein injective dimension, the next equality holds,*

$$\mathrm{width}_R\, Y = \mathrm{depth}\, R + \inf \mathbf{R}\mathrm{Hom}_R\big(\mathrm{E}_R(k), Y\big).$$

In particular, $\mathrm{width}_R\, Y$ and $\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y)$ are simultaneously finite.

Proof. By Theorem 4.4, Y is in $\mathsf{B}(R)$; in particular, $Y \simeq D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)$. Furthermore, we can assume that D is a normalized dualizing complex, in which case we have $\mathbf{R}\Gamma_{\mathfrak{m}} D \simeq \mathrm{E}_R(k)$ by [29, Proposition V.6.1]. We compute as follows:

$$\begin{aligned}
\mathrm{width}_R\, Y &= \mathrm{width}_R\big(D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, Y)\big) \\
&= \mathrm{width}_R\, D + \mathrm{width}_R\, \mathbf{R}\mathrm{Hom}_R(D, Y) \\
&= \inf D + \inf \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(D, Y) \\
&= \mathrm{depth}\, R + \inf \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} D, Y) \\
&= \mathrm{depth}\, R + \inf \mathbf{R}\mathrm{Hom}_R\big(\mathrm{E}_R(k), Y\big).
\end{aligned}$$

The second equality is by [49, Theorem 2.4(b)], the third is by (6.1.1) and [24, Theorem 2.11], while the penultimate one is by adjointness, cf. 5.8. □

6.6. Corollary. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian and local ring with a dualizing complex. If N is a Gorenstein injective module, then*

$$\mathrm{width}_R\, N \geqslant \mathrm{depth}\, R,$$

and equality holds if $\mathrm{width}_R\, N$ is finite.

Proof. We may assume that $\operatorname{width}_R N$ is finite. By Theorem 6.5 the module $\operatorname{Hom}_R(\mathrm{E}_R(k), N) \simeq \mathbf{R}\operatorname{Hom}_R(\mathrm{E}_R(k), N)$ is nonzero and the equality is immediate by the same theorem.   □

6.7. Corollary. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian and local ring, and let D be a normalized dualizing complex for R. If $Y \in \mathsf{D}_\square^{\mathsf{f}}(R)$ has finite Gorenstein injective dimension, then*

$$\operatorname{Gid}_R Y = -\inf \mathbf{R}\operatorname{Hom}_R\big(\mathrm{E}_R(k), Y\big) = -\inf \mathbf{R}\operatorname{Hom}_R(D, Y).$$

Proof. The first equality comes from the computation,

$$\operatorname{Gid}_R Y = \operatorname{depth} R - \inf Y = \operatorname{depth} R - \operatorname{width}_R Y = -\inf \mathbf{R}\operatorname{Hom}_R\big(\mathrm{E}_R(k), Y\big),$$

which uses Theorem 6.3, (6.1.1), and Theorem 6.5. For the second equality in the corollary, we note that

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_R\big(\mathrm{E}_R(k), Y\big) &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\varGamma_{\mathfrak{m}} D, Y) \\
&\simeq \mathbf{L}\varLambda^{\mathfrak{m}}\, \mathbf{R}\operatorname{Hom}_R(D, Y) \\
&\simeq \mathbf{R}\operatorname{Hom}_R(D, Y) \otimes_R^{\mathbf{L}} \hat{R}.
\end{aligned}$$

Here the second isomorphism is by adjointness, cf. 5.8, and the last one is by [24, Proposition 2.7] as $\mathbf{R}\operatorname{Hom}_R(D, Y)$ is in $\mathsf{D}_{\sqsupset}^{\mathsf{f}}(R)$. Since $\hat{R}$ is faithfully flat, the complexes $\mathbf{R}\operatorname{Hom}_R(\mathrm{E}_R(k), Y)$ and $\mathbf{R}\operatorname{Hom}_R(D, Y)$ must have the same infimum.   □

