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PROBABILISTIC NORMS AND STATISTICAL CONVERGENCE OF RANDOM VARIABLES

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Abstract. The paper extends certain stochastic convergence of sequences of $\mathbb{R}^k$ -valued random variables (namely, the convergence in probability, in L^p and almost surely) to the context of E-valued random variables.

1 Introduction

A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. In PN space, the norms of the vectors are represented by probability distribution functions rather than a positive number. Such spaces were first introduced by A. N. Šerstnev in 1963 [20]. Recently, C. Alsina et al [1] introduced a new definition of PN spaces that includes Šerstnev's and leads naturally to the identification of the principle class of PN spaces, the Menger spaces. This definition becomes the standard one and has been adopted by many authors (for instance, [2], [12], [13], [14]) who have investigated the properties of PN spaces. The detailed history and the development of the subject up to 2006 can be found in [19].

On the other hand, statistical convergence was first introduced by Fast [4] as a generalization of ordinary convergence for real sequences. Since then it has been studied by many authors (see [7], [8], [3], [16], [17], [22]). Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number space [18], locally convex spaces [15] and Banach spaces [11]. Karakus [9] has recently introduced and studied statistical convergence on PN spaces and followed by Karakus and Demirci [10] studied statistical convergence of double sequences on PN spaces.

Our work has been inspired by [14] in which the convergence of E-valued random variables is associated with a probabilistic norms. The paper extends certain stochastic convergence (here, statistical convergence) of sequences of $\mathbb{R}^k$ -valued random variables namely, the convergence in probability, in L^p and almost surely to the context of E-valued random variables.

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The paper is organized as follows. In the second section, some preliminary concepts related to PN spaces and statistical convergence are presented. In the third, fourth and fifth sections, the statistical convergence in probability, statistical convergence in L^p and almost surely statistically convergence respectively are investigated. In this context, we obtain some results that replicate those given in [14].

2 Preliminaries

We use the notation and terminology of [23]. Thus Δ^+ is the space of probability distribution functions F that are left-continuous on $\mathbb{R}^+ = (0, +\infty)$, $F(0) = 0$ and such that $F(+\infty) = 1$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions and has both a maximal element ε_0 and a minimal element ε_∞; these are given, respectively, by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) = \begin{cases} 0, & x < +\infty, \\ 1, & x = +\infty. \end{cases}$$

There is a natural topology on Δ^+ that is induced by the modified Lévy metric d_L (see [21] and [23]), i.e.,

$$d_L(F,G) = \inf\{h\colon \text{both } [F,G;h] \text{ and } [G,F;h] \text{ hold}\} \tag{2.1}$$

for all $F, G \in \Delta^+$ and $h \in (0,1]$, where $[F,G;h]$ denote the condition

$$G(x) \leq F(x+h) + h \quad \text{for } x \in (0, \frac{1}{h}). \tag{2.2}$$

Convergence with respect to this metric is equivalent to weak convergence of distribution functions, i.e., $\{F_n\}$ in Δ^+ converges weakly to F in Δ^+ (written $F_n \xrightarrow{w} F$) if and only if $\{F_n(x)\}$ converges to $F(x)$ at every point of continuity of the limit function F. Consequently, we have

$$F_n \xrightarrow{w} F \quad \text{if and only if} \quad d_L(F_n, F) \to 0, \tag{2.3}$$

$$F(t) > 1 - t \quad \text{if and only if} \quad d_L(F, \varepsilon_0) < t \quad \text{for every} \quad t > 0. \tag{2.4}$$

Moreover, the metric space (Δ^+, d_L) is compact.

A triangle function is a binary operation τ on Δ^+ that is commutative, associative, non-decreasing in each place, and has ε_0 as an identity element. Continuity of a triangle function means uniform continuity with respect to the natural product topology on $\Delta^+ \times \Delta^+$.

Definition 1. *A probabilistic normed space (briefly, a PN space) is a quadruple (V, ν, τ, τ^*) in which V is a real linear space, the probabilistic norm ν is a mapping from V into Δ^+ and τ and τ^* are continuous triangle functions so that ν, τ and τ^* are subject to the following conditions:*
(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (the null vector of V);
(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;
(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)q})$ for every $p \in V$ and for every $\alpha \in [0, 1]$.

A PN space is called a *Šerstnev space* if (N1) and (N3) are satisfied along with the following condition:

$$\nu_{\alpha p}(x) = \nu_p(\frac{x}{|\alpha|}), \tag{2.5}$$

for all $\alpha \in \mathbb{R}\backslash\{0\}$ and for all $x > 0$, which implies (N2) and (N4) in the strengthened form:

$$\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)q}), \tag{2.6}$$

for every $p \in V$ and for every $\alpha \in [0, 1]$. A Šerstnev space will be denoted by (V, ν, τ), since the role of τ^* is played by a fixed triangle function τ_M, which satisfies (N2).

