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Research Article

A Family of Heat Functions as Solutions of Indeterminate Moment Problems

Ricardo Gómez and Marcos López-García

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We construct a family of functions satisfying the heat equation and show how they can be used to generate solutions to indeterminate moment problems. The following cases are considered: log-normal, generalized Stieltjes-Wigert, and q-Laguerre.

1. Introduction

For a real-valued, measurable function f defined on $[0,\infty)$, its nth moment is defined as $s_n(f) = \int_0^\infty x^n f(x)dx$, $n \in \mathbb{N} = \{0,1,\dots\}$. Let $(s_n)_{n\geq 0}$ be a sequence of real numbers. If f is a real-valued, measurable function defined on $[0,\infty)$ with moment sequence $(s_n)_{n\geq 0}$, we say that f is a solution to the Stieltjes moment problem (related to $(s_n)_{n\geq 0}$). If the solution is unique, the moment problem is called M-determinate. Otherwise, the moment problem is said to be M-indeterminate. When we replace $\mathbb{N}$ with $\mathbb{Z}$ we can formulate the same problem (the so-called strong Stieltjes moment problem).

In [1–3], Stieltjes was the first to give examples of M-indeterminate moment problems. He showed that the log-normal distribution with density on $(0,\infty)$ given as

$$d_\sigma(x) = (2\pi\sigma^2)^{-1/2} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right), \quad \sigma > 0, \tag{1.1}$$

together with the densities ($a \in [-1,1]$)

$$d_\sigma(x)\left(1 + a\sin\left(2\pi\sigma^{-2}\log x\right)\right) \geq 0, \tag{1.2}$$

have all the moment sequence $(e^{n^2\sigma^2/2})_{n\geq 0}$. So, the log-normal moment problem is M-indeterminate.

In fact, for $\beta \in \mathbb{R}$, we have

$$s_n(x^\beta d_\sigma) = q^{-(n+\beta)^2/2}, \quad n \in \mathbb{Z}, \tag{1.3}$$

where $q = e^{-\sigma^2}$.

The Stieltjes' example and the work in [4] gave rise to the present paper. By looking for real-valued, measurable functions h such that

$$g_{\sigma,\beta}(x) = x^\beta d_\sigma(x)\{1 + h(\sigma^{-2}\log(xq^\beta))\} \tag{1.4}$$

satisfies $s_n(g_{\sigma,\beta}) = s_n(x^\beta d_\sigma)$ for all $n \in \mathbb{Z}$, we are faced (Proposition 2.1) with the problem of characterizing the real-valued, measurable functions h satisfying

$$\int_{\mathbb{R}} \exp\left(\frac{-\sigma^2 x^2}{2}\right) h(x+n)dx = 0, \quad \forall n \in \mathbb{Z}. \tag{1.5}$$

In particular, if h is a 1-periodic, real-valued, measurable function, then the last equality is equivalent to

$$\int_0^1 \theta(x, 2^{-1}\sigma^{-2})h(x)dx = 0, \tag{1.6}$$

where θ is the so-called theta function given by

$$\theta(x,t) = (4\pi t)^{-1/2} \sum_{n\in\mathbb{Z}} e^{-(x+n)^2/4t} \quad \text{(see [5, page 59])}. \tag{1.7}$$

The 1-periodic, positive function θ satisfies the heat equation on $\mathbb{R}_+^2$:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t). \tag{1.8}$$

Notice that if h satisfies (1.5) or (1.6), then so does ah, $a \in \mathbb{R}$. Moreover, when h is bounded below (above), there is $a \in \mathbb{R}$ such that $1 + ah \geq 0$. Hence, in this case a probability density function can be obtained by a standard normalizing procedure.

It only remains for us to find some interesting 1-periodic functions h satisfying (1.6). By setting $h_c(x) = q^{c^2/2}\sigma^2 M_c^{-1}\theta(x+c, 2^{-1}\sigma^{-2})^{-1} - 1$, $c \in [0,1)$, we obtain the well-known classical solution (see, e.g., [4])

$$w_c(x) = d_\sigma(x)(1 + h_c(\sigma^{-2}\log x)) = \frac{x^{c-1}}{M_c(q, -q^{1/2-c}x, -q^{1/2+c}/x; q)_\infty} \tag{1.9}$$

to the log-normal moment problem, where M_c is the constant that makes $\int_0^\infty w_c(x)dx = 1$. Information about theta functions and orthogonal polynomials can be found in [6].

