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Spin fluids and hyperfluids

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Abstract

The general theory of continua with microstructure reviewed in [2] (and, in particular, the theory of gyrocontinua studied in [1]) is adapted to apply to perfect spin fluids and hyperfluids [10]. Obvious changes are needed due to the prevailing interest for solids in [1] and to some extent in [2]; here, in addition, the conservative character of the continuum is exploited. Finally, the rôles of metric, coframe, connection, torsion and curvature in fluids are explored.

1 Introduction

In [1] our study of gyrocontinua was motivated by the possible concrete applications suggested, *e.g.*, by D'Eleuterio et al. (*cfr* [6], [7]). Now we take up the idea, mentioned, *e.g.*, in [10] by Obukhov and Tresguerres, that a theory of gyrofluids may be of interest in cosmological studies. We narrow our developments to a classical setting; however:

(i) we secure a link with a general approach to fluids with microstructure; the link will be particularly detailed for spin fluids (§5);

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(ii) we examine more closely—à la Noll—the consequences of the fluid nature of the continuum (implying invariance of the constitutive laws to unimodular deformations);

(iii) we concentrate at first our attention on hyperfluids (*i.e.* micromorphic fluids) and we import some relevant concepts from the continuum theory of dislocations;

(iv) we take advantage of the occasion for sundry remarks bearing on the general theory.

2 Perfect fluids with microstructure

As shown in [2], one starting point for a theory of perfect fluids with microstructure is the choice of a potential φ per unit mass:

$$\varphi = \varphi(\iota, \nu, \operatorname{grad} \nu) \tag{1}$$

as a function of a scalar taking into account the compressibility of the fluid (*i.e.* $\iota := \det \mathbf{F}$ with $\mathbf{F}$ the placement gradient from a paragon setting) and ν, a variable which specifies the microstructure: formally ν is an element of a manifold $\mathcal{M}$ of finite dimension.

Then from a variational principle we get the balance equations in statics

$$\begin{gathered} \operatorname{div} \mathbf{T} + \rho \mathbf{f} = \mathbf{0}, \ \operatorname{div} \mathcal{S} + \rho\beta - \zeta = 0, \ \text{in}\, \mathcal{B}; \\ \mathbf{T}\mathbf{n} = \mathbf{t}, \ \mathcal{S}\mathbf{n} = \sigma, \ \text{on}\, \partial\mathcal{B}; \end{gathered} \tag{2}$$

where

$$\mathbf{T} := \rho\iota \frac{\partial \varphi}{\partial \iota} \mathbf{I} - (\operatorname{grad} \nu)^T \mathcal{S}\,, \quad \zeta := \rho \frac{\partial \varphi}{\partial \nu}\,, \quad \mathcal{S} := \rho \frac{\partial \varphi}{\partial (\operatorname{grad} \nu)} \tag{3}$$

($\mathbf{I}$, the unit tensor). Here the second addendum in $\mathbf{T}$ could be called the Ericksen stress as in the theory of liquid crystals.

The stress $\mathbf{T}$ needs not be symmetric in general; the invariance of the power of internal actions under changes of observer requires only that:

$$\mathfrak{e}\mathbf{T} = \mathcal{A}^T \zeta + (\operatorname{grad} \mathcal{A}^T) \mathcal{S}, \tag{4}$$

where $\mathbb{e}$ is Ricci's permutation tensor and $\mathcal{A}$ is the infinitesimal generator of rotations on the microstructure, *i.e.* the operator describing the effect of a rotation $\mathbf{q}$ of the observer on the value $\nu_{(\mathbf{q})}$ of the microstructure to the first order in $\mathbf{q}$:

$$\mathcal{A} := \left.\frac{d\nu_{(\mathbf{q})}}{d\mathbf{q}}\right|_{\mathbf{q}=\mathbf{0}} . \tag{5}$$

The constitutive relations (3) are equivalent to (2.1), (2.2) and (2.3) of [10], though we prefer to restrict dependence upon ι, instead than upon the whole tensor $\mathbf{F}^T\mathbf{F}$, due to the fluid nature of the continuum.

No guide is offered in [2] (except through later examples) as to the choice, in general, of the variable ν that best mirrors given physical circumstances, though the way equations (2) are deduced implies at least minimality of the dimension of $\mathcal{M}$. Also the fact that Cauchy's stress may not be symmetric, and hence $\mathcal{A}$ must then be non-null, implies that the appropriate choice is, in such cases, observer-dependent.

