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Research Article

Nontrivial Periodic Solutions for Nonlinear Second-Order Difference Equations

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This paper is concerned with the existence of nontrivial periodic solutions and positive periodic solutions to a nonlinear second-order difference equation. Under some conditions concerning the first positive eigenvalue of the linear equation corresponding to the nonlinear second-order equation, we establish the existence results by using the topological degree and fixed point index theories.

1. Introduction

Let $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{N}$ be the sets of real numbers, integers, and natural numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}[a,b] = \{a, a+1, \ldots, b\}$ when $a \le b$.

In this paper, we deal with the existence of nontrivial periodic solutions and positive periodic solutions for a nonlinear second-order difference equation

$$\Delta^2 u(t-1) + q(t)u(t) = f(t, u(t)), \quad u(t+T) = u(t), \ t \in \mathbf{Z}, \tag{1.1}$$

where T is a positive integer, $q : \mathbf{Z} \to \mathbf{R}$ and $q(t+T) = q(t)$ for any $t \in \mathbf{Z}$, $f : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is continuous in the second variable and $f(t+T, x) = f(t, x)$ for any $(t, x) \in \mathbf{Z} \times \mathbf{R}$, and $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$.

From the T-periodicity of q and f, it is easy to verify that the T-periodic solution to (1.1) is equivalent to the solution to the following periodic boundary value problem (PBVP for short):

$$\begin{gathered}\Delta^2 u(t-1) + q(t)u(t) = f(t, u(t)), \quad t \in \mathbf{Z}[1,T], \\ u(0) = u(T), \qquad \Delta u(0) = \Delta u(T).\end{gathered} \tag{1.2}$$

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, and cybernetics. In recent years, there are many papers to study the existence of periodic solutions for second-order difference equations. By using various methods and techniques, for example, fixed point theorems, the method of upper and lower solutions, coincidence degree theory, critical point theory, a series of existence results of periodic solutions have been obtained. We refer the reader to [1–16] and references therein.

In [2], by using the method of upper and lower solutions, Atici and Cabada investigated the existence and uniqueness of periodic solutions for PBVP (1.2) provided that $q(t) \le 0$, $q(t) \not\equiv 0$. Of course the natural question is what would happen if $q(t) \ge 0$. In this paper, we will assume that

$$0 \le q(t) < 4\sin^2\left(\frac{\pi}{2T}\right), \; q(t) \not\equiv 0. \tag{H}$$

And we will use the topological degree and fixed point index theories to establish the existence of nontrivial periodic solutions and positive periodic solutions for (1.1). We note that some ideas of this paper are from [17–19].

This paper is organized as follows. In Section 2, we give Green's function associated with PBVP (1.2) and then present some preliminary lemmas that will be used to prove our main results. In Sections 4 and 5, by computing the topological degree and fixed point index, we establish some existence results of nontrivial periodic solutions and positive periodic solutions to (1.1). The final section of the paper contains some examples to illustrate our results, and we also remark that the results obtained in previous papers and ours are mutually independent.

2. Preliminaries

In this section, we are going to construct Green's function associated with PBVP (1.2) and then present some preliminary lemmas. Consider T-dimensional Banach space

$$E = \left\{ u = \{u(t)\}_1^T : u(t) \in \mathbf{R}, \; t \in \mathbf{Z}[1,T] \right\} \tag{2.1}$$

equipped with the norm $\|u\| = \max\{|u(t)|, \; t \in \mathbf{Z}[1,T]\}$ for all $u \in E$ and the cone $P = \{u \in E : u(t) \ge 0, \; t \in \mathbf{Z}[1,T]\}$. Then the cone P is normal and has nonempty interiors int P. It is clear that P is also a total cone of E, that is, $E = \overline{P-P}$, which means the set $P-P = \{u-v : u,v \in P\}$ is dense in E. For each $u, v \in E$, we write $u \le v$ if $v-u \in P$. For $r > 0$, let $B_r = \{u \in E : \|u\| < r\}$ and $\partial B_r = \{u \in E : \|u\| = r\}$. Put $Q = \max_{t \in \mathbf{Z}[1,T]} q(t)$.

Lemma 2.1. *If* $0 < Q < 4\sin^2(\pi/2T)$*, then, for each* $v \in E$*, the problem*

$$\begin{gathered}\Delta^2 u(t-1) + Qu(t) = v(t), \quad t \in \mathbf{Z}[1,T], \\ u(0) = u(T), \qquad \Delta u(0) = \Delta u(T).\end{gathered} \tag{2.2}$$

has a unique solution

$$u(t) = \sum_{k=1}^{T} G(t,k)v(k), \quad t \in \mathbf{Z}[0,T+1], \tag{2.3}$$

where $G(t,k)$ *is given by*

$$G(t,k) = \begin{cases} \rho(t-k), & 0 \le k \le t \le T+1, \\ \rho(T+t-k), & 0 \le t \le k \le T+1 \end{cases} \tag{2.4}$$

with $\rho(t) = (1/(2\sin\varphi\sin(\varphi T/2)))\cos\varphi((T/2)-t)$ *and* $\varphi := \arctan(\sqrt{Q(4-Q)}/(2-Q))$.

