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GENERALIZED RANDOM PROCESSES AND CAUCHY'S PROBLEM FOR SOME PARTIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper we consider a parabolic partial differential system of the form $D_t H_t = L(t,x,D) H_t$. The generalized stochastic solutions H_t, corresponding to the generalized stochastic initial conditions H_o, are given. Some properties concerning these generalized stochastic solutions are also obtained.

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1. INTRODUCTION.

Consider the system

$$D_t u = Lu \tag{1.1}$$

where

$$D_t = \frac{\partial}{\partial t}, \quad L = \sum_{|k| \le 2b} L_k(t,x)\, D^k ,$$

$$D^k = (-i)^k D_1^{k_1} \ldots D_n^{k_n}, \quad D_r = \frac{\partial}{\partial x_r}, \quad r = 1, \ldots n,$$

$|k| = k_1 + \ldots + k_n$, $t \in (0,T)$, $T>0$, x is an element of the n-dimensional Euclidean space E_n, and $(L_k(t,x), |k| \le 2b)$ is a family of square matrices of order N.

We assume that (1.1) is a strongly parabolic system on $G_{n+1} = \{(t,x): t \in [0,T], x \in E_n\}$ in the sense that for every complex vector $a = (a_1, \ldots, a_N)$, every $\sigma \in E_n$, and every $(t,x) \in G_{n+1}$;

$$\text{Re}\ [\sum_{|k| = 2b} L_k(t,x)\sigma^k a, \bar{a}] \le -\delta\ |\sigma|^{2b}\ |\alpha|^2$$

where

$$\sigma^k = \sigma_1^{k_1} \ldots \sigma_n^{k_n}, \quad |\sigma|^{2b} = (\sigma_1^2 + \ldots + \sigma_n^2)^b ,$$

$|a|^2 = a_1^2 + \ldots + a_N^2$, and δ is a positive constant (see [1]). In the above inequality and in the following, we denote the scalar product of two N-vector functions u and v by the bracket notation (u,v).

As usual, we denote by $C^m(E_n)$, $0 \le m \le \infty$, the set of all real-valued functions defined on E_n, which have continuous partial derivatives of order up to and including m (of order $< \infty$ if $m = \infty$). By $C_o^m(E_n, N)$ we denote the set of all vector functions $h = (h_1, \ldots, h_N)$ such that every h_r is in $C^m(E_n)$, with compact support, $r = 1, \ldots, N$. We assume that the elements of the matrices $L_k(t,x)$, $|k| \le 2b$, satisfy the following conditions:

(a) They are bounded on G_{n+1} and satisfy a Holder condition of order α with respect to x, $(0 < \alpha \le 1)$.

(b) For every $x \in E_n$, they are continuous functions in $t \in [0, T]$.

(c) For every t in [0, T], they are $C^{\infty}(E_n)$ functions. Let u = $(u_1, \ldots, u_N)$ satisfy the initial condition

$$u(x, o) = u_o(x), \tag{1.2}$$

where $u_o = (u_{o1}, \ldots, u_{oN})$, [$u_{or} \in C(E_n)$ are bounded on E_n, r = 1, ... , N]. We say that u is of the class $S(E_n)$ if for each $t \in (0,T)$, $D_t u_r \in C(E_n)$ and $u_r \in C^{2b}(E_n)$, r = 1, ... , N.

It has been proved [2] that, under conditions (a) and (b), there exists a fundamental matrix Z(t,0,x,y) of the system (1.1) such that

$$u(t,x) = \int_{E_n} Z(t,0,x,y)\, u_o(y)\, dy, \quad dy = dy_1 \ldots dy_r \tag{1.3}$$

represents the unique solution of the Cauchy problem (1.1), (1.2) in the class $S(E_n)$.

