Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 150841, 19 pages
doi:10.1155/2012/150841

Research Article

Partial Extinction, Permanence, and Global Attractivity in Nonautonomous *n*-Species Gilpin-Ayala Competitive Systems with Impulses

Juan Hou and Hanhui Liu

College of Applied Mathematics, Xinjiang University of Finance and Economics, Urumqi 830012, China

Correspondence should be addressed to Juan Hou, hj_xd@yahoo.com.cn

Received 3 May 2012; Accepted 21 June 2012

Academic Editor: Yong Zhou

The qualitative properties of general nonautonomous n-species Gilpin-Ayala competitive systems with impulsive effects are studied. Some new criteria on the permanence, extinction, and global attractivity of partial species are established by using the methods of inequalities estimate and Liapunov functions.

1. Introduction

In [1], the general nonautonomous n-species Lotka-Volterra competitive systems with impulsive effects are investigated. By using the methods of inequalities estimate and constructing the suitable Liapunov functions, the sufficient conditions on the permanence of whole species and global attractivity of systems are established.

In [2], the authors studied the following general nonautonomous n-species Lotka-Volterra competitive systems with impulsive perturbations:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t)\right], \quad t \neq t_k, \\ x_i\left(t_k^+\right) &= h_{ik}x_i(t_k), \quad k = 1, 2, \dots,\ i = 1, 2, \dots, n, \end{aligned} \tag{1.1}$$

and got a series of criteria on the extinction of a part of n-species, the permanence of other part of n-species, and the global attractivity of the systems.

In [3], a periodic n-species Gilpin-Ayala competition system with impulses is studied and obtain some useful behaviors of the system.

In this paper, we investigate the general nonautonomous n-species Gilpin-Ayala competitive systems with impulsive effects.

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t) x_j^{\theta_{ij}}(t)\right], \quad t \neq t_k, \\ x_i\left(t_k^+\right) &= h_{ik} x_i(t_k), \quad k = 1, 2, \ldots, \; i = 1, 2, \ldots, n, \end{aligned} \tag{1.2}$$

where $b_i(t)$ and $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are defined on $R_+ = [0, \infty)$ and are bounded continuous functions, $a_{ij}(t) \geq 0$ for all $t \in R_+$, θ_{ij} and $h_{ik} > 0$ are constants for all $k = 1, 2, \ldots$ and $i, j = 1, 2, \ldots, n$.

2. Preliminaries

Firstly, we introduce the following assumption.

Assumption H. There is a positive constant ω such that for each $i = 1, 2, \ldots, n$

$$\liminf_{t \to \infty} \int_t^{t+\omega} a_{ii}(s) ds > 0, \qquad \liminf_{t \to \infty} \left(\int_t^{t+\omega} b_i(s) ds + \sum_{t \leqslant t_k \leqslant t+\mu} \ln h_{ik} \right) > 0, \tag{2.1}$$

and functions

$$h_i(t, \mu) = \sum_{t \leqslant t_k \leqslant t+\mu} \ln h_{ik}, \quad i = 1, 2, \ldots, n \tag{2.2}$$

are bounded on $t \in R_+$ and $0 \leqslant \mu \leqslant \omega$.

For each $i \in \{1, 2, \ldots, n\}$, we consider the following logistic impulsive equation as the subsystem of system (1.2)

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[b_i(t) - a_{ii}(t) x_i^{\theta_{ii}}(t)\right], \quad t \neq t_k, \\ x_i\left(t_k^+\right) &= h_{ik} x_i(t_k), \quad k = 1, 2, \ldots. \end{aligned} \tag{2.3}$$

From the above assumption, we have the following results.

Lemma 2.1. *Suppose that assumption H holds. Then we have the following:*

(1) *There exist positive constants m and M such that*

$$m \leqslant \liminf_{t\to\infty} x_i(t) \leqslant \limsup_{t\to\infty} x_i(t) \leqslant M, \tag{2.4}$$

for any positive solution $u_i(t)$ of (2.3).

(2) $\lim_{t\to\infty}(x_i^{(1)}(t) - x_i^{(2)}(t)) = 0$ *for any two positive solutions $x_i^{(1)}(t)$ and $x_i^{(2)}(t)$ of* (2.3).

Proof. From assumption H, there are positive constants k_1, k_2, δ and T_0 such that for all $t \geqslant T_0$ we have

$$\int_t^{t+\omega} (b_i(s) - a_{ii}(s)k_1)ds + \sum_{t\le t_k<t+\omega} \ln h_k < -\delta, \tag{2.5}$$

$$\int_t^{t+\omega} (b_i(s) - a_{ii}(s)k_2)ds + \sum_{t\le t_k<t+\omega} \ln h_k > \delta. \tag{2.6}$$

From the boundedness of function $h(t,\mu) = \sum_{t\le t_k<t+\mu} \ln h_k$, there is a positive constant P such that for any $t \in R_+$ and $\mu \in [0,\omega)$

$$|h(t,\mu)| = \left|\sum_{t\le t_k<t+\mu} \ln h_k\right| < P. \tag{2.7}$$

Firstly, we prove that there is a constant $M > 0$ such that

$$\limsup_{t\to\infty} x_i(t) < M, \tag{2.8}$$

for any positive solution $x_i(t)$ of system (2.3). In fact, for any positive solution $x_i(t)$ of system (2.3), we only need to consider the following three cases.

Case I. There is a $t_0 \ge T_0$ such that $x(t) \ge k_1' = \sqrt[\theta_{ii}]{k_1}$ for all $t \ge t_0$.

Case II. There is a $t_0 \ge T_0$ such that $x(t) \le k_1'$ for all $t \ge t_0$.

Case III. $x(t)$ is oscillatory about k_1' for all $t \ge T_0$.