Proof. (i) Taking into account that $Q \in (0, 4\sin^2(\pi/2T))$, an easy computation ensures that $\varphi := \arctan(\sqrt{Q(4-Q)}/(2-Q)) \in (0, (\pi/T))$. Hence, $\rho(t) > 0$, $t \in \mathbf{Z}[0,T]$. It is easy to verify that

$$\Delta^2\rho(t-1) + Q\rho(t) = 0, \qquad \rho(0) = \rho(T), \qquad \rho(1) = \rho(T+1) + 1. \tag{2.5}$$

Let $u(t) = \sum_{k=1}^{T} G(t,k)v(k)$, $t \in \mathbf{Z}[0,T+1]$. Then

$$u(t) = \sum_{k=1}^{t} \rho(t-k)v(k) + \sum_{k=t+1}^{T} \rho(T+t-k)v(k), \quad t \in \mathbf{Z}[1,T], \tag{2.6}$$

where, and in what follows, we denote $\sum_{k=s}^{l} x(k) = 0$ when $l < s$. We have

$$\Delta u(t) = \sum_{k=1}^{t} \Delta\rho(t-k)v(k) + \rho(0)v(t+1) + \sum_{k=t+1}^{T} \Delta\rho(T+t-k)v(k) - \rho(T)v(t+1). \tag{2.7}$$

Then,

$$\begin{aligned}\Delta^2 u(t-1) + Qu(t) &= \sum_{k=1}^{t-1}\left[\Delta^2\rho(t-k-1) + Q\rho(t-k)\right]v(k) + Q\rho(0)v(t) + \Delta\rho(0)v(t) \\ &\quad + \sum_{k=t}^{T}\left[\Delta^2\rho(T+t-k-1) + Q\rho(T+t-k)\right]v(k) - Q\rho(T)v(t) - \Delta\rho(T)v(t) \\ &= v(t), \quad t \in \mathbf{Z}[1,T].\end{aligned} \tag{2.8}$$

From the definitions of u and G, it is easy to see that $u(0) = u(T)$ and $\Delta u(0) = \Delta u(T)$. This completes the proof of the lemma. □

From the expression of G, we see that $G(t,k) > 0$ and $G(t,k) = G(k,t)$ for all $t,k \in \mathbf{Z}[1,T]$. Define operators $K, L : E \to E$, respectively, by

$$(Ku)(t) = \sum_{k=1}^{T} G(t,k)u(k), \quad u \in E,\ t \in \mathbf{Z}[1,T],$$
$$(Lu)(t) = \big(Q - q(t)\big)u(t), \quad u \in E,\ t \in \mathbf{Z}[1,T]. \tag{2.9}$$

Obviously, $K(P) \subseteq P$ and $L(P) \subseteq P$. It is clear that K is strongly positive, that is, $K(u) \in \operatorname{int} P$ for $u \in P \setminus \{\theta\}$.

Lemma 2.2. *Assume that* (H) *holds. Then, $KL : E \to E$ is a linear completely continuous operator with $\|KL\| < 1$, and $(I - KL)^{-1}$, the inverse mapping of $I - KL$, exists and is bounded.*

Proof. It is obvious that $KL : E \to E$ is a linear completely continuous operator. Since $u_0(t) \equiv 1/Q$ is a solution of PBVP (2.2) with $v_0(t) \equiv 1$, we have

$$\sum_{k=1}^{T} G(t,k) = \frac{1}{Q}, \quad t \in \mathbf{Z}[1,T]. \tag{2.10}$$

Then by (H) and the fact that K is strongly positive, one has

$$|(KL)u(t)| = \sum_{k=1}^{T} G(t,k)\big(Q - q(k)\big)|u(k)| \le \|u\| \sum_{k=1}^{T} G(t,k)\big(Q - q(k)\big) = \|u\|\big(1 - \big(Kq\big)(t)\big) < \|u\|, \tag{2.11}$$

where $u \in E$, $t \in \mathbf{Z}[1,T]$. Hence $\|KL\| < 1$, and $(I - KL)^{-1}$, the inverse mapping of $I - KL$, exists and is bounded. The proof of Lemma 2.2 is completed. □

Let

$$S := (I - KL)^{-1}K = \big(I + KL + \cdots + (KL)^n + \cdots\big)K = K + (KL)K + \cdots + (KL)^n K + \cdots. \tag{2.12}$$

The complete continuity of K together with the continuity of $(I - KL)^{-1}$ implies that the operator $S : E \to E$ is completely continuous.

Lemma 2.3. *Assume that* (H) *holds. Then, for each $v \in E$, the following linear periodic boundary value problem*

$$\Delta^2 u(t-1) + q(t)u(t) = v(t), \quad t \in \mathbf{Z}[1,T],$$
$$u(0) = u(T), \qquad \Delta u(0) = \Delta u(T) \tag{2.13}$$

has a unique solution $\{u(t)\}_{t=0}^{T+1}$, *where* $u(t) = (Sv)(t)$, $t \in \mathbf{Z}[1,T]$, *and* $u(0) = u(T)$, $u(1) = u(T+1)$.