One could go beyond and examine if and how a gross deformation of the body influences the reading of ν. More precisely, one could study the group acting on $\mathcal{M}$, each element of which matches a choice of the tensor $\mathbf{F}$ and determine the corresponding infinitesimal generator for the change of ν. Following a suggestion of Noll, we presume that, for a fluid, that generator have a null space that comprises all tensors $\mathbf{F}$ of the form $\mathbf{RU}$, where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ unitary ($\det \mathbf{U} = 1$).

3 Continua with affine microstructure

3.1 General case

To allow a direct comparison with developments and results of [10], we specify now the geometric nature of the microstructure by identifying $\mathcal{M}$ with *Lin*, the linear space of second order tensors $\mathbf{N}$, representing affine transformations, as was done in §21 of [2], without, at first, special constitutive assumptions nor reduction to statics.

Then β and ζ are second order tensors, denoted $\mathbf{B}$ and $\mathbf{Z}$ respectively, $\mathcal{A}$ and $\mathcal{S}$ are tensors of order three $\mathbb{a}$ and $\mathbb{s}$. Introducing also the density of the kinetic co-energy $\chi(\mathbf{N}, \dot{\mathbf{N}})$ (*i.e.* the Legendre transform with respect to $\dot{\mathbf{N}}$ of the kinetic energy's density $\kappa(\mathbf{N}, \dot{\mathbf{N}})$), we may accept, in addition to the usual balances of mass and momentum, the following laws

$$\rho\left(\left(\frac{\partial\chi}{\partial\mathbf{N}}\right)^{\cdot} - \frac{\partial\chi}{\partial\mathbf{N}}\right) = \rho\mathbf{B} - \mathbf{Z} + \operatorname{div}\mathbb{s}, \qquad\qquad (6)$$
$$\mathbb{e}\mathbf{T} = \mathbb{a}^{*}\mathbf{Z} + (\operatorname{grad}\mathbb{a}^{*})\mathbb{s}$$

(with $(\mathbb{a}^{*}\mathbf{C}) \cdot \mathbf{c} = \mathbf{C} \cdot (\mathbb{a}\mathbf{c})$, $\forall \mathbf{C} \in Lin$, $\forall \mathbf{c} \in \mathcal{V}$), ruling the balance of micromomentum and total moment of momentum (see (21.1), (21.2) of [2]).

Many different physical meanings can be attributed to $\mathbf{N}$. In solids it is often possible to propose a paragon setting (*e.g.*, a, perhaps non-existent but easily imaginable, setting of perfect crystallinity); then $\mathbf{N}$ might be interpreted as leading from the local reference in a paragon setting to a current uniform reference, in the sense of Noll, *i.e.* the reference from which one counts the elastic response, supposed to be the same at all points within the body. In other words, if the placement gradient $\mathbf{F}$ is also counted from the paragon setting, then the elastic potential is the same function of $\mathbf{G} := \mathbf{N}\mathbf{F}^{-1}$ at all points of the body. In these cases a rotation $\mathbf{Q} = e^{-\quad \mathbb{q}}$ of the observer changes $\mathbf{N}$ into $\mathbf{QN}$. Therefore the components of the infinitesimal generator $\mathbb{a}$ are

$$\mathbb{a}_{iJk} = \mathbb{e}_{ikh}\mathbf{N}_{hJ} \quad \Rightarrow \quad \mathbb{a}^{*}_{kiJ} = -\mathbb{e}_{kih}\mathbf{N}_{hJ}, \qquad (7)$$

and the second of (6) becomes, in components, (see p. 58 of [2])

$$\operatorname{skw}(T_{ij} + Z_{iL}N_{jL} + \mathbb{s}_{iLh}N_{jL,h}) = 0. \qquad (8)$$

The first equation (6) rules the evolution of the uniform reference (see, *e.g.*, [12] for a recent thorough study), whereas Cauchy's equation together with the constitutive law for the stress deriving from the elastic potential and (8) rule the evolution of the apparent placement.

3.2 Micromorphic hyperelastic continua—case I

The definition of hyperfluid given in [10] leaves open a number of options for the scholar in continuum mechanics. Formally the availability of a tensor field over the body $\mathcal{B}$ (such as $\mathbf{N}$ or $\mathbf{G} := \mathbf{N}\mathbf{F}^{-1}$) allows one to introduce, over $\mathcal{B}$, also a metric tensor ($\mathbf{N}^T\mathbf{N}$ or $\mathbf{G}^T\mathbf{G}$), a coframe obtained through $\mathbf{N}^{-T}$ from the paragon setting, a connection derived from the metric in accordance to standard suggestions (see, again, [10]).