We first consider Case I. Since $x_i(t) \geq k_1'$ for all $t \geq t_0$, then for $t = t_0 + l\omega$, where $l \geq 0$ is any positive integer, integrating system (2.3) from t_0 to t, from (2.5) we have

$$\begin{aligned} x_i(t) &= x_i(t_0)\exp\left(\int_{t_0}^{t}\left(b_i(s) - a_{ii}(s)x_i^{\theta_{ii}}(s)\right)ds + \sum_{t_0\leq t_k<t}\ln h_k\right) \\ &\leq x_i(t_0)\exp\left(\int_{t_0}^{t_0+\omega}(b_i(s) - a_{ii}(s)k_1)ds + \sum_{t_0\leq t_k<t}\ln h_k + \cdots \right. \\ &\qquad \left. + \int_{t_0+(l-1)\omega}^{t_0+l\omega}(b_i(s) - a_{ii}(s)k_1)ds + \sum_{t_0+(l-1)\omega\leq t_k<t_0+l\omega}\ln h_k\right) \\ &\leq x_i(t_0)\exp(-l\delta). \end{aligned} \tag{2.9}$$

Hence, $x_i(t) \to 0$ as $l \to \infty$, which leads a contradiction.

Next, we consider Case III. From the oscillation of $x_i(t)$ about k_1', we can choose two sequences $\{\rho_n\}$ and $\{\rho_n^*\}$ satisfying $T_0 < \rho_1 < \rho_1^* < \cdots < \rho_n < \rho_n^* < \cdots$ and $\lim_{n\to\infty}\rho_n = \lim_{n\to\infty}\rho_n^* = \infty$ such that

$$\begin{gathered} x_i(\rho_n) \leq k_1', \qquad x_i(\rho_n^+) \geq k_1', \qquad x_i(\rho_n^*) \geq k_1', \qquad x_i\left(\rho_n^{*^+}\right) \leq k_1', \\ x_i(t) \geq k_1', \quad \forall t \in (\rho_n, \rho_n^*), \\ x_i(t) \leq k_1', \quad \forall t \in (\rho_n^*, \rho_{n+1}). \end{gathered} \tag{2.10}$$

For any $t \geq T_0$, if $t \in (\rho_n, \rho_n^*]$ for some integer n, then we can choose integer $l \geq 0$ and constant $0 \leq \mu_1 < \omega$ such that $t = \rho_n + l\omega + \mu_1$. Since

$$\dot{x}_i(t) \leq x_i(t)(b_i(t) - a_{ii}(t)k_1), \quad \forall t \in (\rho_n, \rho_n^*),\ t \neq t_k, \tag{2.11}$$

integrating this inequality from ρ_n to t, by (2.5) and (2.7) we obtain

$$\begin{aligned}
x_i(t) &= x_i(\rho_n)\exp\left(\int_{\rho_n}^{t}\Big(b_i(s)-a_{ii}(s)x^{\theta_{ii}}(s)\Big)ds+\sum_{\rho_n\le t_k<t}\ln h_k\right)\\
&\le k_1'\exp\left(\int_{\rho_n}^{\rho_n+\omega}(b_i(s)-a_{ii}(s)k_1)ds+\sum_{\rho_n\le t_k<\rho_n+\omega}\ln h_k+\cdots\right.\\
&\qquad\left.+\int_{\rho_n+l\omega}^{\rho_n+l\omega+\mu_1}(b_i(s)-a_{ii}(s)k_1)ds+\sum_{\rho_n+l\omega\le t_k<\rho_n+l\omega+\mu_1}\ln h_k\right)\\
&\le k_1'\exp\left(-l\delta+\int_{\rho_n+l\omega}^{\rho_n+l\omega+\mu_1}(b_i(s)-a_{ii}(s)k_1)ds+\sum_{\rho_n+l\omega\le t_k<\rho_n+l\omega+\mu_1}\ln h_k\right)\\
&\le k_1'\exp(\alpha_1\omega+P),
\end{aligned}\tag{2.12}$$

where $\alpha_1=\sup_{t\in R_+}\{|b_i(t)|+a_{ii}(t)k_1\}$. If there is an integer n such that $t\in(\rho_n^*,\rho_{n+1}]$, then we obviously have

$$x_i(t)\le k_1'<k_1'\exp(\alpha_1\omega+P).\tag{2.13}$$

Therefore, for Case III we always have

$$x_i(t)\le k_1'\exp(\alpha_1\omega+P),\quad \forall t\ge T_0.\tag{2.14}$$

Lastly, if Case II holds, then we directly have

$$x_i(t)\le k_1'\exp(\alpha_1\omega+P),\quad \forall t\ge T_0.\tag{2.15}$$

Choose constant $M=k_1'\exp(\alpha_1\omega+P)$, then we see that (2.8) holds.

Secondly, a similar argument as in the proof of (2.8) we can prove that there is a constant $m>0$, such that

$$\liminf_{t\to\infty}x(t)>m,\tag{2.16}$$

for any positive solution $x_i(t)$ of system (2.3). Conclusion (1.1) is proved.

Now, we prove conclusion (1.2). Let $x_i^{(1)}(t)$ and $x_i^{(2)}(t)$ be any two positive solutions of system (2.3). From conclusion (1.1), it follows that there are positive constants A and B such that

$$A\leqslant x_i^{(1)}(t),\quad x_i^{(2)}(t)\leqslant B,\quad \forall t\ge 0.\tag{2.17}$$

Choose Liapunov function as follows:

$$V(t) = \left|\ln x_i^{(1)}(t) - \ln x_i^{(2)}(t)\right|. \tag{2.18}$$

For any $k = 1, 2, \ldots$, we have

$$V(t_k^+) = \left|\ln\left(h_k x_i^{(1)}(t_k)\right) - \ln\left(h_k x_i^{(2)}(t_k)\right)\right| = V(t_k). \tag{2.19}$$

