

LINEAR STABILITY ANALYSIS AND HOMOCLINIC ORBIT FOR A GENERALIZED NON-LINEAR HEAT TRANSFER

by

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Short paper

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This paper studies the linear stability and dynamic structure for a generalized non-linear heat equation, and obtains novel analytic solutions such as homoclinic orbit and breather solitary solutions for the first time based on Hirota method.

Key words: non-linear equation, homoclinic orbit, breather solitary

Introduction

The linear heat equation can be expressed in the form:

$$u_t = u_{xx} \quad (1)$$

or equivalently:

$$u_{tt} = u_{xxxx} \quad (2)$$

Introducing a non-linear term to eq. (2) results in:

$$u_{tt} - u_{xxxx} - f(u_t, u_x) = 0 \quad (3)$$

where the non-linear term is a function of u_t and u_x . For simplicity, we write:

$$f(u_t, u_x) = u_t u_x \quad (4)$$

or

$$f(u_t, u_x) = u_x u_{xx} \quad (5)$$

We, therefore, obtain the following non-linear heat equation:

$$u_{tt} - u_x u_{xx} - u_{xxxx} = 0 \quad (6)$$

This equation has some special solution properties, which can explain some thermal phenomena.

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Linear stability analysis

Consider a periodic boundary condition:

$$u(x,t) = u\left(x + \frac{2\pi}{p}, t\right), \quad t \geq 0 \tag{7}$$

where p is a constant.

The periodic condition arises in many thermal problems, for example, heat conduction in an oscillation aerofoil and nuclear reaction.

Assume that u_0 is the solution of eq. (6), and we consider a small perturbation of the solution in the form:

$$u(x,t) = u_0[1 + \varepsilon(x,t)] \tag{8}$$

Substituting (8) into (6) and keeping only the linear term, we obtain:

$$\varepsilon_{tt} - \varepsilon_{xxxx} = 0 \tag{9}$$

Introducing a new variable v defined as $v = \varepsilon_t$, we have:

$$\begin{bmatrix} \varepsilon \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ k^4 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon \\ v \end{bmatrix} \tag{10}$$

where $\varepsilon_{xx} = -k^2\varepsilon$, then the eigenvalues for this matrix are:

$$\lambda = \pm k^2 \tag{11}$$

Exact homoclinic orbits and solitary solutions

Equation (6) can be solved by various methods [1-5], such as the homotopy perturbation method [3], the exp-function method [4, 5]. According to the Hirota method [1, 2], we introduce a transformation in the form:

$$u = 12(\ln \phi)_x \tag{12}$$

Equation (6) can be transformed into the following bi-linear equation:

$$(D_t^2 - D_x^4 + c)\phi \cdot \phi = 0 \tag{13}$$

where c is the integration constant and the Hirota bi-linear operator D is defined by:

$$D_x^m D_t^s = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^s \Bigg|_{x'=x, t'=t} \tag{14}$$

In order to explain the periodic thermal shock, we search for a solution in the form:

$$\phi = 1 + b_1 \cos(px + \delta)e^{\omega t + \gamma} + b_2 e^{2\omega t + 2\gamma} \tag{15}$$

where b_1, b_2, p , and ω are real parameters to be determined and δ and γ are arbitrary constants. Substitution of (15) into (13) leads to the following relations:

$$b_2 = b_1^2, \quad p^2 = \omega^2 \tag{16}$$

We, therefore, obtain the following solitary solution:

$$u = \frac{\pm 12b_1 \omega \sin(px + \delta)e^{\omega t + \gamma}}{1 + b_1 \cos(px + \delta)e^{\omega t + \gamma} + b_1^2 e^{2\omega t + 2\gamma}} \tag{17}$$

where b_1 , and ω are arbitrary constants.

It is obvious that u tends to zero when $t \rightarrow +\infty$ or $t \rightarrow -\infty$, and it is a homoclinic orbit of eq. (6), its dynamic structure is shown in fig. 1.