But the essential step lays in the physical meaning, if any, that can be attributed to those mathematical entities. First of all comes the choice between $\mathbf{N}$ and a tensorial variable, say $\mathbf{G}$, related to it, but independent of unimodular transformations of the paragon setting (as would be expected in fluids). Then the above mentioned definitions, which are usually given when dealing with crystal lattices, need be revisited.

Moreover the theory demands a dependence of φ also on a gradient of $\mathbf{N}$, an issue which is briefly outlined in [2] (as appropriate for a theory of solids) and needs further investigation here.

A final comment is worthwhile: in the brief outline above $\dot{\mathbf{N}}$ and $\mathbf{N}$ were indicated as the variables on which the kinetic energy density κ may depend; however, as noticed in Remark 3 of §6 of [2], such assumption leads to a seemingly preposterous corollary. Perhaps $\dot{\mathbf{G}}$ should be the appropriate kinetic variable. When comparing present developments with those of §21 of [2] notice a discrepancy in notation: our $\mathbf{N}$ here is denoted $\mathbf{G}$ there.

All these provisos notwithstanding we explore first the consequences, in solids, of the existence for all internal actions of a potential of the type

$$\varphi = \varphi(\mathbf{F}, \mathbf{N}, \operatorname{Grad}\mathbf{N}), \tag{9}$$

where the capital letter in Grad means that differentiation is taken over the paragon setting. Usually such existence is dismissed, due to the prevailing interest for cases where almost any change of uniformity is dissipative; however, even if speculative here, the assumption provides us with a convenient preamble to our later developments relative to fluids.

Under the circumstances, the balance laws continue to be equations (6) plus Cauchy's equation, whereas the constitutive laws change into

$$\mathbf{T} = \rho\frac{\partial\varphi}{\partial\mathbf{F}}\mathbf{F}^T, \ \mathbf{Z} = \rho\frac{\partial\varphi}{\partial\mathbf{N}}, \ \mathfrak{s} = \rho\frac{\partial\varphi}{\partial\operatorname{Grad}\mathbf{N}} \tag{10}$$

(the Ericksen's component is now missing).

3.3 Micromorphic hyperelastic continua—case II

Another case, variously studied and interpreted (*cfr*, *e.g.*, §22 of [2]), implies the introduction of a potential φ depending upon the Euler-Lagrange deformation tensor of the gross motion $\mathbf{E}$ (*i.e.* $\mathbf{E} := \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$) and an appropriate variable involving $\mathbf{N}$, as for instance

$$\mathbf{M} := \frac{1}{2}(\mathbf{N}^T\mathbf{F} - \mathbf{I}). \tag{11}$$

From the assumption

$$\varphi = \varphi(\mathbf{E}, \mathbf{M}) \tag{12}$$

it follows that

$$\mathbf{T} = \rho\mathbf{F}\frac{\partial\varphi}{\partial\mathbf{E}}\mathbf{F}^T + \frac{1}{2}\rho\mathbf{N}\frac{\partial\varphi}{\partial\mathbf{M}}\mathbf{F}^T, \quad \mathbf{Z} = \frac{1}{2}\iota\rho\mathbf{F}\left(\frac{\partial\varphi}{\partial\mathbf{M}}\right)^T, \quad \mathfrak{s} = \mathbb{0} \tag{13}$$

and (8) is automatically satisfied.

As we wish here to study fluids, neither of the two cases is of immediate help: there may be doubts about the relevance of a paragon setting (however, think of the case of smectic liquid crystals, where a paragon setting of perfect order exists and is relevant) and, secondly, the dependence of the potential on $\operatorname{grad}\mathbf{N}$ seems to be almost mandatory (again the case of liquid crystals is enlightening). Also, as already mentioned, one must judge if the rôle of $\mathbf{G}$, rather than $\mathbf{N}$, in the kinetic energy of the body is more appropriate.

We take all these reflections into account when we put forward later a definition of hyperfluid.