6.8. Theorem. *Let R be a commutative and noetherian ring with a dualizing complex. If $Y \in \mathsf{D}_\square(R)$ has finite Gorenstein injective dimension, then*

$$\operatorname{Gid}_R Y = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Proof. First, we show "$\geqslant$." For any prime ideal $\mathfrak{p}$ of R, Proposition 5.5 and Theorem 6.3 give the desired inequality,

$$\operatorname{Gid}_R Y \geqslant \operatorname{Gid}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \geqslant \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}.$$

For the converse inequality, "$\leqslant$," we may assume that $\mathrm{H}(Y) \neq 0$. Set $s = \sup Y$ and $g = \operatorname{Gid}_R Y$; by Theorem 3.3 we may assume that Y has the form

$$0 \to I_s \to I_{s-1} \to \cdots \to I_{-g+1} \to B_{-g} \to 0,$$

where the I's are injective and B_{-g} is Gorenstein injective. Proving the inequality amounts to finding a prime ideal $\mathfrak{p}$ of R such that $\operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \leqslant \operatorname{depth} R_{\mathfrak{p}} - g$.

Case $s = -g$. For any integer m,

$$\mathrm{Gid}_R\big(\Sigma^m Y\big) = \mathrm{Gid}_R Y - m \quad \text{and} \quad \mathrm{width}_{R_{\mathfrak{p}}}\big(\Sigma^m Y\big)_{\mathfrak{p}} = \mathrm{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} + m, \qquad (*)$$

so we can assume that $s = -g = 0$, in which case Y is a Gorenstein injective module. By [21, Lemma 2.6] there exists a prime ideal $\mathfrak{p}$, such that the homology of $k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}$ is nontrivial; in particular,

$$\mathrm{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} = \inf\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}\big)$$

is finite. By Proposition 5.5 the $R_{\mathfrak{p}}$-module $Y_{\mathfrak{p}}$ is Gorenstein injective, so $\mathrm{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} = \mathrm{depth}\, R_{\mathfrak{p}}$ by Corollary 6.6.

Case $s = -g + 1$. We may, cf. $(*)$, assume that $s = -1$ and $g = 2$. That is,

$$Y = 0 \to I_{-1} \xrightarrow{\beta} B_{-2} \to 0.$$

From the complete injective resolution of the Gorenstein injective B_{-2}, we get a short exact sequence

$$0 \to B' \to H_{-2} \xrightarrow{\alpha} B_{-2} \to 0,$$

where H_{-2} is injective and B' is Gorenstein injective. Consider the pull-back:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \longrightarrow & B_{-1} & \xrightarrow{\alpha'} & I_{-1} & \longrightarrow & 0 \\
  &   & \Big\| &   & \Big\downarrow{\beta'} &   & \Big\downarrow{\beta} &   &   \\
0 & \longrightarrow & B' & \longrightarrow & H_{-2} & \xrightarrow{\alpha} & B_{-2} & \longrightarrow & 0
\end{array}
$$

The rows are exact sequences, and the top one shows that the module B_{-1} is Gorenstein injective. By the snake lemma β' and β have isomorphic kernels and cokernels; that is, $\mathrm{Ker}\,\beta' \cong \mathrm{Ker}\,\beta = \mathrm{H}_{-1}(Y)$ and $\mathrm{Coker}\,\beta' \cong \mathrm{Coker}\,\beta = \mathrm{H}_{-2}(Y)$. Thus, the homomorphisms α' and α make up a quasi-isomorphism of complexes, and we have $Y \simeq 0 \to B_{-1} \xrightarrow{\beta'} H_{-2} \to 0$. Similarly, there is a short exact sequence of modules $0 \to B \to H_{-1} \xrightarrow{\gamma} B_{-1} \to 0$, where H_{-1} is injective and B is Gorenstein injective. The diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & H_{-1} & \xrightarrow{\beta'\gamma} & H_{-2} & \longrightarrow & 0 \\
  &   & \Big\downarrow &   & \Big\downarrow{\gamma} &   & \Big\| &   &   \\
0 & \longrightarrow & B_{-1} & \longrightarrow & H_{-2} & \xrightarrow{\beta'} & H_{-2} & \longrightarrow & 0
\end{array}
$$