To get more examples, for $\alpha > -1$, we define the following function:

$$\theta_\alpha(x,t) = \sum_{n\in\mathbb{Z}} (2\pi n)^{2(1+\alpha)} e^{-4\pi^2 n^2 t + 2\pi n i x}. \tag{1.10}$$

Clearly, θ_α is a 1-periodic function in the variable x and satisfies the heat equation on $\mathbb{R}^2_+$. In addition, for $\alpha \geq -1$ and $t_1, t_2 > 0$, we show that

$$\int_0^1 \theta(y,t_1)\theta_\alpha(x-y,t_2)\,dy = \theta_\alpha(x,t_1+t_2). \tag{1.11}$$

Therefore, the following 1-periodic, continuous function satisfies the condition (1.6):

$$h_{y,t,\alpha}(x) = \theta_\alpha(y-x,t) - \theta_\alpha(y,2^{-1}\sigma^{-2}+t), \quad y \in [0,1),\ t > 0. \tag{1.12}$$

Furthermore, for $\alpha > -1$, we have $\int_0^1 \theta_\alpha(x,t)dx = 0$ for all $t > 0$, thus the following 1-periodic, continuous function satisfies the condition (1.6):

$$h_{t,\alpha}(x) = \theta_\alpha(x,t)\theta(x,2^{-1}\sigma^{-2})^{-1}, \quad t > 0,\ \alpha > -1. \tag{1.13}$$

In [7, 8], Christiansen also generates new measures from old ones. The similarity of his work with the one developed here comes from the quasiperiodicity of the theta function.

The paper is organized as follows. Preliminaries are given in Section 2. We define the family $\{\theta_\alpha\}_{\alpha\geq -1}$ of heat functions in Section 3, where more functions h satisfying (1.6) are shown. The last two sections refer to the generalized Stieltjes-Wigert and the q-Laguerre moment problems, respectively. Finally, we show a nonperiodic, continuous function h fulfilling the condition (1.5).

2. Notation and preliminaries

For $(x,t) \in \mathbb{R}^2_+$, let (see [5, pages 33, 59])

$$\begin{gathered} K(x,t) = (4\pi t)^{-1/2}e^{-x^2/4t}, \\ \theta(x,t) = \sum_{n\in\mathbb{Z}} K(x+n,t) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t + 2\pi n i x}. \end{gathered} \tag{2.1}$$

The positive functions K, θ satisfy the heat equation on $\mathbb{R}^2_+$. Clearly, θ is a 1-periodic function in the variable x. Moreover,

$$\int_{\mathbb{R}} K(x,t)\,dx = 1, \qquad \int_0^1 \theta(x,t)\,dx = 1, \quad \forall t > 0. \tag{2.2}$$

For $c \in [0,1)$, we set

$$\mathcal{M}_c := \int_0^1 \frac{\theta(x,2^{-1}\sigma^{-2})}{\theta(x+c,2^{-1}\sigma^{-2})}\,dx. \tag{2.3}$$

Throughout this paper we will write $q = e^{-\sigma^2}$, $\sigma > 0$ fixed. The density of the log-normal distribution with parameter σ^2 can be written as

$$d_\sigma(x) = \frac{1}{x}K(\log x, 2^{-1}\sigma^2). \tag{2.4}$$

For $\beta \in \mathbb{R}$, we have

$$x^{\beta} d_{\sigma}(x) = q^{\beta-\beta^2/2} d_{\sigma}(xq^{\beta}). \tag{2.5}$$

Therefore,

$$\begin{aligned} s_{\beta}(d_{\sigma}) &:= \int_0^{\infty} x^{\beta} d_{\sigma}(x)dx = q^{-\beta^2/2}\int_0^{\infty} d_{\sigma}(x)dx \\ &= q^{-\beta^2/2}\int_{\mathbb{R}} K(x,2^{-1}\sigma^2)dx = q^{-\beta^2/2}. \end{aligned} \tag{2.6}$$

In particular, the strong Stieltjes moment sequence of $x^{\beta} d_{\sigma}$ is given by

$$s_n(x^{\beta} d_{\sigma}) = q^{-(n+\beta)^2/2}, \quad n \in \mathbb{Z}. \tag{2.7}$$

