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**ON ELLIPTIC PROBLEMS  
WITH A NONLINEARITY  
DEPENDING ON THE GRADIENT**

**Abstract.** We investigate the solvability of the Neumann problem (1.1) involving the nonlinearity depending on the gradient. We prove the existence of a solution when the right hand side  $f$  of the equation belongs to  $L^m(\Omega)$  with  $1 \leq m < 2$ .

**Keywords:** Neumann problem, nonlinearity depending on the gradient,  $L^1$  data.

**Mathematics Subject Classification:** 35D05, 35J25, 35J60.

## 1. INTRODUCTION

In this paper we investigate the solvability of the nonlinear Neumann problem with a nonlinearity depending on the gradient. First we consider the following problem

$$\begin{cases} -\Delta u + |\nabla u|^q + \lambda u = f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\lambda > 0$  is a parameter,  $1 \leq q \leq 2$  and  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 3$ , is a bounded domain with a smooth boundary  $\partial\Omega$ . It is assumed that  $f \in L^1(\Omega)$ . If  $f > 0$  on  $\Omega$ , then solutions, if they exist, are positive. In Section 3 we consider problem (1.1) with  $|\nabla u|^q$  replaced by a nonlinearity satisfying a sign condition. The boundary value problems with data in  $L^1$  has been studied quite extensively in recent years. The Dirichlet problem with a nonlinearity depending only on  $u$  has been considered in papers [7, 10]. Some extensions to the Neumann problem can be found in paper [12]. These results has been extended to the case where a nonlinearity depends on the gradient. In particular, more general elliptic operators with more general nonlinearities with  $f \in L^1(\Omega)$  or being a Radon measure have been investigated in [3–6, 11]. Further extensions to the Dirichlet problem with  $L^2$  boundary data can be found in [11]. We refer to paper [2] for the bibliographical references. It seems that less is known for the Neumann problem.

By  $W^{1,p}(\Omega)$ ,  $1 \leq p < \infty$ , we denote the Sobolev space equipped with norm

$$\|u\|_{W^{1,p}}^p = \int_{\Omega} (|\nabla u|^p + |u|^p) dx.$$

Throughout this paper, in a given Banach space  $X$ , we denote strong convergence by “ $\rightarrow$ ” and weak convergence by “ $\rightharpoonup$ ”. The norms in the Lebesgue spaces  $L^p(\Omega)$ ,  $1 \leq p < \infty$ , are denoted by  $\|\cdot\|_{L^p}$ .

The paper is organized as follows. In Section 2 we prove the existence of positive solutions of (1.1) assuming that  $f$  is positive and belongs to  $L^1(\Omega)$ . Section 3 is devoted to the problem with a nonlinearity satisfying a sign condition, where we do not assume that  $f$  is positive. The crucial point in our approach are estimates of  $W^{1,q}$ -norm of solutions of (1.1) in terms of  $L^m$ -norm of  $f$  (see Lemmas 2.1, 3.1, 3.3). The estimates in terms of  $L^m$ -norm of  $f$  (see Lemmas 3.1, 3.3) in a linear case were given in [8] and are extended in this paper to solutions of (1.1). In these two lemmas the important assumption is that  $q \neq \frac{N}{N-1}$ , which is due to the use of special test functions in the proofs. We were unable to show whether these lemmas continue to hold for  $q = \frac{N}{N-1}$ . In Section 4 we establish the higher integrability property for positive solutions of (1.1).

The main results of this paper are Theorems 2.2, 3.2, 3.4. In the proofs we use some ideas from paper [4].

## 2. EXISTENCE OF POSITIVE SOLUTIONS

In this section consider problem (1.1) assuming that  $f > 0$  on  $\Omega$ . Then a solution, if it exists, is positive on  $\Omega$ . We need the following definition of a solution of (1.1): let  $f \in L^1(\Omega)$ , then a function  $u \in W^{1,q}(\Omega)$  is a solution of (1.1) if

$$\int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} |\nabla u|^q v dx + \lambda \int_{\Omega} uv dx = \int_{\Omega} f v dx \quad (2.1)$$

for every function  $v \in W^{1,\infty}(\Omega)$ .

**Lemma 2.1.** *Let  $1 \leq q \leq 2$  and  $f \in L^\infty(\Omega)$  with  $f > 0$  on  $\Omega$ . If  $u \in W^{1,2}(\Omega)$  is a positive solution of (1.1), then*

$$\int_{\Omega} (|\nabla u|^q + u^q) dx \leq C_1 \int_{\Omega} f dx + C_2 \left( \int_{\Omega} f dx \right)^q, \quad (2.2)$$

where  $C_1, C_2 > 0$  are constants independent of  $u$  and  $f$ .