is commutative, as $\mathrm{Ker}\,\gamma = B$, and shows that the vertical maps form a surjective morphism of complexes. The kernel of this morphism $(0 \to B \to B \to 0)$ is exact, so it is

a homology isomorphism, and we may replace Y with the top row of the diagram. Set $H = 0 \to H_{-1} \to H_{-2} \to 0$, then the natural inclusion of H into Y yields a short exact sequence of complexes $0 \to H \to Y \to B \to 0$. By construction $\mathrm{id}_R H \leqslant 2$. To see that equality holds, let J be an R-module such that $\mathrm{H}_{-2}(\mathbf{R}\mathrm{Hom}_R(J, Y)) \neq 0$, cf. Theorem 3.3, and inspect the exact sequence of homology modules

$$\mathrm{H}_{-2}\big(\mathbf{R}\mathrm{Hom}_R(J, H)\big) \to \mathrm{H}_{-2}\big(\mathbf{R}\mathrm{Hom}_R(J, Y)\big) \to \mathrm{H}_{-2}\big(\mathbf{R}\mathrm{Hom}_R(J, B)\big) = 0.$$

By the Chouinard formula for injective dimension [49, Theorem 2.10] we can now choose a prime ideal $\mathfrak{p}$ such that $\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} H_\mathfrak{p} = 2$. Set $d = \mathrm{depth}\, R_\mathfrak{p}$ and consider the exact sequence:

$$\mathrm{H}_{d-1}\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} B_\mathfrak{p}\big) \to \mathrm{H}_{d-2}\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} H_\mathfrak{p}\big) \to \mathrm{H}_{d-2}\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y_\mathfrak{p}\big).$$

The left-hand module is 0 by 5.5 and 6.6, while the middle one is nonzero by choice of $\mathfrak{p}$. This forces $\mathrm{H}_{d-2}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y_\mathfrak{p}) \neq 0$ and therefore $\mathrm{width}_{R_\mathfrak{p}} Y_\mathfrak{p} \leqslant d - 2$ as desired.

Case $s > -g + 1$. We may assume that $g = 1$ and $s > 0$; i.e., Y has the form

$$0 \to I_s \to \cdots \to I_1 \to I_0 \to B_{-1} \to 0,$$

where the I's are injective and B_{-1} is Gorenstein injective. Set $I = Y_1 \sqsupset$ and $Y' = \sqsubset_0 Y$, then we have an exact sequence of complexes $0 \to Y' \to Y \to I \to 0$. Recycling the argument applied to H above, we see that $\mathrm{Gid}_R Y' = 1$. As $-1 \leqslant \sup Y' \leqslant 0$, it follows by the preceding cases, that we can choose a prime ideal $\mathfrak{p}$ such that $\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} Y'_\mathfrak{p} = 1$. Set $d = \mathrm{depth}\, R_\mathfrak{p}$ and consider the exact sequence of homology modules

$$\mathrm{H}_d\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} I_\mathfrak{p}\big) \to \mathrm{H}_{d-1}\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y'_\mathfrak{p}\big) \to \mathrm{H}_{d-1}\big(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y_\mathfrak{p}\big).$$

By construction $\mathrm{id}_{R_\mathfrak{p}} I_\mathfrak{p} \leqslant -1$, so the left-hand module is 0 by the classical Chouinard formula. The module in the middle is nonzero by choice of $\mathfrak{p}$, and this forces $\mathrm{H}_{d-1}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y_\mathfrak{p}) \neq 0$, which again implies $\mathrm{width}_{R_\mathfrak{p}} Y_\mathfrak{p} \leqslant d - 1$ as desired. $\quad\square$

The final result is parallel to Theorem 5.7.

