

ANDRUSYAK I.V.<sup>1</sup>, FILEVYCH P.V.<sup>2</sup>

## RADIAL BOUNDARY VALUES OF LACUNARY POWER SERIES

We strengthened MacLane's theorem concerning radial boundary values of lacunary power series.

*Key words and phrases:* analytic function, lacunary power series, radial boundary value, asymptotic value.

<sup>1</sup> Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine

<sup>2</sup> Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine

E-mail: andrusyak.ivanna@gmail.com (Andrusyak I.V.), filevych@mail.ru (Filevych P.V.)

## INTRODUCTION

Denote by  $\mathcal{H}$  the class of analytic functions on the unite disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  and let  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . As usual, a value  $v \in \overline{\mathbb{C}}$  is called the radial boundary value of a function  $f \in \mathcal{H}$  at a point  $e^{i\theta} \in \partial\mathbb{D}$  if

$$\lim_{r \uparrow 1} f(re^{i\theta}) = v.$$

By  $\mathcal{R}$  we denote the class of functions  $f \in \mathcal{H}$  having radial boundary values on a dense set of points  $e^{i\theta}$  of  $\partial\mathbb{D}$ . A value  $v \in \overline{\mathbb{C}}$  is called an asymptotic value of a function  $f \in \mathcal{H}$  at a point  $\omega \in \partial\mathbb{D}$  if there exists a path  $\gamma : z = z(t), t \in [0, 1]$ , such that  $z(t) \in \mathbb{D}$  for all  $t \in [0, 1)$ ,  $z(1) = \omega$  and

$$\lim_{t \uparrow 1} f(z(t)) = v.$$

By  $\mathcal{A}$  we denote the MacLane class, i.e. the class of functions  $f \in \mathcal{H}$  having asymptotic values on a dense set of points  $\omega$  of  $\partial\mathbb{D}$ . Clearly,  $\mathcal{R} \subset \mathcal{A}$ . It is well known that this inclusion is strict. Recall that, by the classical Fatou theorem, for any bounded function  $f \in \mathcal{H}$  we have  $f \in \mathcal{R}$  and therefore  $f \in \mathcal{A}$ .

Let  $\Lambda$  be the class of increasing sequences that consists of nonnegative integers  $\lambda = (\lambda_n)$ . For any sequence  $\lambda \in \Lambda$ , let

$$q(\lambda) = \liminf_{n \rightarrow \infty} \frac{\lambda_{n+1}}{\lambda_n}.$$

Denote by  $\mathcal{H}(\lambda)$  the class of functions  $f \in \mathcal{H}$  of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad z \in \mathbb{D}. \quad (1)$$

G.R. MacLane has proved the following theorems (see [1, Theorem 19]).

YΔK 517.53

2010 Mathematics Subject Classification: 30B10, 30B30.

**Theorem A.** Let  $\lambda \in \Lambda$ . If  $q(\lambda) > 3$ , then  $\mathcal{H}(\lambda) \subset \mathcal{R}$ .

**Theorem B.** Let  $\lambda \in \Lambda$ . If  $q(\lambda) > 3$ , then for any function  $f \in \mathcal{H}(\lambda)$  of the form (1) such that

$$\sum_{n=0}^{\infty} |a_n| = +\infty \quad (2)$$

there exists a dense set  $\Theta$  in  $[0, 2\pi]$  such that for any  $\theta \in \Theta$  the following relation holds

$$\lim_{r \uparrow 1} \operatorname{Re} f(re^{i\theta}) = +\infty. \quad (3)$$

Note, that if for a function  $f$  condition (2) is not satisfied, then this function is bounded in  $\mathbb{D}$ . Therefore Theorem A is a consequence of the Fatou theorem and Theorem B. It is also clear that in Theorem B the value  $\operatorname{Re} f(re^{i\theta})$  can be replaced by one of the values  $-\operatorname{Re} f(re^{i\theta})$ ,  $\operatorname{Im} f(re^{i\theta})$  or  $-\operatorname{Im} f(re^{i\theta})$  (it is sufficient to apply this theorem to the functions  $-f$ ,  $-if$  or  $if$  respectively).

If we require only the inclusion  $\mathcal{H}(\lambda) \subset \mathcal{A}$ , then the condition  $q(\lambda) > 3$  can be essentially weakened. This fact follows from the following Murai theorem [2].