For $0 < q < 1$, $n \in \mathbb{N}$, we introduce some notation from q-calculus (see [9, page 233]):

$$(p;q)_0 := 1, \quad (p;q)_n := \prod_{k=0}^{n-1}(1-pq^k), \quad n \geq 1, \qquad (p;q)_{\infty} := \prod_{k=0}^{\infty}(1-pq^k). \tag{2.8}$$

For $\beta \in \mathbb{R}$, we set

$$(p;q)_{\beta} := \frac{(p;q)_{\infty}}{(pq^{\beta};q)_{\infty}}. \tag{2.9}$$

The following easily verified identities will be used:

$$(p;q)_n = \frac{(p;q)_{\infty}}{(pq^n;q)_{\infty}}, \qquad (p;q)_{n+\beta} = (pq^n;q)_{\beta}(p;q)_n. \tag{2.10}$$

We use the following notation:

$$\begin{aligned} (p_1,p_2,\dots,p_k;q)_n &= (p_1;q)_n(p_2;q)_n\cdots(p_k;q)_n, \\ (p_1,p_2,\dots,p_k;q)_{\infty} &= (p_1;q)_{\infty}(p_2;q)_{\infty}\cdots(p_k;q)_{\infty}. \end{aligned} \tag{2.11}$$

For $z \in \mathbb{C}$, we consider the two q-exponential functions

$$\begin{aligned} e_q(z) &= \sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_{\infty}}, \quad |z| < 1, \\ E_q(z) &= \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}z^k}{(q;q)_k} = (-z;q)_{\infty}. \end{aligned} \tag{2.12}$$

For $x \in \mathbb{R}$, we define

$$L_q(x) = \sum_{n\in\mathbb{Z}} q^{(1/2)n^2}x^n. \tag{2.13}$$

The value of the sum $L_q(x)$ is known by Jacobi's triple product identity

$$L_q(x) = (q, -\sqrt{q}x, -\sqrt{q}/x; q)_\infty. \tag{2.14}$$

It is easy to check the identity

$$\sqrt{\frac{\sigma^2}{2\pi}} q^{x^2/2} L_q(q^{-x}) = \theta(x, 2^{-1}\sigma^{-2}), \quad \forall x \in \mathbb{R}. \tag{2.15}$$

For $c \in [0,1)$, we introduce the following constant (see [4]):

$$M_c := \int_0^\infty \frac{x^{c-1}}{L_q(xq^{-c})} dx = \frac{\pi q^{c(c-1/2)}}{\sin(\pi c)} \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty^2} \tag{2.16}$$

for $c > 0$, and $M_0 = \log(q^{-1})$. By the monotone convergence theorem and equality (2.15) we have

$$\mathcal{M}_c = \sum_{n\in\mathbb{Z}} \int_0^1 \frac{K(x+n, 2^{-1}\sigma^{-2})}{\theta(x+c, 2^{-1}\sigma^{-2})} dx = \int_{\mathbb{R}} \frac{q^{x^2/2}}{q^{(x+c)^2/2} L_q(q^{-(x+c)})} dx = q^{-c^2/2}\sigma^{-2} M_c. \tag{2.17}$$

Proposition 2.1. *Let $h \in L^1(\mathbb{R}, e^{-\sigma^2(x-n)^2/2}dx)$ for all $n \in \mathbb{N}$ $(\mathbb{Z})$. The function $g_{\sigma,\beta}(x) = x^\beta d_\sigma(x)\{1 + h(\sigma^{-2}\log(xq^\beta))\}$ has the same (strong) Stieltjes moment sequence as $x^\beta d_\sigma$ if and only if*

$$\int_{\mathbb{R}} K(x, 2^{-1}\sigma^{-2}) h(x+n) dx = 0 \quad \text{for every } n \in \mathbb{N}(\mathbb{Z}). \tag{2.18}$$

