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Research Article

Least Squares Estimators for Unit Root Processes with Locally Stationary Disturbance

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The random walk is used as a model expressing equitableness and the effectiveness of various finance phenomena. Random walk is included in unit root process which is a class of nonstationary processes. Due to its nonstationarity, the least squares estimator (LSE) of random walk does not satisfy asymptotic normality. However, it is well known that the sequence of partial sum processes of random walk weakly converges to standard Brownian motion. This result is so-called functional central limit theorem (FCLT). We can derive the limiting distribution of LSE of unit root process from the FCLT result. The FCLT result has been extended to unit root process with locally stationary process (LSP) innovation. This model includes different two types of nonstationarity. Since the LSP innovation has time-varying spectral structure, it is suitable for describing the empirical financial time series data. Here we will derive the limiting distributions of LSE of unit root, near unit root and general integrated processes with LSP innovation. Testing problem between unit root and near unit root will be also discussed. Furthermore, we will suggest two kind of extensions for LSE, which include various famous estimators as special cases.

1. Introduction

Since the random walk is a martingale sequence, the best predictor of the next term becomes the value of this term. In this sense, the random walk is used as a model expressing equitableness and the effectiveness of various finance phenomena in economics. Furthermore, because the random walk is a unit root process, taking the difference of the random walk, we can recover the independent sequence. However, the information of the original sequence will be lost by taking the difference when it does not include a unit root. Therefore, the testing of the existence of unit root in the original sequence becomes important.

In this section, we review the fundamental asymptotic results for unit root processes. Let $\{\varepsilon_j\}$ be i.i.d. $(0, \sigma^2)$ random variables, where $\sigma^2 > 0$, and define the partial sum

$$\begin{aligned} r_j &= r_{j-1} + \varepsilon_j \quad (r_0 = 0) \\ &= \sum_{i=1}^{j} \varepsilon_i, \quad (j = 1, \ldots, T), \end{aligned} \tag{1.1}$$

which is the so-called random walk process. Random walk corresponds to the first-order autoregressive (AR(1)) model with unit coefficient. Therefore, random walk is included in unit root (I(1)) processes which is a class of nonstationary processes. Let $\mathcal{C} = \mathcal{C}[0,1]$ be the space of all real-valued continuous functions defined on $[0,1]$. For random walk process, we construct the sequence of the processes of the partial sum $\{R_T\}$ in $\mathcal{C}$ as

$$R_T(t) = \frac{1}{\sigma\sqrt{T}} r_j + T\left(t - \frac{j}{T}\right)\frac{1}{\sigma\sqrt{T}}\varepsilon_j, \quad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right). \tag{1.2}$$

It is well known that the partial sum process $\{R_T\}$ converge weakly to a standard Brownian motion on $[0,1]$, namely,

$$\mathcal{L}(R_T) \longrightarrow \mathcal{L}(W) \quad \text{as } T \longrightarrow \infty, \tag{1.3}$$

where $\mathcal{L}(\cdot)$ denotes the distribution law of the corresponding random elements. This result is the so-called functional central limit theorem (FCLT) (see Billingsley [1]).

The FCLT result can be extended to the unit root process where the innovation is general linear process. We consider a sequence $\{\widetilde{R}_T\}$ of a stochastic process in $\mathcal{C}$ defined by

$$\widetilde{R}_T(t) = \frac{1}{\sqrt{T}}\widetilde{r}_j + T\left(t - \frac{j}{T}\right)\frac{1}{\sqrt{T}}u_j, \quad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right), \tag{1.4}$$

where $\widetilde{r}_j = \sum_{i=1}^{j} u_i$ and $\{u_j\}$ is assumed to be generated by

$$u_j = \sum_{l=0}^{\infty} \alpha_l \varepsilon_{j-l}, \quad \alpha_0 = 1. \tag{1.5}$$

Here, $\{\varepsilon_j\}$ is a sequence of i.i.d. $(0, \sigma^2)$ random variables, and $\{\alpha_j\}$ is a sequence of constants which satisfies $\sum_{l=0}^{\infty} l|\alpha_l| < \infty$; therefore, $\{u_j\}$ becomes stationary process. Using the Beveridge and Nelson [2] decomposition, it holds (see, e.g., Tanaka [3])

$$\mathcal{L}\left(\widetilde{R}_T\right) \longrightarrow \mathcal{L}(\alpha W), \quad \alpha = \sum_{l=0}^{\infty} \alpha_l. \tag{1.6}$$

The asymptotic property of LSE for stationary autoregressive models has been well established (see, e.g., Hannan [4]). On the other hand, due to its nonstationarity, the

LSE of random walk does not satisfy asymptotic normality. However, we can derive the limiting distribution of LSE of unit root process from the FCLT result. For more detailed understanding about unit root process with i.i.d. or stationary innovation, refer to, for example, Billingsley [1] and Tanaka [3].

In the above case, the $\{u_j\}$'s are stationary and hence, have constant variance, while covariances depend on only time differences. This is referred to as the homogeneous case, which is too restrictive to interpret empirical data, for example, empirical financial data. Recently, an important class of nonstationary processes have been proposed by Dahlhaus (see, e.g., Dahlhaus [5, 6]), called locally stationary processes. In this paper, we alternatively adopt locally stationary innovation process, which has smoothly changing variance. Since the LSP innovation has time-varying spectral structure, it is suitable for describing the empirical financial time series data.