3.4 Continua with dislocations

The attributes of the affine field $\mathbf{N}$ are unlike those of the field $\mathbf{F}$, because $\mathbf{N}$ need not be *compatible*. Only if $\operatorname{Grad}\mathbf{N}$ coincides throughout the body with the minor right transpose $(\operatorname{Grad}\mathbf{N})^t$, there may exist a vector field $\mathbf{a}$ such that $\mathbf{N} = \operatorname{Grad}\mathbf{a}$ and so that $\mathbf{N}$ becomes compatible; but, actually, the interest lays in the *incompatibility* of $\mathbf{N}$. In reality, most often a paragon set of three independent, perhaps even orthonormal directors exists, $\{\mathbf{d}^*_{(R)} \mid R \in \{1,2,3\}\}$, having an immediate physical significance in the description of particles (as it would be for the edges in a perfect orthorombic crystal lattice), the evolution of which is ruled by $\mathbf{N}$:

$$\mathbf{d}_{(R)} = \mathbf{N}\mathbf{d}^*_{(R)}, \tag{14}$$

and the directors $\mathbf{d}_{(R)}$ have the same significance but in the present setting, where the crystal lattice may be, and usually is, defective.

The reciprocal directors are then defined through the relations

$$\sum_{R=1}^{3} \mathbf{d}^{(R)} \otimes \mathbf{d}_{(R)} = \mathbf{I}, \tag{15}$$

and hence

$$\mathbf{d}^{(R)} = \mathbf{N}^{-T}\mathbf{d}^{*(R)}, \ \mathbf{N} = \sum_{R=1}^{3} \mathbf{d}_{(R)} \otimes \mathbf{d}^{*(R)}. \tag{16}$$

Correspondingly the concepts of *linear connection* Γ^T_{RS}, of *wryness* $\mathbb{w}$ and of *inhomogeneity* $\mathbb{h}$ are expedient (in Noll's terminology—*cfr* [9]—our body would be materially uniform and simple; $\mathbf{N}$ would correspond to a *uniform reference* for $\mathcal{B}$; Γ would then be the unique affine connection called the *material connection* associated with the material uniformity; the Riemann curvature associated with this connection being null, $\mathbb{h}$ would be the Cartan *torsion* of the material connection):

$$\begin{gathered} \mathbb{w} := \textstyle\sum_{R,S=1}^{3} \mathbf{d}_{(S),R} \otimes \mathbf{d}^{(S)} \otimes \mathbf{d}^{(R)}, \\ \mathbb{h} := \frac{1}{2}(\mathbb{w} - \mathbb{w}^t), \ \Gamma^T_{RS} := \mathbf{d}^{(T)} \cdot ((\mathbb{w}\mathbf{d}_{(R)})\mathbf{d}_{(S)}); \end{gathered} \tag{17}$$

thus Γ^T_{RS} are the anholonomic components of $\mathbb{w}$.

$\mathbb{h}$ being right-skew-symmetric, there exists a second order tensor $\mathbf{B}$, Kröner's *dislocation density* (*cfr* [8]), such that $\mathbf{B}^T = \mathbb{e}\mathbb{h}^T$, $\mathbb{h}^T = \frac{1}{2}\mathbb{e}\mathbf{B}^T$ and the Burgers vector $\mathbf{b}$ relative to a plane of normal $\mathbf{n}$ is given by $\mathbf{b} = \mathbf{B}\mathbf{n}$. Furthermore, as $\mathbb{h}^T = \frac{1}{2}\mathbb{e}\mathbb{e}\mathbb{w}^T$, it can be proved that $\mathbf{B}^T = \mathbb{e}\mathbb{w}^T$.

As shown in [9] (equation (15.34)), the nine components of $\mathbf{B}$ are related through the three conditions:

$$(\operatorname{Grad}\mathbf{B})\mathbf{N}^{-1} + \mathbf{B}(\mathbb{e}\mathbf{B}) = 0 \tag{18}$$

expressing the first Bianchi identity. Condition (18) allows us to discard, formally at least, the gradient of the dislocation density from the list of state variables when **B** and **N** are included.

To make clear the dependence of $\mathtt{w}$ upon $\operatorname{Grad}\mathbf{N}$ it is sufficient to recall definition (17) and relation (16):

$$\begin{gathered} \mathtt{w}_{ijh} = \textstyle\sum_{R,S=1}^{3} N_{iS,R} d_j^{(S)} d_h^{(R)} = N_{iS,R} N_{Sj}^{-1} N_{Rh}^{-1} \,, \\ B_{ij} = \mathbb{e}_{jpq} N_{iS,R} N_{Sp}^{-1} N_{Rq}^{-1} . \end{gathered} \tag{19}$$

Thus the dislocation density **B** (or the inhomogeneity $\mathbb{h}$) measures the incompatibility of the local deformation.