**Theorem C.** Let  $\lambda \in \Lambda$ . If  $q(\lambda) > 1$ , then  $\mathcal{H}(\lambda) \subset \mathcal{A}$ .

In connection with the stated results there is a question: *does there exist  $q \in [1, 3]$  such that the condition  $q(\lambda) > q$  is sufficient for the inclusion  $\mathcal{H}(\lambda) \subset \mathcal{R}$ ?*

From our results we can conclude that the condition  $q(\lambda) > 3$  in Theorem A is far from being final. Despite this, an answer to the question posed above is not obtained.

**Theorem 1.** For any  $q > 1$  there exists a sequence  $\lambda \in \Lambda$  such that  $q(\lambda) = q$  and  $\mathcal{H}(\lambda) \subset \mathcal{R}$ .

For a sequence  $\lambda \in \Lambda$  let

$$q_1(\lambda) = \min \left\{ \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}} \right\}, \quad q_2(\lambda) = \max \left\{ \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}} \right\}.$$

Theorem 1 is a direct consequence of the Fatou theorem and Theorem 2 stated below, which strengthens Theorem B.

**Theorem 2.** Let  $\lambda \in \Lambda$ . If

$$(q_1(\lambda) - 1)q_2(\lambda) > 6, \quad (4)$$

then for any function  $f \in \mathcal{H}(\lambda)$  of the form (1) which satisfies condition (2) there exists a dense set  $\Theta$  in  $[0, 2\pi]$  such that for any  $\theta \in \Theta$  equality (3) holds.

#### PROOF OF THEOREM 2

Let for any sequence  $\lambda \in \Lambda$  inequality (4) holds. Put

$$p_1 = \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \quad p_2 = \liminf_{k \rightarrow \infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}}.$$

Suppose that  $p_1 \leq p_2$  (in the case  $p_1 \geq p_2$  our considerations are similar). Then  $q_1(\lambda) = p_1$ ,  $q_2(\lambda) = p_2$ , and condition (4) can be written as  $(p_1 - 1)p_2 > 6$ . It is clear that  $p_1 > 1$  and  $p_2 > 3$ , therefore there exist constants  $q_1 \in (1, p_1)$  and  $q_2 \in (3, p_2)$  such that  $(q_1 - 1)q_2 > 6$ ,

moreover  $q_1 < 3$ . From the definitions of variables  $p_1$  and  $p_2$  it follows that there exists an integer  $k_0 \in \mathbb{N}_0$  such that for all integers  $k \geq k_0$  the following inequalities  $\lambda_{2k+1} \geq q_1 \lambda_{2k}$  and  $\lambda_{2k+2} \geq q_2 \lambda_{2k+1}$  hold.

In what follows for each segment  $I \subset \mathbb{R}$  we denote by  $|I|$ ,  $a(I)$ , and  $b(I)$  its length, the left end and right end respectively.

Consider any segment  $I \subset [0, 2\pi]$  and a function  $f \in \mathcal{H}(\lambda)$  of the form (1), which satisfies condition (2). Let us prove that there exists a point  $\theta$  in the segment  $I$  such that relation (3) holds. Let  $\Theta$  be the set of all  $\theta \in [0, 2\pi]$ , for which (3) holds. Then the set  $\Theta$  is dense in  $[0, 2\pi]$ .

Put

$$\varepsilon = \frac{((q_1 - 1)q_2 - 6)\pi}{(q_1 + 1)q_2 - 2}. \quad (5)$$

It is easy to check that

$$\varepsilon < \frac{(q_1 - 1)\pi}{q_1 + 1}. \quad (6)$$

Take  $\delta = \cos \frac{\pi - \varepsilon}{2}$ . Since  $\varepsilon \in (0, \pi)$ , we have  $\delta > 0$ .

Let  $n \in \mathbb{N}$ ,  $\alpha_n = \arg a_n$ . Then we have  $\cos(\lambda_n \theta + \alpha_n) \geq \delta$  on the union of segments

$$\left[ -\frac{\pi - \varepsilon}{2\lambda_n} + \frac{2\pi m - \alpha_n}{\lambda_n}, \frac{\pi - \varepsilon}{2\lambda_n} + \frac{2\pi m - \alpha_n}{\lambda_n} \right], \quad m \in \mathbb{Z}, \quad (7)$$

of length  $\frac{\pi - \varepsilon}{\lambda_n}$ . Obviously, if  $n_0 = \min \left\{ n \in \mathbb{N} : |I| \geq \frac{3\pi - \varepsilon}{\lambda_n} \right\}$ , then for every integer  $n \geq n_0$  the segment  $I$  contains at least one of the segments (7).