*Proof.* Testing (2.1) with the constant function 1 we get

$$\int_{\Omega} |\nabla u|^q dx + \lambda \int_{\Omega} u dx = \int_{\Omega} f dx. \quad (2.3)$$

It is clear that equality (2.3) yields (2.2) if  $q = 1$ . To proceed further we use a decomposition  $W^{1,2}(\Omega) = V \oplus \text{span } 1$ , where

$$V = \{v \in W^{1,2}(\Omega); \int_{\Omega} v \, dx = 0\}.$$

Then  $u = v + t$ , with  $v \in V$  and  $t = \frac{1}{|\Omega|} \int_{\Omega} u \, dx > 0$ , because  $u$  is positive. From (2.3) we deduce

$$t \leq \frac{1}{\lambda|\Omega|} \int_{\Omega} f \, dx. \tag{2.4}$$

We now observe that the Poincaré inequality is valid in  $V$ , that is, there exists a constant  $C(\Omega) > 0$  such that

$$\int_{\Omega} |v|^q \, dx \leq C(\Omega) \int_{\Omega} |\nabla v|^q \, dx$$

for every  $v \in V$ . Consequently, using (2.4), we can estimate the norm of  $u$  in  $W^{1,q}(\Omega)$  as follows

$$\begin{aligned} \int_{\Omega} (|\nabla u|^q + u^q) \, dx &\leq \int_{\Omega} |\nabla v|^q \, dx + 2^{q-1} \int_{\Omega} (v^q + t^q) \, dx \leq \\ &\leq \int_{\Omega} |\nabla v|^q \, dx + 2^{q-1} C(\Omega) \int_{\Omega} |\nabla v|^q \, dx + 2^{q-1} |\Omega| t^q. \end{aligned}$$

This combined with (2.4) and (2.3) implies (2.2). □

We are now in a position to formulate the first existence result.

**Theorem 2.2.** *Let  $1 \leq q \leq 2$  and  $f$  be a positive function in  $L^1(\Omega)$ . Then problem (1.1) admits a positive solution in  $W^{1,q}(\Omega)$ .*

*Proof.* The proof will be given in 2 steps.

**Step 1.** Assume  $f \in L^\infty(\Omega)$ . Consider the problem

$$\begin{cases} -\Delta u + \lambda u = f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{on } \Omega. \end{cases} \tag{2.5}$$

This problem has a unique positive solution  $v \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  (see [1]). We now use some ideas from papers [5] and [6]. For each  $n \in \mathbb{N}$  we consider the following problem

$$\begin{cases} -\Delta w_n + \frac{|\nabla w_n|^q}{1 + \frac{1}{n} |\nabla w_n|^q} + \lambda w_n = f(x) & \text{in } \Omega, \\ \frac{\partial w_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ w_n > 0 & \text{on } \Omega. \end{cases} \tag{2.6}$$

It is clear that  $v$  is a super-solution to problem (2.6) and 0 is a sub-solution. Thus problem (2.6) admits a solution  $0 \leq w_n \leq v$ . This fact is known for equation (2.6) with the Dirichlet boundary conditions (see [5]). The result from [5] can be easily extended to the Neumann problem (2.6). The sequence  $\{w_n\}$  is uniformly bounded in  $L^\infty(\Omega)$ . Testing (2.6) with  $w_n$  we obtain

$$\int_{\Omega} (|\nabla w_n|^2 + \lambda w_n^2) dx \leq \|f\|_{L^2} \|w_n\|_{L^2},$$

which shows that the sequence  $\{w_n\}$  is bounded in  $W^{1,2}(\Omega)$ . We may assume that  $w_n \rightharpoonup w$  in  $W^{1,2}(\Omega)$ ,  $w_n \rightarrow w$  in  $L^2(\Omega)$  and  $w_n \rightarrow w$  a.e. on  $\Omega$ . We now show that  $w_n \rightarrow w$  in  $W^{1,2}(\Omega)$ . We put  $\phi(s) = s \exp(\frac{s^2}{4})$  for  $s \in \mathbb{R}$ . We introduce notation  $H_n(s) = \frac{|s|^q}{1 + \frac{1}{n}|s|^q}$ . The function  $\phi$  satisfies  $\phi'(s) - |\phi(s)| \geq \frac{1}{2}$  for  $s \in \mathbb{R}$ . Testing (2.6) with  $\phi(w_n - w)$  we obtain

$$\begin{aligned} \int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) dx + \int_{\Omega} H_n(|\nabla w_n|) \phi(w_n - w) dx + \\ + \lambda \int_{\Omega} w_n \phi(w_n - w) dx = \int_{\Omega} f(x) \phi(w_n - w) dx. \end{aligned} \quad (2.7)$$

It is easy to check that

$$\int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) dx = \int_{\Omega} |\nabla(w_n - w)|^2 \phi'(w_n - w) dx + o(1). \quad (2.8)$$

To estimate the second term on the left side of (2.7) we use the inequality: if  $1 \leq q < 2$ , then for every  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that

$$s^q \leq \epsilon s^2 + C_\epsilon \quad \text{for every } s \geq 0. \quad (2.9)$$

We then have

$$\begin{aligned} \int_{\Omega} H_n(|\nabla w_n|) |\phi(w_n - w)| dx &\leq \epsilon \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| dx + C_\epsilon \int_{\Omega} |\phi(w_n - w)| dx = \\ &= \epsilon \int_{\Omega} |\nabla(w_n - w)|^2 |\phi(w_n - w)| dx - \\ &\quad - \epsilon \int_{\Omega} |\nabla w|^2 |\phi(w_n - w)| dx + \\ &\quad + 2\epsilon \int_{\Omega} \nabla w_n \nabla w |\phi(w_n - w)| dx + \\ &\quad + C_\epsilon \int_{\Omega} |\phi(w_n - w)| dx. \end{aligned} \quad (2.10)$$