The field of $\mathtt{w}$ or, rather, of **B** could take the place of $\operatorname{Grad}\mathbf{N}$ in the potential φ; an appropriate interpretation of the derivative $\rho\frac{\partial\hat{\varphi}}{\partial\mathbf{B}}$ (where $\hat{\varphi}$ is obtained from φ, merely by expressing in it $\operatorname{Grad}\mathbf{N}$ in terms of **B**) is valuable.

4 The micromorphic hyperelastic fluid or hyperfluid

4.1 General case

In a non crystalline fluid no privileged settings exist; in particular, no paragon setting may be summoned, even though molecular number densities, at least, need be compared (but they are simply determined by the value of ι^{-1}); hence, at first glance, the concepts reviewed in the previous section seem to be wholly unsuitable. However, no paragon setting is needed if, formally, we substitute the tensor **N** in φ with the tensor $\mathbf{G} := \mathbf{N}\mathbf{F}^{-1}$. Said differently, even if we were to imagine a setting, chosen at random, as paragon, a successive change of that setting through any unimodular deformation would have no effect on the potential, hence on all internal actions; that circumstance characterises fluids even if in a circuitous way. Again, the choice of **G** en lieu of **N** as an argument in φ could be said to mean that the paragon setting is the present one.

In conclusion, for fluids, we may take for the potential a function of ι, $\mathbf{G}$, $\operatorname{grad}\mathbf{G}$

$$\varphi = \varphi(\iota, \mathbf{G}, \operatorname{grad}\mathbf{G}). \tag{20}$$

Under a change of observer by a rotation of axial vector $\mathbf{q}$, the function φ changes into

$$\varphi_{(\mathbf{q})} = \varphi(\iota_{(\mathbf{q})}, \mathbf{G}_{(\mathbf{q})}, (\operatorname{grad}\mathbf{G})_{(\mathbf{q})}), \tag{21}$$

with (let $\mathbf{Q} = e^{-\ \ \mathbf{q}}$)

$$\begin{gathered}\iota_{(\mathbf{q})} = \iota,\ \mathbf{G}_{(\mathbf{q})} = \mathbf{Q}\mathbf{G}\mathbf{Q}^T, \\ \Big[(\operatorname{grad}\mathbf{G})_{(\mathbf{q})}\Big]_{ijk} = Q_{is}G_{sr,p}Q_{jr}Q_{kp}.\end{gathered} \tag{22}$$

However, being a scalar, φ should not be influenced by such changes: $\varphi_{(\mathbf{q})} = \varphi$, $\forall\mathbf{q}$. Hence φ cannot be any function of its arguments, but rather depend on $\mathbf{G}$ and $\operatorname{grad}\mathbf{G}$ a peculiar way, through an invariant base.

To derive the consequences of assumption (20) we may follow §12 of [2]; thus the conditions of static equilibrium are (*cfr* (2)):

$$\begin{gathered}\operatorname{div}\mathbf{T} + \rho\mathbf{f} = 0\,, \quad \operatorname{div}\mathbb{s} + \rho\mathbf{B} - \mathbf{Z} = 0\,, \quad \text{in}\,\mathcal{B}; \\ \mathbf{T}\mathbf{n} = \mathbf{t}\,, \quad \mathbb{s}\mathbf{n} = \mathbf{S}\,, \quad \text{on}\,\partial\mathcal{B};\end{gathered} \tag{23}$$

with definitions (take $\mathbf{C}\cdot[(\operatorname{grad}\mathbf{G})\mathbf{c}] = [(\operatorname{grad}\mathbf{G})^*\mathbf{C}]\cdot\mathbf{c}$, $\forall\mathbf{C}\in Lin$, $\forall\mathbf{c}\in\mathcal{V}$)

$$\mathbf{T} := \rho\iota\frac{\partial\varphi}{\partial\iota}\mathbf{I} - (\operatorname{grad}\mathbf{G})^*\mathbb{s},\ \mathbf{Z} := \rho\frac{\partial\varphi}{\partial\mathbf{G}},\ \mathbb{s} := \rho\frac{\partial\varphi}{\partial(\operatorname{grad}\mathbf{G})}. \tag{24}$$