This paper is organized as follows. In the appendix, we review the extension of the FCLT results to the cases that the innovations are locally stationary process. Namely, we explain the FCLT for unit root, near unit root, and general integrated processes with LSP innovations. In Section 2, we obtain the asymptotic distribution of the least squares estimator for each case of the appendix. In Section 3, we also consider the testing problem for unit root with LSP innovation. Finally, in Section 4, we discuss the extensions of LSE, which include various famous estimators as special cases.

2. The Property of Least Squares Estimator

In this section, we investigate the asymptotic properties of least squares estimators for unit root, near unit root, and $I(d)$ processes with locally stationary process innovations. Testing problem for unit root is also discussed. For the notations which are not defined in this section, refer to the appendix.

2.1. Least Squares Estimator for Unit Root Process

Here, we consider the following statistics:

$$\widehat{\rho} = \frac{\sum_{j=2}^{T} x_{j-1,T}\, x_{j,T}}{\sum_{j=2}^{T} x_{j-1,T}^2}, \tag{2.1}$$

obtained from model (A.3), which can be regarded as the least squares estimator (LSE) of autoregressive coefficient in the first-order autoregressive (AR(1)) model $x_{j,T} = \rho x_{j-1,T} + u_{j,T}$. Define

$$\begin{aligned} U_{1,T} &= \frac{1}{T\sigma^2}\sum_{j=2}^{T} x_{j-1,T}\left(x_{j,T} - x_{j-1,T}\right) \\ &= \frac{1}{2}X_T(1)^2 - \frac{1}{2}X(0)^2 - \frac{1}{2T\sigma^2}\sum_{j=1}^{T} u_{j,T}^2 - \frac{X(0)u_{1,T}}{\sqrt{T}\sigma}, \\ V_{1,T} &= \frac{1}{T^2\sigma^2}\sum_{j=2}^{T} x_{j-1,T}^2 = \frac{1}{T}\sum_{j=1}^{T} X_T\left(\frac{j}{T}\right)^2 - \frac{1}{T}X_T(1)^2, \end{aligned} \tag{2.2}$$

then we have

$$S_{1,T} \equiv T\left(\widehat{\rho}-1\right) = \frac{U_{1,T}}{V_{1,T}}. \tag{2.3}$$

Let us define a continuous function $H_1(x) = (H_{11}(x), H_{12}(x))$ for $x \in \mathcal{C}$, where

$$H_{11}(x) = \frac{1}{2}\left\{x(1)^2 - x(0)^2 - \int_0^1 \sum_{l=0}^{\infty} \alpha_l(v)^2 dv\right\}, \qquad H_{12}(x) = \int_0^1 x(v)^2 dv. \tag{2.4}$$

It is easy to check

$$U_{1,T} = H_{11}(X_T) + o_P(1), \qquad V_{1,T} = H_{12}(X_T) + o_P(1). \tag{2.5}$$

Therefore, the continuous mapping theorem (CMT) leads to $\mathcal{L}(U_{1,T}, V_{1,T}) \longrightarrow \mathcal{L}(H_1(X))$ and

$$\begin{aligned}\mathcal{L}(S_{1,T}) &= \mathcal{L}\left(T\left(\widehat{\rho}-1\right)\right)\\ &\longrightarrow \mathcal{L}\left(\frac{H_{11}(X)}{H_{12}(X)}\right) = \mathcal{L}\left(\frac{(1/2)\left\{X(1)^2 - X(0)^2 - \int_0^1 \sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\}}{\int_0^1 X(v)^2 dv}\right)\\ &= \mathcal{L}\left(\frac{\int_0^1 X(v)dX(v) + (1/2)\int_0^1\left[\left\{\sum_{l=0}^{\infty}\alpha_l(v)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(v)^2\right]dv}{\int_0^1 X(v)^2 dv}\right).\end{aligned} \tag{2.6}$$

2.2. Least Squares Estimator for Near Unit Root Process

We next consider the least squares estimator $\widehat{\rho}_T$ for model (A.11) in the case that $\beta(t) \equiv \beta$ is a constant on $[0,1]$, namely,

$$y_{j,T} = \rho_T y_{j-1,T} + u_{j,T}, \quad (j = 1, \ldots, T), \tag{2.7}$$

with $\rho_T = 1 - \beta/T$. Then, we have

$$\widehat{\rho}_T = 1 - \frac{\widehat{\beta}}{T} = \frac{\sum_{j=2}^{T} y_{j-1,T} y_{j,T}}{\sum_{j=2}^{T} y_{j-1,T}^2}, \qquad S_{2,T} \equiv T\left(\widehat{\rho}_T - 1\right) = -\widehat{\beta} = \frac{U_{2,T}}{V_{2,T}}, \tag{2.8}$$

where

$$U_{2,T} = \frac{1}{T\sigma^2}\sum_{j=2}^{T} y_{j-1,T}\left(y_{j,T} - y_{j-1,T}\right)$$
$$= \frac{1}{2}Y_T(1)^2 - \frac{1}{2}Y(0)^2 - \frac{1}{2T\sigma^2}\sum_{j=1}^{T}\left(u_{j,T} - \frac{\beta}{T}y_{j-1,T}\right)^2 - \frac{1}{\sqrt{T}\sigma}Y(0)\left(u_{1,T} - \frac{\beta}{T}y_{0,T}\right), \quad (2.9)$$
$$V_{2,T} = \frac{1}{T^2\sigma^2}\sum_{j=2}^{T} y_{j-1,T}^2 = \frac{1}{T}\sum_{j=1}^{T} Y_T\left(\frac{j}{T}\right)^2 - \frac{1}{T}Y_T(1)^2.$$