Hence, $V(t)$ is continuous for all $t \in R_+$ and from the Mean-Value Theorem we can obtain

$$\frac{1}{B}\left|x_i^{(1)}(t) - x_i^{(2)}(t)\right| \leqslant V(t) \leqslant \frac{1}{A}\left|x_i^{(1)}(t) - x_i^{(2)}(t)\right|. \tag{2.20}$$

Calculating the upper right derivative of $V(t)$, then from (2.20) we obtain

$$\begin{aligned}
D^+V &= \operatorname{sign}\left(x_i^{(1)}(t) - x_i^{(2)}(t)\right)\left(\frac{\dot{x}_i^{(1)}(t)}{x_i^{(1)}(t)} - \frac{\dot{x}_i^{(2)}(t)}{x_i^{(2)}(t)}\right) \\
&= -a_{ii}(t)\left|x_i^{(1)\theta_{ii}}(t) - x_i^{(2)\theta_{ii}}(t)\right| \\
&\leq -a_{ii}(t)[\theta_{ii}]A_{ii}^{\theta}\left|x_i^{(1)}(t) - x_i^{(2)}(t)\right| \\
&\leq -a_{ii}(t)[\theta_{ii}]A^{\theta_{ii}}V(t), \quad t \neq t_k,\ k = 1, 2, \ldots,
\end{aligned} \tag{2.21}$$

where $[\theta_{ii}] \leq \theta_{ii}$ is the integer part of θ_{ii}.

From this, we further have for any $t > 0$

$$V(t) \leqslant V(0)\exp\left(-[\theta_{ii}]A^{\theta_{ii}}\int_0^t a_{ii}(s)ds\right). \tag{2.22}$$

From condition (2.5) we can obtain $\int_0^t a_{ii}(t)dt \to \infty$ as $t \to \infty$. Hence, $V(t) \to 0$ as $t \to \infty$. Further from (2.20) we finally obtain $\lim_{t\to\infty}(x_i^{(1)}(t) - x_i^{(2)}(t)) = 0$. Conclusion (1.2) is proved. This completes the proof of Lemma 2.1. □

Applying Lemma 2.1 and the comparison theorem of impulsive differential equations, we easily prove the following result.

Lemma 2.2. *Suppose that assumption H holds then there is a constant $B > 0$ such that*

$$\limsup_{t\to\infty} x_i(t) \leq B, \quad i = 1, 2, \ldots, n, \tag{2.23}$$

for any positive solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of system (1.2).

3. Extinction

On the partial extinction of system (1.2), we have the following result.

Theorem 3.1. *Suppose that assumption H holds. Let r be a given integer and $1 \le r < n$. If for any $l > r$ there is a $i_l < l$ such that for any $j \le l$*

$$\theta_{i_l j} = \theta_{lj}, \tag{3.1}$$

$$\limsup_{t\to\infty} \frac{\int_t^{t+\omega} b_l(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{lk}}{\int_t^{t+\omega} b_{i_l}(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{i_l k}} < \liminf_{t\to\infty} \frac{a_{lj}(t)}{a_{i_l j}(t)}, \quad \forall j \le l, \tag{3.2}$$

or

$$\liminf_{t\to\infty} \frac{\int_t^{t+\omega} b_{i_l}(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{i_l k}}{\int_t^{t+\omega} b_l(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{lk}} > \limsup_{t\to\infty} \frac{a_{i_l j}(t)}{a_{lj}(t)}, \quad \forall j \le l, \tag{3.3}$$

then species x_i $(i = r+1, r+2, \dots, n)$ are extinction, that is, for any positive solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of system (1.2),

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = r+1, r+2, \dots, n. \tag{3.4}$$

Proof. Firstly, from assumption H, that (2.7) still holds and there are constants $\eta_0 > 0$ and $T_0 > 0$ such that

$$\int_t^{t+\omega} b_i(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{ik} \ge \eta_0, \tag{3.5}$$

for all $t \ge T_0$ and $i = 1, 2, \dots, n$.

We first prove $x_n(t) \to 0$ as $t \to \infty$. Without loss of generality, we assume that condition (3.2) holds. When condition (3.3) holds, a similar argument can be given. Since

$$\limsup_{t\to\infty} \frac{\int_t^{t+\omega} b_n(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{nk}}{\int_t^{t+\omega} b_p(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{pk}} < \liminf_{t\to\infty} \frac{a_{nj}(t)}{a_{pj}(t)}, \quad j = 1, 2, \dots, n, \tag{3.6}$$

where $p = i_n$. Hence, we can choose positive constants α, β, ε and $T_n \ge T_0$ such that

$$\frac{\int_t^{t+\omega} b_n(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{nk}}{\int_t^{t+\omega} b_p(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{pk}} < \frac{\alpha}{\beta} - \varepsilon < \frac{\alpha}{\beta} < \frac{a_{nj}(t)}{a_{pj}(t)}, \tag{3.7}$$

for all $t \geq T_n$ and $j = 1, 2, \ldots, n$. Hence, from (3.5) we further obtain

$$\begin{aligned}
&\int_t^{t+\omega} \left(-\alpha b_p(s) + \beta b_n(s)\right) ds + \beta \sum_{t \leq t_k < t+\omega} \ln h_{nk} - \alpha \sum_{t \leq t_k < t+\omega} \ln h_{pk} \\
&\quad < -\beta\varepsilon \left(\int_t^{t+\omega} b_p(s) ds + \sum_{t \leq t_k < t+\omega} \ln h_{pk} \right) \\
&\quad \leq -\beta\varepsilon\eta_0,
\end{aligned} \tag{3.8}$$

$$\alpha a_{pj}(t) - \beta a_{nj}(t) = \beta a_{pj} \left[\frac{\alpha}{\beta} - \frac{a_{nj}(t)}{a_{pj}(t)} \right] < 0, \tag{3.9}$$

for all $t \geq T_n$ and $j = 1, 2, \ldots, n$.

