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Bessel potential space on the Laguerre hypergroup

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Abstract

In this article, we define the fractional differentiation D_δ of order δ, $\delta > 0$, induced by the Laguerre operator L and associated with respect to the Haar measure dm_α. We obtain a characterization of the Bessel potential space $L^p_\delta(\mathbb{K})$ using D_δ and different equivalent norms.

Keywords: Heat-diffusion Poisson semigroups, Fractional power, Riesz potential, Fractional differentiation

1 Introduction

During the second half of the twentieth century (until the 1990s), the Continuous Time Random Walk (CTRW) method was practically the only tool available to describe subdiffusive and/or superdiffusive phenomena associated with complex systems for many groups of research. The main reason behind the usefulness of fractional derivatives have been until this moment the close link that exists between fractional models and the so called Jump stochastic models, such as the CTRW or those of the multiple trapping type.

Note that fractional operators also provide a method for reflecting the memory properties and non-locality of many anomalous processes. In any case, at the moment it is not clear what is the best fractional time derivative or the spatial fractional derivative to be used in the different models.

Fractional calculus deals with the study of so-called fractional order integral and derivative operators over real or complex domains and their applications.

Since 1990, there has been a spectacular increase in the use of fractional models to simulate the dynamics of many different anomalous processes, especially those involving ultraslow diffusion. We hereby propose a few examples of fields where the fractional models have been used: materials theory, transport theory, fluid of contaminant flow phenomena through heterogeneous porous media, physics theory, electromagnetic theory, thermodynamics or mechanics, signal theory, chaos theory and/or fractals, geology and astrophysics, biology and other life sciences, economics or chemistry, etc.

As one would expect, since a fractional derivative is a generalization of an ordinary derivative, it is going to lose many of its basic properties. For example, it loses its geometric or physical interpretation but the index law is only valid when working on very specific function spaces and the derivative of the product of two functions is difficult to obtain and the chain rule is not straightforward to apply.

It is natural to ask then, what properties fractional derivatives have that make them so suitable for modeling certain complex systems. The answer lies in the property exhibited by many of the aforementioned systems of non-local dynamics, that is, the processes dynamics have a certain degree of memory. While fractional operators naturally incorporate the interesting property of no locality. They do lose some of the typical, basic properties of ordinary differential operators. The ordinary derivative is clearly, by definition, local [1].

According to the ideas presented by Stein [2], the fundamental operators of the harmonic analysis (fractional integrals, Riesz transformation, g-functions, ...) can be considered in the context of the Laguerre operator L.

It is important to mention that this way of describing harmonic operators in the Laguerre context was initiated by Muckenhoupt [3].

The organization of the article is as follows. Section 2 contains some basic facts needed in the sequel about the Laguerre hypergroup. Section 3 is devoted to some generation and representation for the semigroups also we define the fractional power, the heat-diffusion and the Poisson-Laguerre semigroups based on a Laguerre operator. Finally, Sect. 4 is devoted to proving the main result of this article (Theorem 1) where we establish that $||D_\delta f||_p$ and $||f||_{\delta,p}$ are equivalent when the fractional differentiation D_δ is defined for $\delta > 0$.

2 Preliminary

In this section we set some notations and we recall some basic results in harmonic analysis related to Laguerre hypergroups (see [4-6]).

First we begin with some notation.

• We denote by $\mathbb{K} = [0, \infty] \times \mathbb{R}$ equipped with the weighted Lebesgue measure m_α on $\mathbb{K}$ given by

$$\mathrm{d}m_\alpha(x,t) = \frac{x^{2\alpha+1}\mathrm{d}x\mathrm{d}t}{\pi\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

For every $1 \leq p \leq \infty$,we denote by $L^p(\mathbb{K}) = L^p(\mathbb{K}, \mathrm{d}m_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that:

$$||f||_{L^p(\mathbb{K})} = \left(\int_{\mathbb{K}} |f(x,t)|^p \mathrm{d}m_\alpha(x,t)\right)^{\frac{1}{p}} < \infty, \text{ if } p \in [1, \infty[.$$

and

$$||f||_{L^\infty(\mathbb{K})} = ess \sup_{(x,t)\in\mathbb{K}} |f(x,t)|.$$

• $\mathcal{D}(\mathbb{K})$ the subspace of $S(\mathbb{K})$ of functions ψ satisfying the following:

(i) There exists $m_0 \in \mathbb{N}$ satisfying $\psi(\lambda, m) = 0$, for all $(\lambda, m) \in \mathbb{K}$ such that $m > m_0$.