Let us define a continuous function $H_2(x) = (H_{21}(x), H_{22}(x))$ for $x \in \mathcal{C}$, where

$$H_{21}(x) = \frac{1}{2}\left\{x(1)^2 - x(0)^2 - \int_0^1 \sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\}, \qquad H_{22}(x) = \int_0^1 x(v)^2 dv. \quad (2.10)$$

It is easy to check

$$U_{2,T} = H_{21}(Y_T) + o_P(1), \qquad V_{2,T} = H_{22}(Y_T) + o_P(1). \quad (2.11)$$

Therefore, the CMT leads to $\mathcal{L}(U_{2,T}, V_{2,T}) \to \mathcal{L}(H_2(Y))$ and

$$\mathcal{L}(S_{2,T}) = \mathcal{L}(T(\widehat{\rho} - 1)) = \mathcal{L}\left(-\widehat{\beta}\right)$$
$$\longrightarrow \mathcal{L}\left(\frac{H_{21}(Y)}{H_{22}(Y)}\right) = \mathcal{L}\left(\frac{(1/2)\left\{Y(1)^2 - Y(0)^2 - \int_0^1 \sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\}}{\int_0^1 Y(v)^2 dv}\right) \quad (2.12)$$
$$= \mathcal{L}\left(\frac{\int_0^1 Y(v)dY(v) + (1/2)\int_0^1\left[\left\{\sum_{l=0}^{\infty}\alpha_l(v)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(v)^2\right]dv}{\int_0^1 Y(v)^2 dv}\right).$$

2.3. Least Squares Estimator for $I(d)$ Process

Furthermore, we consider the least squares estimator

$$\widehat{\rho}^{\{d\}} = \frac{\sum_{j=2}^{T} x_{j-1,T}^{\{d\}} x_{j,T}^{\{d\}}}{\sum_{j=2}^{T}\left(x_{j-1,T}^{\{d\}}\right)^2}, \qquad S_{3,T} \equiv T\left(\widehat{\rho}^{\{d\}} - 1\right) = \frac{U_{3,T}}{V_{3,T}}, \quad (2.13)$$

obtained from model $x_{j,T}^{\{d\}} = \rho x_{j-1,T}^{\{d\}} + x_{j,T}^{\{d-1\}}$, where

$$\begin{aligned}
U_{3,T} &= \frac{1}{T^{2d-1}\sigma^2}\sum_{j=2}^{T} x_{j-1,T}^{\{d\}}\left(x_{j,T}^{\{d\}} - x_{j-1,T}^{\{d\}}\right) \\
&= \frac{1}{2}X_T^{\{d\}}(1)^2 - \frac{1}{2T^2}\sum_{j=1}^{T}\left\{X_T^{\{d-1\}}\left(\frac{j}{T}\right)\right\}^2 - \frac{1}{T}X_T^{\{d\}}(0)X_T^{\{d-1\}}\left(\frac{1}{T}\right) \\
V_{3,T} &= \frac{1}{T^{2d}\sigma^2}\sum_{j=2}^{T}\left(x_{j-1,T}^{\{d\}}\right)^2 = \frac{1}{T}\sum_{j=1}^{T}\left\{X_T^{\{d\}}\left(\frac{j}{T}\right)\right\}^2 - \frac{1}{T}\left\{X_T^{\{d\}}(1)\right\}^2.
\end{aligned} \tag{2.14}$$

Let us define a continuous function $H_3(x) = (H_{31}(x), H_{32}(x))$ for $x \in \mathcal{C}$, where

$$H_{31}(x) = \frac{1}{2}x(1)^2, \qquad H_{32}(x) = \int_0^1 x(v)^2 dv. \tag{2.15}$$

It is easy to check

$$U_{3,T} = H_{31}\left(X_T^{\{d\}}\right) + o_P(1), \qquad V_{3,T} = H_{32}\left(X_T^{\{d\}}\right) + o_P(1). \tag{2.16}$$

Therefore, the CMT leads to $\mathcal{L}(U_{3,T}, V_{3,T}) \to \mathcal{L}(H_3(X^{\{d-1\}}))$ and

$$\begin{aligned}
\mathcal{L}(S_{3,T}) &= \mathcal{L}\left(T\left(\hat{\rho}^{\{d\}} - 1\right)\right) \\
&\longrightarrow \mathcal{L}\left(\frac{H_{31}(X^{\{d-1\}})}{H_{32}(X^{\{d-1\}})}\right) \\
&= \mathcal{L}\left(\frac{(1/2)\left\{X^{\{d-1\}}(1)\right\}^2}{\int_0^1 \left\{X^{\{d-1\}}(v)\right\}^2 dv}\right) \\
&= \mathcal{L}\left(\frac{\int_0^1 X^{\{d-1\}}(v)dX^{\{d-1\}}(v)}{\int_0^1 \left\{X^{\{d-1\}}(v)\right\}^2 dv}\right).
\end{aligned} \tag{2.17}$$

The equality above is due to $(d-1)$-times differentiability of $X^{\{d-1\}}$.