Fix an integer  $m \geq \max \left\{ k_0, \frac{n_0}{2} \right\}$  and let  $I_{2m} \subset I$  be a segment of length  $\frac{\pi - \varepsilon}{\lambda_{2m}}$  such that  $\cos(\lambda_{2m} \theta + \alpha_{2m}) \geq \delta$  for all  $\theta \in I_{2m}$ . By  $\theta_{2m}$  we denote the midpoint of the segment  $I_{2m}$ . Then  $I_{2m} = \left[ \theta_{2m} - \frac{\pi - \varepsilon}{2\lambda_{2m}}, \theta_{2m} + \frac{\pi - \varepsilon}{2\lambda_{2m}} \right]$ .

Let  $\theta_{2m+1}$  be a point in the set  $\{\theta \in \mathbb{R} : \cos(\lambda_{2m+1} \theta + \alpha_{2m+1}) = -1\}$  that is closest to  $\theta_{2m}$ . Clearly,  $|\theta_{2m+1} - \theta_{2m}| \leq \frac{\pi}{\lambda_{2m+1}}$  and  $\cos(\lambda_{2m+1} \theta + \alpha_{2m+1}) \geq \delta$  for each segments

$$S_1 = \left[ \theta_{2m+1} - \frac{3\pi - \varepsilon}{2\lambda_{2m+1}}, \theta_{2m+1} - \frac{\pi + \varepsilon}{2\lambda_{2m+1}} \right], \quad S_2 = \left[ \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}}, \theta_{2m+1} + \frac{3\pi - \varepsilon}{2\lambda_{2m+1}} \right].$$

Put

$$x = \frac{(q_1 - 1)\pi - (q_1 + 1)\varepsilon}{2\lambda_{2m+1}}.$$

Then, according to (6),  $x > 0$ . Let us show that there exists a segment  $I_{2m+1} \subset I_{2m}$  of length  $x$  such that  $\cos(\lambda_{2m+1} \theta + \alpha_{2m+1}) \geq \delta$  for all  $\theta \in I_{2m+1}$ .

If  $\theta_{2m} - \frac{\pi}{\lambda_{2m+1}} \leq \theta_{2m+1} \leq \theta_{2m}$ , then let  $I_{2m+1} = \left[ \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}}, \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}} + x \right]$ . It is clear that  $|I_{2m+1}| = x$  and  $a(I_{2m+1}) = a(S_2)$ . Since  $1 < q_1 < 3$ , we have

$$b(I_{2m+1}) = \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}} + x = \theta_{2m+1} + \frac{q_1(\pi - \varepsilon)}{2\lambda_{2m+1}} < \theta_{2m+1} + \frac{3\pi - \varepsilon}{2\lambda_{2m+1}} = b(S_2).$$

Thus  $I_{2m+1} \subset S_2$ , therefore  $\cos(\lambda_{2m+1} \theta + \alpha_{2m+1}) \geq \delta$  for all  $\theta \in I_{2m+1}$ . In addition,  $I_{2m+1} \subset I_{2m}$ , because

$$\begin{aligned} a(I_{2m+1}) &= \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}} \geq \theta_{2m} - \frac{\pi}{\lambda_{2m+1}} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}} > \theta_{2m} - \frac{\pi - \varepsilon}{2\lambda_{2m}} = a(I_{2m}), \\ b(I_{2m+1}) &= \theta_{2m+1} + \frac{\pi + \varepsilon}{2\lambda_{2m+1}} + x \leq \theta_{2m} + \frac{q_1(\pi - \varepsilon)}{2\lambda_{2m+1}} \leq \theta_{2m} + \frac{\pi - \varepsilon}{2\lambda_{2m}} = b(I_{2m}). \end{aligned}$$