Consider the Liapunov function as follows:

$$V_n(t) = \left(x_p(t)\right)^{-\alpha} \left(x_n(t)\right)^{\beta}. \tag{3.10}$$

Calculating the derivative, and from (3.1), we can obtain for any $t \geq 0$

$$\begin{aligned}
\frac{dV_n(t)}{dt} &= V_n(t) \left[-\alpha \left(b_p(t) - \sum_{j=1}^{n} a_{pj}(t) x_j^{\theta_{pj}}(t) \right) + \beta \left(b_n(t) - \sum_{j=1}^{n} a_{nj}(t) x_j^{\theta_{nj}}(t) \right) \right] \\
&= V_n(t) \left[-\alpha b_p(t) + \beta b_n(t) + \sum_{j=1}^{n} \left(\alpha a_{pj}(t) - \beta a_{nj}(t) \right) x_j^{\theta_{pj}}(t) \right],
\end{aligned} \tag{3.11}$$

for all $t \neq t_k$ and

$$V_n(t_k^+) = h_{pk}^{-\alpha} h_{nk}^{\beta} V_n(t_k), \tag{3.12}$$

for all $k = 1, 2, \ldots$. From (3.9), we further have

$$\begin{aligned}
\frac{dV_n(t)}{dt} &\leq V_n(t) \left(-\alpha b_p(t) + \beta b_n(t) \right), \quad t \geq T_n, \ t \neq t_k, \\
V_n(t_k^+) &= h_{pk}^{-\alpha} h_{nk}^{\beta} V_n(t_k), \quad k = 1, 2, \ldots.
\end{aligned} \tag{3.13}$$

For any $t > T_n$, there is an integer $q_t \geq 0$ such that $t \in [T_n + q_t\omega, T_n + (q_t + 1)\omega]$. Hence, by integrating (3.13) from T_n to t, we obtain

$$\begin{aligned} V_n(t) &\leq V_n(T_n) \exp\left(\int_{T_n}^{t} \left[-\alpha b_p(s) + \beta b_n(s)\right] ds + \sum_{T_n \leq t_k < t} \ln\left(h_{pk}^{-\alpha} h_{nk}^{\beta}\right)\right) \\ &= V_n(T_n) \exp\left\{ \int_{T_n}^{T_n+\omega} \left[-\alpha b_p(s) + \beta b_n(s)\right] ds \right. \\ &\qquad + \sum_{T_n \leq t_k < T_n+\omega} \ln\left(h_{pk}^{-\alpha} h_{nk}^{\beta}\right) + \cdots + \int_{T_n+q_t\omega}^{t} \left[-\alpha b_p(s) + \beta b_n(s)\right] ds \\ &\qquad \left. + \sum_{T_n+q_t\omega \leq t_k < t} \ln\left(h_{pk}^{-\alpha} h_{nk}^{\beta}\right) \right\} \\ &\leq M_n \exp(-\varepsilon\beta\eta_0 q_t), \end{aligned} \tag{3.14}$$

where

$$M_n = V_n(T_n) \exp\left(\omega \sup_{t \geq 0}\left\{\alpha\left|b_p(t)\right| + \beta|b_n(t)|\right\} + (\alpha + \beta)P\right). \tag{3.15}$$

Since $q_t \to \infty$ as $t \to \infty$, it follows that from (3.14)

$$V_n(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.16}$$

Since

$$\begin{aligned} (x_n(t))^{\beta} &= V_n(t)\left(x_p(t)\right)^{\alpha}, \\ (h_{nk} x_n(t_k))^{\beta} &= h_{pk}^{\alpha} x_p(t_k) h_{pk}^{-\alpha} h_{nk}^{\beta} V_n(t_k), \end{aligned} \tag{3.17}$$

by the boundedness of $x(t)$ on $[0, \infty)$ (see Lemma 2.2), we have

$$x_n(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.18}$$

For any integer $l > r$, assume that we have obtained $x_i(t) \to 0$ as $t \to \infty$ for all $i > l$. Now, we prove that $x_l(t) \to 0$ as $t \to \infty$. Suppose that condition (3.3) holds. When condition (3.2) holds, the argument is similar. Let $i = i_l$, by (3.1), we have $\theta_{i_l j} = \theta_{lj}$, then for $j \leq l$, we have $\theta_{ij} = \theta_{lj}$. Then we can choose positive constants λ, η, δ and $T_l \geq T_0$ such that

$$\frac{\int_t^{t+\omega} b_q(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{qk}}{\int_t^{t+\omega} b_l(s)ds + \sum_{t\leqslant t_k<t+\omega} \ln h_{lk}} > \frac{\lambda}{\eta} + \delta > \frac{\lambda}{\eta} > \frac{a_{qj}(t)}{a_{lj}(t)}, \tag{3.19}$$

for all $t \geq T_l$, and $j = 1, 2, \ldots, l$, where $q = i_l$.

Consider the Liapunov function as follows:

$$V_l(t) = \left(x_q(t)\right)^{-\eta}\left(x_l(t)\right)^{\lambda}. \tag{3.20}$$