3. Testing for Unit Root

In the analysis of empirical financial data, the existence of the unit root is an important problem. However, as we see in the previous section, the asymptotic results between unit root and near unit root processes are quite different (the drift term appeared in the limiting process of near unit root). Therefore, we consider the following testing problem against the local alternative hypothesis:

$$H_0 : \rho = 1 \qquad H_1 : \rho = 1 - \frac{\beta}{T}. \tag{3.1}$$

We should assume that σ^2 is a unit to identify the models. Let the statistics $S_{1,T}$ be constructed in (2.3). Recall that, as $T \to \infty$, under H_0,

$$\begin{aligned}\mathcal{L}(S_{1,T}) &\longrightarrow \mathcal{L}\left(\frac{\int_0^1 X(\nu)dX(\nu) + (1/2)\int_0^1\left[\left\{\sum_{l=0}^{\infty}\alpha_l(\nu)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(\nu)^2\right]d\nu}{\int_0^1 X(\nu)^2 d\nu}\right)\\ &= \mathcal{L}\left(\frac{U}{V} + \frac{\int_0^1\left[\left\{\sum_{l=0}^{\infty}\alpha_l(\nu)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(\nu)^2\right]d\nu}{2\int_0^1 X(\nu)^2 d\nu}\right),\end{aligned} \tag{3.2}$$

where

$$U = \int_0^1 X(\nu)dX(\nu), \qquad V = \int_0^1 X(\nu)^2 d\nu. \tag{3.3}$$

Since $\{\sum_{l=0}^{\infty}\alpha_l(\nu)\}^2,\ \sum_{l=0}^{\infty}\alpha_l(\nu)^2$ are unknown, we construct a test statistic

$$Z_\rho = T(\hat{\rho} - 1) + \frac{(1/T)\sum_{j=1}^{T}\hat{u}_{j,T}^2 - (1/T)\sum_{t=1}^{T}\hat{f}(t/T,0)}{2(1/T^2)\sum_{j=2}^{T}x_{j-1,T}^2}, \tag{3.4}$$

where $\hat{u}_{j,T} = x_{j,T} - x_{j-1,T}$. A nonparametric time-varying spectral density estimator $\hat{f}(u,\lambda)$ is given by

$$\begin{aligned}\hat{f}(u,\lambda_l) &= M\int K(M(\lambda_l - \mu_k))I_N(u,\mu_k)d\mu_k\\ &\approx \frac{2\pi M}{T}\sum_{k=-T/4\pi M+l}^{T/4\pi M+l} K(M(\lambda_l - \mu_k))I_N(u,\mu_k),\end{aligned} \tag{3.5}$$

where $\lambda_l = (2\pi/T)l - \pi$, $l = 1,\ldots,T-1$ and $\mu_k = (2\pi/T)k - \pi$, $k = 1,\ldots,T-1$. Here, $I_N(u,\lambda)$ is the local periodogram around time u given by

$$I_N(u,\lambda) = \frac{1}{2\pi N}\left|\sum_{s=1}^{N} h\left(\frac{s}{N}\right)\hat{u}_{[uT]-N/2+s,T}e^{-i\lambda s}\right|^2, \tag{3.6}$$

where $[\cdot]$ denotes Gauss symbol and, for real number a, $[a]$ is the greatest integer that is less than or equal to a. Furthermore, we employ the following kernel functions and the orders of bandwidth for smoothing in time and frequency domain, respectively,

$$K(x) = 6\left(\frac{1}{4} - x^2\right), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \quad h(x) = \{6x(1-x)\}^{1/2}, \quad x \in [0,1],$$
$$M = T^{1/6}, \qquad N = T^{5/6}, \tag{3.7}$$

which are optimal in the sense that they minimize the mean squared error of nonparametric estimator (see Dahlhaus [6]); however, we simply multiply the orders of bandwidth by the constants equal to one. Then, it can be established that, under H_0,

$$\mathcal{L}(Z_\rho) \longrightarrow \mathcal{L}\left(\frac{U}{V}\right). \tag{3.8}$$

We now have to deal with statistics for which numerical integration must be elaborated. Let R be such a statistic, which takes the form $R = U/V$. Using Imhof's [7] formula gives us distribution function of R,

$$F_R(x) = P(R \le x) = P(xV - U \ge 0) = \frac{1}{2} + \frac{1}{\pi}\int_0^1 \frac{1}{s}\,\mathrm{Im}\{\phi(s;x)\}ds, \tag{3.9}$$

where $\phi(s;x)$ is the characteristic function of $xV - U$, namely,

$$\phi(-is;x) = E\left[\exp\{s(xV-U)\}\right] = E\left[\exp\left\{s\left(x\int_0^1 X(v)^2dv - \int_0^1 X(v)dX(v)\right)\right\}\right]. \tag{3.10}$$

However, so far we do not have the explicit form of the distribution function of the estimator. Therefore, we cannot perform a numerical experiment except for the clear simple cases. It includes the complicated problem in the differential equation and requires one further paper for solution.

4. Extensions of LSE

In this section, we consider the extensions of LSE $\hat{\rho}_T$ for near random walk model $y_{j,T} = \rho_T y_{j-1,T} + u_{j,T}$, $\rho_T = 1 - \beta/T$.