6.9. Theorem. *Let R be a commutative and noetherian ring with a dualizing complex. A colimit of Gorenstein injective R-modules indexed by a filtered set is Gorenstein injective.*

Proof. Let $B_i \to B_j$ be a filtered, direct system of Gorenstein injective modules. By Theorem 4.4 all the B_i's belong to $\mathsf{B}(R)$, so by Lemma 5.6 also the colimit $\varinjlim B_i$ is in $\mathsf{B}(R)$. That is, $\mathrm{Gid}_R \varinjlim B_i < \infty$. Since tensor products [39, Theorem A1] and homology commute with filtered colimits [46, Theorem 2.6.15], we have

$$\mathrm{width}_{R_\mathfrak{p}} (\varinjlim B_i)_\mathfrak{p} \geqslant \inf_i \big\{ \mathrm{width}_{R_\mathfrak{p}} (B_i)_\mathfrak{p} \big\},$$

for each prime ideal $\mathfrak{p}$. By Theorem 6.8 we now have

$$
\begin{aligned}
\mathrm{Gid}_R \varinjlim B_i &= \sup\big\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}}(\varinjlim B_i)_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\big\} \\
&\leqslant \sup\big\{\mathrm{depth}\, R_\mathfrak{p} - \inf_i\big\{\mathrm{width}_{R_\mathfrak{p}}(B_i)_\mathfrak{p}\big\} \mid \mathfrak{p} \in \mathrm{Spec}\, R\big\} \\
&= \sup_i\big\{\sup\big\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}}(B_i)_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\big\}\big\} \\
&= \sup_i\{\mathrm{Gid}_R B_i\} = 0. \qquad \square
\end{aligned}
$$

The Chouinard formula, Theorem 6.8, plays a crucial role in the proof above. Indeed, it is not clear from the formulas in 3.3 and 3.4 that $\mathrm{Gid}_R \varinjlim B_i \leqslant \sup_i\{\mathrm{Gid}_R B_i\}$. In [7] Chouinard proved a similar formula for modules of finite flat dimension. Also this has been extended to Gorenstein flat dimension: for modules in [30, Theorem 3.19] and [10, Theorem 2.4(b)], and for complexes in [32, Theorem 8.8].

Acknowledgments

Appendix A. Dualizing complexes

Dualizing complexes over noncommutative rings are a delicate matter. The literature contains a number of different, although related, extensions of Grothendieck's original definition [29, V.§2] to the noncommutative realm. Yekutieli [50] introduced dualizing complexes for associative $\mathbb{Z}$-graded algebras over a field. Later, Yekutieli–Zhang [51] gave a definition for pairs of noncommutative algebras over a field which has been used by, among others, Jørgensen [35] and Wu–Zhang [47]. Related definitions can be found in Frankild–Iyengar–Jørgensen [25] and Miyachi [40].

Definition 1.1 is inspired by Miyachi [40, p. 156] and constitutes an extension of Yekutieli–Zhang's [51, Definition 1.1]: They consider a noetherian pair $\langle S, R \rangle$ of algebras over a field k; a complex $D \in \mathsf{D}_\square(S \otimes_k R^{\mathrm{opp}})$ is said to be dualizing for $\langle S, R \rangle$ if:

(i) D has finite injective dimension over S and R^{opp}.
(ii) The homology of D is degreewise finitely generated over S and R^{opp}.
(iii) The homothety morphisms $S \to \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}(D, D)$ in $\mathsf{D}(S \otimes_k S^{\mathrm{opp}})$ and $R \to \mathbf{R}\mathrm{Hom}_S(D, D)$ in $\mathsf{D}(R \otimes_k R^{\mathrm{opp}})$ are isomorphisms.

As also noted in [51], condition (i) is equivalent to:

(i′) There exists a quasi-isomorphism $D \xrightarrow{\simeq} I$ in $\mathsf{D}_\square(S \otimes_k R^{\mathrm{opp}})$ such that each I_ℓ is injective over S and R^{opp}.