Proof. It is easy to see that PBVP (2.13) is equivalent to the operator equation $u = KLu + Kv$. Therefore, PBVP (2.13) has a unique solution $\{u(t)\}_{t=0}^{T+1}$, where $u(t) = (Sv)(t) = ((I - KL)^{-1}Kv)(t)$, $t \in \mathbf{Z}[1,T]$, and $u(0) = u(T)$, $u(T+1) = u(1)$. □

Lemma 2.4. *Assume that* (H) *holds. Then, for the operator S defined by* (2.12), *the spectral radius $r(S) > 0$ and there exists $\xi \in E$ with $\xi > 0$ on $\mathbf{Z}[1,T]$ such that $S\xi = r(S)\xi$ and $\sum_{t=1}^{T}\xi(t) = 1/r(S)$. Moreover, $\lambda_1 = 1/r(S)$ is the first positive eigenvalue of the linear PBVP corresponding to PBVP* (1.2) *and*

$$\sum_{t=1}^{T}(Su)(t)\xi(t) = \frac{1}{\lambda_1}\sum_{t=1}^{T}u(t)\xi(t), \quad \forall u \in E. \tag{2.14}$$

Proof. An obvious modification of the proof of [13, Lemma 2.3] yields this result. We omit the details here. □

Lemma 2.5. *Assume that* (H) *holds. Then,* $S(P) \subseteq P_1$, *where*

$$P_1 = \left\{u \in P : \sum_{t=1}^{T}u(t)\xi(t) \geq \delta\|u\|\right\}, \quad \delta = \frac{2\sin\varphi\sin(\varphi T/2)\min_{t\in\mathbf{Z}[1,T]}\xi(t)}{\lambda_1\left\|(I-KL)^{-1}\right\|}, \tag{2.15}$$

and λ_1, ξ are given in Lemma 2.4, φ is given in Lemma 2.1.

Proof. By (2.14), we have, for any $u \in P$,

$$\sum_{t=1}^{T}(Su)(t)\xi(t) = \frac{1}{\lambda_1}\sum_{t=1}^{T}u(t)\xi(t) \geq \frac{\min_{t\in\mathbf{Z}[1,T]}\xi(t)}{\lambda_1}\sum_{t=1}^{T}u(t). \tag{2.16}$$

On the other hand, one has

$$\begin{aligned}\|Su\| &= \left\|(I-KL)^{-1}Ku\right\| \leq \left\|(I-KL)^{-1}\right\|\|Ku\| = \left\|(I-KL)^{-1}\right\|\max_{t\in\mathbf{Z}[1,T]}\sum_{k=1}^{T}G(t,k)u(k) \\ &\leq \left\|(I-KL)^{-1}\right\|\max_{t,k\in\mathbf{Z}[1,T]}G(t,k)\sum_{k=1}^{T}u(k) = \frac{\left\|(I-KL)^{-1}\right\|}{2\sin\varphi\sin(\varphi T/2)}\sum_{k=1}^{T}u(k).\end{aligned} \tag{2.17}$$

Then,

$$\sum_{t=1}^{T}(Su)(t)\xi(t) \geq \frac{2\sin\varphi\sin(\varphi T/2)\min_{t\in\mathbf{Z}[1,T]}\xi(t)}{\lambda_1\left\|(I-KL)^{-1}\right\|}\|Su\| = \delta\|Su\|. \tag{2.18}$$

Hence, $S(P) \subseteq P_1$. The proof is complete. □

Define operators f, $A : E \to E$, respectively, by

$$(\mathrm{f}u)(t) = f(t, u(t)), \quad u \in E, \ t \in \mathbf{Z}[1, T], \qquad A = S\mathrm{f}. \tag{2.19}$$

It follows from the continuity of f together with the complete continuity of S that $A : E \to E$ is completely continuous.

Remark 2.6. By Lemma 2.3, it is easy to see that $u = \{u(t)\}_{t=1}^{T} \in E$ is a fixed point of the operator A if and only if $u = \{u(t)\}_{t=0}^{T+1}$ is a solution of PBVP (1.2), where $u(0) = u(T), \ u(1) = u(T+1)$.

The proofs of the main theorems of this paper are based on the topological degree and fixed point index theories. The following four well-known lemmas in [20–22] are needed in our argument.

Lemma 2.7. *Let Ω be a bounded open set in a real Banach space E with $\theta \in \Omega$, let and $A : \overline{\Omega} \to E$ be completely continuous. If there exists $x_0 \in E \setminus \{\theta\}$ such that $x - Ax \neq \mu x_0$ for all $x \in \partial\Omega$ and $\mu \geq 0$, then the topological degree* $\deg(I - A, \Omega, \theta) = 0$.

Lemma 2.8. *Let Ω be a bounded open set in a real Banach space E with $\theta \in \Omega$, and let $A : \overline{\Omega} \to E$ be completely continuous. If $Ax \neq \mu x$ for all $x \in \partial\Omega$ and $\mu \geq 1$, then the topological degree* $\deg(I - A, \Omega, \theta) = 1$.