4.1. Ochi Estimator

Ochi [8] proposed the class of estimators of the following form, which are the extensions of LSE for autoregressive coefficient:

$$\hat{\rho}_T^{(\theta_1,\theta_2)} = 1 - \frac{\hat{\beta}_{(\theta_1,\theta_2)}}{T} = \frac{\sum_{j=2}^{T} y_{j-1,T}\, y_{j,T}}{\sum_{j=2}^{T-1} y_{j,T}^2 + \theta_1 y_{1,T}^2 + \theta_2 y_{T,T}^2}, \quad \theta_1, \theta_2 \geq 0,$$
$$S_{4,T} = T\left(\hat{\rho}_T^{(\theta_1,\theta_2)} - 1\right) = -\hat{\beta}_{(\theta_1,\theta_2)} = \frac{U_{4,T}}{V_{4,T}}, \tag{4.1}$$

where

$$\begin{aligned} U_{4,T} &= \frac{1}{T\sigma^2}\left\{ \sum_{j=2}^{T} y_{j-1,T}\, y_{j,T} - \sum_{j=2}^{T-1} y_{j,T}^2 - \theta_1 y_{1,T}^2 - \theta_2 y_{T,T}^2 \right\} \\ &= \left\{ \frac{1}{2}(1-2\theta_1) + \frac{\beta}{T}(2\theta_1 - 1) + \frac{\beta^2}{T^2}(1-\theta_1) \right\} Y(0)^2 \\ &\quad + \frac{1}{2}(1-\theta_2) Y_T(1)^2 - \frac{1}{2}\frac{1}{T\sigma^2}\sum_{j=1}^{T}\left(u_{j,T} - \frac{\beta}{T} y_{j-1,T} \right)^2 \\ &\quad + \frac{1}{\sqrt{T}\sigma}\left\{ 1 - 2\theta_1 + \frac{2\beta}{T}(\theta_1 - 1) \right\} u_{1,T} Y(0) + \frac{1}{T\sigma^2}(1-\theta_1) u_{1,T}' 2, \\ V_{4,T} &= \frac{1}{T^2\sigma^2}\left\{ \sum_{j=2}^{T-1} y_{j,T}^2 + \theta_1 y_{1,T}^2 + \theta_2 y_{T,T}^2 \right\} \\ &= \frac{1}{T}\sum_{j=1}^{T} Y_T\left(\frac{j}{T}\right)^2 + (\theta_1 - 1)\frac{1}{T} Y_T\left(\frac{1}{T}\right)^2 + (\theta_2 - 1)\frac{1}{T} Y_T(1)^2. \end{aligned} \tag{4.2}$$

This class of estimators includes LSE $\hat{\rho}_T^{(1,0)}$, Daniels's estimator $\hat{\rho}_T^{(1/2,1/2)}$, and Yule-Walker estimator $\hat{\rho}_T^{(1,1)}$ as the special cases.

Define for $x \in \mathcal{C}$, $H_4(x) = (H_{41}(x), H_{42}(x))$,

$$H_{41}(x) = \frac{1}{2}\left\{ (1-2\theta_1)x(0)^2 + (1-2\theta_2)x(1)^2 - \int_0^1 \sum_{l=0}^{\infty} \alpha_l(v)^2 dv \right\},$$
$$H_{42}(x) = \int_0^1 x(v)^2 dv, \tag{4.3}$$

then we see that $H_4(x)$ is continuous and

$$U_{4,T} = H_{41}(Y_T) + o_P(1), \qquad V_{4,T} = H_{42}(Y_T) + o_P(1). \tag{4.4}$$

From the CMT, we obtain $\mathcal{L}(U_{4,T}, V_{4,T}) \rightarrow \mathcal{L}(H_4(Y))$, and therefore,

$$\mathcal{L}(S_{4,T}) = \mathcal{L}\left(T\left(\hat{\rho}_T^{(\theta_1,\theta_2)} - 1\right)\right) = \mathcal{L}\left(-\hat{\beta}_{(\theta_1,\theta_2)}\right) \longrightarrow \mathcal{L}\left(\frac{H_{41}(Y)}{H_{42}(Y)}\right), \tag{4.5}$$

where

$$\begin{aligned} H_{41}(Y) &= \frac{1}{2}\left\{(1-2\theta_1)Y(0)^2 + (1-2\theta_2)Y(1)^2 - \int_0^1 \sum_{l=0}^{\infty} \alpha_l(v)^2 dv\right\} \\ &= (1-2\theta_2)\int_0^1 Y(v)dY(v) + (1-\theta_1-\theta_2)Y(0)^2 \\ &\quad + \frac{1}{2}\int_0^1 \left[(1-2\theta_2)\left\{\sum_{l=0}^{\infty}\alpha_l(v)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(v)^2\right] dv, \\ H_{42}(Y) &= \int_0^1 Y(v)^2 dv. \end{aligned} \tag{4.6}$$

