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A note on the damped vibrating systems

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Abstract

The presence of pure imaginary eigenvalues of the partially damped vibrating systems is treated. The number of such eigenvalues is determined using the rank of a matrix which is directly related to the system matrices.

Keywords: linear system, incomplete dissipation, pure imaginary eigenvalue

1 Introduction and preliminaries

In this note a "damped vibrating system" is understood to be the classical model of a linear, viscously damped elastic system with n degrees of freedom. This system has equations of motion

$$A\ddot{q} + B\dot{q} + Cq = 0, \quad q \in \Re^n \tag{1}$$

where A, B and C are $n \times n$ constant real symmetric matrices. The inertia matrix A and stiffness matrix C are positive definite (> 0), and the damping matrix B may be positive definite or positive semi-definite (≥ 0). In the case $B > 0$ dissipation is complete, and the case $B \geq 0$ corresponds to incomplete dissipation. In the latter case the system is called partially dissipative (damped).

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It is convenient, although not necessary, to rewrite equation (1) in the form

$$\ddot{x} + D\dot{x} + Kx = 0, \tag{2}$$

using the congruent transformation $x = A^{1/2}q$, where $A^{1/2}$ denotes the unique positive definite square root of the matrix A, and $D = A^{-1/2}BA^{-1/2}$, and $K = A^{-1/2}CA^{-1/2}$.

All solutions $x(t)$ of the equation (2) (or $q(t)$ of (1)) can be characterized algebraically using properties of the quadratic matrix polynomial

$$L(\lambda) = \lambda^2 I + \lambda D + K, \tag{3}$$

where I is the identity matrix. The eigenvalues of the system are zeros of the characteristic polynomial

$$\Delta(\lambda) = \det(L(\lambda)) \tag{4}$$

Since (4) is a polynomial of degree $2n$ with respect to λ, there are $2n$ eigenvalues, counting multiplicities. If λ is an eigenvalue, the nonzero vectors X in the nullspace of $L(\lambda)$ are the eigenvectors associated with λ, i. e.,

$$L(\lambda)X = 0 \tag{5}$$

In general, eigenvalues and corresponding eigenvectors may be real or may appear in complex conjugate pairs.

If the dissipation is complete, it is well-known that the system (2) (or (1)) is asymptotically stable ($x(t) \to 0$ as $t \to \infty$ for all solutions $x(t)$). On the other hand, the partially damped system (2) may or may not be asymptotically stable, although it is obviously stable in the Lyapunov sense (any solution of equation (2) remains bounded). Consequently, all eigenvalues of this system lie in the *closed* left-half of the complex plane ($Re\lambda \leq 0$). Notice that if the system is asymptotically stable, then $Re\lambda < 0$.

Recently some attention has been paid to the question whether or not a damped system has pure imaginary eigenvalues, i. e., in the terminology of the mechanical vibrations, whether or not undamped motions (also called "residual motions") are possible in such system (see [1] and quoted references). From the above discussion it is clear that nonexistence of undamped motions is equivalent to the asymptotic stability of the system, and consequently, any test for asymptotic stability gives the answer of the question. A survey of the stability criteria for linear second order systems is given in [2]. Also, it should be mentioned that the paper [1] rediscovered an old criterion for asymptotic stability of the system [3], as was recently stressed in [4].

In this note we are interested in the determination of the number of pure imaginary eigenvalues of the system without computing the zeros of the characteristic polynomial (4). The main result given in the next section (Theorem 1) is based on the well-known condition of asymptotic stability [5], which coincides with the rank condition of controllability of a linear system (see [6]).

2 Results

Introduce the $n \times n^2$ matrix

$$\Phi = \left(\begin{array}{ccccc} D \vdots & KD & \vdots \ \vdots \ \vdots & K^{n-1}D \end{array} \right) \tag{6}$$

which plays key role in a test for asymptotic stability of the system [5].

Theorem 1. The system (2) has $r = n - rank\Phi$ conjugate pairs of purely imaginary eigenvalues, including multiplicity.

Corollary. If $rankD = m$, then $0 \leq r \leq n - m$.

This follows immediately from $rankD \leq rank\Phi \leq n$.

To prove Theorem 1 we need the following lemmas.

Lemma 1. Let $(i\omega, X), \omega \in \Re, i = \sqrt{-1}$, be an eigenpair of $L(\lambda)$. Then (ω^2, X) and $(0, X)$are eigenpairs of the matrices K and D, respectively.

Proof. From

$$L(i\omega)X = (-\omega^2 I + i\omega D + K)X = 0, \tag{7}$$

we obtain

$$< X, (K - \omega^2 I)X > + i\omega < X, DX >= 0, \tag{8}$$

where $< .,. >$ denotes the inner product, and $< X, (K - \omega^2 I)X >$, and $< X, DX >$ are real quantities, since K and D are real symmetric matrices. Then $< X, DX >= 0$, which implies $DX = 0$, since $D \geq 0$. This together with $L(i\omega)X = 0$ gives $KX = \omega^2 X$. $\square$

It is clear that the eigenvector X in Lemma 1 can be taken to be *unit* ($< X, X >= 1$) and *real*.