Lemma 2.9. *Let E be a Banach space and $X \subset E$ a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If there exists $x_0 \in X \setminus \{\theta\}$ such that $x - Ax \neq \mu x_0$ for all $x \in X \cap \partial\Omega$ and $\mu \geq 0$, then the fixed point index* $i(A, X \cap \Omega, X) = 0$.

Lemma 2.10. *Let E be a Banach space and $X \subset E$ a cone in E. Assume that Ω is a bounded open subset of E with $\theta \in \Omega$. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If $Ax \neq \mu x$ for all $x \in X \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index* $i(A, X \cap \Omega, X) = 1$.

3. Existence of Nontrivial Periodic Solutions

Theorem 3.1. *Assume that* (H) *holds. If the following conditions are satisfied*

$$\limsup_{x \to \infty} \max_{t \in \mathbf{Z}[1,T]} \left| \frac{f(t,x)}{x} \right| < \lambda_1, \tag{3.1}$$

$$\liminf_{x \to 0^+} \min_{t \in \mathbf{Z}[1,T]} \frac{f(t,x)}{x} > \lambda_1, \tag{3.2}$$

$$\limsup_{x \to 0^-} \max_{t \in \mathbf{Z}[1,T]} \frac{f(t,x)}{x} < \lambda_1, \tag{3.3}$$

where λ_1 is the first positive eigenvalue of the linear operator S given in Lemma 2.4, then (1.1) *has at least one nontrivial periodic solution.*

Proof. In view of Remark 2.6, it suffices to prove that the operator A has at least fixed point in $E \setminus \{\theta\}$. It follows from (3.2) and (3.3) that there exist $r > 0$ and $\sigma \in (0,1)$ such that

$$\begin{aligned} f(t,x) &\geq \lambda_1(1+\sigma)x \geq \lambda_1(1-\sigma)x, \quad \forall x \in [0,r],\ t \in \mathbf{Z}[1,T], \\ f(t,x) &\geq \lambda_1(1-\sigma)x \geq \lambda_1(1+\sigma)x, \quad \forall x \in [-r,0],\ t \in \mathbf{Z}[1,T]. \end{aligned} \tag{3.4}$$

By the above two inequalities, we have

$$f(t,x) \geq \lambda_1(1+\sigma)x, \quad \forall |x| \leq r,\ t \in \mathbf{Z}[1,T], \tag{3.5}$$

$$f(t,x) \geq \lambda_1(1-\sigma)x, \quad \forall |x| \leq r,\ t \in \mathbf{Z}[1,T]. \tag{3.6}$$

We may suppose that A has no fixed point on ∂B_r. Otherwise, the proof is finished. Now we will prove

$$u \neq Au + \mu\xi, \quad \forall u \in \partial B_r,\ \mu \geq 0, \tag{3.7}$$

where ξ is given in Lemma 2.4. Suppose the contrary; then there exist $u_0 \in \partial B_r$ and $\mu_0 \geq 0$ such that $u_0 = Au_0 + \mu_0\xi$. Then $\mu_0 > 0$. Multiplying the equality $u_0 = Au_0 + \mu_0\xi$ by ξ on its both sides, summing from 1 to T, and using (2.14) and (3.5), it follows that

$$\begin{aligned} \sum_{t=1}^{T} u_0(t)\xi(t) &= \sum_{t=1}^{T}(Au_0)(t)\xi(t) + \mu_0\sum_{t=1}^{T}\xi^2(t) = \frac{1}{\lambda_1}\sum_{t=1}^{T} f(t,u_0(t))\xi(t) + \mu_0\sum_{t=1}^{T}\xi^2(t) \\ &\geq (1+\sigma)\sum_{t=1}^{T} u_0(t)\xi(t) + \mu_0\sum_{t=1}^{T}\xi^2(t). \end{aligned} \tag{3.8}$$

Similarly, by (3.6), we know also that

$$\sum_{t=1}^{T} u_0(t)\xi(t) \geq (1-\sigma)\sum_{t=1}^{T} u_0(t)\xi(t) + \mu_0\sum_{t=1}^{T}\xi^2(t). \tag{3.9}$$

If $\sum_{t=1}^{T} u_0(t)\xi(t) \geq 0$, then (3.8) implies that $\sum_{t=1}^{T}\xi^2(t) \leq 0$, which contradicts $\xi > 0$ on $\mathbf{Z}[1,T]$. If $\sum_{t=1}^{T} u_0(t)\xi(t) < 0$, then (3.9) also implies that $\sum_{t=1}^{T}\xi^2(t) < 0$, which is a contradiction. Thus, (3.7) holds. On the basis of Lemma 2.7, we have

$$\deg(I - A, B_r, \theta) = 0. \tag{3.10}$$

From (3.1) it follows that there exist $G > 0$ and $\varepsilon \in (0,1)$ such that $|f(t,x)| < \lambda_1(1-\varepsilon)|x|$ for $|x| > G,\ t \in \mathbf{Z}[1,T]$. Let $C = \sup_{t\in\mathbf{Z}[1,T],|x|\leq G} f(t,x)$. Obviously,

$$\left|f(t,x)\right| \leq \lambda_1(1-\varepsilon)|x| + C, \quad \forall x \in \mathbf{R},\ t \in \mathbf{Z}[1,T]. \tag{3.11}$$