The non-linear heat equation behaves sometimes also soliton in the form:

$$\phi = 1 + b_1 \cos(p_1x + c_1t)e^{p_2x + c_2t} + b_2e^{2p_2x + 2c_2t} \tag{18}$$

where b_1, b_2, p_1, p_2, c_1 , and c_2 are real parameters to be determined. Substitution (18) into (6) leads to the following relations:

$$\begin{aligned} c_1 &= -2p_1p_2, \quad c_2 = p_1^2 - p_2^2, \\ b_2 &= \frac{b_1^2 p_1^2}{p_1^2 - 3p_2^2} \end{aligned} \tag{19}$$

or

$$c_1 = 2p_1p_2, \quad c_2 = p_2^2 - p_1^2, \quad b_2 = \frac{b_1^2 p_1^2}{p_1^2 - 3p_2^2} \tag{20}$$

We can get a new class of solitary solutions in the form:

$$u = \frac{12b_1e^{p_2x + c_2t} [p_2 \cos(p_1x + c_1t) - p_1 \sin(p_1x + c_1t)] + 24b_2p_2e^{2p_2x + 2c_2t}}{1 + b_1 \cos(p_1x + c_1t)e^{p_2x + c_2t} + b_2e^{2p_2x + 2c_2t}} \tag{21}$$

where p_1, p_2 , and b_1 are arbitrary constants. Its dynamic structure is illustrated in fig. 2.

The criterion for a non-singular solution is $p_1^2 > 3p_2^2$.

Conclusions

The non-linear heat equation reveals some special thermal properties, the thermal shock can be either periodic solitary wave (fig. 1) or various solitary waves (fig. 2).

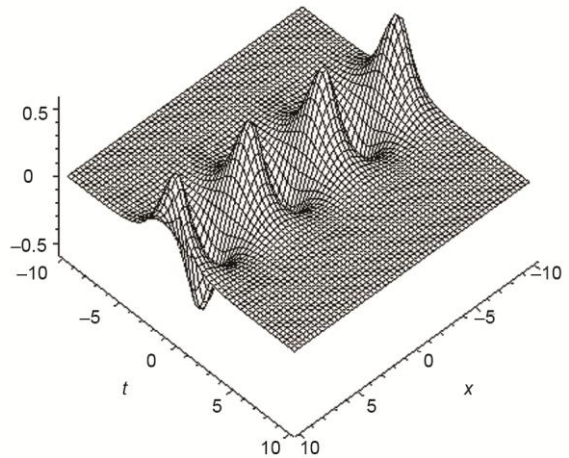


Figure 1. Homoclinic orbit of the generalized non-linear heat equation

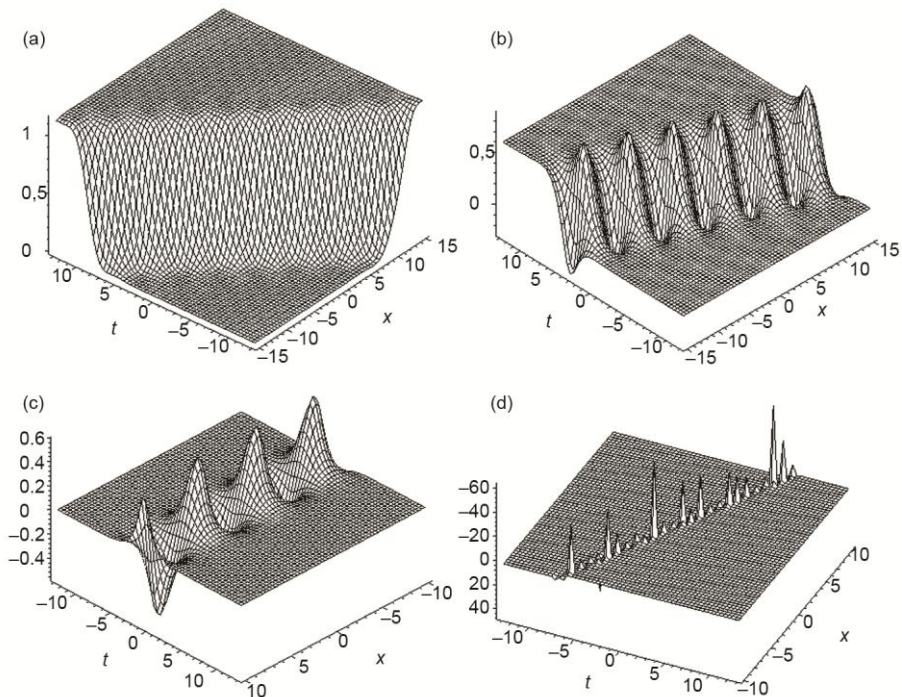


Figure 2. Solitary solution of the generalized non-linear heat equation; (a) kink solution with $b_1 = 4$, $p_1 = 1$, $p_2 = 0.57$, (b) kink-breath solution $b_1 = 4$, $p_1 = 1$, $p_2 = 0.3$, (c) breath solution with $b_1 = 4$, $p_1 = 1$, $p_2 = 0.01$, (d) singular solution with $b_1 = 4$, $p_1 = 1$, $p_2 = 0.6$

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