By calculating, we obtain for any $t \geq 0$

$$\begin{aligned} \frac{dV_l(t)}{dt} = V_l(t)\Bigg[&-\eta b_q(t) + \lambda b_l(t) + \sum_{j=1}^{l}\left(\eta a_{qj}(t) - \lambda a_{lj}(t)\right)x_j^{\theta_{lj}}(t) \\ &+ \sum_{j=l+1}^{n} \eta a_{qj}(t) x_j^{\theta_{qj}}(t) - \sum_{j=l+1}^{n} \lambda a_{lj}(t) x_j^{\theta_{lj}}(t)\Bigg], \end{aligned} \tag{3.21}$$

for all $t \neq t_k$ and

$$V_l\left(t_k^+\right) = h_{qk}^{-\eta} h_{lk}^{\lambda} V_l(t_k), \tag{3.22}$$

for all $k = 1, 2, \ldots$. From (3.3) and (3.19), we have

$$\begin{aligned} &\int_t^{t+\omega}\left(-\eta b_q(s) + \lambda b_l(s)\right)ds + \lambda \sum_{t\leq t_k<t+\omega} \ln h_{lk} - \eta \sum_{t\leq t_k<t+\omega} \ln h_{qk} \\ &\quad < -\eta\delta\left(\int_t^{t+\omega} b_l(s)ds + \sum_{t\leq t_k<t+\omega} \ln h_{lk}\right) \\ &\quad \leq -\delta\eta\eta_0, \end{aligned} \tag{3.23}$$

$$\eta a_{qj}(t) - \lambda a_{lj}(t) < 0, \tag{3.24}$$

for all $t \geq T_l$ and $j = 1, 2, \ldots, l$. Hence, from (3.21), it follows that

$$\frac{dV_l(t)}{dt} \leq V_l(t)\Bigg[-\eta b_q(t) + \lambda b_l(t) + \sum_{j=1}^{l}\left(\eta a_{qj}(t) - \lambda a_{lj}(t)\right)x_j^{\theta_{lj}}(t) + \sum_{j=l+1}^{n}\eta a_{qj}(t)x_j^{\theta_{qj}}(t) - \sum_{j=l+1}^{n}\lambda a_{lj}(t)x_j^{\theta_{lj}}(t)\Bigg], \quad t \geq T_l,\ t \neq t_k,$$
$$V_n\left(t_k^+\right) = h_{pk}^{-\alpha}h_{nk}^{\beta}V_n(t_k), \quad k = 1, 2, \ldots. \tag{3.25}$$

Since $x_i(t) \to 0$ as $t \to \infty$ for all $i > l$, by the boundedness of $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ on $[0, \infty)$, we obtain

$$\lim_{t\to\infty}\int_t^{t+\omega}\sum_{j=l+1}^{n}\left(\eta a_{qj}(s)x_j^{\theta_{qj}}(s) - \lambda a_{lj}(s)\right)x_j^{\theta_{lj}}(s)ds = 0. \tag{3.26}$$

Hence, for any small $\varepsilon > 0$, there is a $T_l' > 0$, such that

$$\int_t^{t+\omega}\sum_{j=l+1}^{n}\left(\eta a_{qj}(s)x_j^{\theta_{qj}}(s) - \lambda a_{lj}(s)\right)x_j^{\theta_{lj}}(s)ds < \varepsilon, \quad t > T_l'. \tag{3.27}$$

Combining (3.23), it follows that there is enough large $T_l^* > \max\{T_l, T_l'\}$ such that for all $t \geq T_l^*$,

$$\int_t^{t+\omega}\left[-\eta b_q(s) + \lambda b_l(s) + \sum_{j=l+1}^{n}\left(\eta a_{qj}(s)x_j^{\theta_{qj}}(s) - \sum_{j=l+1}^{n}\lambda a_{lj}(s)\right)x_j^{\theta_{lj}}(s)\right]ds - \eta\sum_{t\leq t_k<t+\omega}\ln h_{qk} + \lambda\sum_{t\leq t_k<t+\omega}\ln h_{lk} \leq -\frac{1}{2}\delta\eta\eta_0,$$
$$x_i(t) \leq \delta \quad \forall i > l. \tag{3.28}$$

For any $t > T_l^*$, we firstly choose an integer $q_t \geq 0$ such that $t \in (T_l^* + q_t\omega, T_l^* + (q_t + 1)\omega]$. Integrating (3.25) from T_l^* to t, then from (3.3) and (3.28), we have

$$\begin{aligned}
V_l(t) \leq{}& V_l(T_l^*) \exp\left\{ \int_{T_l^*}^{t} \left[-\eta b_q(s) + \lambda b_l(s) + \sum_{j=l+1}^{n} \left(\eta a_{qj}(s) x_j^{\theta_{qj}}(s) - \sum_{j=l+1}^{n} \lambda a_{lj}(s) \right) x_j^{\theta_{lj}}(s) \right] ds \right. \\
&\left. + \sum_{T_l^* \leq t_k < t} \ln\left(h_{qk}^{-\eta} h_{lk}^{\lambda} \right) \right\} \\
={}& V_l(T_l^*) \exp\left\{ \left(\int_{T_l^*}^{T_l^*+\omega} \left[-\eta b_q(s) + \lambda b_l(s) + \sum_{j=l+1}^{n} \eta a_{qj}(s) x_j^{\theta_{qj}}(s) - \sum_{j=l+1}^{n} \lambda a_{lj}(s) x_j^{\theta_{lj}}(s) \right] ds \right. \right. \\
&\left. + \sum_{T_l^* \leq t_k < T_l^*+\omega} \ln\left(h_{qk}^{-\eta} h_{lk}^{\lambda} \right) \right) + \cdots \\
&+ \left(\int_{T_l^*+(q_t-1)\omega}^{T_l^*+q_t\omega} \left[-\eta b_q(s) + \lambda b_l(s) + \sum_{j=l+1}^{n} \eta a_{qj}(s) x_j^{\theta_{qj}}(s) \right. \right. \\
&\left. \left. - \sum_{j=l+1}^{n} \lambda a_{lj}(s) x_j^{\theta_{lj}}(s) \right] ds + \sum_{T_l^*+(q_t-1)\omega \leq t_k < T_l^*+q_t\omega} \ln\left(h_{qk}^{-\eta} h_{lk}^{\lambda} \right) \right) \\
&+ \left(\int_{T_l^*+q_t\omega}^{t} \left[-\eta b_q(s) + \lambda b_l(s) + \sum_{j=l+1}^{n} \eta a_{qj}(s) x_j^{\theta_{qj}}(s) \right. \right. \\
&\left. \left. \left. - \sum_{j=l+1}^{n} \lambda a_{lj}(s) x_j^{\theta_{lj}}(s) \right] ds + \sum_{T_l^*+q_t\omega \leq t_k < t} \ln\left(h_{qk}^{-\eta} h_{lk}^{\lambda} \right) \right) \right\} \\
\leq{}& M_l \exp\left(-\frac{1}{2} \delta \eta \eta_0 q_t \right),
\end{aligned} \tag{3.29}$$

where

$$M_l = V_l(T_l^*) \exp\left\{ \left(\omega \sup_{t \geq 0} \left\{ \eta \left| b_q(t) \right| + \lambda |b_l(t)| + \sum_{j=l+1}^{n} (\eta a_{qj}(t) + \lambda a_{lj}(t)) \delta^{\theta_{lj}} \right\} + (\lambda + \eta) P \right) \right\}, \tag{3.30}$$