Remark. Following §13 of [2], the invariance of φ under changes of the observer implies and is implied by the balance of moment of momentum; here, as the components of the infinitesimal generator of rotations are $\mathbb{a}_{ijk} = \mathbb{e}_{ihk}G_{hj} + \mathbb{e}_{jhk}G_{ih}$, equation (13.3) of [2] gives

$$\mathbb{e}\left(\mathbf{G}\mathbf{Z}^T + \mathbf{G}^T\mathbf{Z} - \mathbb{s}^t(\operatorname{grad}\mathbf{G})^T -^t (\operatorname{grad}\mathbf{G})\mathbb{s}^t - (\operatorname{grad}\mathbf{G})^{Tt}\mathbb{s}\right) = \mathbf{0}\,. \tag{25}$$

This result is at distinct variance with the result (8) above and those of §14 of [2]. The discrepancy is due to the fact that the reading of $\mathbf{G}$ is not affected by a superposed rigid motion of the body, whereas $\mathbf{N}$ is.

4.2 Fluids with dislocations

Perhaps the major interest lays in the subcase when the independent variables in φ are (the scalar ι and) the metric represented by the symmetric tensor $\mathbf{H}$ and the tensor density $\mathbf{A}$ of *fluid dislocations*

$$\varphi = \varphi(\iota, \mathbf{H}, \mathbf{A}) , \quad \mathbf{H} := \mathbf{G}^T\mathbf{G} , \quad A_{ij} = \mathbb{e}_{jpq} G_{is,r} G^{-1}_{sp} G^{-1}_{rq} . \tag{26}$$

Again an invariant base must be determined, but involving now only two second order tensors, one of them symmetric.

Because $\mathbf{H}$ and $\mathbf{A}$ transform under rotation in the standard way ($\mathbf{H}_{(\mathbf{Q})} = \mathbf{QHQ}^T$ and $\mathbf{A}_{(\mathbf{Q})} = \mathbf{QAQ}^T$), the search of that base coincides with that of a solution of the equation

$$\mathbb{e}\left[2\frac{\partial\varphi}{\partial\mathbf{H}}\mathbf{H} + \frac{\partial\varphi}{\partial\mathbf{A}}\mathbf{A}^T + \left(\frac{\partial\varphi}{\partial\mathbf{A}}\right)^T \mathbf{A}\right] = \mathbf{0}. \tag{27}$$

Again, when (27) applies, the balance of moment of momentum is assured. the other balance equations are Cauchy's equation and

$$\rho\left(\left(\frac{\partial\chi}{\partial\dot{\mathbf{G}}}\right)^{\cdot} - \frac{\partial\chi}{\partial\mathbf{G}}\right) = \rho\mathbf{B} - \mathbf{Z} + \operatorname{div}\mathbb{s}, \tag{28}$$

where $\mathbf{T}$, $\mathbf{Z}$, $\mathbb{s}$ obey the constitutive laws

$$\begin{gathered} \mathbf{T} = \rho\iota\tfrac{\partial\varphi}{\partial\iota}\mathbf{I} - (\operatorname{grad}\mathbf{G})^*\mathbb{s}, \\ Z_{hk} := 2\rho G_{hp}\tfrac{\partial\varphi}{\partial H_{pk}} \\ +\rho\tfrac{\partial\varphi}{\partial A_{ij}} G_{il,m}\mathbb{e}_{jpq}(G^{-1}_{mh}G^{-1}_{kq}G^{-1}_{lp} - G^{-1}_{mq}G^{-1}_{lh}G^{-1}_{kp}), \\ \mathbb{s}_{ihk} := \rho\tfrac{\partial\varphi}{\partial A_{ij}}\mathbb{e}_{jpq}G^{-1}_{hp}G^{-1}_{kq}. \end{gathered} \tag{29}$$

Notice that in (28), acceding to an intimation in §3.2 above, we have chosen $\mathbf{G}$ and $\dot{\mathbf{G}}$ as the relevant variables in χ.

5 Spin fluids

One issue left open sofar concerns the explicit expression of the microkinetic coenergy density χ; the issue is of pressing relevance for some spin fluids, therefore we address it here, at first in general.

It was shown in [3] that, even when the expression of the microkinetic energy density κ is fully known, the coenergy is determined only to within an additive homogeneous function of $\dot{\nu}$ of degree one. In particular, if physical circumstances suggest as appropriate for κ a homogeneous dependence on $\dot{\nu}$ of degree two

$$\kappa = \frac{1}{2}\dot{\nu}^{\alpha}\Omega_{\alpha\beta}\dot{\nu}^{\beta}\,, \tag{30}$$

then χ differs from κ only by an addendum of the form $\lambda_{\alpha}\dot{\nu}^{\alpha}$, where λ is an arbitrary covariant vector within the tangent fibre at ν. In general the term involving κ, and fully determined by κ, is more complex, but the indetermination is again additive and of the same type.