Let (V_r : r = 1, ... , N) be a family of Gaussian random measures in the sense of Gelfand and Vilenkin [2]. Let g_r be a complex-valued function defined on E_1. We say that g_r is of the class K_r if the integral $\int_{E_1} |g_r(s)|^2\, dF_r(s)$ exists, where F_r is a positive measure such that

$$E\,[V_r(B_1)\, \overline{V_r(B_2)}] = F_r(B_1 \cap B_2)$$

for any two Borel sets B_1 and B_2 on the real line [r = 1, ..., N and E (.) denotes the expectation of (.)].

Let H be an N-vector of generalized stochastic processes, which associates with every h in $C_o^{\infty}(E_n, N)$ an N-vector of random variables defined by

$$H(h) = (H_1(h), \ldots, H_N(h)),$$

$$H_r(h) = \int_{E_1} g_{ro}(s)\, d\, V_r(s), \tag{1.4}$$

$$g_{ro}(s) = \int_{E_n} (I_r(x,s),\ h(x))\ dx,$$

where $(I_r;\ r = 1, \ldots, N)$ is a family of N-vectors of continuous functions on E_{n+1}.

It is assumed also that all the components of I_r are bounded on E_n, independently of s. Clearly, g_{ro} is of the class K_r.

The theoretical development in section 2 exhibits the use of formula (1.3) in order to integrate (1.1) when the initial condition is an N-vector of generalized stochastic processes, which is defined by (1.4). Also, some essential properties are derived in section 3.

2. GENERALIZED STOCHASTIC SOLUTIONS.

An N-vector w(t,x,s) of functions is said to be of the class $C(E_{n+1}, N)$ if, for each t in (0,T), the components of w(t,x,s) represent continuous functions of (x,s) on E_{n+1} and they are bounded on E_n, independently of s. We say that the generalized stochastic vector H_t is of the class V if there exists a family $[S_r(t,x,s) : S_r \in C(E_{n+1}, N),\ r = 1, \ldots, N]$ such that, for each h in $C_o^\infty(E_n, N)$, $H_t(h)$ can be represented in the form

$$H_t(h) = \int_{E_1} g(t,s)\ dV(s),$$

$$g = (g_1,\ldots,g_N),\ g_r(t,s) = \int_{E_n} (S_r(t,x,s),\ h(x))\,dx,$$

$$H_t(h) = (H_{1t}(h),\ldots,\ H_{Nt}(h)),\ H_{rt}(h) = \int_{E_1} g_r\ (t,s)dV_r(s).$$

It is clear that, for each t in (0,T), $g_r \in K_r$. The expectation of $|H_{rt}|^2$ is given by

$$E\,|H_{rt}|^2 = \int_{E_1} |g_r(t,s)|^2\, d\,F_r(s).$$

If $D_t g_r(t,s)$ exists and belongs to K_r for each t in (0,T), then we define $\frac{d}{dt} H_{rt}(h)$ by

$$\frac{d}{dt} H_{rt}(h) = \underset{t \to 0}{\text{l.i.m}} \int_{E_1} \frac{\Delta g_r(t,s)}{\Delta t}\, d\,V_r(s) = \int_{E_1} D_t\, g_r(t,s)\, d\,V_r(s)\ ,$$

where $\Delta g_r(t,s) = g_r(t + \Delta t,s) - g_r(t,s)$ and l.i.m. denotes limit in the mean, i.e.

$$\lim_{t \to o} \int_{E_1} \left|\frac{\Delta g_r(t,s)}{\Delta t} - D_t g_r(t,s)\right|^2 dF_r(s) = 0.$$

Let $L^* = \sum_{|k| \le 2b} (-1)^{|k|} D^k\, L_k^*$, where $(L_k^*,\ |k| \le 2b)$ is the family of adjoint matrices to $(L_k,\ |k| \le 2b)$. Since the coefficients of the operator L are $C^\infty\,(E_n)$ functions, it follows that, for every h in $C_o^\infty\,(E_n,\ N)$, $L^*h = h_t$ is also in $C_o^\infty\,(E_n,N)$. We call H_t a generalized stochastic solution of the system (1.1) if H_t and $\frac{dH_t}{dt}$ are of the class V and

$$\frac{dH_t\,(h)}{dt} = H_t(h_t^*) \tag{2.1}$$

for every h in $C_o^\infty\,(E_n,N)$ and t in (0,T). We assume that

$$H_o(h) = H(h) \tag{2.2}$$

where H is defined by (1.4).