Since

$$\int_{\Omega} |\nabla w|^2 |\phi(w_n - w)| dx \rightarrow 0, \quad \int_{\Omega} \nabla w_n \nabla w |\phi(w_n - w)| dx \rightarrow 0$$

and

$$\int_{\Omega} |\phi(w_n - w)| dx \rightarrow 0$$

as  $n \rightarrow \infty$ , we derive from (2.10) that

$$\int_{\Omega} H_n(|\nabla w_n|) |\phi(w_n - w)| dx \leq \epsilon \int_{\Omega} |\nabla w_n - \nabla w|^2 |\phi(w_n - w)| dx + o(1). \quad (2.11)$$

If  $q = 2$ , then instead of (2.10) we have

$$\int_{\Omega} H_n(|\nabla w_n|) |\phi(w_n - w)| dx \leq \int_{\Omega} |\nabla w_n|^2 \phi(w_n - w) dx$$

and (2.11) holds with  $\epsilon = 1$ . We also have

$$\int_{\Omega} f(x) \phi(w_n - w) dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} w_n \phi(w_n - w) dx \rightarrow 0 \quad (2.12)$$

as  $n \rightarrow \infty$ . If  $1 \leq q < 2$  we derive from (2.7), (2.8), (2.11) and (2.12) that

$$\frac{1}{2} \int_{\Omega} |\nabla(w_n - w)|^2 dx \leq \int_{\Omega} (\phi'(w_n - w) - \epsilon |\phi(w_n - w)|) |\nabla(w_n - w)|^2 dx = o(1).$$

Thus  $w_n \rightarrow w$  in  $W^{1,2}(\Omega)$ . If  $q = 2$ , the above inequality continues to hold with  $\epsilon = 1$ . In this case we also have that  $w_n \rightarrow w$  in  $W^{1,2}(\Omega)$ . Since  $1 \leq q \leq 2$ ,  $\nabla w_n \rightarrow \nabla w$  in  $L^q(\Omega)$ . For each  $\phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  and for each  $n$  we have

$$\int_{\Omega} \nabla w_n \nabla \phi dx + \int_{\Omega} \frac{|\nabla w_n|^q}{1 + \frac{1}{n} |\nabla w_n|^q} \phi dx + \lambda \int_{\Omega} w_n \phi dx = \int_{\Omega} f \phi dx.$$

Letting  $n \rightarrow \infty$  we get

$$\int_{\Omega} \nabla w \nabla \phi dx + \int_{\Omega} |\nabla w|^q \phi dx + \lambda \int_{\Omega} w \phi dx = \int_{\Omega} f \phi dx.$$

So  $w \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a weak solution of (1.1).

**Step 2.** First we consider the case  $1 \leq q < 2$ . Let  $f \in L^1(\Omega)$  and let  $\{f_n\} \subset L^\infty(\Omega)$  such that  $f_n \rightarrow f$  in  $L^1(\Omega)$ . By Step 1 for each  $n \in \mathbb{N}$  there exists a solution  $u_n \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  to problem (1.1) with  $f = f_n$ . For each  $k > 1$  we put  $T_k(s) = \min(s, k)$  for  $0 \leq s$ . Taking  $T_k u_n$  as a test function in (1.1) we get

$$\int_{\Omega} |\nabla T_k u_n|^2 dx + \lambda \int_{\Omega} |T_k u_n|^2 dx \leq \int_{\Omega} f_n T_k u_n dx \leq k \|f_n\|_{L^1}.$$

Consequently,  $\{T_k u_n\}$  is bounded in  $W^{1,2}(\Omega)$ . By Lemma 2.1 we may assume that  $u_n \rightharpoonup u$  in  $W^{1,q}(\Omega)$ . We may also assume that  $T_k u_n \rightharpoonup T_k u$  in  $W^{1,2}(\Omega)$  and  $T_k u_n \rightarrow T_k u$  in  $L^2(\Omega)$ . Let  $G_k(s) = s - T_k(s)$  and put  $\psi_{k-1}(s) = T_1(G_{k-1}(s))$ . Thus

$$\psi_{k-1}(u_n) |\nabla u_n|^q \geq |\nabla u_n|^q \chi_{(u_n > k)}.$$

Using  $\psi_{k-1}(u_n)$  as a test function in (2.1) (with  $f = f_n$ ) we get

$$\int_{\Omega} |\nabla \psi_{k-1}(u_n)|^2 dx + \int_{\Omega} \psi_{k-1}(u_n) |\nabla u_n|^q dx + \lambda \int_{\Omega} u_n \psi_{k-1}(u_n) dx = \int_{\Omega} f_n \psi_{k-1}(u_n) dx.$$

Since  $\{u_n\}$  is bounded in  $L^p(\Omega)$  for each  $p \leq q^* = \frac{Nq}{N-q}$  we see that

$$|\{x \in \Omega; k-1 < u_n(x) < k\}| \rightarrow 0 \text{ and } |\{x \in \Omega; k < u_n(x)\}| \rightarrow 0$$

as  $k \rightarrow \infty$  uniformly in  $n$ . So

$$\lim_{\substack{k \rightarrow \infty \\ u_n > k}} \int |\nabla u_n|^q dx = 0 \tag{2.13}$$

uniformly in  $n$ . Using as a test function  $\phi(T_k u_n - T_k u)$  and repeating the argument from Step 1 we show that  $T_k u_n \rightarrow T_k u$  in  $W^{1,2}(\Omega)$ . We now use this to show that the sequence  $\{|\nabla u_n|^q\}$  is equi-integrable. This follows from (2.13) and the following inequality: for every measurable subset  $E \subset \Omega$  we have

$$\int_E |\nabla u_n|^q dx \leq \int_E |\nabla T_k u_n|^q dx + \int_{(u_n \geq k) \cap E} |\nabla u_n|^q dx.$$