Proof. By using (2.5) and changing variables $y = -n + \sigma^{-2}\log(xq^\beta)$ we obtain

$$s_n(g_{\sigma,\beta}) = s_n(x^\beta d_\sigma) + q^{-(\beta+n)^2/2} \int_{-\infty}^{\infty} K(y, 2^{-1}\sigma^{-2}) h(y+n) dy, \tag{2.19}$$

and the result follows. □

In particular, if h is a 1-periodic function in $L^1((0,1))$, then $s_n(g_{\sigma,\beta}) = s_n(x^\beta d_\sigma)$ for all $n \in \mathbb{Z}$ if and only if

$$\int_0^1 \theta(x, 2^{-1}\sigma^{-2}) h(x) dx = 0. \tag{2.20}$$

Remark 2.2. If h satisfies (2.18) or (2.20), then so does ah, $a \in \mathbb{R}$. Moreover, when h is bounded below (above), there is $a \in \mathbb{R}$ such that $1 + ah \geq 0$.

Definition 2.3. For $\beta \in \mathbb{R}$, let $\widetilde{V}_\beta$ denote the set of real-valued, measurable functions f defined on $[0,\infty)$ solving the strong moment problem

$$s_n(f) = q^{-(n+\beta)^2/2} := s_{n,\beta}, \quad n \in \mathbb{Z}. \tag{2.21}$$

Example 2.4. For $\beta \in \mathbb{R}$, $x^\beta d_\sigma \in \widetilde{V}_\beta$.

Now we want to find some interesting 1-periodic functions h satisfying (2.20).

Example 2.5. By setting $h_c(x) = \mathcal{M}_c^{-1}\theta(x+c, 2^{-1}\sigma^{-2})^{-1} - 1$, with $c \in [0,1)$ and using (2.15), (2.17), and (2.20), we obtain the classical solution

$$w_c(x) = d_\sigma(x)\{1 + h_c(\sigma^{-2}\log x)\} = \frac{x^{c-1}}{M_c L_q(xq^{-c})} \in \widetilde{V}_0. \tag{2.22}$$

Example 2.6. If $f \in \widetilde{V}_0$, then an easy calculation shows that

$$q^{\beta-\beta^2/2} f(q^\beta x) \in \widetilde{V}_\beta, \quad \beta \in \mathbb{R}. \tag{2.23}$$

Example 2.7. Let f be a 1-periodic function integrable on $(0,1)$. Then the function

$$h(x) = \left(f(x) - \int_0^1 f(x)dx\right)\theta(x, 2^{-1}\sigma^{-2})^{-1} \tag{2.24}$$

satisfies (2.20). In particular, we can put $f_t(x) = \theta(x,t)$ with $t > 0$.

To get more examples, in the next section we introduce a family of functions satisfying the heat equation for which functions fulfilling the condition (2.20) can be defined.

3. The families of heat functions $\{K_\alpha\}_\alpha$, $\{\theta_\alpha\}_\alpha$

We follow the notation in [10, Chapter 9]. For $f \in L^1(\mathbb{R})$, we define its Fourier transform as

$$(\Phi f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\frac{dx}{\sqrt{2\pi}}. \tag{3.1}$$

For $\alpha \geq -1$ and $t > 0$ fixed, the function $\xi^{2(1+\alpha)}e^{-\xi^2 t}$ is in $L^1(\mathbb{R})$, so we define

$$K_\alpha(x,t) = \frac{1}{\sqrt{2\pi}}\Phi^{-1}(\xi^{2(1+\alpha)}e^{-\xi^2 t})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\xi^{2(1+\alpha)}e^{-\xi^2 t}\cos(x\xi)\frac{d\xi}{\sqrt{2\pi}}. \tag{3.2}$$

Then K_α is a real-valued function that satisfies the heat equation on $\mathbb{R}_+^2$. We can rewrite

$$\begin{aligned} K_\alpha(x,t) &= \frac{1}{\sqrt{\pi}}t^{-1-\alpha}K(x,t)\int_{-\infty}^{\infty}\xi^{2(1+\alpha)}e^{-(\xi - ix/2\sqrt{t})^2}d\xi \\ &= \frac{1}{\sqrt{\pi}}t^{-1-\alpha}K(x,t)\int_{-\infty}^{\infty}\left(\xi + \frac{ix}{2\sqrt{t}}\right)^{2(1+\alpha)}e^{-\xi^2}d\xi. \end{aligned} \tag{3.3}$$