(ii) for all $m \leq m_0$, the function $\lambda \mapsto \psi(\lambda,m)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$ with compact support and vanishes in a neighborhood of zero.

• $\mathcal{D}'(\mathbb{K})$ the topological dual space of $\mathcal{D}(\mathbb{K})$.

• $\hat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$ the dual space of $\mathbb{K}$.

• $L^p(\hat{\mathbb{K}}) = L^p(\hat{\mathbb{K}}), \mathrm{d}\gamma_\alpha)$ the spaces of complex-valued functions *f*, measurable on $\hat{\mathbb{K}}$ such that:

$$||f||_{L^p(\hat{\mathbb{K}})} = (\int_{\hat{\mathbb{K}}} |f(\lambda, m)|^p \mathrm{d}\gamma_\alpha(\lambda, m))^{\frac{1}{p}} < \infty, \quad \text{if } 1 \le p < \infty$$

and

$$||f||_{L^\infty(\hat{\mathbb{K}})} = ess \sup_{(\lambda, m) \in \hat{\mathbb{K}}} |f(\lambda, m)|$$

where $\mathrm{d}\gamma_\alpha(\lambda, m)$ being the positive measure defined on $\hat{\mathbb{K}}$ by:

$$\int_{\hat{\mathbb{K}}} f(\lambda, m)\mathrm{d}\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} f(\lambda, m)|\lambda|^{\alpha+1}\mathrm{d}\lambda.$$

For (x, t) $]0, \infty[\times\mathbb{R}$ and $\alpha \in [0, \infty[$, we consider the following partial differential operator, named the Laguerre operator:

$$L = \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2} \tag{1}$$

Remark 1. *For $\alpha = n - 1$, $n \in \mathbb{N}^*$, the operator L is the radial part of the sublaplacian on the Heisenberg group $\mathbb{H}^n$.*

For $(\lambda, m) \in \hat{\mathbb{K}}$ and $(x, t) \in \mathbb{K}$, we put $\varphi_{\lambda,m}(x, t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}(|\lambda|x^2)$, where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre function defined on $[0, \infty]$ by $\mathcal{L}_m^{(\alpha)}(x) = e^{\frac{-x}{2}} L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$ and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree *m* and order α.

Proposition 1. *For $(\lambda, m) \in \hat{\mathbb{K}}$, the function $\phi_{\lambda,m}$, is the unique solution of the following problem:*

$$\begin{cases} Lu = -4|\lambda|(m + \frac{\alpha+1}{2})u, \\ u(0,0) = 1, \\ \partial_x u(0,t) = 0, \ \forall t \in \mathbb{R}, \end{cases}$$

We denote by: $c_{\lambda,m} = 4|\lambda|(m + \frac{\alpha+1}{2}) = |(\lambda, m)|_{\hat{\mathbb{K}}}$.

Definition 1. (i) *The generalized Fourier transform F is defined on $L^1(\mathbb{K})$by:*

$$F(f)(\lambda, m) = \int_{\mathbb{K}} f(x,t)\varphi_{-\lambda,m}(x,t)\mathrm{d}m_\alpha(x,t), (\lambda, m) \in \hat{\mathbb{K}}$$

(ii) *We have also the inverse formula of the generalized Fourier transform F^{-1} on by:*

$$F^{-1}(f)(x,t) = \int_{\hat{\mathbb{K}}} f(\lambda, m)\varphi_{\lambda,m}(x,t)\mathrm{d}\gamma_\alpha(\lambda, m), (x,t) \in \mathbb{K}.$$

For $(\lambda, m) \in \hat{\mathbb{K}}$, we denote by: $P_{(\lambda,m)}f = F(f)(\lambda, m)\phi_{\lambda,m}$.