Since $q_t \rightarrow \infty$ as $t \rightarrow \infty$, we obtain from (3.29)

$$V_l(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \tag{3.31}$$

Since

$$(x_l(t))^{\eta} = V_l(t)\left(x_q(t)\right)^{\lambda},$$
$$(h_{lk}x_l(t_k))^{\eta} = h_{lk}^{\eta}h_{qk}^{-\lambda}V_l(t_k)\left(h_{qk}x_q(t_k)\right)^{\lambda}, \tag{3.32}$$

by the boundedness of $x(t)$ on $[0,\infty)$, it follows that

$$x_l(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.33}$$

Finally, by the induction principle, we obtain that $x_i(t) \to 0$ as $t \to \infty$ for all $i > r$. This completes the proof of Theorem 3.1. □

4. Permanence

In this section, we study the permanence of partial species $x_i(t)$ $(i = 1,2,\ldots,r)$ of system (1.2). We state and prove the following result.

Theorem 4.1. *Suppose that all the conditions of Theorem 3.1 hold. If for each $i = 1,2,\ldots,r$*

$$\liminf_{t\to\infty}\left(\int_t^{t+\omega}\left[b_i(s) - \sum_{j\neq i}^{r} a_{ij}(s)u_{j0}^{\theta_{ij}}(t)\right]ds + \sum_{t\le t_k<t+\omega}\ln h_{ik}\right) > 0, \tag{4.1}$$

where u_{i0} is some fixed positive solution of (2.3), *then species x_i $(i = 1,2,\ldots,r)$ are permanent, that is, there are positive constants m and M such that for any positive solution $x(t) = (x_1(t),x_2(t),\ldots,x_n(t))$ of system* (1.2)

$$m \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M, \quad i = 1,2,\ldots,r. \tag{4.2}$$

Proof. From (4.1) and the boundedness of functions $a_{ij}(t)$ $(i,j=1,2,\ldots,n)$ on R_+, there are constants $\varepsilon_0>0$ and $T_1>0$ such that for any $t\ge T_1$ and $i=1,2,\ldots,r$.

$$\int_t^{t+\omega}\left[b_i(s)-\sum_{j=1}^{n}a_{ij}(s)\varepsilon_0-\sum_{j\neq i}^{r}a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right]ds+\sum_{t\le t_k<t+\omega}\ln h_{ik}>\varepsilon_0. \tag{4.3}$$

For any $i\le r$, from system (1.2), we have

$$\begin{aligned}\frac{dx_i(t)}{dt}&\le x_i(t)\left[b_i(t)-\sum_{j=1}^{n}a_{ij}(t)x_j^{\theta_{ij}}(t)\right],\\&\le x_i(t)\left[a_i(t)-b_{ii}(t)x_i^{\theta_{ii}}(t)\right],\quad t\neq t_k,\ t\ge 0,\\x_i\left(t_k^+\right)&=h_{ik}x_i(t_k),\quad k=1,2,\ldots,\end{aligned} \tag{4.4}$$

we have

$$x_i(t)\le u_i(t)\quad \forall t\ge 0, \tag{4.5}$$

where $u_i(t)$ is the solution of (2.3) with initial condition $u_i(0)\ge x_i(0)$. From Lemma 2.1 and Theorem 3.1, for the above constant ε_0 there is a $T_2\ge T_1$ such that for all $t\ge T_2$

$$x_i(t)\le u_i(t)\le u_{i0}(t)+\varepsilon_0,\quad i=1,2,\ldots,r, \tag{4.6}$$

$$x_i(t)<\varepsilon_0,\quad i=r+1,r+2,\cdots,n. \tag{4.7}$$

Let

$$\begin{aligned}\gamma_i&=\sup_{t\ge 0}\left\{|b_i(t)|+\sum_{j=1}^{n}a_{ij}(t)\varepsilon_0+\sum_{j\neq i}^{r}a_{ij}(t)u_{j0}^{\theta_{ij}}(t)\right\},\\m&=\min_{1\le i\le r}\left\{\varepsilon_0\exp\left(-\gamma_i\omega-P\right)\right\},\end{aligned} \tag{4.8}$$

where constant $P>0$ is given in (2.7). Obviously, $m>0$ and m is independent of any positive solution of system (1.2).

Now, we prove that there is a $T_3\ge T_2$ such that

$$x_i(t)\ge m\quad \forall t\ge T_3,\ i=1,2,\ldots,r. \tag{4.9}$$

We only need to consider the following three cases for each $i = 1, 2, \ldots, r$.

Case I. There is a $t_1 \geq T_2$ such that $x_i(t) \leq \varepsilon_0' = \sqrt[\theta_{ii}]{\varepsilon_0}$ for all $t \geq t_1$.

Case II. There is a $t_2 \geq T_2$ such that $x_i(t) \geq \varepsilon_0'$ for all $t \geq t_2$.

Case III. $x_i(t)$ oscillates about ε_0' for all $t \geq T_2$.