4.2. Another Extension of LSE

Next, we suggest another class of estimators which are also the extensions of LSE. Define for $\theta(u)(\in \mathcal{C})$ with continuous derivative $\theta'(u) = (\partial/\partial u)\theta(u)$,

$$\hat{\rho}_T^{\theta} = 1 - \frac{\hat{\beta}_\theta}{T} = \frac{\sum_{j=2}^{T}\theta((j-1)/T)y_{j-1,T}y_{j,T}}{\sum_{j=2}^{T}\theta((j-1)/T)y_{j-1,T}^2}, \qquad S_{5,T} = T\left(\hat{\rho}_T^{\theta} - 1\right) = -\hat{\beta}_\theta = \frac{U_{5,T}}{V_{5,T}}, \tag{4.7}$$

where

$$\begin{aligned} U_{5,T} &= \frac{1}{T\sigma^2}\sum_{j=2}^{T}\theta\left(\frac{j-1}{T}\right)y_{j-1,T}(y_{j,T} - y_{j-1,T}) \\ &= -\frac{1}{2}\sum_{j=1}^{T}\left\{\theta\left(\frac{j}{T}\right) - \theta\left(\frac{j-1}{T}\right)\right\}Y_T\left(\frac{j}{T}\right)^2 + \frac{1}{2}\theta(1)Y_T(1)^2 - \frac{1}{2}\theta(0)Y(0)^2 \\ &\quad - \frac{1}{2}\frac{1}{T\sigma^2}\sum_{j=1}^{T}\theta\left(\frac{j}{T}\right)\left(u_{j,T} - \frac{\beta}{T}y_{j-1,T}\right)^2 + \frac{1}{2T\sigma^2}\theta\left(\frac{1}{T}\right)\left(u_{1,T} - \frac{\beta}{T}y_{0,T}\right)^2 \\ &\quad + \frac{1}{2T\sigma^2}\theta(0)\left\{u_{1,T}(u_{1,T} + 2y_{0,T}) - \frac{2\beta}{T}y_{0,T}(y_{0,T} + u_{1,T}) + \frac{\beta^2}{T^2}y_{0,T}^2\right\}, \\ V_{5,T} &= \frac{1}{T^2\sigma^2}\sum_{j=2}^{T}\theta\left(\frac{j-1}{T}\right)y_{j-1,T}^2 = \frac{1}{T}\sum_{j=1}^{T}\theta\left(\frac{j}{T}\right)Y_T\left(\frac{j}{T}\right)^2 - \frac{1}{T}\theta(1)Y_T(1)^2. \end{aligned} \tag{4.8}$$

If we take the taper function as $\theta(u)$, this estimator corresponds to the local LSE.

Define for $x \in \mathcal{C}$, $H_5(x) = (H_{51}(x), H_{52}(x))$,

$$H_{51}(x) = -\frac{1}{2}\left\{\int_0^1 \theta'(v)x(v)^2 dv - \theta(1)x(1)^2 + \theta(0)x(0)^2\right\} - \frac{1}{2}\left\{\int_0^1 \theta(v)\sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\},$$
$$H_{52}(x) = \int_0^1 \theta(v)x(v)^2 dv, \tag{4.9}$$

where $\theta'(u) = (\partial/\partial u)\theta(u)$, then we see that $H_5(x)$ is continuous and

$$U_{5,T} = H_{51}(Y_T) + o_P(1), \qquad V_{5,T} = H_{52}(Y_T) + o_P(1). \tag{4.10}$$

From the CMT, we obtain $\mathcal{L}(U_{5,T}, V_{5,T}) \to \mathcal{L}(H_5(Y))$, and therefore,

$$\mathcal{L}(S_{5,T}) = \mathcal{L}\left(T\left(\widehat{\rho}_T^{\theta} - 1\right)\right) = \mathcal{L}\left(-\widehat{\beta}_\theta\right) \longrightarrow \mathcal{L}\left(\frac{H_{51}(Y)}{H_{52}(Y)}\right) \equiv \mathcal{L}\left(Y^{\theta}\right), \tag{4.11}$$

where

$$H_{51}(Y) = -\frac{1}{2}\left\{\int_0^1 \theta'(v)Y(v)^2 dv - \theta(1)Y(1)^2 + \theta(0)Y(0)^2\right\} - \frac{1}{2}\left\{\int_0^1 \theta(v)\sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\},$$
$$H_{52}(Y) = \int_0^1 \theta(v)Y(v)^2 dv. \tag{4.12}$$

The integration by part leads to

$$Y^{\theta} = \frac{(1/2)\left\{\int_0^1 \theta(v)dY^{(1)}(v) - \int_0^1 \theta(v)\sum_{l=0}^{\infty}\alpha_l(v)^2 dv\right\}}{\int_0^1 \theta(v)Y(v)^2 dv}, \tag{4.13}$$

with $Y^{(1)}(t) = Y(t)^2$. Hence, using Ito's formula,

$$dY^{(1)}(t) = d\left\{Y(t)^2\right\} = 2Y(t)dY(t) + \left\{\sum_{l=0}^{\infty}\alpha_l(t)\right\}^2 dt, \tag{4.14}$$

we have

$$\Upsilon^{\theta} = \frac{\int_0^1 \theta(v)\Upsilon(v)d\Upsilon(v) + (1/2)\int_0^1 \theta(v)\left[\left\{\sum_{l=0}^{\infty}\alpha_l(v)\right\}^2 - \sum_{l=0}^{\infty}\alpha_l(v)^2\right]dv}{\int_0^1 \theta(v)\Upsilon(v)^2 dv}. \tag{4.15}$$

Appendices

In this appendix, we review the extensions of functional central limit theorem to the cases that innovations are locally stationary processes, which are used for the main results of this paper.