3 The heat-diffusion and the Poisson-Laguerre semigroups

3.1 The heat-diffusion semigroup

The heat-diffusion semigroup $\{T_t\}_{t\geq 0}$, associated to (-L), is then defined by

$$\begin{aligned} T_t f(\gamma,s) &:= e^{-tL} f(\gamma,s) \\ &= \int_{\hat{\mathbb{K}}} e^{-tc_{\lambda,m}} P_{(\lambda,m)} f(\gamma,s) \mathrm{d}\gamma_\alpha(\lambda,m) \\ &= \int_{\hat{\mathbb{K}}} \int_{\mathbb{K}} e^{-tc_{\lambda,m}} f(u,v) \varphi_{-\lambda,m}(u,v) \varphi_{\lambda,m}(\gamma,s) \mathrm{d}m_\alpha(u,v) \mathrm{d}\gamma_\alpha(\lambda,m) \\ &= \int_{\mathbb{K}} f(u,v) [\int_{\hat{\mathbb{K}}} e^{-tc_{\lambda,m}} \varphi_{-\lambda,m}(u,v) \varphi_{\lambda,m}(\gamma,s) \mathrm{d}\gamma_\alpha(\lambda,m)] \mathrm{d}m_\alpha(u,v) \\ &= \int_{\mathbb{K}} f(u,v) T_t((u,v),(\gamma,s)) \mathrm{d}m_\alpha(u,v). \end{aligned}$$

where

$$T_t((u,v),(\gamma,s)) = \int_{\hat{\mathbb{K}}} e^{-tc_{\lambda,m}} \varphi_{-\lambda,m}(u,v) \varphi_{\lambda,m}(\gamma,s) \mathrm{d}\gamma_\alpha(\lambda,m)$$

is the heat kernel of the integral representation $T_t f$.

Proposition 2. *This semigroup* $\{T_t\}_{t\geq 0}$ *is a strongly continuous semigroup on* $L^p(\mathbb{K})$ *with infinitesimal generator L(see* [7]).

Proof. Let $f \in L^p(\mathbb{K})$ then

$$\lim_{s\to t} \|T(s)f - T(t)f\|_{L^p(\mathbb{K})} = \lim_{s\to t} \|(T(s)-T(t))f\|_{L^p(\mathbb{K})} \leq \lim_{s\to t} \|T(s)-T(t)\|_{L^p(\mathbb{K})} \|f\|_{L^p(\mathbb{K})} = 0.$$

By the definition of the heat-diffusion semigroup $\{T_t\}_{t\geq 0}$, we establish the following result.

Corollary 1. *For* $(\mu,\eta) \in \hat{\mathbb{K}}$, *we have*

$$T_t \varphi_{\mu,\eta}(\gamma,s) = e^{-tc_{\mu,\eta}} \varphi_{\mu,\eta}(\gamma,s),$$

Proof. we have

$$\begin{aligned} T_t \varphi_{\mu,\eta}(\gamma,s) &= \int_{\hat{\mathbb{K}}} \int_{\mathbb{K}} e^{-tc_{\lambda,m}} \varphi_{\mu,\eta}(u,v) \varphi_{-\lambda,m}(u,v) \varphi_{\lambda,m}(\gamma,s) \mathrm{d}m_\alpha(u,v) \mathrm{d}\gamma_\alpha(\lambda,m) \\ &= \int_{\mathbb{K}} \varphi_{\mu,\eta}(u,v) (\int_{\hat{\mathbb{K}}} e^{-tc_{\lambda,m}} \varphi_{-\lambda,m}(u,v) \varphi_{\lambda,m}(\gamma,s) \mathrm{d}\gamma_\alpha(\lambda,m)) \mathrm{d}m_\alpha(u,v) \\ &= \int_{\mathbb{K}} \varphi_{\mu,\eta}(u,v) F^{-1}(e^{-tc_{.,.}} \varphi_{-.,.}(u,v))(\gamma,s) \mathrm{d}m_\alpha(u,v) \\ &= F(F^{-1}(e^{-tc_{.,.}} \varphi_{.,.}(\gamma,s)))(-\mu,\eta) \\ &= e^{-tc_{\mu,\eta}} \varphi_{\mu,\eta}(\gamma,s). \end{aligned}$$