Therefore,

$$|K_\alpha(x,t)| \leq C_\alpha t^{-1-\alpha}K(x,t)\int_{-\infty}^{\infty}\left(\xi^{2(1+\alpha)} + \frac{x^{2(1+\alpha)}}{t^{1+\alpha}}\right)e^{-\xi^2}d\xi \leq C_\alpha t^{-1-\alpha}K(x,t)\left[1 + \frac{x^{2(1+\alpha)}}{t^{1+\alpha}}\right]. \tag{3.4}$$

Since $x^\lambda e^{-x} \leq C_\lambda e^{-x/2}$ for $x, \lambda > 0$, we have

$$\frac{x^{2(1+\alpha)}}{(4t)^{1+\alpha}}K(x,t) = \frac{1}{\sqrt{4\pi t}}\left(\frac{x^{2(1+\alpha)}}{(4t)^{1+\alpha}}e^{-x^2/4t}\right) \leq \frac{C_\alpha}{\sqrt{4\pi t}}e^{-x^2/8t} = C_\alpha K(x,2t). \tag{3.5}$$

Whence,

$$|K_\alpha(x,t)| \le C_\alpha t^{-1-\alpha}(K(x,t)+K(x,2t)), \tag{3.6}$$

and $K_\alpha(\cdot,t) \in L^1(\mathbb{R})$ for all $t>0$.

The convolution of $f,g \in L^1(\mathbb{R})$ is given by

$$(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\frac{dy}{\sqrt{2\pi}}. \tag{3.7}$$

Using the fact that $\Phi(f*g)=\Phi(f)\Phi(g)$, the definition of K_α, and the inversion formula, we get

$$\int_{\mathbb{R}} K_\alpha(y,t_1)K_\beta(x-y,t_2)\,dy = K_{\alpha+\beta+1}(x,t_1+t_2), \tag{3.8}$$

whenever $\alpha,\beta \ge -1$, $t_1,t_2>0$.

Next, for $\alpha \ge -1$, we introduce the function

$$\theta_\alpha(x,t) = \sum_{n\in\mathbb{Z}} K_\alpha(x+n,t). \tag{3.9}$$

From (3.6) we have the estimate

$$|\theta_\alpha(x,t)| \le C_\alpha t^{-1-\alpha}(\theta(x,t)+\theta(x,2t)). \tag{3.10}$$

Remark 3.1. From (3.2) we have that $K_{-1}\equiv K$, $\theta_{-1}\equiv\theta$. If $\alpha\in\mathbb{N}$, then

$$K_\alpha = (-1)^{1+\alpha}\frac{\partial^{1+\alpha}K}{\partial t^{1+\alpha}}, \qquad \theta_\alpha = (-1)^{1+\alpha}\frac{\partial^{1+\alpha}\theta}{\partial t^{1+\alpha}}. \tag{3.11}$$

For $\alpha>-1$, the inversion formula implies

$$\int_0^1 \theta_\alpha(x,t)\,dx = \int_{-\infty}^{\infty}\Phi^{-1}\left(\xi^{2(1+\alpha)}e^{-\xi^2 t}\right)(x)\frac{dx}{\sqrt{2\pi}} = \xi^{2(1+\alpha)}e^{-\xi^2 t}\,\big|_{\xi=0} = 0, \quad \forall t>0. \tag{3.12}$$

Example 3.2. For $\alpha>-1$, $t>0$, the function

$$h_{t,\alpha}(x) = \theta_\alpha(x,t)\theta(x,2^{-1}\sigma^{-2})^{-1} \tag{3.13}$$

satisfies (2.20).

For $\alpha,\beta\ge -1$, $t_1,t_2>0$, the equality (3.8) implies

$$\int_0^1 \theta_\alpha(y,t_1)\theta_\beta(x-y,t_2)\,dy = \theta_{\alpha+\beta+1}(x,t_1+t_2). \tag{3.14}$$

In particular, for $\alpha\ge -1$, $\beta=-1$, we obtain

$$\int_0^1 \theta(y,t_2)\theta_\alpha(x-y,t_1)\,dy = \int_0^1 \theta_\alpha(y,t_1)\theta(x-y,t_2)\,dy = \theta_\alpha(x,t_1+t_2). \tag{3.15}$$

Example 3.3. For $y \in [0,1)$, $t > 0$, the following function fulfills (2.20):

$$h_{y,t,\alpha}(x) = \theta_\alpha(y-x,t) - \theta_\alpha(y, 2^{-1}\sigma^{-2}+t). \tag{3.16}$$

The following proposition gives an explicit formula for θ_α.