Even more is true: The canonical ring homomorphisms $S \to S \otimes_k R^{\mathrm{opp}} \leftarrow R^{\mathrm{opp}}$ give restriction functors,

$$\mathsf{C}(S) \leftarrow \mathsf{C}\big(S \otimes_k R^{\mathrm{opp}}\big) \to \mathsf{C}\big(R^{\mathrm{opp}}\big).$$
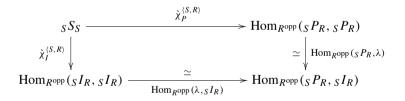
Since k is a field, these restriction functors are exact, cf. [50, p. 45], and thus they send quasi-isomorphisms to quasi-isomorphisms. They also send projective/injective modules to projective/injective modules, cf. [50, Lemma 2.1]. Consequently, a projective/injective resolution of D in $\mathsf{C}(S \otimes_k R^{\mathrm{opp}})$ restricts to a projective/injective resolution of D in $\mathsf{C}(S)$ and in $\mathsf{C}(R^{\mathrm{opp}})$. Thus, in the setting of [51], there automatically exist quasi-isomorphisms of complexes of (S, R^{opp})-bimodules,

$$\mathsf{C}_{\sqsupset}\big(S \otimes_k R^{\mathrm{opp}}\big) \ni P \xrightarrow{\simeq} D \quad \text{and} \quad D \xrightarrow{\simeq} I \in \mathsf{C}_{\square}\big(S \otimes_k R^{\mathrm{opp}}\big)$$

such that each P_ℓ (respectively, I_ℓ) is projective (respectively, injective) over S and over R^{opp}. It is also by virtue of these biresolutions of D that the morphisms from (iii) above make sense, cf. [50, p. 52].

In this paper, we work with a noetherian pair of rings, not just a noetherian pair of algebras over a field. Without the underlying field k, the existence of appropriate biresolutions does not come for free. Therefore, the existence of such resolutions has been made part of the very definition of a dualizing complex, cf. 1.1(2, 3).

Also a few remarks about Definition 1.1(4) are in order.[5] Let ${}_S P_R \xrightarrow{\simeq} {}_S D_R$ and ${}_S D_R \xrightarrow{\simeq} {}_S I_R$ be as in 1.1(2, 3) and let $\lambda : {}_S P_R \xrightarrow{\simeq} {}_S I_R$ be the composite; note that λ is S- and R^{opp}-linear. Consider the diagram of complexes of (S, S^{opp})-bimodules,

$$
\begin{array}{ccc}
{}_S S_S & \xrightarrow{\ \hat{\chi}_P^{\langle S,R\rangle}\ } & \mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, {}_S P_R) \\
{\scriptstyle \hat{\chi}_I^{\langle S,R\rangle}}\Big\downarrow & & \simeq\Big\downarrow{\scriptstyle \mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, \lambda)} \\
\mathrm{Hom}_{R^{\mathrm{opp}}}({}_S I_R, {}_S I_R) & \xrightarrow[\ \mathrm{Hom}_{R^{\mathrm{opp}}}(\lambda, {}_S I_R)\ ]{\simeq} & \mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, {}_S I_R)
\end{array}
$$

Note that the following facts:

- $\hat{\chi}_P^{\langle S,R\rangle}$ and $\hat{\chi}_I^{\langle S,R\rangle}$ are both S-linear and S^{opp}-linear,
- $\mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, \lambda)$ is S^{opp}-linear, and
- $\mathrm{Hom}_{R^{\mathrm{opp}}}(\lambda, {}_S I_R)$ is S-linear

are immediate consequences of the (S, S^{opp})-bistructures on

$$\mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, {}_S P_R), \quad \mathrm{Hom}_{R^{\mathrm{opp}}}({}_S P_R, {}_S I_R) \quad \text{and} \quad \mathrm{Hom}_{R^{\mathrm{opp}}}({}_S I_R, {}_S I_R).$$

---

[5] These remarks are parallel to Yekutieli's considerations [50, p. 52] about well-definedness of derived functors between derived categories of bimodules.