Within the simpler circumstances and in terms of components the dynamic version of the second balance equation (2) becomes

$$\rho\left[\left(\frac{\partial\kappa}{\partial\dot{\nu}^{\alpha}}\right)^{\cdot} - \frac{\partial\kappa}{\partial\nu^{\alpha}} + \Lambda_{\alpha\beta}\dot{\nu}^{\beta}\right] = \rho\beta_{\alpha} - \zeta_{\alpha} + \mathcal{S}_{\alpha i,i}\,. \tag{31}$$

The extra term within the square brackets, linear in $\dot{\nu}$, is powerless as the tensor

$$\Lambda_{\alpha\beta} = \frac{\partial\lambda_{\alpha}}{\partial\nu^{\beta}} - \frac{\partial\lambda_{\beta}}{\partial\nu^{\alpha}} \tag{32}$$

is skew.

Before we proceed to specialize the general balance equation (31) into one valid for spin continua, we must resolve some pertinent riddles, otherwise misunderstandings coud arise easily. If the basic microstructural variable for the continuum with spin were the vector specifying an independent microrotation of a subelement within each material element (a rotation occurring around an axis fixed materially by the macromotion and hence measured from the current macro-setting spelled through the macro-placement gradient $\mathbf{F}$) then one would have to follow developments as in §19 of [2] (see p. 50) and [1], leading to a theory of gyrocontinua. But such an approach seems inappropriate to achieve a physically motivated model of spin fluids, where the relevant variable is kinetic (*i.e.*, the spinning velocity) rather than geometric (*i.e.*, the rotation). In fact, even at the macroscopic level, the essential fields for fluid are kinetic and the molecular identification of each *element* in the macromotion is a vacuous goal. Nevertheless, as in the macroscopic theory, also here a, to some extent forced, link with the standard theory of [2] may be found.

Follow the developments at p. 38 of [2] and introduce, for the microstructure, the anholomic constraint

$$\dot{\nu} = \mathcal{A}\mathbf{m}\,; \tag{33}$$

$\mathbf{m}$ an arbitrary vector in Euclidean space, so that the microkinetics has rotational character. Then one is led to (14.7) of [2] as a pure balance equation

$$\rho\left(\mathcal{A}_i^\alpha \frac{\partial\chi}{\partial\dot{\nu}^\alpha}\right)^{\cdot} = \mathcal{A}_i^\alpha(\rho\beta_\alpha - \zeta_\alpha + \mathcal{S}_{\alpha j,j})\,, \tag{34}$$

where ζ and $\mathcal{S}$ indicate now only the active parts of the corresponding quantitites. Taking into account the expression of χ which led also to (31), we get

$$\begin{gathered}\rho\left[\mathcal{A}_i^\alpha\left(\tfrac{\partial\kappa}{\partial\dot{\nu}^\alpha} - \lambda_\alpha\right)\right]^{\cdot} + \rho\tfrac{\partial\lambda_\alpha}{\partial\nu^\beta}\left(\mathcal{A}_i^\alpha\mathcal{A}_j^\beta - \mathcal{A}_j^\alpha\mathcal{A}_i^\beta\right)m_j = \\ \mathcal{A}_i^\alpha\left(\rho\beta_\alpha - \zeta_\alpha + \mathcal{S}_{\alpha j,j}\right)\,.\end{gathered} \tag{35}$$

This equation can be written in terms of Euclidean vectors and tensors as follows

$$\rho\dot{\mathbf{h}} + \rho\mathbf{m}\times\mathbf{p} = \rho\mathbf{b} - \mathbf{z} + \operatorname{div}\mathbf{S}\,. \tag{36}$$

In this balance equation the reduced microforce $\mathbf{z}$ and microstress $\mathbf{S}$:

$$z_i := \mathcal{A}_i^\alpha\zeta_\alpha + \mathcal{A}_{i,j}^\alpha\mathcal{S}_{\alpha j}\,,\ S_{ij} := \mathcal{A}_i^\alpha\mathcal{S}_{\alpha j}\,, \tag{37}$$

are the transforms of ζ and $\mathcal{S}$ in accordance with the formal rules of change consequent to the change of variable specified by a Jacobian matrix: here the matrix $\mathcal{A}_i^\alpha$, see rules (10.8) of [2]. The remarks preceeding that equation in [2] suggest introduction of the tensor $\mathbf{Y}$ of microinertia density

$$Y_{ij} = \mathcal{A}_i^\alpha\Omega_{\alpha\beta}\mathcal{A}_j^\beta \tag{38}$$