THEOREM 1: The Cauchy problem (2.1), (2.2) has a unique generalized stochastic solution H_t in the class V.

PROOF: Let $(S_r(t,x,s) : r = 1, \ldots , N)$ be a family of solutions of the system (1.1) with the initial conditions:

$$S_r(0,x,s) = I_r(x,s),\ r = 1, \ldots, N.$$

Using formula (1.3), one gets

$$S_r(t,x,s) = \int_{E_n} Z(t,0,x,y)\ I_r(y,s)\ dy. \tag{2.3}$$

According to the properties of the fundamental matrix Z, we find $S_r \in C(E_{n+1}, N)$, $r = 1, \ldots, N$. Set,

$$H_t(h) = \int_{E_1} g(t,s)\ dV\ (s)$$

and

$$g_r(t,s) = \int_{E_n} (S_r(t,x,s),\ h(x))dx \text{ with } h \in C_o^\infty\ (E_n,\ N),$$

where $S_1(t,x,s), \ldots, S_N(t,x,s)$ are defined by (2.3). Since $S_r \in C(E_{n+1}, N)$, it follows that H_t is of the class V. Using again the properties of Z, we get

$$D_t \int_{E_n} (S_r(t,x,s),\ h(x))dx = \int_{E_n} (D_t S_r(t,x,s),\ h(x))dx$$

$$= \int_{E_n} (S_r(t,x,s),\ h_t^*(x))\ dx.$$

The last formula proves that $D_t\ g_r \in K_r$.

Now we already have

$$\frac{d}{dt} H_t(h) = \int_{E_1} \int_{E_n} (S_r(t,x,s),\ h_t^*(x))\ dx\ dV(s) = H_t(h_t^*),$$

where $\frac{d}{dt} H_t$ is of the class V.

We also have

$$H_o(h) = \int_{E_1} g(0,s)\ dV(s)\ ,$$

where

$$g_r(0,s) = \int_{E_n} (I_r(x,s),\ h(x))\ dx.$$

Thus the existence of the generalized stochastic solution H_t with the initial condition $H_o = H$ is proved. To prove the uniqueness of H_t, it is sufficient to show that the only solution of (2.1) with the initial condition $H_o(h) = H(h) = 0$ is $H_t(h) = 0$ for every h in $C_o^\infty(E_n,N)$ and t in (0,T). If $H_o = 0$, then $E\,|H_{ro}|^2 = \int_{E_1} |g_{ro}(s)|^2\ dF(s) = 0$, and hence $g_{ro}(s) = 0$ on E_1.

Therefore,

$$g_{ro}(s) = \int_{E_n} (I_r(x,s),\ h(x))\ dx = 0,$$

which is true for any arbitrary h in $C_o^\infty(E_n, N)$, and hence $I_r(x,s) = 0$ on E_{n+1}. Since $\frac{d}{dt} H_t(h) = H_t(h_t^*)$, it follows that

$$E\ \left|\frac{d}{dt} H_{rt}(h) - H_{rt}\ (h_t^*)\right|^2 = 0;$$

therefore,

$$\int_{E_n} (D_t S_r(t,x,s) - L\ S_r(t,x,s),\ h(x))\ dx = 0,$$

which implies

$$D_t S_r(t,x,s) = L\ S_r(t,x,s). \tag{2.4}$$

We also have

$$S_r(0,x,s) = 0. \tag{2.5}$$

The uniqueness of the problem (2.4), (2.5) gives

$$S_r(t,x,s) = 0, \tag{2.6}$$

$$t \in (0,T),\ (x,s) \in E_{n+1},\ (r = 1,\dots, N).$$

Using (2.6), one gets $H_t(h) = 0$, for every h in $C_o^\infty (E_n,N)$ and t in (0,T). This completes the proof.