Choose R such that $R > \max\{r, (a\varepsilon)^{-1}C\}$, where $a = \min_{t\in\mathbf{Z}[1,T]}\xi(t)$. We next show $Au \neq \mu u$, for all $u \in \partial B_R$, $\mu \geq 1$. In fact, if there exist $u_1 \in \partial B_R$ and $\mu_1 \geq 1$ such that $Au_1 = \mu_1 u_1$, then, by the definition of A and (3.11), we obtain

$$\begin{aligned}|u_1(t)| \leq |Au_1(t)| &\leq (I-KL)^{-1}\left(\sum_{k=1}^{T} G(t,k)\left|f(k,u_1(k))\right|\right)\\ &\leq \lambda_1(1-\varepsilon)(I-KL)^{-1}\left(\sum_{k=1}^{T} G(t,k)|u_1(k)|\right) + C(I-KL)^{-1}\sum_{k=1}^{T} G(t,k).\end{aligned} \tag{3.12}$$

Set $u_2(t) = |u_1(t)|$. Then $u_2 \in P \setminus \{\theta\}$, and, for any $t \in \mathbf{Z}[1,T]$, $\lambda_1(1-\varepsilon)(Su_2)(t) + C(Sv_0)(t) \geq u_2(t)$, where $v_0(t) \equiv 1$. Then, by (2.14), we have

$$\sum_{t=1}^{T}\left[(1-\varepsilon)u_2(t) + \frac{C}{\lambda_1}v_0(t)\right]\xi(t) = \lambda_1(1-\varepsilon)\sum_{t=1}^{T}(Su_2)(t)\xi(t) + C\sum_{t=1}^{T}(Sv_0)(t)\xi(t) \geq \sum_{t=1}^{T}u_2(t)\xi(t). \tag{3.13}$$

Using the above inequality and noticing that $\sum_{t=1}^{T}\xi(t) = \lambda_1$ (see Lemma 2.4), we have that $\varepsilon^{-1}C \geq \sum_{t=1}^{T}u_2(t)\xi(t) \geq a\sum_{t=1}^{T}u_2(t)$. This implies that $R = \|u_2\| \leq \sum_{t=1}^{T}u_2(t) \leq (a\varepsilon)^{-1}C$, which contradicts the choice of R. It follows from Lemma 2.8 that

$$\deg(I-A, B_R, \theta) = 1. \tag{3.14}$$

According to the additivity of Leray-Schauder degree, by (3.14) and (3.10), we get

$$\deg\left(I-A, B_R \setminus \overline{B_r}, \theta\right) = \deg(I-A, B_R, \theta) - \deg(I-A, B_r, \theta) = 1, \tag{3.15}$$

which implies that the nonlinear operator A has at least one fixed point in $B_R \setminus \overline{B_r}$. Thus, (1.1) has at least one nontrivial periodic solution. The proof is complete. □

Theorem 3.2. *Assume that* (H) *holds. If the following conditions are satisfied*

$$\limsup_{x\to 0}\max_{t\in\mathbf{Z}[1,T]}\left|\frac{f(t,x)}{x}\right| < \lambda_1, \tag{3.16}$$

$$\liminf_{x\to +\infty}\min_{t\in\mathbf{Z}[1,T]}\frac{f(t,x)}{x} > \lambda_1, \tag{3.17}$$

$$\limsup_{x\to -\infty}\max_{t\in\mathbf{Z}[1,T]}\frac{f(t,x)}{x} < \lambda_1, \tag{3.18}$$

where λ_1 is the first positive eigenvalue of the linear operator S given in Lemma 2.4, then (1.1) *has at least one nontrivial periodic solution.*

Proof. It suffices to prove that the operator A has at least fixed point in $E \setminus \{\theta\}$. From (3.16), we find that there exist $\varepsilon \in (0,1)$ and $r > 0$ such that

$$\left|f(t,x)\right| \le \lambda_1(1-\varepsilon)|x|, \quad \forall |x| \le r,\ t \in \mathbf{Z}[1,T], \tag{3.19}$$

Now we prove

$$Au \ne \mu u, \quad \forall u \in \partial B_r,\ \mu \ge 1. \tag{3.20}$$

If (3.20) does hold, there exist $\mu_0 \ge 1$ and $u_0 \in \partial B_r$ such that $Au_0 = \mu_0 u_0$. Then, by (3.19), we have

$$\begin{aligned} |u_0(t)| \le |Au_0(t)| &\le (I-KL)^{-1}\left(\sum_{k=1}^{T} G(t,k)\left|f(k,u_0(k))\right|\right) \\ &\le \lambda_1(1-\varepsilon)(I-KL)^{-1}\left(\sum_{k=1}^{T} G(t,k)|u_0(k)|\right), \quad t \in \mathbf{Z}[1,T]. \end{aligned} \tag{3.21}$$