Indeed, given  $\epsilon > 0$ , according to (2.13), we can find  $k$  large enough such that

$$\int_{u_n \geq k} |\nabla u_n|^q dx < \frac{\epsilon}{2}$$

for all  $n$ . Since  $\nabla T_k(u_n) \rightarrow T_k(u)$  in  $L^2(\Omega)$  there exists  $\delta > 0$  such that

$$\int_E |\nabla T_k(u_n)|^q dx < \frac{\epsilon}{2}$$

provided  $|E| \leq \delta$  and for all  $n$ . By Vitali's theorem  $\nabla u_n \rightarrow \nabla u$  in  $L^q(\Omega)$ . Thus  $u$  is a weak solution of (1.1). If  $q = 2$ , then by Lemma 2.1 the sequence  $\{u_n\}$  is bounded in  $W^{1,2}(\Omega)$ . An obvious modification of Step 2 completes the proof.  $\square$

3. NONLINEARITY WITH A SIGN CONDITION

In this section we discuss the solvability of the following problem

$$\begin{cases} -\Delta u + g(x, u, \nabla u) + \lambda u = f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We assume that the nonlinearity  $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function, that is,  $g(\cdot, s, \xi)$  is measurable on  $\Omega$  for every  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$  and  $g(x, \cdot, \cdot)$  is continuous on  $\mathbb{R} \times \mathbb{R}^N$  for a.e.  $x \in \Omega$ . Moreover, we assume that

- ( $g_1$ ) there exist an increasing and continuous function  $b : [0, \infty) \rightarrow [0, \infty)$  with  $b(0) = 0$  and a positive function  $a \in L^1(\Omega)$  such that

$$|g(x, s, \xi)| \leq b(|s|)(|\xi|^q + a(x))$$

for a.e.  $x \in \Omega$  and for every  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ .

- ( $g_2$ )  $g(x, s, \xi)$  sgn  $s \geq 0$  for a.e.  $x \in \Omega$  and for every  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ .

A typical example of a nonlinearity satisfying ( $g_1$ ) and ( $g_2$ ) is  $g(x, s, \xi) = s|\xi|^q$ .

We now consider equation (3.1) without assumption that  $f$  is positive on  $\Omega$ . Obviously, it is assumed that  $f \not\equiv 0$  on  $\Omega$ . We assume that  $\frac{N}{N-1} < q < 2$ . Then there exists  $1 < m < \frac{2N}{N+q}$  such that  $q = m^* = \frac{Nm}{N-m}$ . In this case  $m$  is given by  $m = \frac{Nq}{N+q}$ . We also use notation  $q^* = \frac{Nq}{N-q}$ . With these notations we establish the estimates of norms  $\|u\|_{L^{q^*}}$  and  $\|u\|_{W^{1,q}}$  of a solution  $u$  of (1.1) in terms of the norm  $\|f\|_{L^m}$ .

**Lemma 3.1.** *Let  $f \in L^\infty(\Omega)$  and  $\frac{N}{N-1} < q < 2$ . If  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a solution of (3.1), then*

$$\begin{aligned} \int_{\Omega} |u|^{q^*} dx &\leq C_1 \left( \int_{\Omega} (|\nabla u|^q + |u|^q) dx \right)^{\frac{q^*}{q}} \leq \\ &\leq C_2 \|f\|_{L^m}^{\frac{q^*}{2}} \left( \int_{\Omega} |u|^{q^*} dx \right)^{\frac{(1-r)}{2}} \left( \int_{\Omega} (1 + u^2)^{\frac{q^*}{2}} dx \right)^{\frac{r}{2}}, \end{aligned} \tag{3.2}$$

where  $r = \frac{N(2-q)}{N-q}$  and  $C_1 > 0$  and  $C_2 > 0$  are constants independent of  $u$  and  $f$ .

*Proof.* We follow some ideas from [8], where the same estimate was proved for the linear problem. Put  $\varphi(x) = \frac{u}{(1+u^2)^{\frac{r}{2}}}$ . Since  $\frac{N}{N-1} < q < 2$ , we have  $0 < r < 1$ . Since  $u \in L^\infty(\Omega)$ ,  $\varphi$  is a legitimate test function. Upon the substitution we obtain

$$\begin{aligned} (1-r) \int_{\Omega} \frac{|\nabla u|^2}{(1+u^2)^{\frac{r}{2}}} dx + \lambda \int_{\Omega} \frac{u^2}{(1+u^2)^{\frac{r}{2}}} dx &\leq \int_{\Omega} \frac{|fu|}{(1+u^2)^{\frac{r}{2}}} dx \leq \\ &\leq \|f\|_{L^m} \left( \int_{\Omega} |u|^{(1-r)m'} dx \right)^{\frac{1}{m'}}, \end{aligned} \quad (3.3)$$

where  $m' = \frac{m}{m-1}$ . Here we used the fact that

$$\int_{\Omega} \frac{ug(x, u, \nabla u)}{(1+u^2)^{\frac{r}{2}}} dx \geq 0$$

due to assumption  $(g_2)$ . In what follows we denote by  $C > 0$  a constant which is independent of  $u$  and  $f$  and may vary from line to line. By the Sobolev inequality we have