3. A CONVERGENCE THEOREM.

Let $h_m = (h_{m_1}, \dots, h_{m_N})$, m = 1, 2, .. be a sequence in C_o^∞ (G, N), where G is a bounded open domain of E_n. Suppose that

$$\lim_{m\to\infty} \int (h_{m_r}(x) - w_r(x))^2\, dx = 0, \tag{3.1}$$

where $w_r \in L_2(G)$, r = 1, ... , N and L_2 (G) denotes the set of all Lebesgue measurable square integrable functions on G. It is assumed that $w_r(x) = 0$ for $x \notin G$ where r = 1, ... N.

THEOREM 2: If $H_t(h_m) = \int g_m(t,s)\, dV(s)$,

then

$$\underset{m\to\infty}{\text{l.i.m.}}\ H_t(h_m) = \int \eta(t,s)\, dV(s),$$

where $g_m(t,s) = (g_{m_1}(t,s), \dots, g_{m_N}(t,s))$,

$$g_{m_r}(t,s) = \int (S_r(t,x,s),\ h_m(x))\, dx,\ \eta = (\eta_1,\dots, \eta_N),$$

$\eta_r(t,s) = \int (S_r(t,x,s),\ w(x))dx$, and the family ($S_r$, r = 1, ..., N) is defined by (2.3).

PROOF: A straight forward application of the Cauchy - Schwarz inequality establishes that

$$\lim_{m\to\infty} \int_G (S_r(t,x,s),\ h_m(x))\, dx = \int_G (S_r(t,x,s),\ w(x))dx \tag{3.2}$$

According to the conditions imposed on the family $(I_r(x,s),\ r = 1, \ldots, N)$ and according to the properties of the fundamental matrix Z, we can find a constant A such that

$$|g_{m_r}(t,s)| \le A, \tag{3.3}$$

for all m, s, $t \in (0,T)$ and $r = 1, \ldots, N$. For any positive integers ℓ and m, we have

$$E\ |H_{rt}(h_m) - H_{rt}(h_\ell)|^2 = \int |g_{m_r}(t,s) - g_{\ell_r}(t,s)|^2\ dF_r(s). \tag{3.4}$$

By a standard argument based on (3.2) and (3.3), the righthand side of (3.4) can be shown to go to zero. Thus, $H_t(h_m)$ is a Cauchy sequence. We deduce also that

$$\lim_{m\to\infty} \int |g_{m_r}(t,s) - \eta_r(t,s)|^2\ dF_r(s) = 0.$$

The last argument leads to the fact that there exists a stochastic process $R_r(t)$ such that $E\ |R(t)|^2 < \infty$ and that

$$\lim_{m\to\infty}\ E\ |H_{rt}(h_m) - R_r(t)|^2 = 0\ .$$

Following Doob [3], we find

$$R_r(t) = \int \eta_r(t,s)\ dV_r(s),$$

$$\eta_r(t,s) = \int (S_r(t,x,s),\ w(x))d\mathbf{x}.$$

This completes the proof.

COROLLARY: For vector functions $(w = w_1, \ldots, w_N)$ where $w_r \in L_2(Q)$ and $w_r(x) = 0$ for $x \notin G$), there exists a sequence (h_m) in $C_o^\infty(E_n, N)$ such that

$$\underset{m\to\infty}{\text{l.i.m.}}\ H_o(h_m) = H_o(w),$$

$$\underset{m \to \infty}{\text{l.i.m.}} \; H_t(h_m)=H_t(w).$$

The proof can be deduced directly by using theorem 2. (Compare [4]).

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