Set $u_1(t) = |u_0(t)|$. Then $u_1 \in P \setminus \{\theta\}$ and $\lambda_1(1-\varepsilon)Su_1 \ge u_1$. Multiplying this inequality by ξ and summing from 1 to T, it follows from (2.14) that

$$(1-\varepsilon)\sum_{t=1}^{T} u_1(t)\xi(t) = \lambda_1(1-\varepsilon)\sum_{t=1}^{T}(Su_1)(t)\xi(t) \ge \sum_{t=1}^{T} u_1(t)\xi(t). \tag{3.22}$$

This together with $\sum_{t=1}^{T} u_1(t)\xi(t) > 0$ implies that $1-\varepsilon \ge 1$, which contradicts the choice of ε, and so (3.20) holds. It follows from Lemma 2.8 that

$$\deg(I-A, B_r, \theta) = 1. \tag{3.23}$$

By (3.17), (3.18), and the continuity of $f(t,x)$ with respect to x, we know that there exist $\sigma \in (0,1)$ and $C > 0$ such that

$$\begin{aligned} f(t,x) &\ge \lambda_1(1+\sigma)x - C, \quad \forall x \ge 0,\ t \in \mathbf{Z}[1,T], \\ f(t,x) &\ge \lambda_1(1-\sigma)x - C, \quad \forall x \le 0,\ t \in \mathbf{Z}[1,T]. \end{aligned} \tag{3.24}$$

Then,

$$\begin{aligned} f(t,x) &\ge \lambda_1(1+\sigma)x - C \ge \lambda_1(1-\sigma)x - C, \quad \forall x \ge 0,\ t \in \mathbf{Z}[1,T], \\ f(t,x) &\ge \lambda_1(1-\sigma)x - C \ge \lambda_1(1+\sigma)x - C, \quad \forall x \le 0,\ t \in \mathbf{Z}[1,T]. \end{aligned} \tag{3.25}$$

By the above two inequalities, we have

$$f(t,x) \geq \lambda_1(1+\sigma)x - C, \quad \forall x \in \mathbf{R},\ t \in \mathbf{Z}[1,T], \tag{3.26}$$

$$f(t,x) \geq \lambda_1(1-\sigma)x - C, \quad \forall x \in \mathbf{R},\ t \in \mathbf{Z}[1,T]. \tag{3.27}$$

Set

$$\Omega = \{u \in E : u = Au + \tau\xi \text{ for some } \tau \geq 0\}, \tag{3.28}$$

where ξ is given in Lemma 2.4. We claim that Ω is bounded in E. In fact, for any $u \in \Omega$, there exists $\tau \geq 0$ such that $u = Au + \tau\xi \geq Au$. Then, by (3.26), we have

$$u(t) \geq \lambda_1(1+\sigma)(Su)(t) - C(Sv_0)(t), \quad t \in \mathbf{Z}[1,T], \tag{3.29}$$

where $v_0(t) \equiv 1$. Multiplying the above inequality by $\xi(t)$ on both sides and summing from 1 to T, it follows from (2.14) that

$$\sum_{t=1}^{T} u(t)\xi(t) \geq \lambda_1(1+\sigma)\sum_{t=1}^{T}(Su)(t)\xi(t) - C\sum_{t=1}^{T}(Sv_0)(t)\xi(t) = (1+\sigma)\sum_{t=1}^{T}u(t)\xi(t) - \frac{C}{\lambda_1}\sum_{t=1}^{T}\xi(t). \tag{3.30}$$

Then, noticing that $\sum_{t=1}^{T}\xi(t) = \lambda_1$, we have

$$\sigma\sum_{t=1}^{T}u(t)\xi(t) \leq C. \tag{3.31}$$

On the other hand, bearing in mind that $\xi = \lambda_1 S\xi$, we obtain that, for $u \in \Omega$,

$$u - \lambda_1(1-\sigma)Su + CSv_0 = S\mathrm{f}u - \lambda_1(1-\sigma)Su + CSv_0 + \tau\xi = S[\mathrm{f}u - \lambda_1(1-\sigma)u + Cv_0 + \tau\lambda_1\xi]. \tag{3.32}$$

By (3.27), we obtain that $\mathrm{f}u - \lambda_1(1-\sigma)u + Cv_0 + \tau\lambda_1\xi \in P$. Lemma 2.5 yields that $u - \lambda_1(1-\sigma)Su + CSv_0 \in P_1$. Then by (2.14) and (3.31), we obtain that

$$\begin{aligned}\|u - \lambda_1(1-\sigma)Su + CSv_0\| &\leq \frac{1}{\delta}\sum_{t=1}^{T}[u(t) - \lambda_1(1-\sigma)(Su)(t) + C(Sv_0)(t)]\xi(t)\\ &= \frac{1}{\delta}\sum_{t=1}^{T}\left[u(t)\xi(t) - (1-\sigma)u(t)\xi(t) + \frac{C}{\lambda_1}\xi(t)\right] \leq \frac{2C}{\delta}.\end{aligned} \tag{3.33}$$

This gives

$$\|u - \lambda_1(1-\sigma)Su\| \le \frac{2C}{\delta} + C\|Sv_0\|, \quad \forall u \in \Omega. \tag{3.34}$$