Moreover, the S-linearity of λ makes the above diagram commutative. To see this, observe that $(\mathrm{Hom}_{R^{\mathrm{opp}}}(\lambda, {}_SI_R) \circ \grave{\chi}_I^{\langle S,R \rangle})(s)$ and $(\mathrm{Hom}_{R^{\mathrm{opp}}}({}_SP_R, \lambda) \circ \grave{\chi}_P^{\langle S,R \rangle})(s)$ yield maps

$$ {}_SP_R \xrightarrow{\ \lambda\ } {}_SI_R \xrightarrow{\ s\cdot-\ } {}_SI_R \quad \text{and} \quad {}_SP_R \xrightarrow{\ s\cdot-\ } {}_SP_R \xrightarrow{\ \lambda\ } {}_SI_R, $$

respectively, where $s \cdot -$ denotes left-multiplication with a generic element in S. A similar analysis shows that the S-linearity of λ implies both S-linearity of $\mathrm{Hom}_{R^{\mathrm{opp}}}({}_SP_R, \lambda)$ and S^{opp}-linearity of $\mathrm{Hom}_{R^{\mathrm{opp}}}(\lambda, {}_SI_R)$.

Since ${}_SP_R$ is a right-bounded complex of projective R^{opp}-modules, $\mathrm{Hom}_{R^{\mathrm{opp}}}({}_SP_R, \lambda)$ is a quasi-isomorphism. Similarly, $\mathrm{Hom}_{R^{\mathrm{opp}}}(\lambda, {}_SI_R)$ is a quasi-isomorphism. Consequently,

$$ \grave{\chi}_P^{\langle S,R \rangle} \text{ is a quasi-isomorphism} \quad \Leftrightarrow \quad \grave{\chi}_I^{\langle S,R \rangle} \text{ is a quasi-isomorphism.} $$

When we in 1.1(4) require that $\grave{\chi}_D^{\langle S,R \rangle} : {}_SS_S \to \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}({}_SD_R, {}_SD_R)$ is invertible in $\mathsf{D}(S)$ (equivalently, invertible in $\mathsf{D}(S^{\mathrm{opp}})$), it means that $\grave{\chi}_P^{\langle S,R \rangle}$ is a quasi-isomorphism of S-complexes (equivalently, of S^{opp}-complexes).

Similar remarks apply to the morphism $\acute{\chi}_D^{\langle S,R \rangle} : {}_RR_R \to \mathbf{R}\mathrm{Hom}_S({}_SD_R, {}_SD_R)$; here it becomes important that λ is R^{opp}-linear.

Proof of 1.2. By symmetry it suffices to prove that a dualizing complex D for a noetherian pair $\langle S, R \rangle$ is dualizing for $\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle$ as well. Obviously, R^{opp} is left noetherian and S^{opp} is right noetherian, so $\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle$ is a noetherian pair. Furthermore, since (S, R^{opp})-bimodules are naturally identified with $(R^{\mathrm{opp}}, (S^{\mathrm{opp}})^{\mathrm{opp}})$-bimodules, it is clear that D satisfies conditions (1), (2), and (3) in 1.1 relative to $\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle$. Finally, we need to see that the homothety morphisms are invertible. The morphism

$$ \acute{\chi}_P^{\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle} : S^{\mathrm{opp}} \to \mathrm{Hom}_{R^{\mathrm{opp}}}(P, P) $$

is identical to $\grave{\chi}_P^{\langle S,R \rangle} : S \to \mathrm{Hom}_{R^{\mathrm{opp}}}(P, P)$ through the identification $S = S^{\mathrm{opp}}$ (as (S, S^{opp})-bimodules, not as rings). By assumption $\grave{\chi}_P^{\langle S,R \rangle}$ is a quasi-isomorphism, and hence so is $\acute{\chi}_P^{\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle}$. Similarly, $\grave{\chi}_P^{\langle R^{\mathrm{opp}}, S^{\mathrm{opp}} \rangle}$ is identified with $\acute{\chi}_P^{\langle S,R \rangle}$. $\square$

The next result was also stated in Section 1 (Proposition 1.5); this time we prove it.