A. FCLT for Locally Stationary Processes

Hirukawa and Sadakata [9] extended the FCLT results to the unit root processes which have locally stationary process innovations. Namely, they derived the FCLT for unit root, near unit root, and general integrated processes with LSP innovations. In this section, we briefly review these results which are applied in previous sections.

A.1. Unit Root Process with Locally Stationary Disturbance

First, we introduce locally stationary process innovation. Let $\{u_{j,T}\}$ be generated by the following time-varying MA (∞) model:

$$u_{j,T} = \sum_{l=0}^{\infty}\alpha_l\left(\frac{j}{T}\right)\varepsilon_{j-l} := \sum_{l=0}^{\infty}\alpha_l\left(\frac{j}{T}\right)L^l\varepsilon_j = \alpha\left(\frac{j}{T}, L\right)\varepsilon_j, \tag{A.1}$$

where L is the lag-operator which acts as $L\varepsilon_j = \varepsilon_{j-1}$ and $\alpha(u, L) = \sum_{l=0}^{\infty}\alpha_l(u)L^l$, and time-varying MA coefficients satisfy

$$\sum_{l=0}^{\infty} l \sup_{0\le u\le 1}|\alpha_l(u)| < \infty, \qquad \sum_{l=0}^{\infty} l \sup_{0\le u\le 1}\left|\frac{\partial}{\partial u}\alpha_l(u)\right| < \infty. \tag{A.2}$$

Then, these $\{u_{j,T}\}$'s become locally stationary processes (see Dahlhaus [5], Hirukawa and Taniguchi [10]). Using this innovation process, define the partial sum $\{x_{j,T}\}$ as

$$x_{j,T} = x_{j-1,T} + u_{j,T} = x_{0,T} + \sum_{i=1}^{j} u_{i,T}, \tag{A.3}$$

where $x_{0,T} = \sigma\sqrt{T}X(0)$, $X(0) \sim N(\gamma_X, \delta_X^2)$ and is independent of $\{\varepsilon_j\}$.

We consider a sequence $\{X_T\}$ of partial sum stochastic processes in $\mathcal{C}$ defined by

$$X_T(t) = \frac{1}{\sigma\sqrt{T}}x_{j,T} + T\left(t - \frac{j}{T}\right)\frac{1}{\sigma\sqrt{T}}u_{j,T}, \quad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right). \tag{A.4}$$

Now, we define on $\mathbb{R} \times \mathcal{C}$

$$h_t^{(1)}(x, y) = x + \alpha(t, 1)y(t) - \int_0^t \alpha'(v, 1)y(v)dv,$$
$$\alpha(t, 1) = \sum_{l=0}^{\infty} \alpha_l(t), \qquad \alpha'(t, 1) = \frac{\partial}{\partial t}\alpha(t, 1) = \sum_{l=0}^{\infty} \frac{\partial}{\partial t}\alpha_l(t). \tag{A.5}$$

Then, we can obtain

$$\mathcal{L}(X_T) \longrightarrow \mathcal{L}\left\{h^{(1)}(X(0), W)\right\} \equiv \mathcal{L}(X). \tag{A.6}$$

The integration by parts leads to

$$\begin{aligned} X(t) &= X(0) + \alpha(t, 1)W(t) - \int_0^t \alpha'(v, 1)W(v)\,dv \\ &= X(0) + \int_0^t \alpha(v, 1)\,dW(v), \\ dX(t) &= \alpha(t, 1)\,dW(t). \end{aligned} \tag{A.7}$$

Note that the time-varying MA (∞) process $u_{j,T}$ in (A.1) has the spectral representation

$$u_{j,T} = \int_{-\pi}^{\pi} A\left(\frac{j}{T}, \lambda\right) e^{ij\lambda} d\xi(\lambda), \tag{A.8}$$

where $\xi(\lambda)$ is the spectral measure of i.i.d. process $\{\varepsilon_j\}$ which satisfies $\varepsilon_j = \int_{-\pi}^{\pi} e^{ij\lambda} d\xi(\lambda)$, and the transfer function $A(t, \lambda)$ is given by

$$A(t, \lambda) = \sum_{l=0}^{\infty} \alpha_l(t) e^{-il\lambda}, \qquad A(t, 0) = \sum_{l=0}^{\infty} \alpha_l(t) = \alpha(t, 1). \tag{A.9}$$

Therefore, stochastic differential in (A.7) can be written as

$$dX(t) = A(t, 0)dW(t). \tag{A.10}$$

A.2. Near Unit Root Process with Locally Stationary Disturbance

In this section, we consider the following near unit root process $\{y_{j,T}\}$ with locally stationary disturbance:

$$\begin{aligned} y_{j,T} &= \rho_{j,T} y_{j-1,T} + u_{j,T}, \quad (j = 1, \ldots, T) \\ &= \prod_{i=1}^{j} \rho_{i,T} y_{0,T} + \sum_{i=1}^{j} \left(\prod_{k=i+1}^{j} \rho_{k,T} \right) u_{i,T}, \end{aligned} \tag{A.11}$$

where $\{u_{j,T}\}$ is generated from the time-varying MA (∞) model in (A.1), $\rho_{j,T} = 1 - (1/T)\beta(j/T)$, $\beta(t) \in \mathcal{C}[0,1]$, $y_{0,T} = \sqrt{T}\sigma Y(0)$, and $Y(0) \sim N(\gamma_Y, \delta_Y)$ is independent of $\{\varepsilon_j\}$ and $X(0)$. Then, we define a sequence $\{Y_T\}$ of partial sum processes in $\mathcal{C}$ as

$$Y_T(t) = \frac{1}{\sigma\sqrt{T}} y_{j,T} + T\left(t - \frac{j}{T}\right)\frac{y_{j,T} - y_{j-1,T}}{\sigma\sqrt{T}}, \quad \left(\frac{j-1}{T} \leq t \leq \frac{j}{T}\right). \tag{A.12}$$