Hence, $(I-\lambda_1(1-\sigma)S)(\Omega) \subset B_{R_1}$, where $R_1 = 2C/\delta + C\|Sv_0\| > 0$. It follows from $\lambda_1(1-\sigma)r(S) < 1$ that $I - \lambda_1(1-\sigma)S$ has a linear bounded inverse $(I - \lambda_1(1-\sigma)S)^{-1}$. Therefore, there exists $R_2 > 0$ such that

$$\Omega \subset (I - \lambda_1(1-\sigma)S)^{-1}(B_{R_1}) \subset B_{R_2}. \tag{3.35}$$

Then, we can conclude that Ω is bounded in E, proving our claim. Thus, there exists $R > \max\{r, R_2\}$ such that

$$u \ne Au + \tau\xi, \quad \forall u \in \partial B_R, \ \tau \ge 0. \tag{3.36}$$

This and Lemma 2.7 give $\deg(I - A, B_R, \theta) = 0$. Taking (3.23) into account, we have $\deg(I - A, B_R \setminus \overline{B_r}, \theta) = -1$. Then, A has at least one fixed point in $B_R \setminus \overline{B_r}$, which means that (1.1) has at least one nontrivial periodic solution. The proof is completed. □

4. Existence of Positive Periodic Solutions

Theorem 4.1. *Assume that* (H) *holds. If the following conditions are satisfied*

$$xf(t,x) \ge 0, \quad \forall x \in \mathbf{R}, \ t \in \mathbf{Z}[1,T], \tag{4.1}$$

$$\limsup_{x\to\infty} \max_{t\in\mathbf{Z}[1,T]} \frac{f(t,x)}{x} < \lambda_1, \tag{4.2}$$

$$\liminf_{x\to 0} \min_{t\in\mathbf{Z}[1,T]} \frac{f(t,x)}{x} > \lambda_1, \tag{4.3}$$

where λ_1 is the first positive eigenvalue of the linear operator S given in Lemma 2.4, then (1.1) *has at least one positive periodic solution and one negative periodic solution.*

Proof. From (4.1), we know that $A(P) \subset P$. Similar to the proof of Theorem 3.1, it follows from (4.1)–(4.3) and Lemmas 2.9 and 2.10 that there exist $0 < r < R$ such that

$$i(A, B_r \cap P, P) = 0, \qquad i(A, B_R \cap P, P) = 1. \tag{4.4}$$

Hence, by the additivity of the fixed point index, we have

$$i\left(A, \left(B_R \setminus \overline{B_r}\right) \cap P, P\right) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = 1. \tag{4.5}$$

Then, the nonlinear operator A has at least one fixed point on $(B_R \setminus \overline{B_r}) \cap P$. So (1.1) has at least one positive periodic solution.

Put $f_1(t,x) = -f(t,-x)$, for all $(t,x) \in \mathbf{Z}[1,T] \times \mathbf{R}$. Define operators $\mathrm{f}_1, A_1 : E \to E$, respectively, by

$$\begin{gathered}(\mathrm{f}_1 u)(t) = f_1(t, u(t)), \quad u \in E,\ t \in \mathbf{Z}[1,T], \\ A_1 = S\mathrm{f}_1.\end{gathered} \tag{4.6}$$

Obviously, $A_1(P) \subset P$. Following almost the same procedure as above, from $\limsup_{x\to\infty} \max_{t\in\mathbf{Z}[1,T]}(f_1(t,x)/x) < \lambda_1$ and $\liminf_{x\to 0}\min_{t\in\mathbf{Z}[1,T]}(f_1(t,x)/x) > \lambda_1$, we know also that the nonlinear operator A_1 has at least one fixed point $\zeta \in P \setminus \{\theta\}$. Then $A_1\zeta = \zeta$. This means that $A(-\zeta) = S(\mathrm{f}(-\zeta)) = S(-\mathrm{f}_1(\zeta)) = -A_1(\zeta) = -\zeta$. Hence, (1.1) has at least one negative periodic solution $-\zeta$, and the conclusion is achieved. □

Theorem 4.2. *Assume that* (H) *and* (4.1) *hold. If the following conditions are satisfied*

$$\begin{gathered}\limsup_{x\to 0} \max_{t\in\mathbf{Z}[1,T]} \frac{f(t,x)}{x} < \lambda_1, \\ \liminf_{x\to \infty} \min_{t\in\mathbf{Z}[1,T]} \frac{f(t,x)}{x} > \lambda_1,\end{gathered} \tag{4.7}$$

where λ_1 is the first positive eigenvalue of the linear operator S given in Lemma 2.4, then (1.1) *has at least one positive periodic solution and one negative periodic solution.*

The proof is similar to that of Theorem 4.1 and so we omit it here.

5. Examples

Example 5.1. Let $f(t,x) = \sqrt{|x|}$. It is easy to see that $\limsup_{x\to\infty}\max_{t\in\mathbf{Z}[1,T]}|f(t,x)/x| = 0 < \lambda_1$, $\liminf_{x\to 0^+}\min_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = +\infty > \lambda_1$, and $\limsup_{x\to 0^-}\max_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = -\infty < \lambda_1$. Then, it follows from Theorem 3.1 that (1.1) has at least one nontrivial periodic solution.