$$\begin{aligned} \left( \int_{\Omega} |u|^{q^*} dx \right)^{\frac{q}{q^*}} &\leq C \int_{\Omega} (|\nabla u|^q + |u|^q) dx = \\ &= C \int_{\Omega} \frac{|\nabla u|^q}{(1+u^2)^{\frac{rq}{4}}} (1+u^2)^{\frac{rq}{4}} dx + \\ &\quad + C \int_{\Omega} \frac{|u|^q}{(1+u^2)^{\frac{rq}{4}}} (1+u^2)^{\frac{rq}{4}} dx \leq \\ &\leq C \left( \int_{\Omega} \frac{|\nabla u|^2}{(1+u^2)^{\frac{r}{2}}} dx \right)^{\frac{q}{2}} \left( \int_{\Omega} (1+u^2)^{\frac{rq}{2(2-q)}} dx \right)^{\frac{2-q}{2}} + \\ &\quad + C \left( \int_{\Omega} \frac{u^2}{(1+u^2)^{\frac{r}{2}}} dx \right)^{\frac{q}{2}} \left( \int_{\Omega} (1+u^2)^{\frac{rq}{2(2-q)}} dx \right)^{\frac{2-q}{2}}. \end{aligned} \quad (3.4)$$

Inserting (3.3) into (3.4) we derive

$$\begin{aligned} \left( \int_{\Omega} |u|^{q^*} dx \right)^{\frac{q}{q^*}} &\leq C \int_{\Omega} (|\nabla u|^q + |u|^q) dx \leq \\ &\leq C \|f\|_{L^m}^{\frac{q}{2}} \left( \int_{\Omega} |u|^{(1-r)m'} dx \right)^{\frac{q}{2m'}} \left( \int_{\Omega} (1+u^2)^{\frac{rq}{2(2-q)}} dx \right)^{\frac{2-q}{2}}. \end{aligned}$$



Since  $r = \frac{N(2-q)}{N-q}$ , we have  $\frac{rq}{2-q} = q^*$  and  $(1-r)m' = q^*$ . Therefore the above inequality becomes

$$\begin{aligned} \int_{\Omega} |u|^{q^*} dx &\leq C \left( \int_{\Omega} (|\nabla u|^q + |u|^q) dx \right)^{\frac{q^*}{q}} \leq \\ &\leq C \|f\|_{L^m}^{\frac{q^*}{2}} \left( \int_{\Omega} |u|^{q^*} dx \right)^{\frac{q^*}{2m'}} \left( \int_{\Omega} (1+u^2)^{\frac{q^*}{2}} dx \right)^{\frac{(2-q)q^*}{2q}}. \end{aligned}$$

Since  $\frac{q^*}{2m'} = \frac{1-r}{2}$  and  $\frac{(2-q)q^*}{2q} = \frac{r}{2}$ , the result follows. □

We are now in a position to formulate the second existence result.

**Theorem 3.2.** *Let  $\frac{N}{N-1} < q < 2$  and  $f \in L^m(\Omega)$  with  $m = \frac{Nq}{N+q}$ . Suppose that assumptions  $(g_1)$  and  $(g_2)$  hold. Then problem (1.1) admits a solution in  $W^{1,q}(\Omega)$ .*

*Proof.* The proof is similar to that of Theorem 2.2 except some technical modifications. First we assume that  $f \in L^\infty(\Omega)$ . For every  $n \in \mathbb{N}$  we put

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}$$

and consider the following problem

$$\begin{cases} -\Delta u + g_n(x, u, \nabla u) + \lambda u = f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Then the functions  $v_1 = \frac{\|f\|_\infty}{\lambda}$  and  $v_2 = -\frac{\|f\|_\infty}{\lambda}$  are a super-solution and a sub-solution to problem (3.5), respectively. For every  $n$  problem (3.5) has a solution  $w_n$  satisfying  $v_1 \leq w_n \leq v_2$  on  $\Omega$ . Hence the sequence  $\{w_n\}$  is bounded in  $L^\infty(\Omega)$ , that is,  $\|w_n\|_\infty \leq M$  for some constant  $M > 0$  and for all  $n \in \mathbb{N}$ . Testing (3.5) with  $w_n$  we show that  $\{w_n\}$  is bounded in  $W^{1,2}(\Omega)$ . So we may assume that  $w_n \rightharpoonup w$  in  $W^{1,2}(\Omega)$ ,  $w_n \rightarrow w$  in  $L^2(\Omega)$  and  $w_n \rightarrow w$  a.e. on  $\Omega$ . Let  $\phi$  be a function introduced in the proof of Theorem 2.2. Testing (3.5) with  $\phi(w_n - w)$  we obtain

$$\begin{aligned} \int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) dx + \int_{\Omega} g_n(x, w_n, \nabla w_n) \phi(w_n - w) dx + \\ + \lambda \int_{\Omega} w_n \phi(w_n - w) dx = \int_{\Omega} f(x) \phi(w_n - w) dx. \end{aligned} \tag{3.6}$$

It is clear that

$$\int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) dx = \int_{\Omega} |\nabla(w_n - w)|^2 \phi'(w_n - w) dx + o(1). \tag{3.7}$$