Proposition 3.4. *For $\alpha > -1$,*

$$\theta_\alpha(x,t) = \sum_{n\in\mathbb{Z}} (2\pi n)^{2(1+\alpha)} e^{-4\pi^2 n^2 t + 2\pi n i x}. \tag{3.17}$$

Proof. From (3.6) we have

$$\left| \sum_{m\in\mathbb{Z}} K_\alpha(x+m,t) \right| \le \frac{C_\alpha}{t^{1+\alpha}} (\theta(x,t) + \theta(x,2t)). \tag{3.18}$$

Hence, the series converges uniformly on compact subsets of $\mathbb{R}_+^2$ and therefore it is continuous. Since the series is 1-periodic in x, it admits a representation as a Fourier series,

$$\sum_{m\in\mathbb{Z}} K_\alpha(x+m,t) = \sum_{m\in\mathbb{Z}} a_m(t) e^{2\pi m i x}, \tag{3.19}$$

where convergence is in $L^2([0,1])$. Moreover,

$$\int_0^1 \sum_{m\in\mathbb{Z}} |K_\alpha(x+m,t)| \, dx \le \frac{C_\alpha}{t^{1+\alpha}} \int_{-\infty}^{\infty} (K(x,t) + K(x,2t)) \, dx = \frac{C_\alpha}{t^{1+\alpha}}. \tag{3.20}$$

By the dominated convergence theorem we have

$$\begin{aligned} a_m(t) &= \int_0^1 \left[\sum_{n\in\mathbb{Z}} K_\alpha(x+n,t) \right] e^{-2\pi m i x} dx = \int_{-\infty}^{\infty} \sqrt{2\pi} K_\alpha(x,t) e^{-2\pi m i x} \frac{dx}{\sqrt{2\pi}} \\ &= (\Phi[\sqrt{2\pi} K_\alpha(\cdot,t)])(2\pi m) = (2\pi m)^{2(1+\alpha)} e^{-4\pi^2 m^2 t}. \end{aligned} \tag{3.21}$$

□

The last result implies that θ_α satisfies the heat equation on $\mathbb{R}_+^2$.

Example 3.5. If $h(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi n i x} \in L^2([0,1])$, then h satisfies (2.20) if and only if

$$\sum_{n\in\mathbb{Z}} c_n e^{-2\pi^2 n^2/\sigma^2} = 0. \tag{3.22}$$

4. Generalized Stieltjes-Wigert

For $0 \leq p < 1$, the generalized Stieltjes-Wigert moment problem has the following weight function on $(0,\infty)$:

$$g(x;p,q) := \left(p, -\frac{p\sqrt{q}}{x}; q\right)_\infty d_\sigma(x). \tag{4.1}$$

When $p = 0$, the function g is the log-normal density. The next result is based on ideas in [4].

Proposition 4.1. *For every positive function $f \in \widetilde{V}_\beta$, $\beta \geq 0$, the function $(p, -p\sqrt{q}/x; q)_\infty f(x)$ has the Stieltjes moment sequence*

$$s_{n,\beta,p} := (p;q)_{n+\beta} q^{-(n+\beta)^2/2}, \quad n \in \mathbb{N}. \tag{4.2}$$

Proof. Since all the functions below are positive, using (2.12) and (2.9) we have

$$\begin{aligned}\int_0^\infty x^n g(x;p,q) f(x) dx &= (p;q)_\infty \sum_{k=0}^\infty \frac{q^{k^2/2} p^k}{(q;q)_k} \int_0^\infty x^{n-k} f(x) dx \\ &= (p;q)_\infty q^{-(n+\beta)^2/2} \sum_{k=0}^\infty \frac{\left(pq^{n+\beta}\right)^k}{(q;q)_k} \\ &= (p;q)_{n+\beta} q^{-(n+\beta)^2/2}, \quad n \in \mathbb{N}.\end{aligned} \tag{4.3}$$

In fact, the last inequality holds for $n \in \mathbb{Z}$, $\beta \in \mathbb{R}$ as long as $pq^{n+\beta} < 1$. In particular, for $p = q^{1/2}$ we obtain

$$\int_0^\infty x^n \left(q^{1/2}, -q/x; q\right)_\infty f(x) dx = \left(q^{1/2}; q\right)_{n+\beta} q^{-(n+\beta)^2/2} \tag{4.4}$$

for $n \in \mathbb{Z}$, $\beta \in \mathbb{R}$ whenever $q^{n+\beta+1/2} < 1$. □

Example 4.2. For $\beta \geq 0$, $q^{-\beta^2/2+\beta} w_c(q^\beta x)(p, -p\sqrt{q}/x; q)_\infty$ has the moment sequence $(s_{n,\beta,p})$.