3.2 The fractional power

For $\delta > 0$, the negative power $L^{-\delta}$ of L with respect to the measure $\mathrm{d}m_\alpha$ is defined, as in [8], by

$$L^{-\delta} f(\gamma,s) := \int_{\hat{\mathbb{K}}} \frac{P_{(\lambda,m)} f(\gamma,s)}{c_{\lambda,m}^{\delta}} \varphi_{\lambda,m}(\gamma,s) \mathrm{d}\gamma_\alpha(\lambda,m), \quad f \in L^2(\mathbb{K}, \mathrm{d}m_\alpha).$$

It is not hard to prove that $L^{-\delta}$ can be expressed, for $f \in L^2(\mathbb{K}, \mathrm{d}m_\alpha)$, by means of the following integral

$$L^{-\delta}f(\gamma, s) = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} T_t f(\gamma, s)\mathrm{d}t.$$

$L^{-\delta}$ is also called δth fractional integral associated with L. This kind of fractional integrals has been investigated by several authors ([9-12]).

Corollary 2. *If* $f(\gamma, s) = \phi_{\lambda,m}(\gamma, s)$, *we have:*

$$L^{-\delta}\varphi_{\lambda,m}(\gamma, s) = \frac{1}{c_{\lambda,m}^{\delta}}\varphi_{\lambda,m}(\gamma, s).$$

Proof. The proof is trivial by using $\Gamma(\delta) = \int_0^\infty t^{\delta-1}e^{-t}\mathrm{d}t$ and the change of variable $u = t\sqrt{c_{\lambda,m}}$.

3.3 The Poisson-Laguerre semigroup

The Poisson-Laguerre semigroup $\{P_t\}_{t\geq 0}$, associated to $(-L)$, is given by

$$\begin{aligned} P_t f(\gamma, s) &:= e^{-tL^{1/2}} f(\gamma, s) \\ &= \int_{\hat{\mathbb{K}}} e^{-tc_{\lambda,m}^{1/2}} [P_{(\lambda,m)}f](\gamma, s)\mathrm{d}\gamma_\alpha(\lambda, m). \end{aligned}$$

where $L^{1/2}$ is defined by using the spectral theorem.

Now, by using the Bochner subordination formula

$$e^{-\beta} = \frac{\beta}{\sqrt{4\pi}} \int_0^\infty s^{-3/2}e^{-s}e^{-\beta^2/4s}\mathrm{d}s.$$

After the change of variable $w = \frac{t^2 c_{\lambda,m}}{4s}$, we obtain:

$$P_t f(\gamma, s') = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-w}}{\sqrt{w}} T_{\frac{t^2}{4w}} f(\gamma, s')\mathrm{d}w.$$

Proposition 3. *This semigroup* $\{P_t\}_{t\geq 0}$ *is also a strongly continuous semigroup on* $L^p(\mathbb{K})$,*with infinitesimal generator* $L^{1/2}$.

Proof. We use the fact that $T_{\frac{t^2}{4w}}$ is strongly continuous.

By the definition of the Poisson-Laguerre semigroup $\{P_t\}_{t\geq 0}$,we establish also the following result

Corollary 3. *For* $(\mu, \eta) \in \hat{\mathbb{K}}$, *we have*

$$P_t\varphi_{\mu,\eta}(\gamma, s) = e^{-t\sqrt{c_{\mu,\eta}}}\varphi_{\mu,\eta}(\gamma, s).$$

Proof. We replace $c_{\mu,\eta}$ by $\sqrt{c_{\mu,\eta}}$ in the proof of Corollary 1, then the result is immediate.