Lemma 2. a) If $(i\omega_1, X^{(1)})$ and $(i\omega_2, X^{(2)})$ are eigenpairs of $L(\lambda)$ with $\omega_1^2 \neq \omega_2^2$, then $< X^{(1)}, X^{(2)} >= 0$.
b) If the eigenvalue $i\omega$ of $L(\lambda)$ has multiplicity k, it possesses k eigenvectors which are mutually orthogonal.

Proof. a) The result follows from Lemma 1 and the additional fact that eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal.
b) Since the system (2) is stable, the multiple eigenvalue $i\omega$ must be semi-simple, which means that the eigenvalue has k linearly independent eigenvectors. Since a linear combination of these k vectors is also an eigenvector of $L(\lambda)$ associated with $i\omega$, the Gram-Schmidt process (see [7]) can be used to obtain k mutually orthogonal eigenvectors. $\square$

Lemma 3. Let $\pm i\omega_1, ..., \pm i\omega_r$ be eigenvalues of $L(\lambda)$. Then there exists an orthogonal matrix Q such that

$$Q^T D Q = \hat{D} = \begin{pmatrix} 0_r & 0 \\ 0 & \hat{D}_{n-r} \end{pmatrix}, \tag{9}$$

and

$$Q^T K Q = \hat{K} = \begin{pmatrix} \Omega_r & 0 \\ 0 & \hat{K}_{n-r} \end{pmatrix}, \tag{10}$$

where 0_r is the zero square matrix of order r, and $\Omega_r = diag(\omega_1^2, ..., \omega_r^2)$.

Proof. By lemmas 1 and 2, there exists an orthonormal set of r vectors $X^{(1)}, ..., X^{(r)}$, such that

$$DX^{(j)} = 0, \quad KX^{(j)} = \omega_j^2 X^{(j)}, \quad j = 1, \ldots, r \tag{11}$$

Now, consider an orthogonal matrix Q having the vectors $X^{(1)}, ..., X^{(r)}$ as its first r columns,

$$Q = (X^{(1)}, ..., X^{(n)}) \tag{12}$$

The matrices D and K are then orthogonally congruent to matrices $\hat{D}$ and $\hat{K}$, respectively, described by

$$\hat{D} = Q^T DQ = (< X^{(i)}, DX^{(j)} >) \tag{13}$$

and

$$\hat{K} = Q^T KQ = (< X^{(i)}, KX^{(j)} >), \tag{14}$$

where $i, j = 1, \ldots, n$. Using (11) and $< X^{(i)}, X^{(j)} >= \delta_{ij}$, where δ_{ij} is the Kronecker delta and $i, j = 1, \ldots, n$, we compute

$$< X^{(i)}, DX^{(j)} >= 0 \tag{15}$$

and

$$< X^{(i)}, KX^{(j)} >= \omega_j^2 \delta_{ij}, \tag{16}$$

where $i = 1, \ldots, nandj = 1, \ldots, r$. The relations (15) and (16) show that $\hat{D}$ and $\hat{K}$ have the partitioned forms (9) and (10). $\Box$

Proof of Theorem 1. Suppose that $\Delta(\pm i\omega_j) = 0, \omega_j \in \Re, j = 1, \ldots, r$ and that remaining zeros of $\Delta(\lambda)$ take places on the open left-half of the complex plane. Then from Lemma 3 it follows that there exists an orthogonal coordinate transformation

$$x = Q \begin{pmatrix} y \\ z \end{pmatrix}, \quad y \in \Re^r, \quad z \in \Re^{n-r}, \tag{17}$$

which transforms equation (2) to the form

$$\begin{pmatrix} \ddot{y} \\ \ddot{z} \end{pmatrix} + \hat{D} \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} + \hat{K} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

where $\hat{D}$ and $\hat{K}$ have the partitioned forms (9) and (10). Under the above assumptions it is clear that the $(n-r)$ dimensional subsystem of (18)

$$\ddot{z} + \hat{D}_{n-r}\dot{z} + \hat{K}_{n-r}z = 0, \quad z \in \Re^{n-r} \tag{19}$$

is asymptotically stable and, according to well-known result [5], we have

$$rank \begin{pmatrix} \hat{D}_{n-r} & \vdots & \hat{K}_{n-r}\hat{D}_{n-r} & \vdots & \vdots & \vdots & \hat{K}_{n-r}^{n-r-1}\hat{D}_{n-r} \end{pmatrix} = n - r \tag{20}$$