More examples can be obtained if we combine (2.20) and the results in Section 3.

5. q-Laguerre

The normalized q-Laguerre polynomials $L_m^{(\alpha)}(x;q)$ (see [11]) belong to an M-indeterminate moment problem with moments:

$$S_{n,\alpha} := q^{-\alpha n - \binom{n+1}{2}} \left(q^{\alpha+1}; q\right)_n, \quad n \in \mathbb{N}, \tag{5.1}$$

with $0 < q < 1$, $\alpha > -1$.

Proposition 5.1. *Let $\alpha > -1$. For every positive function $f \in \widetilde{V}_{\alpha+1/2}$, the function $q^{(\alpha+1/2)^2/2}(q^{\alpha+1}, -q/x; q)_\infty f(x)$ has the Stieltjes moment sequence $S_{n,\alpha}$.*

Proof. Let $f \in \widetilde{V}_{\alpha+1/2}$. It follows from (4.4) with $\beta = \alpha + 1/2$, (2.9), and (2.10) that the function $q^{(\alpha+1/2)^2/2}(q^{1/2};q)^{-1}_{\alpha+1/2}(q^{1/2},-q/x;q)_\infty f(x)$ has the moment sequence

$$q^{(\alpha+1/2)^2/2}\left(q^{1/2};q\right)^{-1}_{\alpha+1/2}\left(q^{1/2};q\right)_{n+\alpha+1/2}q^{-(n+\alpha+1/2)^2/2} = S_{n,\alpha}. \tag{5.2}$$

The result follows since $(q^{1/2};q)^{-1}_{\alpha+1/2}(q^{1/2};q)_\infty = (q^{\alpha+1};q)_\infty$. □

Example 5.2. Example 2.6 with $\beta = \alpha + 1/2 > -1/2$ implies that the function

$$q^{\alpha+1/2}\left(q^{\alpha+1},-q/x;q\right)_\infty w_c\left(q^{\alpha+1/2}x\right) = \frac{q^{c(\alpha+1/2)}\left(q^{\alpha+1},-q/x;q\right)_\infty x^{c-1}}{M_c\left(q,-q^{\alpha+1-c}x,-q^{c-\alpha}/x;q\right)_\infty} \tag{5.3}$$

with $c \in (0,1]$ has moment sequence $S_{n,\alpha}$. In particular, when $\alpha = c - 1$ and using (2.16), we obtain the function

$$-\frac{\sin(\pi\alpha)}{\pi}\frac{(q;q)_\infty}{\left(q^{-\alpha};q\right)_\infty}\frac{x^\alpha}{(-x;q)_\infty}. \tag{5.4}$$

More examples can be obtained if we combine (2.20) and the results in Section 3. Finally, we show a nonperiodic, continuous function h fulfilling condition (2.18).

Example 5.3. For $\gamma \in \mathbb{R}\backslash 2\pi\mathbb{Q}$, consider $h(x) = (1 + k\cos(2\pi x))\cos\gamma x$, where

$$k = \frac{-\int_{\mathbb{R}} e^{-\sigma^2x^2/2}\cos(\gamma x)dx}{\int_{\mathbb{R}} e^{-\sigma^2x^2/2}\cos(\gamma x)\cos(2\pi x)dx} < 0. \tag{5.5}$$

So, h is not periodic at all and satisfies (2.18).

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Ricardo Gómez: Instituto de Matemáticas, Universidad Nacional Autónoma de México,
CP 04510 México DF, Mexico
Email address: rgomez@matem.unam.mx

Marcos López-García: Instituto de Matemáticas, Universidad Nacional Autónoma de México,
CP 04510 México DF, Mexico
Email address: flopez@matem.unam.mx