3.4 The Riesz potential

For $\delta > 0$, the Riesz potential of order δ, I_δ, with respect to the measure $\mathrm{d}m_\alpha$ is defined, as in the classical case [13], by

$$I_\delta := (-L)^{-\delta/2}.$$

Proposition 4. *The Riesz potential can be also writed as*

$$I_\delta f(y,s) = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} P_t f(y,s)dt.$$

Proof. By using $(-L)^{-\delta}$, we have

$$I_{\delta/2} f(y,s) = \frac{1}{\Gamma(\delta/2)} \int_0^\infty t^{\delta/2-1} T_t f(y,s)\mathrm{d}t.$$

After to replace $P_t f(y, s)$ with his expression, the change of variable $t' = \frac{t^2}{4u}$ and the property of the function Gamma, we obtain:

$$\frac{1}{\Gamma(\delta/2)} \int_0^\infty t^{\delta/2-1} T_t f(y,s)\mathrm{d}t - \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} P_t f(y,s)\mathrm{d}t = 0.$$

Corollary 4. *If* $f(y, s) = \phi_{\lambda,m}(y, s)$, *we have*

$$I_\delta \varphi_{\lambda,m}(y,s) = \frac{1}{c_{\lambda,m}^{\delta/2}} \varphi_{\lambda,m}(y,s).$$

Proof. The proof is trivial by using $\Gamma(\delta) = \int_0^\infty t^{\delta-1} e^{-t}\mathrm{d}t$ and the change of variable $u = t\sqrt{c_{\lambda,m}}$.

4 Characterization of the potential spaces $L_\delta^p(\mathbb{K})$

4.1 The fractional differentiation

Following the classical case, the fractional differentiation D_δ of order $\delta > 0$ on the Laguerrre hypergroup is defined formally by

$$D_\delta := (-L)^{\frac{\delta}{2}}.$$

Corollary 5. *In the case of* $0 < \delta < 1$, *we have*

$$D_\delta \varphi_{\lambda,m}(y,s) = c_{\lambda,m}^{\delta/2} \varphi_{\lambda,m}(y,s).$$

Proof. In the case of $0 < \delta < 1$, we can write using [13] that

$$D_\delta f(y,s) = \frac{1}{c_\delta} \int_0^\infty t^{-\delta-1} (P_t f - f)(y,s)\mathrm{d}t. \tag{2}$$

where

$$c_\delta = \int_0^\infty u^{-\delta-1}(e^{-u} - 1)\mathrm{d}u.$$

By a change of variable $u = t\sqrt{c_{\lambda,m}}$ and the definition of c_δ, we have again:

$$D_\delta \varphi_{\lambda,m}(y,s) = c_{\lambda,m}^{\delta/2} \varphi_{\lambda,m}(y,s).$$

Remark 2. *Observe that:*

$$I_\delta(D_\delta f) = D_\delta(I_\delta f) = f.$$

As an application of the operator fractional derivative D_δ, we will give a characterization of the potential spaces $L^p_\delta(\mathbb{K})$, which is simpler and more powerful, valid for any $1 < p < \infty$ and $\delta \geq 0$.