For Case I, let $t = t_1 + l\omega$, where $l \geq 0$ is any integer. From (4.3)–(4.7) we obtain

$$
\begin{aligned}
x_i(t) &= x_i(t_1)\exp\left(\int_{t_1}^{t}\left(b_i(s) - a_{ii}(s)x_i^{\theta_{ii}}(s) - \sum_{j\neq i}^{n} a_{ij}(s)x_j^{\theta_{ij}}(s)\right)ds + \sum_{t_1\leq t_k<t}\ln h_{ik}\right)\\
&\geq x_i(t_1)\exp\left(\int_{t_1}^{t_1+\omega}\left(b_i(s) - \sum_{j=1}^{n} a_{ij}(s)\varepsilon_0 - \sum_{j\neq i}^{r} a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds\right.\\
&\qquad + \sum_{t_1\leq t_k<t_1+\omega}\ln h_{ik} + \cdots + \int_{t_1+(l-1)\omega}^{t_1+l\omega}\left(b_i(s) - \sum_{j=1}^{n} a_{ij}(s)\varepsilon_0 - \sum_{j\neq i}^{r} a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds\\
&\qquad \left. + \sum_{t_1+(l-1)\omega\leq t_k<t_1+l\omega}\ln h_{ik}\right)\\
&\geq x_i(t_1)\exp(l\varepsilon_0).
\end{aligned}
\tag{4.10}
$$

Therefore, $x_i(t) \to \infty$ as $l \to \infty$ which leads to a contradiction.

For Case III, we choose two sequences $\{\rho_n\}$ and $\{\rho_n^*\}$ satisfying $T_2 \leq \rho_1 < \rho_1^* < \cdots < \rho_n < \rho_n^* < \cdots$ and $\lim_{n\to\infty}\rho_n = \lim_{n\to\infty}\rho_n^* = \infty$ such that

$$
\begin{gathered}
x_i(\rho_n) \geq \varepsilon_0', \qquad x_i(\rho_n^+) \leq \varepsilon_0', \qquad x_i(\rho_n^*) \leq \varepsilon_0', \qquad x_i\left(\rho_n^{*^+}\right) \geq \varepsilon_0',\\
x_i(t) \leq \varepsilon_0' \quad \forall t \in (\rho_n, \rho_n^*),\\
x_i(t) \geq \varepsilon_0' \quad \forall t \in (\rho_n^*, \rho_{n+1}).
\end{gathered}
\tag{4.11}
$$

For any $t \geq T_2$, if $t \in (\rho_n, \rho_n^*]$ for some integer n, then we can choose an integer $l \geq 0$ such that $t = \rho_n + l\omega + \nu_i$, where $\nu_i \in [0, \omega)$ is a constant. Since for any $t \in (\rho_n, \rho_n^*)$ from (4.6) and (4.7) we have

$$
\dot{x}_i(t) \geq x_i(t)\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\varepsilon_0 - \sum_{j\neq i}^{r} b_{ij}(t)u_{j0}^{\theta_{ij}}(t)\right), \quad t \neq t_k.
\tag{4.12}
$$

Integrating this inequality from ρ_n to t, then from (4.7) and (3.28)-(3.29) we have

$$\begin{aligned}
x_i(t) &\geq x(\rho_n)\exp\left(\int_{\rho_n}^{t}\left(b_i(s)-\sum_{j=1}^{n}a_{ij}(s)\varepsilon_0-\sum_{j\neq i}^{r}a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds+\sum_{\rho_n\leq t_k<t}\ln h_{ik}\right)\\
&\geq \varepsilon_0\exp\left(\int_{\rho_n}^{\rho_n+\omega}\left(b_i(s)-\sum_{j=1}^{n}a_{ij}(s)\varepsilon_0-\sum_{j\neq i}^{r}a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds\right.\\
&\qquad\qquad+\sum_{\rho_n\leq t_k<\rho_n+\omega}\ln h_{ik}+\cdots+\int_{\rho_n+(l-1)\omega}^{\rho_n+l\omega}\left(b_i(s)-\sum_{j=1}^{n}a_{ij}(s)\varepsilon_0-\sum_{j\neq i}^{r}a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds\\
&\qquad\qquad\left.+\sum_{\rho_n+(l-1)\omega\leq t_k<\rho_n+l\omega}\ln h_{ik}\right)\\
&\quad+\int_{\rho_n+l\omega}^{\rho_n+l\omega+\nu_i}\left(b_i(s)-\sum_{j=1}^{n}a_{ij}(s)\varepsilon_0-\sum_{j\neq i}^{r}a_{ij}(s)u_{j0}^{\theta_{ij}}(s)\right)ds+\sum_{\rho_n+l\omega\leq t_k<\rho_n+l\omega+\nu_i}\ln h_{ik}\\
&\geq \varepsilon_0\exp(-\gamma_i\omega-P).
\end{aligned}\tag{4.13}$$

If there exists an integer n such that $t\in(\rho_n^*,\rho_{n+1}]$, then we obviously have

$$x_i(t)\geq\varepsilon_0>\varepsilon_0\exp(-\gamma_i\omega-P).\tag{4.14}$$

This shows that for Case III we always have

$$x_i(t)\geq\varepsilon_0\exp(-\gamma_i\omega-P),\quad \forall t\geq T_2.\tag{4.15}$$

Finally, if Case II holds, then from $x_i(t)\geq\varepsilon_0'$ for all $t\geq t_1$, we can directly obtain that (4.9) holds.