We use inequality (2.9) and assumption  $(g_1)$  to estimate the second integral on the left side of (3.6)

$$\begin{aligned} \int_{\Omega} |g_n \phi(w_n - w)| \, dx &\leq b(M) \int_{\Omega} |\nabla w_n|^q |\phi(w_n - w)| \, dx + \int_{\Omega} a(x) |\phi(w_n - w)| \, dx \leq \\ &\leq b(M) \epsilon \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx + C_{\epsilon} \int_{\Omega} |\phi(w_n - w)| \, dx + \\ &\quad + \int_{\Omega} a(x) |\phi(w_n - w)| \, dx. \end{aligned}$$

Since  $\phi(w_n - w) \rightarrow 0$  a.e. on  $\Omega$  and  $\sup_n |\phi(w_n - w)| < \infty$  by the Lebesgue dominated convergence theorem we get

$$\int_{\Omega} |g_n \phi(w_n - w)| \, dx \leq b(M) \epsilon \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx + o(1).$$

As in the proof of Theorem 2.2 we deduce from this that

$$\int_{\Omega} |g_n \phi(w_n - w)| \, dx \leq b(M) \epsilon \int_{\Omega} |\nabla w_n - \nabla w|^2 |\phi(w_n - w)| \, dx + o(1). \quad (3.8)$$

Taking  $\epsilon b(M) \leq 1$  we deduce from (3.6), (3.7) and (3.8) that

$$\int_{\Omega} |\nabla w_n - \nabla w|^2 \, dx \leq \int_{\Omega} (\phi'(w_n - w) - \epsilon b(M) |\phi(w_n - w)|) |\nabla w_n - \nabla w|^2 \, dx = o(1).$$

Thus  $w_n \rightarrow w$  in  $W^{1,2}(\Omega)$ . It is clear that  $w$  is a solution of (3.1). In the final step we choose a sequence  $\{f_n\} \subset L^{\infty}(\Omega)$  such that  $f_n \rightarrow f$  in  $L^m(\Omega)$ . Then for every  $n \in \mathbb{N}$  problem (3.1) with  $f = f_n$  admits a solution  $u_n \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ . We now define a sequence of truncations  $\{T_k(u_n)\}$  for every  $k > 0$ , where  $T_k = \max(-k, \min(s, k))$ . Let  $G_k(s) = s - T_k(s)$  and put  $\psi_{k-1}(s) = T_1(G_{k-1}(s))$ . Thus

$$\psi_{k-1}(u_n) |\nabla u_n|^2 \geq |\nabla u_n|^2 \chi_{|u_n| \geq k}.$$

As in the proof of Theorem 2.2 we show that the sequence  $\{T_k(u_n)\}$  is bounded in  $W^{1,2}(\Omega)$ . Hence we can assume that  $T_k(u_n) \rightharpoonup T_k u$  in  $W^{1,2}(\Omega)$ ,  $T_k(u_n) \rightarrow T_k u$  in  $L^2(\Omega)$  and  $T_k(u_n) \rightarrow T_k(u)$  a.e. on  $\Omega$ . By Lemma 3.1 we may also assume that  $u_n \rightharpoonup u$  in  $W^{1,q}(\Omega)$ . Using as a test function  $\psi_{k-1}(u_n)$  we show that  $\nabla u_n \rightarrow \nabla u$  in  $L^q(\Omega)$  and  $u$  is a weak solution of (3.1).  $\square$

We now turn our attention to positive solutions of (3.1). If  $f > 0$  on  $\Omega$ , then a solution obtained in Theorem 4.3 is positive. In this case we can also consider the interval  $1 \leq q < \frac{N}{N-1}$ . We commence with an apriori estimate.

**Lemma 3.3.** *Suppose that  $1 \leq q < \frac{N}{N-1}$ ,  $f > 0$  on  $\Omega$  and  $f \in L^\infty(\Omega)$ . If  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a positive solution of problem (3.1), then*

$$\begin{aligned} \int_{\Omega} u^{q^*} dx &\leq C_1 \left( \int_{\Omega} (|\nabla u|^q + u^q) dx \right)^{\frac{q^*}{q}} \leq \\ &\leq C_2 \left( \int_{\Omega} (1 + u)^{q^*} dx \right)^{\frac{(2-q)q^*}{2q}} \left( \|f\|_{L^1}^{\frac{q^*}{2}} + \|f\|_{L^1}^{\frac{(2-r)q^*}{2}} \right) \end{aligned}$$

where  $C_1, C_2 > 0$  are constants independent of  $f$  and  $u$  and  $r = \frac{N(2-q)}{N-q}$ .

*Proof.* The proof is a modification of the argument used in the proof of Lemma 2.5 in [8]. We take as a test function  $\phi(x) = (1 + u)^{1-r}$ . Since  $q < \frac{N}{N-1}$ , we have  $r > 1$ . Also  $r < 2$  because  $N \geq 3$ . Hence  $\phi(x) \leq 1$  on  $\Omega$  and upon a substitution we obtain

$$\begin{aligned} (r - 1) \int_{\Omega} \frac{|\nabla u|^2}{(1 + u)^r} dx &= \int_{\Omega} g(x, u, \nabla u)(1 + u)^{1-r} dx + \\ &+ \lambda \int_{\Omega} u(1 + u)^{1-r} dx - \\ &- \int_{\Omega} f(1 + u)^{1-r} dx \leq \\ &\leq \int_{\Omega} g(x, u, \nabla u) dx + \lambda \int_{\Omega} u dx. \end{aligned} \tag{3.9}$$