4.2 Bessel potential space on $\mathbb{K}$

We mention that the Laguerre potential spaces is defined as

$$L^p_\delta(\mathbb{K}) := \{f : (I-L)^{\delta/2} f \in L^p(\mathbb{K}), \quad 1 < p < \infty, \quad \delta \geq 0\}$$

equipped with the norm

$$||f||_{p,\delta} = ||(I-L)^{\delta/2} f||_{L^p(\mathbb{K})}.$$

Let us define the Laguerre Bessel operator as

$$(I-L)^{-\delta/2} f(\gamma, s) := \int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{-\delta/2} P_{(\lambda,m)} f(\gamma, s) \mathrm{d}\gamma_\alpha(\lambda, m)$$

where $c_{\lambda,m}$ is the homogenous norm of $(\lambda, m) \in \hat{\mathbb{K}}$

Proposition 5. *If* $0 \leq \delta_1 < \delta_2$ *then* $L^p_{\delta_2}(\mathbb{K}) \subset L^p_{\delta_1}(\mathbb{K})$ *for each* $1 < p < \infty$

Proof. We have

$$\begin{aligned} ||f||_{p,\delta_1} &= \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{\delta_1/2} P_{(\lambda,m)} f(\gamma, s) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(\gamma, s) \\ &\leq \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{\delta_2/2} |P_{(\lambda,m)} f(\gamma, s)| \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(\gamma, s) \\ &= ||f||p, \delta_2. \end{aligned}$$

Now, let us establish a relation among different norms of potential spaces.

Proposition 6. *Given* $1 < p < \infty$ *and* $\delta \geq 1$, *if* $f \in L^p_\delta(\mathbb{K})$ *then*

(i) $f \in L^p_{\delta-1}(\mathbb{K})$.

(ii) $Lf \in L^p_{\delta-1}(\mathbb{K})$.

Moreover,

$$||f||_{p,\delta-1} + ||Lf||_{p,\delta-1} \leq C_p ||f||_{p,\delta}.$$

Proof. (i) is immediate, since $L^p_{\delta_2} \subset L^p_{\delta_1}$ such that $\delta_1 < \delta_2$.

(ii) We use the fact that L is symmetric, $F(Lf) = -c_{\lambda,m} F(f) = -c_{\lambda,m} F(f)$ and $L^p_{\delta+1}(\mathbb{K}) \subset L^p_\delta(\mathbb{K})$, then:

$$\begin{aligned} ||Lf||^p_{p,\delta-1} &= ||(I-L)^{(\delta-1)/2} Lf||^p_{p,\delta-1} \\ &= \int_{\mathbb{K}} |(I-L)^{(\delta-1)/2} Lf|^p \mathrm{d}m_\alpha \\ &= \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{(\delta-1)/2} P_{(\lambda,m)}(Lf) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(x, t) \\ &= \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{(\delta-1)/2} F(Lf)(\lambda, m) \varphi_{\lambda,m}(x, t) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(x, t) \\ &= \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{(\delta-1)/2} c_{\lambda,m} F(f)(\lambda, m) \varphi_{\lambda,m}(x, t) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(x, t) \\ &\leq \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{(\delta-1)/2} F(f)(\lambda, m) \varphi_{\lambda,m}(x, t) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(x, t) \\ &= \int_{\mathbb{K}} |\int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{(\delta-1)/2} P_{(\lambda,m)}(f)(x, t) \mathrm{d}\gamma_\alpha(\lambda, m)|^p \mathrm{d}m_\alpha(x, t) \\ &= ||Lf||^p_{p,\delta+1} \\ &\leq ||f||^p_{p,\delta}. \end{aligned}$$

Then, we get

$$||f||_{p,\delta-1} + ||Lf||_{p,\delta-1} \leq C_p||f||_{p,\delta}.$$

Next we show that if $f \in L^p_\delta(\mathbb{K})$ is equivalent to $D_\delta f \in L^p(\mathbb{K})$. The main tool is Meyer's multiplier theorem and let us underline that the definition of D_δ on all the spaces $L^p_\delta(\mathbb{K})$, $1 < p < \infty$, is also based on an application of Meyer's theorem [13].

Theorem 1. *Let* $\delta \geq 0$ *and* $1 < p < \infty$, *we have:*

$f \in L^p_\delta(\mathbb{K})$*if and only if* $D_\delta f \in L^p(\mathbb{K})$*Moreover, there exist a constant* $B_{p,\delta}$ *and* $A_{p,\delta}$ *such that:*

$$B_{p,\delta}||f||_{p,\delta} \leq ||D_\delta f||_p(\mathbb{K}) \leq A_{p,\delta}||f||_{p,\delta}.$$

To prove this result we need the following lemma.