(see also (6.15) of [2]) and the vector $\mathbf{b}$ of external actions

$$b_i = \mathcal{A}_i^\alpha\beta_\alpha\,. \tag{39}$$

Finally three other quantities occur in (36): the skew tensor

$$\mathbb{e}_{ijk}p_k = \frac{\partial \lambda_\alpha}{\partial \nu^\beta}\left(\mathcal{A}_i^\alpha \mathcal{A}_j^\beta - \mathcal{A}_j^\alpha \mathcal{A}_i^\beta\right), \tag{40}$$

the vector

$$l_i = \mathcal{A}_i^\alpha \lambda_\alpha \tag{41}$$

and the vector $\mathbf{h}$ of moment of micromomentum of sorts

$$\mathbf{h} = \mathbf{Y}\mathbf{m} + \mathbf{l}. \tag{42}$$

In §14 of [2] it was shown that Cauchy's equation still applies. In addition it was found that the active macroscopic symmetric stress $\mathbf{T}$ is altered by the addition of a skew reactive term: see (14.6) of [2]. However, in the present instance, that second result does not hold as it is based on the relation (9.7) of [2] which relies on the assumption, invalid here, that $\operatorname{skw}(\dot{\mathbf{F}}\mathbf{F}^{-1})$ and $\dot{\nu}$ be both expressible linearly in terms of the vector $\mathbf{r} = -\frac{1}{2}\mathbb{e}(\dot{\mathbf{F}}\mathbf{F}^{-1})$: rather here $\dot{\nu}$ is linear in the independent vector $\mathbf{m}$, not affected by $\mathbf{r}$. Hence in analogy with our Remark in §4.1 we conclude that $\mathbf{T}$ is symmetric and wholly active and enters Cauchy's equation in the usual way.

Now we may forget how (36) was derived as the steps taken, though suggestive, were to some extent formal; we may simply take (36) as the fundamental balance equation ruling the evolution of the microstructure for a special, though still wide, class of continua which we presume would include spin fluids. Of course, to come to anything relevant, we must add substance to form by exploring reasonable constitutive choices for the many quantities involved.

We quote here one of the simplest choice for the inertia terms

(i) to presume λ to belong to the null space of $\mathcal{A}$ and hence $\mathbf{l}$ to be null;

(ii) the microinertia density tensor to be spherical and constant

$$\mathbf{Y} = \gamma^{-1}\mathbf{I}, \tag{43}$$

leading to an equation resembling in part Gilbert's equation for ferromagnets (γ, the gyromagnetic ratio; $\mathbf{m}$ the magnetization)

$$\rho\gamma^{-1}\dot{\mathbf{m}} + \rho\mathbf{m}\times\mathbf{p} = \rho\mathbf{b} - \mathbf{z} + \operatorname{div}\mathbf{S} \tag{44}$$

(see. *e.g.*, [11]).

Many other deep choices remain open (the constitutive choices for **p**, **z**, **S** to begin with); they are beyond the scope of the present paper. Also the specification of the external action **b** may pose hurdles; **h** is a spin velocity read with reference to a frame bound to the macroscopic element and, as remarked already, one must *invent* such a frame in a fluid. A step in that direction was taken in [4], Sect. 14, with the suggestion that the relevant frame be that of minimal peculiar macroscopic moment of momentum; but then another balance equation beyond Cauchy's needs be called upon.

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Fluidi sa spinom i hiperfluidi

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Opšta teorija kontinuuma sa mikrostructurom revijalno izložena u [2] (kao i, u specijalnom slučaju, teorija girokontinuuma proučena u [1]) je tako prilagodjena da bude primenjiva na idealne spin-fluide i hiperfluide [10]. Očigledne promene su, pritom, neophodne zbog preovladjućeg interesovanja za čvrsta tela u [1] i, do izvesne mere u [2]. Dodatno se ovde iskorišćava konzervativni karakter posmatranog kontinuuma. Konačno, uloge metrike, ko-sistema referencije, povezanosti, torzije i krivine su istražene.