Testing equation (3.1) with a constant function 1 we obtain

$$\int_{\Omega} g(x, u, \nabla u) dx + \lambda \int_{\Omega} u dx = \int_{\Omega} f dx. \tag{3.10}$$

From (3.9) and (3.10) we derive

$$\int_{\Omega} \frac{|\nabla u|^2}{(1 + u)^r} dx \leq \frac{1}{r - 1} \int_{\Omega} f dx \quad \text{and} \quad \int_{\Omega} u dx \leq \frac{1}{\lambda} \int_{\Omega} f dx. \tag{3.11}$$

By the Sobolev inequality we obtain

$$\begin{aligned}
\left(\int_{\Omega} u^{q^*} dx\right)^{\frac{q}{q^*}} &\leq C \int_{\Omega} (|\nabla u|^q + u^q) dx = \\
&= C \int_{\Omega} \frac{|\nabla u|^q}{(1+u)^{\frac{rq}{2}}} (1+u)^{\frac{rq}{2}} dx + C \int_{\Omega} \frac{u^q}{(1+u)^{\frac{rq}{2}}} (1+u)^{\frac{rq}{2}} dx \leq \\
&\leq C \left(\int_{\Omega} \frac{|\nabla u|^2}{(1+u)^r} dx\right)^{\frac{q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}} + \\
&\quad + C \left(\int_{\Omega} \frac{u^2}{(1+u)^r} dx\right)^{\frac{q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}} \leq \\
&\leq C \left(\int_{\Omega} \frac{|\nabla u|^2}{(1+u)^r} dx\right)^{\frac{q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}} + \\
&\quad + C \left(\int_{\Omega} u^{2-r} dx\right)^{\frac{q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}} \leq \\
&\leq C \left(\int_{\Omega} \frac{|\nabla u|^2}{(1+u)^r} dx\right)^{\frac{q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}} + \\
&\quad + C |\Omega|^{\frac{q(r-1)}{2}} \left(\int_{\Omega} |u| dx\right)^{\frac{(2-r)q}{2}} \left(\int_{\Omega} (1+u)^{\frac{rq}{2-q}} dx\right)^{\frac{2-q}{2}}.
\end{aligned}$$

We now observe that  $q^* = \frac{rq}{2-q}$ . Hence combining the above estimate with (3.11) the result follows.  $\square$

It is clear that Lemma 3.3 leads to the following existence result.

**Theorem 3.4.** *Suppose that  $1 \leq q < \frac{N}{N-1}$ ,  $f > 0$  on  $\Omega$  and  $f \in L^1(\Omega)$ . The problem (3.1) has a positive solution  $u \in W^{1,q}(\Omega)$ .*

#### 4. HIGHER INTEGRABILITY PROPERTY FOR SOLUTIONS OF (1.1)

The method used in the proof of Lemma 2.1 allows only to estimate the norm  $W^{1,q}$  of a positive solution, where  $q$  is the exponent appearing in the equation. In the case  $1 \leq q < 2$ , a question arises whether a solution to (1.1) belongs to  $W^{1,\bar{q}}(\Omega)$  with  $q < \bar{q}$ . We distinguish two cases: (i)  $1 \leq q < \frac{N}{N-1}$  and (ii)  $\frac{N}{N-1} < q < 2$ . In the case (i) assuming that  $f \in L^1(\Omega)$  we show that a solution belongs to  $W^{1,\bar{q}}(\Omega)$  or every  $q < \bar{q} < \frac{N}{N-1}$ . In the case (ii) we show that a solution belongs  $W^{1,\bar{q}}(\Omega)$  for some  $q < \bar{q} < 2$  under some additional assumption on  $f$ . According to Step 1 of the proof of Theorem 2.2, if  $f \in L^\infty(\Omega)$ , then problem (1.1) has a solution  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ .

**Lemma 4.1.** *Suppose that  $f > 0$  on  $\Omega$ ,  $f \in L^\infty(\Omega)$  and  $1 \leq q < \bar{q} < \frac{N}{N-1}$ . If  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a positive solution of (1.1), then there exist constants  $C_1, C_2 > 0$ , independent of  $u$  and  $f$  such that*

$$\begin{aligned} \int_{\Omega} u^{\bar{q}^*} dx &\leq C_1 \left( \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) dx \right)^{\frac{\bar{q}^*}{\bar{q}}} \leq \\ &\leq C_2 \left( \int_{\Omega} (1 + u^{\bar{q}^*}) dx \right)^{\frac{(2-\bar{q})\bar{q}^*}{2\bar{q}}} \left( \|f\|_{L^1}^{\frac{\bar{q}^*}{2}} + \|f\|_{L^1}^{\frac{(2-\bar{r})\bar{q}^*}{2}} \right), \end{aligned}$$

where  $\bar{r} = \frac{N(2-\bar{q})}{N-\bar{q}}$  and  $\bar{q}^* = \frac{N\bar{q}}{N-\bar{q}}$ .