Lemma 1. *Let* $f \in L^p_\delta(\mathbb{K})$*and* $\psi = (I - L)^{\delta/2}f$, *for* $\delta \geq 0$ *and* $1 < p < \infty$, *then:*

(i) $P_{\lambda,m}D_\delta f = c^{\delta/2}_{\lambda,m}P_{\lambda,m}f$.

(ii) $P_{\lambda,m}\psi = (1 + c_{\lambda,m})^{-\delta/2} P_{\lambda,m}f$.

Proof.

(i) We have

$$\begin{aligned} F(D_\delta f) &= F((-L)^{\delta/2}f) \\ &= < (-L)^{\delta/2}f, \varphi_{-\lambda,m} > \mathrm{d}m_\alpha \\ &= < f, (-L)^{\delta/2}\varphi_{-\lambda,m} > \mathrm{d}m_\alpha \\ &= c^{\delta/2}_{\lambda,m}F(f). \end{aligned}$$

Then

$$P_{\lambda,m}(D_\delta f) = c^{\delta/2}_{\lambda,m}P_{\lambda,m}f.$$

(ii) We know that

$$\begin{aligned} \psi &= (I - L)^{\delta/2}f \\ &= F^{-1}[(1 + c_{\lambda,m})^{-\delta/2}F(f)] \end{aligned}$$

then

$$F(\psi) = (1 + c_{\lambda,m})^{-\delta/2}F(f).$$

Using the definition of $P_{\lambda,m}$, we obtain

$$P_{\lambda,m}\psi = (1 + c_{\lambda,m})^{-\delta/2}P_{\lambda,m}f.$$

Now let to prove the Theorem 1

Proof. Let $f \in L^p_\delta(\mathbb{K})$ and $\psi = (I - L)^{\delta/2}f$, then:

$$\begin{aligned} D_\delta f &= \int_{\hat{\mathbb{K}}} c^{\delta/2}_{\lambda,m}P_{\lambda,m}f\,\mathrm{d}\gamma_\alpha(\lambda, m) \\ &= \int_{\hat{\mathbb{K}}} \left(\frac{c_{\lambda,m}}{1 + c_{\lambda,m}}\right)^{\delta/2}P_{\lambda,m}\psi\,\mathrm{d}\gamma_\alpha(\lambda, m). \end{aligned}$$

Since $||f||_{p,\delta} = ||\psi||_p$, by Meyer's multipliers theorem and using the multipliers $h(z) = (1 + z)^{-\delta/2}$, we obtain that:

$$||D_\delta f||_p \leq A_{p,\delta}||\psi||_p = A_{p,\delta}||f||_{p,\delta}.$$

To prove the converse, suppose $D_\delta f \in L^p(\mathbb{K})$ and consider

$$\begin{aligned}\psi &= (I - L)^{\delta/2} f \\ &= \int_{\hat{\mathbb{K}}} (1 + c_{\lambda,m})^{\delta/2} P_{\lambda,m} f \, d\gamma_\alpha(\lambda, m) \\ &= \int_{\hat{\mathbb{K}}} \left(\frac{1 + c_{\lambda,m}}{c_{\lambda,m}}\right)^{\delta/2} P_{\lambda,m}(D_\delta f) \, d\gamma_\alpha(\lambda, m).\end{aligned}$$

so by Meyer's multipliers theorem, using the multiplier $h(z) = (z + 1)^{\delta/2}$, we have:

$$||f||_{p,\delta} = ||\psi||_p \leq B_{p,\delta}||D_\delta f||_p.$$

Finally, we can write that

$$L^p_\delta(\mathbb{K}) = \{f : D_\delta f \in L^p(\mathbb{K}), \quad \delta \geq 0, \quad 1 < p < \infty\}.$$

Competing interests
The author declares that they have no competing interests.

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