If  $\theta_{2m} \leq \theta_{2m+1} \leq \theta_{2m} + \frac{\pi}{\lambda_{2m+1}}$ , then let  $I_{2m+1} = \left[ \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} - x, \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} \right]$ . It is clear that  $|I_{2m+1}| = x$  and  $b(I_{2m+1}) = b(S_1)$ . Since  $1 < q_1 < 3$ , we obtain

$$a(I_{2m+1}) = \theta_{2m+1} - \frac{\pi + \varepsilon}{2\lambda_{2m+1}} - x = \theta_{2m+1} - \frac{q_1(\pi - \varepsilon)}{2\lambda_{2m+1}} > \theta_{2m+1} - \frac{3\pi - \varepsilon}{2\lambda_{2m+1}} = a(S_1).$$

Thus  $I_{2m+1} \subset S_1$ , therefore  $\cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) \geq \delta$  for all  $\theta \in I_{2m+1}$ . In addition,  $I_{2m+1} \subset I_{2m}$ , because

$$b(I_{2m+1}) = \theta_{2m+1} - \frac{\pi + \varepsilon}{2\lambda_{2m+1}} \leq \theta_{2m} + \frac{\pi}{\lambda_{2m+1}} - \frac{\pi + \varepsilon}{2\lambda_{2m+1}} < \theta_{2m} + \frac{\pi - \varepsilon}{2\lambda_{2m}} = b(I_{2m}),$$

$$a(I_{2m+1}) = \theta_{2m+1} - \frac{\pi + \varepsilon}{2\lambda_{2m+1}} - x \geq \theta_{2m} - \frac{q_1(\pi - \varepsilon)}{2\lambda_{2m+1}} \geq \theta_{2m} - \frac{\pi - \varepsilon}{2\lambda_{2m}} = a(I_{2m}).$$

From the aforementioned properties it follows the existence of segment  $I_{2m+1}$ .

Further, using the inequality  $\lambda_{2m+1} \leq \lambda_{2m+2}/q_2$  and equality (5) we obtain

$$|I_{2m+1}| = x \geq \frac{(q_1 - 1)q_2\pi - (q_1 + 1)q_2\varepsilon}{2\lambda_{2m+2}} = \frac{3\pi - \varepsilon}{\lambda_{2m+2}},$$

whence we see that there exists a segment  $I_{2m+2} \subset I_{2m+1}$  of length  $\frac{\pi-\varepsilon}{\lambda_{2m+2}}$  such that the inequality  $\cos(\lambda_{2m+2}\theta + \alpha_{2m+2}) \geq \delta$  holds for all  $\theta \in I_{2m+2}$ .

The analysis of our considerations shows that by induction it is possible to construct a system of embedded segments  $I_{2m} \supset I_{2m+1} \supset I_{2m+2} \supset I_{2m+3} \supset \dots$  such that for every integer  $n \geq 2m$  the inequality  $\cos(\lambda_n\theta + \alpha_n) \geq \delta$  holds for all  $\theta \in I_n$ . Let  $\theta$  be the common point of all segments  $I_n$ ,  $n \geq 2m$ . Then  $\theta \in I$  and

$$\operatorname{Re} f(re^{i\theta}) = \sum_{n=0}^{\infty} |a_n|r^n \cos(\lambda_n\theta + \alpha_n) \geq - \sum_{n < 2m} |a_n|r^n + \delta \sum_{n \geq 2m} |a_n|r^n,$$

whence, according to (2), we obtain (3). Theorem is proved.

#### REFERENCES

- [1] MacLane G.R. Asumptotic values of holomorphic functions. In: Norbeck E. (Ed.) Monograph in Mathematics, 49 (1). William Marsh Rice University, Houston, 1963.
- [2] Murai T. *The boundary behaviour of Hadamard lacunary series*. Nagoya Math. J. 1983, 89, 65–76.

Received 16.03.2014

Андрусак І.В., Філевич П.В. *Радіальні граничні значення лакунарних степеневих рядів* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 4–7.

Підсилено теорему Мак-Лейна про граничні значення лакунарних степеневих рядів.

*Ключові слова і фрази:* аналітична функція, лакунарний степеневий ряд, радіальне граничне значення, асимптотичне значення.

Андрусак И.В., Филевич П.В. *Радіальні граничні значення лакунарних степеневих рядів* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 4–7.

Усилена теорема Мак-Лейна о радиальных граничных значениях лакунарных степеневых рядов.

*Ключевые слова и фразы:* аналитическая функция, лакунарный степеневый ряд, радиальное граничное значение, асимптотическое значение.