Example 5.2. Let $f(t,x) = 2x^4 + x^3$. It is not difficult to see that $\limsup_{x\to 0}\max_{t\in\mathbf{Z}[1,T]} |f(t,x)/x| = 0 < \lambda_1$, $\liminf_{x\to +\infty}\min_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = +\infty > \lambda_1$, and $\limsup_{x\to -\infty} \max_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = -\infty < \lambda_1$. Then, it follows from Theorem 3.2 that (1.1) has at least one nontrivial periodic solution.

Example 5.3. Let $f(t,x) = 3x^5 e^{x^6}$. Obviously, $xf(t,x) \geq 0$ for all $x \in \mathbf{R}$ and $t \in \mathbf{Z}[1,T]$. Moreover, $\limsup_{x\to 0}\max_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = 0 < \lambda_1$ and $\liminf_{x\to\infty}\min_{t\in\mathbf{Z}[1,T]}(f(t,x)/x) = +\infty > \lambda_1$. Then it follows from Theorem 4.2 that (1.1) has at least one positive periodic solution and one negative periodic solution.

Remark 5.4. It is easy to see that the existence of nontrivial periodic solutions in Examples 5.1–5.3 could not be obtained by any theorems in [1–16, 19].

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References

[1] F. M. Atici and G. S. Guseinov, "Positive periodic solutions for nonlinear difference equations with periodic coefficients," *Journal of Mathematical Analysis and Applications*, vol. 232, no. 1, pp. 166–182, 1999.

[2] F. M. Atici and A. Cabada, "Existence and uniqueness results for discrete second-order periodic boundary value problems," *Computers & Mathematics with Applications*, vol. 45, no. 6-9, pp. 1417–1427, 2003.

[3] J. Yu, Z. Guo, and X. Zou, "Periodic solutions of second order self-adjoint difference equations," *Journal of the London Mathematical Society. Second Series*, vol. 71, no. 1, pp. 146–160, 2005.

[4] H. Bin, J. Yu, and Z. Guo, "Nontrivial periodic solutions for asymptotically linear resonant difference problem," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 477–488, 2006.

[5] J. Yu and B. Zheng, "Multiplicity of periodic solutions for second-order discrete Hamiltonian systems with a small forcing term," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 9, pp. 3016–3029, 2008.

[6] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society. Second Series*, vol. 68, no. 2, pp. 419–430, 2003.

[7] J. Yu, X. Deng, and Z. Guo, "Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1140–1151, 2006.

[8] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China. Series A. Mathematics*, vol. 46, no. 4, pp. 506–515, 2003.

[9] Z. Zhou, J. Yu, and Y. Chen, "Periodic solutions of a $2n$th-order nonlinear difference equation," *Science China. Mathematics*, vol. 53, no. 1, pp. 41–50, 2010.

[10] Y. Xue and C. Tang, "Multiple periodic solutions for superquadratic second-order discrete Hamiltonian systems," *Applied Mathematics and Computation*, vol. 196, no. 2, pp. 494–500, 2008.

[11] Y. Xue and C. Tang, "Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 7, pp. 2072–2080, 2007.

[12] T. He and W. Chen, "Periodic solutions of second order discrete convex systems involving the p-Laplacian," *Applied Mathematics and Computation*, vol. 206, no. 1, pp. 124–132, 2008.

[13] T. He and Y. Xu, "Positive solutions for nonlinear discrete second-order boundary value problems with parameter dependence," *Journal of Mathematical Analysis and Applications*, vol. 379, no. 2, pp. 627–636, 2011.

[14] F. Lian and Y. Xu, "Multiple solutions for boundary value problems of a discrete generalized Emden-Fowler equation," *Applied Mathematics Letters*, vol. 23, no. 1, pp. 8–12, 2010.

[15] R. Ma and H. Ma, "Positive solutions for nonlinear discrete periodic boundary value problems," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 136–141, 2010.

[16] X. He and X. Wu, "Existence and multiplicity of solutions for nonlinear second order difference boundary value problems," *Computers & Mathematics with Applications*, vol. 57, no. 1, pp. 1–8, 2009.

[17] F. Li and Z. Liang, "Existence of positive periodic solutions to nonlinear second order differential equations," *Applied Mathematics Letters*, vol. 18, no. 11, pp. 1256–1264, 2005.

[18] Z. Yang, "Existence of nontrivial solutions for a nonlinear Sturm-Liouville problem with integral boundary conditions," *Nonlinear Analysis*, vol. 68, no. 1, pp. 216–225, 2008.

[19] A. Cabada and V. Otero-Espinar, "Optimal existence results for nth order periodic boundary value difference equations," *Journal of Mathematical Analysis and Applications*, vol. 247, no. 1, pp. 67–86, 2000.

[20] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[21] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, Boston, Mass, USA, 1988.

[22] D. J. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology Press, Ji'nan, China, 2nd edition, 2001.