*Proof.* As in the proof of Lemma 3.3 we take as a test function  $\phi(x) = (1 + u)^{1-\bar{r}}$ . Since  $\bar{q} < \frac{N}{N-1}$ , we have  $\bar{r} > 1$ . Also  $\bar{r} < 2$  because  $N \geq 3$ . Hence  $\phi(x) \leq 1$  on  $\Omega$  and upon a substitution we obtain

$$\begin{aligned} (\bar{r} - 1) \int_{\Omega} \frac{|\nabla u|^2}{(1 + u)^{\bar{r}}} dx &= \int_{\Omega} |\nabla u|^q (1 + u)^{1-\bar{r}} dx + \lambda \int_{\Omega} u(1 + u)^{1-\bar{r}} dx - \\ &- \int_{\Omega} f(1 + u)^{1-\bar{r}} dx \leq \int_{\Omega} |\nabla u|^q dx + \lambda \int_{\Omega} u dx. \end{aligned} \tag{4.1}$$

Testing (1.1) with a constant function 1 we obtain

$$\int_{\Omega} |\nabla u|^q + \lambda \int_{\Omega} u dx = \int_{\Omega} f dx. \tag{4.2}$$

By the Sobolev inequality we obtain

$$\begin{aligned} \left( \int_{\Omega} u^{\bar{q}^*} dx \right)^{\frac{\bar{q}}{\bar{q}^*}} &\leq C \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) dx = \\ &= C \int_{\Omega} \frac{|\nabla u|^{\bar{q}}}{(1 + u)^{\frac{\bar{r}\bar{q}}{2}}} (1 + u)^{\frac{\bar{r}\bar{q}}{2}} dx + C \int_{\Omega} \frac{u^{\bar{q}}}{(1 + u)^{\frac{\bar{r}\bar{q}}{2}}} (1 + u)^{\frac{\bar{r}\bar{q}}{2}} dx \leq \\ &\leq C \left( \int_{\Omega} \frac{|\nabla u|^2}{(1 + u)^{\bar{r}}} dx \right)^{\frac{\bar{q}}{2}} \left( \int_{\Omega} (1 + u)^{\frac{\bar{r}\bar{q}}{2-\bar{q}}} dx \right)^{\frac{2-\bar{q}}{2}} + \\ &+ C \left( \int_{\Omega} \frac{u^2}{(1 + u)^{\bar{r}}} dx \right)^{\frac{\bar{q}}{2}} \left( \int_{\Omega} (1 + u)^{\frac{\bar{r}\bar{q}}{2-\bar{q}}} dx \right)^{\frac{2-\bar{q}}{2}}. \end{aligned}$$

Combining the above inequality with (4.1) and (4.2) we obtain

$$\begin{aligned} \left( \int_{\Omega} u^{\bar{q}^*} dx \right)^{\frac{\bar{q}}{\bar{q}^*}} &\leq C \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) dx \leq \\ &\leq C \left( \int_{\Omega} f dx \right)^{\frac{\bar{q}}{2}} \left( \int_{\Omega} (1+u)^{\bar{q}^*} dx \right)^{\frac{2-\bar{q}}{2}} + \\ &\quad + C \left( \int_{\Omega} (1+u)^{\bar{q}^*} dx \right)^{\frac{2-\bar{q}}{2}} \left( \int_{\Omega} u^{2-\bar{r}} dx \right)^{\frac{\bar{q}}{2}} \leq \\ &\leq C \left( \int_{\Omega} (1+u)^{\bar{q}^*} dx \right)^{\frac{2-\bar{q}}{2}} \left[ \|f\|_{L^1}^{\frac{\bar{q}}{2}} + \|f\|_{L^1}^{(2-\bar{r})\frac{\bar{q}}{2}} \right]. \end{aligned}$$

This yields the desired estimate.  $\square$

**Lemma 4.2.** *Let  $f > 0$  on  $\Omega$ ,  $f \in L^\infty(\Omega)$  and  $\frac{N}{N-1} < q < \bar{q} < 2$ . If  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a positive solution of (1.1), then*

$$\begin{aligned} \int_{\Omega} u^{\bar{q}^*} dx &\leq C_1 \left( \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) dx \right)^{\frac{\bar{q}^*}{\bar{q}}} \leq \\ &\leq C_2 \|f\|_{L^{\bar{m}}}^{\frac{\bar{q}^*}{2}} \left( \int_{\Omega} u^{\bar{q}^*} dx \right)^{\frac{1-\bar{r}}{2}} \left( \int_{\Omega} (1+u^2)^{\frac{\bar{q}^*}{2}} dx \right)^{\frac{\bar{r}}{2}}, \end{aligned}$$

where  $C_1, C_2 > 0$  are positive constants independent of  $u$  and  $f$ , and  $\bar{r} = \frac{N(2-\bar{q})}{N-\bar{q}}$ ,  $\bar{m} = \frac{N\bar{q}}{N+\bar{q}}$ .

The proof is similar to that of Lemma 3.1 and is omitted.

These two lemmas yield the following result.

**Theorem 4.3.** *Suppose that  $f > 0$  on  $\Omega$ .*

- (i) *If  $f \in L^1(\Omega)$  and  $1 \leq q < \frac{N}{N-1}$ , then problem (1.1) has a solution that belongs to  $W^{1,\bar{q}}(\Omega)$  for every  $q \leq \bar{q} < \frac{N}{N-1}$ .*
- (ii) *If  $f \in L^{\bar{m}}(\Omega)$  with  $\bar{m} = \frac{N\bar{q}}{N+\bar{q}}$ ,  $\frac{N}{N-1} \leq q < \bar{q} < 2$ , then problem (1.1) has a solution belonging to  $W^{1,\bar{q}}(\Omega)$ .*

Higher integrability property can also be established to solutions of problem (3.1).

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