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HIGHER ORDER SCHWARZIAN DERIVATIVES FOR CONVEX UNIVALENT FUNCTIOS

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We observe that in contrast to the class S, the extremal functions for the bound of higher order Schwarzian derivatives for the class S^c of convex univalent functions are different. We prove the sharp bound for three first consecutive derivatives.

Let S denote the class of holomorphic and univalent functions in the unit disk $\mathbb{D} = z : |z| < 1$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ..., \quad z \in \mathbb{D},$$

and $S^c \subset S$ the class consisting of convex functions.

Let

$$S(f)(z) = \Big(\frac{f^{''}(z)}{f'(z)}\Big)^{'} - \frac{1}{2}\Big(\frac{f^{''}(z)}{f'(z)}\Big)^2, \quad z \in \mathbb{D}$$

denote the Schwarzian derivative for f. Let us denote $\sigma_3(f) = S(f)$ and let the higher order Schwarzian derivative be defined inductively (see [5]) as:

$$\sigma_{n+1}(f) = (\sigma_n(f))^{'} - (n-1)\sigma_n(f) \cdot \frac{f^{''}}{f'}, \quad n \geq 4 \tag{1}$$

In [5] it was proved that the upper bound for $|\sigma_n(f)|, f \in S$ is attained for the Koebe function for each $n = 3, 4, ...$

In this note we show that situation is different when we deal with the class of convex univalent functions. Because of linear invariance of the class S^c one can restrict the considerations to $\sigma_n(f)(0) := S_n$. We have the following

THEOREM 1. *If $f \in S^c$, then the following sharp estimates hold:*

$$|S_3| = |6(a_3 - a_2^2)| \le 2,$$
$$|S_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 4,$$
$$|S_5| = 24|5a_5 - 20a_4a_2 - 9a_3^2 + 48a_3a_2^2 - 24a_2^4| \le 12.$$

The extremal functions (up to rotations) have the form

$$f_n(z) = \int_0^z (1 - t^{n-1})^{-\frac{2}{n-1}}\,dt, \quad n = 3, 4, 5, \tag{2}$$

respectively.

PROOF. From (1) one can easily find

$$\begin{aligned} \sigma_4(f) &= \frac{f''''}{f'} - 6\frac{f'''f''}{f'^2} + 6\Big(\frac{f''}{f'}\Big)^3, \\ \sigma_5(f) &= \frac{f'''''}{f'} - 10\frac{f''''f''}{f'^2} - 6\Big(\frac{f'''}{f'}\Big)^2 + 48\frac{f'''f''^2}{f'^3} - 36\Big(\frac{f''}{f'}\Big)^4. \end{aligned} \tag{3}$$

Note that in [5] there are two misprints in the last formula.

Therefore we have from (3):

$$\begin{aligned} S_3 &= 6(a_3 - a_2^2), \\ S_4 &= 24(a_4 - 3a_2a_3 + 2a_2^3), \\ S_5 &= 24(5a_5 - 20a_4a_2 - 9a_3^2 + 48a_3a_2^2 - 24a_2^2). \end{aligned} \tag{4}$$

We are going to use the connection of the class S^c and functions with positive real part in $\mathbb{D}$, as well as the functions satisfying the Schwarz lemma conditions.

Namely we have

$$f \in S^c \Leftrightarrow 1 + \frac{zf''(z)}{f'z} = p(z) = \frac{1+\omega(z)}{1-\omega(z)}, \quad z \in \mathbb{D}, \tag{5}$$

where $p(z) = 1 + p_1z + p_2z^2 + \dots$, $Re\{p(z)\} > 0$, $z \in \mathbb{D}$ (i.e., $p \in P$, the class of functions with positive real part) and $\omega(z) = c_1z + c_2z^2 + \dots$, $|\omega(z)| < 1$, $z \in \mathbb{D}$ (i.e., $\omega \in \Omega$, the class of Schwarz functions).

From (5) we find

$$S_3 = 2c_2$$

and because $|c_2| \leq 1 - |c_1|^2$,

$$|S_3| \leq 2$$

which is as well the well-known result of Hummel [1]. The extremal function is

$$f_3(z) = \int_0^z \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+z}{1-z}.$$

The functional S_4 has a special form of the functional $|a_4 + sa_2a_3 + ua_2^3|, u, s \in \mathbb{R}$ which was estimated sharply for each $s, u \in \mathbb{R}$ in [4] and therefore the result follows by taking $s = -3, u = 2$ in Theorem 1 in [4].

The extremal function is determined by taking $\omega(z) = z^3$ in (5) which gives (2). Finally in order to get the bound for $|S_5|$ which is complicated we transform it to the class Ω of Schwarz functions $\omega(z)$.

By equating the coefficients in (5) one can find the relations:

$$a_2 = c_1,$$
$$a_3 = \frac{1}{3}(c_2 + 3c_1^2),$$
$$a_4 = \frac{1}{6}(c_3 + 5c_1c_2 + 6c + 1^3),$$
$$a_5 = \frac{1}{10}(c_4 + \frac{14}{3}c_3c_1 + \frac{43}{3}c_2c_1^2 + 2c_2^2 + 10c_1^4),$$

witch transform S_5 as given by (4) to a nicer form

$$S_5 = 12(c_4 - 2c_3c_1 + c_2c_1^2). \tag{6}$$

Now we can try to estimate (6) by the use of the Carathéodory inequalities applied to the class Ω as it was done in [4]. However, this leads to very complicated calculations. But one can observe that within the class Ω the functional $|c_4 - 2c_3c_1 + c_2c_1^2|$ and $|c_4 + 2c_3c_1 + c_2c_1^2|$ have the same upper bound, because if $\omega(z) \in \Omega$, then $\omega_1(z) = -\omega(-z) \in \Omega$.

On the other hand, comparing the coefficients p_k and c_k in (5) one gets

$$p_1 = 2c_1,$$
$$p_2 = 2(c_2 + c_1^2),$$
$$p_3 = 2(c_3 + 2c_1c_2 + c_1^3),$$
$$p_4 = 2(c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4),$$

from which we obtain that

$$2(c_4 + 2c_3c_1 + c_2c_1^2) = p_4 - \frac{1}{2}p_2^2.$$

Leutwiller and Schober [3] gave the precise bound for $|p_4 - \frac{1}{2}p_2^2| \leq 2$, which implies that $|c_4 + 2c_3c_1 + c_1^3| = |c_4 - 2c_3c_1 + c_1^3| \leq 1$. This completes the proof. The extremal function is obtained by taking $\omega(z) = z^4$ in (5).

Note that writing S_5 with the coefficients of p_k leads to another "bad" expression

REMARK 1. *We conjecture that for every* $n = 6, ...$ *the maximal value of* $|S_n|$ *is attained by the function given by* (2).

REMARK 2. *The application of the general approach to the bound* S_4 *and* S_5 *would lead within the class* P *to consideration of functions of the form*

$$p(z) = \sum_{k=1}^{n} \lambda_k \frac{1 + ze^{-i\theta_k}}{1 - ze^{-i\theta_k}}, n \leq 4$$

or 5, which is very difficult to handle because it involves long and tedious calculations.

REMARK 3. *One can observe that the bound for* $|\sigma_n(f)|$ *given in [5] follows directly from the formula* (1) *in [5] and the result of R.Klouth and K.-J.Wirths[2].*

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