Therefore, from Lemma 2.2 and (4.9), it follows that species $x_i(t)$ $(i=1,2,\ldots,r)$ are permanent. This proof of Theorem 4.1 is completed. □

5. Global Attractivity

In this section, we further discuss the global attractivity of species $x_i(t)$ $(i\leq r)$. In order to obtain our results, we first consider the following subsystem which is composed of the species

$x_i(t)$ $(i \le r)$ of system (1.2) and for convenience of statement we use the variable $u_i(t)$ $(i \le r)$ to denote the species of this subsystem,

$$\frac{du_i(t)}{dt} = u_i(t)\left[b_i(t) - \sum_{j=1}^{r} a_{ij}(t)u_j^{\theta_{ij}}(t)\right], \quad t \ne t_k,$$
$$u_i(t_k^+) = h_{ik}u_i(t_k), \quad i = 1,2,\ldots,r, \; k = 1,2,\ldots. \tag{5.1}$$

We need the following lemma.

Lemma 5.1. *Suppose that assumption H and condition* (4.1) *of Theorem 4.1 hold. Then subsystem* (5.1) *is permanent.*

Lemma 5.1 can be proved by using the same method given in the proof of Theorem 4.1. We now state and prove the main result of this section.

Theorem 5.2. *Suppose that all conditions of Theorem 3.1 and Theorem 4.1 hold. If there are positive constants ρ, D and d_i $(i = 1,2,\ldots,r)$ and nonnegative integrable function $\mu(t)$ defined on R_+, satisfying $\int_s^t \mu(\tau)d\tau \ge -D + \rho(t-s)$ for all $t \ge s \ge 0$, such that*

$$d_i a_{ii}(t) - \sum_{j \ne i}^{r} d_j a_{ji}(t) \ge \mu(t), \quad i = 1,2,\ldots,r, \tag{5.2}$$

for all $t \ge 0$, then for any positive solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of system (1.2) *and any positive solution $u(t) = (u_1(t), u_2(t), \ldots, u_r(t))$ of subsystem* (5.1)

$$\lim_{t\to\infty}(x_i(t) - u_i(t)) = 0, \quad i = 1,2,\ldots,r. \tag{5.3}$$

Proof. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ be a positive solution of system (1.2) and $u(t) = (u_1(t), u_2(t), \ldots, u_r(t))$ be a positive solution of subsystem (5.1). By Theorem 3.1, we have $x_i(t) \to 0$ as $t \to \infty$ for all $i > r$. From Theorem 4.1 and Lemma 5.1, there are positive constants m and M such that

$$m \le x_i(t), \quad u_i(t) \le M, \quad i = 1,2,\ldots,r, \tag{5.4}$$

for all $t \ge 0$. Choose the Liapunov function as follows:

$$V_r(t) = \sum_{i=1}^{r} d_i |\ln x_i(t) - \ln u_i(t)|. \tag{5.5}$$

Since

$$\begin{aligned} V_r(t_k^+) &= \sum_{i=1}^{r} d_i \left| \ln x_i(t_k^+) - \ln u_i(t_k^+) \right| \\ &= \sum_{i=1}^{r} d_i |\ln h_{ik} x_i(t_k) - \ln h_{ik} u_i(t_k)| \\ &= V_r(t_k), \end{aligned} \tag{5.6}$$

then $V(t)$ is continuous for all $t \geq 0$. Calculating the upper right derivative of $V_r(t)$, we have

$$\begin{aligned} D^+V_r(t) &\leq \sum_{i=1}^{r} d_i \left(-a_{ii} \left| x_i^{\theta_{ii}}(t) - u_i^{\theta_{ii}}(t) \right| + \sum_{j\neq i}^{r} a_{ij}(t) \left| x_j^{\theta_{ij}}(t) - u_j^{\theta_{ij}}(t) \right| \right) + g(t) \\ &= -\sum_{i=1}^{r} \left(d_i a_{ii} - \sum_{j\neq i}^{r} d_j a_{ji}(t) \right) \left| x_i^{\theta_{ji}}(t) - u_i^{\theta_{ji}}(t) \right| + g(t), \end{aligned} \tag{5.7}$$

for all $t \geq 0$, where

$$g(t) = \sum_{i=1}^{r} d_i \sum_{j=r+1}^{n} a_{ji}(t) x_j^{\theta_{ji}}(t). \tag{5.8}$$

By (5.2), we have

$$D^+V_r(t) \leq -\mu(t) \sum_{i=1}^{r} \left| x_i^{\theta_{ji}}(t) - u_i^{\theta_{ji}}(t) \right| + g(t), \quad \forall t \geq 0. \tag{5.9}$$

By (5.4), we further obtain

$$D^+V_r(t) \leq -\lambda\mu(t)V_r(t) + g(t), \quad \forall t \geq 0, \tag{5.10}$$

where $\lambda = \min_{1\leq i\leq r} d_i^{-1} m > 0$. Applying the comparison theorem and the variation of constants formula of first-order linear differential equation, we have

$$V_r(t) \leq e^{-\int_0^t \lambda\mu(s)ds} \left(\int_0^t g(s) e^{\int_0^s \lambda\mu(\tau)d\tau} ds + V_r(0) \right), \tag{5.11}$$

for all $t \geq 0$. Since $g(t) \to 0$ as $t \to \infty$, from the properties of function $\mu(t)$ and (5.11), it is not hard to obtain $V_r(t) \to 0$ as $t \to \infty$. That shows

$$\lim_{t\to\infty} (x_i(t) - u_i(t)) = 0, \quad i = 1, 2, \ldots, r. \tag{5.12}$$

This completes the proof of Theorem 5.2. □

References

[1] J. Hou, Z. Teng, and S. Gao, "Permanence and global stability for nonautonomous *N*-species Lotka-Valterra competitive system with impulses," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1882–1896, 2010.

[2] J. Hou, Z. D. Teng, and S. J. Gao, "Partial extinction, permanence and global attractivity in nonautonomous *n*-species Lotka-Volterra competitive systems with impulses," *Applied Mathematics: A Journal of Chinese Universities*, vol. 26, no. 2, pp. 161–176, 2011.

[3] M. He, Z. Li, and F. Chen, "Permanence, extinction and global attractivity of the periodic Gilpin-Ayala competition system with impulses," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1537–1551, 2010.