On the other hand, the matrix Φ coincides with the matrix

$$Q\left(\ \hat{D} \vdots\ \ \hat{K}\hat{D}\ \ \vdots\ \vdots\ \vdots\ \ \hat{K}^{n-1}\hat{D}\ \right)P, \tag{21}$$

where $P = diag(Q^T, ..., Q^T)$. Then

$$rank\Phi = rank\left(\ \hat{D}_{n-r} \vdots\ \ \hat{K}_{n-r}\hat{D}_{n-r}\ \ \vdots\ \vdots\ \vdots\ \ \hat{K}_{n-r}^{n-1}\hat{D}_{n-r}\ \right), \tag{22}$$

since Q and P are nonsingular, and $\hat{D} = diag(0_r, \hat{D}_{n-r})$, and $\hat{K}^j\hat{D} = diag(0_r, \quad \hat{K}_{n-r}^j\hat{D}_{n-r})$. Now, according to the Cayley-Hamilton theorem (see [7]), every matrix $\hat{K}_{n-r}^j\hat{D}_{n-r}$ with integer $j \geq n-r$ can be represented by a linear combination of the matrices $\hat{D}_{n-r}, \hat{K}_{n-r}\hat{D}_{n-r}, \ldots, \hat{K}_{n-r}^{n-r-1}\hat{D}_{n-r}$, and, consequently

$$\begin{aligned} &rank\left(\ \hat{D}_{n-r}\ \ \vdots\ \vdots\ \vdots\ \ \hat{K}_{n-r}^{n-1}\hat{D}_{n-r}\ \right) = \\ &rank\left(\ \hat{D}_{n-r}\ \ \vdots\ \vdots\ \vdots\ \ \hat{K}_{n-r}^{n-r-1}\hat{D}_{n-r}\ \right). \end{aligned} \tag{23}$$

The result then follows from (20), (22) and (23). □

Remark 1. The proof of theorem 1 is based on a transformation converting the system (2) into two uncoupled subsystems; one of them is r-dimensional undamped subsystem, where r is the number of conjugate pairs of purely imaginary eigenvalues of the system *including multiplicity*, the second is (n – r)-dimensional damped asymptotically stable subsystem. When the matrix K has *distinct* eigenvalues, and r its eigenvectors lie in the nullspace of the damping matrix, the decomposability of the system in *modal* coordinates was observed in [3].

Remark 2. The matrix (6) can be expressed in terms of the original matrices as

$$\Phi = A^{-1/2}\tilde{\Phi}\ diag(A^{-1/2}, ..., A^{-1/2}), \tag{24}$$

where

$$\tilde{\Phi} = \left(\ B \vdots\ \ (CA^{-1})B\ \ \vdots\ \vdots\ \vdots\ \ (CA^{-1})^{n-1}B\ \right). \tag{25}$$

Consequently, $rank\Phi = rank\tilde{\Phi}$, since A is nonsingular.

In the case of “classical damping” in which D and K commute the following result as a consequence of Theorem 1 can be obtained.

Theorem 2. If $DK = KD$, then the system has $r = n - rankD$ conjugate pairs of purely imaginary eigenvalues.

Proof. Since D and K commute there exists an orthogonal matrix such that both D and K are orthogonally congruent to diagonal matrices [4]. Then, evidently, $rank\Phi = \quad rankD$, and Theorem 2 follows from Theorem 1. □

3 Illustrative examples

Example 1. Consider the two-degree-of-freedom system shown in Fig.1, where $c_i > 0$ and $\beta > 0$ stand for the spring constants and coefficient of viscous damping, respectively, and q_1 and q_2 are the displacements from equilibrium positions of masses m_1 and m_2.

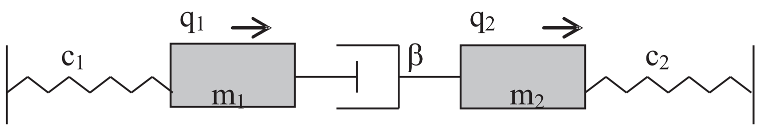

Figure 1: The system of example

The inertia, damping and stiffness matrices of this system are as follows

$$A = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad B = \beta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \tag{26}$$

It is clear that $rankB = 1$, and consequently, the system is partially damped. The matrix (25) takes the form

$$\tilde{\Phi} = \begin{pmatrix} B & CA^{-1}B \end{pmatrix} \quad = \quad \beta \begin{pmatrix} 1 & -1 & \frac{c_1}{m_1} & -\frac{c_1}{m_1} \\ -1 & 1 & -\frac{c_2}{m_2} & \frac{c_2}{m_2} \end{pmatrix} \tag{27}$$

Thus, by Theorem 1, we have

$$r = 2 - rank\tilde{\Phi} = \begin{cases} 0, & c_1m_2 \neq c_2m_1, \\ 1, & c_1m_2 = c_2m_1. \end{cases} \tag{28}$$

In the case $c_1m_2 = c_2m_1$, the system can oscillate such that relative motion between the masses is absent, so that the damper dissipates no energy. If $c_1 m_2 \neq c_2 m_1$, the system does not have pure imaginary eigenvalues, and all motions lead up to dissipation of energy.

Example 2. Consider the three-degree-of-freedom system (2) with

$$D = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{29}$$

previously studied in [5].

It can be easily verified that $rankD = 1$, and that $DK = KD$. Thus, by Theorem 2, system of this example has two conjugate pairs of purely imaginary eigenvalues.

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O prigušeno oscilujućim sistemima

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Razmatra se prisustvo čisto imaginarnih sopstvenih vrednosti delimično prigušeno oscilujućih sistema. Broj takvih sopstvenih vrednosti se određuje pomoću ranga matrice koja je direktno vezana za matrice sistema.