A.1. Proposition. *Assume that the noetherian pair $\langle S, R \rangle$ has a dualizing complex ${}_SD_R$. If $X \in \mathsf{D}(R)$ has finite $\mathrm{fd}_R X$, then there is an inequality,*

$$ \mathrm{pd}_R X \leqslant \max\{\mathrm{id}_S({}_SD_R) + \sup({}_SD_R \otimes_R^{\mathbf{L}} X),\ \sup X\} < \infty. $$

Moreover, $\mathrm{FPD}(R)$ *is finite if and only if* $\mathrm{FFD}(R)$ *is finite.*

Proof. Define the integer n by

$$ n = \max\{\mathrm{id}_S({}_SD_R) + \sup({}_SD_R \otimes_R^{\mathbf{L}} X),\ \sup X\}. $$

Let $Q \xrightarrow{\simeq} X$ be a projective resolution of X. Since $\infty > n \geqslant \sup X = \sup Q$, we have a quasi-isomorphism $Q \xrightarrow{\simeq} \subset_n Q$, and hence it suffices to prove that the R-module $C_n^Q = \text{Coker}(Q_{n+1} \to Q_n)$ is projective. This is tantamount to showing that $\text{Ext}_R^1(C_n^Q, C_{n+1}^Q) = 0$. We have the following isomorphisms of abelian groups:

$$\text{Ext}_R^1\big(C_n^Q, C_{n+1}^Q\big) \cong \text{H}_{-(n+1)} \, \textbf{R}\text{Hom}_R\big(X, C_{n+1}^Q\big)$$

$$\cong \text{H}_{-(n+1)} \, \textbf{R}\text{Hom}_R\big(X, \textbf{R}\text{Hom}_S\big({}_S D_R, {}_S D_R \otimes_R^{\textbf{L}} C_{n+1}^Q\big)\big)$$

$$\cong \text{H}_{-(n+1)} \, \textbf{R}\text{Hom}_S\big({}_S D_R \otimes_R^{\textbf{L}} X, {}_S D_R \otimes_R^{\textbf{L}} C_{n+1}^Q\big).$$

The first isomorphism follows as $n \geqslant \sup X$, and the second one follows as $\text{fd}_R \, C_{n+1}^Q$ is finite, and hence $C_{n+1}^Q \in \textsf{A}(R)$ by 1.3. The third isomorphism is by adjointness. It is now sufficient to show that

$$-\inf \textbf{R}\text{Hom}_S\big({}_S D_R \otimes_R^{\textbf{L}} X, {}_S D_R \otimes_R^{\textbf{L}} C_{n+1}^Q\big) \leqslant n,$$

and this follows as:

$$-\inf \textbf{R}\text{Hom}_S\big({}_S D_R \otimes_R^{\textbf{L}} X, {}_S D_R \otimes_R^{\textbf{L}} C_{n+1}^Q\big)$$

$$\leqslant \text{id}_S\big({}_S D_R \otimes_R^{\textbf{L}} C_{n+1}^Q\big) + \sup\big({}_S D_R \otimes_R^{\textbf{L}} X\big)$$

$$\leqslant \text{id}_S({}_S D_R) + \sup\big({}_S D_R \otimes_R^{\textbf{L}} X\big)$$

$$\leqslant n.$$

The first inequality is by [4, Theorem 2.4.I], and the second one is by [4, Theorem 4.5(F)], as S is left noetherian and $\text{fd}_R \, C_{n+1}^Q$ is finite.

The last claim follows as

$$\text{FFD}(R) \leqslant \text{FPD}(R) \leqslant \text{FFD}(R) + \text{id}_S({}_S D_R) + \sup({}_S D_R).$$

The first inequality is by [33, Proposition 6]. To verify the second one, let M be a module with $\text{pd}_R \, M$ finite. We have already seen that

$$\text{pd}_R \, M \leqslant \max\big\{\text{id}_S({}_S D_R) + \sup\big({}_S D_R \otimes_R^{\textbf{L}} M\big), 0\big\},$$

so it suffices to see that

$$\sup\big({}_S D_R \otimes_R^{\textbf{L}} M\big) \leqslant \text{fd}_R \, M + \sup({}_S D_R),$$

and that follows from [4, Theorem 2.4.F]. $\quad\square$

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