Define on $\mathbb{R}^2 \times \mathcal{C}$

$$h_t^{(2)}(x, y, z) = e^{-\int_0^t \beta(v)dv}(y - x) - \int_0^t \beta(v) e^{-\int_v^t \beta(s)ds} z(v)dv + z(t). \tag{A.13}$$

Then, we can obtain

$$\mathcal{L}(Y_T) \longrightarrow \mathcal{L}\left\{h^{(2)}(\mathrm{X}(0), Y(0), \mathrm{X})\right\} \equiv \mathcal{L}(Y). \tag{A.14}$$

The integration by parts and Ito's formula lead to

$$\begin{aligned}
Y(t) &= e^{-\int_0^t \beta(s)ds}\left(Y(0) - X(0) - \int_0^t \beta(v) e^{\int_0^v \beta(s)ds} X(v)dv\right) + X(t) \\
&= e^{-\int_0^t \beta(s)ds}\left(Y(0) + \int_0^t e^{\int_0^v \beta(\mu)d\mu} dX(v)\right) \\
&= e^{-\int_0^t \beta(s)ds}\left(Y(0) + \int_0^t e^{\int_0^v \beta(\mu)d\mu} \alpha(v,1)dW(v)\right), \qquad \text{(A.15)} \\
dY(t) &= -\beta(t)Y(t) + \alpha(t,1)dW(t) \\
&= -\beta(t)Y(t) + A(t,0)dW(t) \\
&= -\beta(t)Y(t) + dX(t).
\end{aligned}$$

A.3. $I(d)$ Process with Locally Stationary Disturbance

Let $I(d)$ process $\{x_{j,T}^{\{d\}}\}$ be generated by

$$(1 - L)^d x_{j,T}^{\{d\}} = u_{j,T}, \quad (j = 1, \ldots, T), \tag{A.16}$$

with $x_{-d+1,T}^{\{d\}} = \cdots = x_{0,T}^{\{d\}} = 0$, and $\{u_{j,T}\}$ being the time-varying MA (∞) process in (A.1). Note that the relation (A.16) can be rewritten as

$$(1 - L)x_{j,T}^{\{d\}} = x_{j,T}^{\{d-1\}}. \tag{A.17}$$

Then, we construct the partial sum process $\{X_T^{\{d\}}\}$ as

$$X_T^{\{d\}}(t) = \frac{1}{T^{d-1}}\left\{\frac{1}{\sigma\sqrt{T}}x_{j,T}^{\{d\}} + T\left(t - \frac{j}{T}\right)\frac{1}{\sigma\sqrt{T}}x_{j,T}^{\{d-1\}}\right\}, \tag{A.18}$$

for $(j-1)/T \le t \le j/T$, $d \ge 2$, and $X_T^{\{1\}}(t) \equiv X_T(t)$, where the partial sum process $\{X_T\}$ is defined in (A.4). Let us first discuss weak convergence to the onefold integrated process $\{X^{\{1\}}\}$ defined by

$$X^{\{1\}}(t) = \int_0^t X(\nu)d\nu = \int_0^t \left\{X(0) + \int_0^{\nu} \alpha(\mu, 1)dW(\mu)\right\}d\nu. \tag{A.19}$$

For $d = 2$, the partial sum process in (A.18) becomes

$$X_T^{\{2\}}(t) = \frac{1}{T}\left\{\sum_{i=1}^{j} X_T\left(\frac{i}{T}\right) + T\left(t - \frac{j}{T}\right)X_T\left(\frac{j}{T}\right)\right\}, \quad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right). \tag{A.20}$$

Define on $\mathcal{C}$

$$h_t^{(3)}(x) = \int_0^t x(\nu)d\nu. \tag{A.21}$$

Then, we can see that

$$\mathcal{L}\left(X_T^{\{2\}}\right) \longrightarrow \mathcal{L}\left\{h^{(3)}(X)\right\} = \mathcal{L}\left\{X^{\{1\}}\right\}. \tag{A.22}$$

For the general integer d, define the d-fold integrated process $\{X^{\{d\}}\}$ by

$$X^{\{d\}}(t) = \int_0^t X^{\{d-1\}}(\nu)d\nu, \qquad X^{\{0\}}(t) = X(t). \tag{A.23}$$

From the similar argument in the case of $d = 2$, we can see that the partial sum process $\{X_T^{\{d\}}\}$ satisfies

$$\mathcal{L}\left(X_T^{\{d\}}\right) \longrightarrow \mathcal{L}\left\{h^{(3)}\left(X^{\{d-1\}}\right)\right\} = \mathcal{L}\left\{X^{\{d-1\}}\right\}. \tag{A.24}$$

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