A PN space is endowed with the *strong topology* generated by the strong neighborhood system $\{\mathcal{N}_\theta(\lambda) \colon \lambda > 0\}$, where

$$\mathcal{N}_\theta(\lambda) = \{p \in V \colon d_L(\nu_p, \varepsilon_0) < \lambda\} \tag{2.7}$$

(see [23]) and the latter is metrizable. A sequence $\{p_n\}$ of elements of V converges to θ_V, the null element of V, in the strong topology (written $p_n \longrightarrow \theta_V$) if, and only if,

$$\lim_{n \to +\infty} d_L(\nu_{p_n}, \varepsilon_0) = 0, \tag{2.8}$$

i.e., for every $\lambda > 0$ there is an $m = m(\lambda) \in \mathbb{N}$ such that $d_L(\nu_{p_n}, \varepsilon_0) < \lambda$ for all $n \geq m$, where d_L denotes the modified Lévy metric (2.1). In terms of neighborhoods, we have $p_n \longrightarrow \theta_V$ provided that for any $\lambda > 0$ there is an $N(\lambda) \in \mathbb{N}$ such that $p_n \in \mathcal{N}_{\theta_V}(\lambda)$ (i.e., $\nu_{p_n}(\lambda) > 1 - \lambda$) whenever $n \geq N$.

Of course, there is nothing special about θ_V as a limit; if one wishes to consider the convergence of the sequence $\{p_n\}$ to the vector p, then it suffices to consider the sequence $\{p_n - p\}$ and its convergence to θ_V.

An important class of PN spaces is that of E-normed spaces (see [12]). Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(V, \|\cdot\|)$ a normed space, and S a linear space of

V-valued random variables (possibly, the entire space). For every $p \in S$ and for every $\lambda \in \overline{\mathbb{R}}_+$, let $\nu\colon S \to \Delta^+$ be defined by

$$\nu_p(\lambda) := P\{\omega \in \Omega\colon \|p(\omega)\| < \lambda\}; \tag{2.9}$$

then (S, ν) is an E-normed space (briefly, EN space) with base $(\Omega, \mathcal{A}, P)$ and target $(V, \|\cdot\|)$.

Example 2. *Let $L^0 = L^0(\Omega, \mathcal{A}, P)$, the linear space of (equivalence classes of) random variable $f\colon \Omega \to \mathbb{R}$. Let $\nu\colon S \to \Delta^+$ be defined, for every $f \in L^0$ and for every $\lambda \in \overline{\mathbb{R}}_+$, by*

$$\nu_f(\lambda) := P\{\omega \in \Omega\colon |f(\omega)| < \lambda\}.$$

Then, the couple (L^0, ν) is an EN space. It is a PN space under the triangle function τ_W and τ_M [23].

In what follows, we list some of the basic concepts related to the theory of statistical convergence and we refer to [5] and [6] for more details.

Definition 3. *The natural density of a set K of positive integers is defined by*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \in K\colon k \leq n\}|$$

where $|\{k \in K\colon k \leq n\}|$ denotes the number of elements of K not exceeding n. It is clear that for finite set K, we have $\delta(K) = 0$.

Remark 4. *We will be particularly concerned with integer sets having natural density zero. Thus, if $\{x_k\}$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $\{x_k\}$ satisfies P for "almost all k", and we abbreviate this by "a.a.k".*

In PN space, one can consider the statistical convergence of sequences in the following manner.

Definition 5. *Let $(V, \|\cdot\|)$ be a normed space. A sequence $\{p_k\}$ in V is said to be statistically convergent to $l \in V$ provided that, for every $\lambda > 0$,*

$$\delta(\{k \in \mathbb{N}\colon \|p_k - l\| \geq \lambda\}) = 0$$

holds, viz. $\|p_k - l\| < \lambda$ for a.a.k. In this case we write $p_k \stackrel{stat}{\longrightarrow} l$

Remark 6. *We note that for every $\lambda > 0$, $\|p_k - l\| < \lambda$ implies that $\|p_k - l\| \stackrel{n\to\infty}{\longrightarrow} 0$. Thus, one can say that, for every $\lambda > 0$, $\|p_k - l\| < \lambda$ for a.a.k implies that $\|p_k - l\| \stackrel{stat}{\longrightarrow} 0$*

Using these concepts, we extend the statistical convergence of sequences in PN spaces endowed with the strong topology. We begin with defining the convergence of probability distribution functions.

Definition 7. *Let (Δ^+, d_L) be a metric space. Then, a sequence $\{F_k\}$ of Δ^+ is said to be statistically convergent (weakly) to $F \in \Delta^+$, if and only if, for every $\lambda > 0$,*

$$\delta(\{k \in \mathbb{N}\colon d_L(F_k, F) \geq \lambda\} = 0, \tag{2.10}$$

viz. $d_L(F_k, F) < \lambda$ for a.a.k. In this case we write $F_k \xrightarrow{stat(w)} F$.

By (2.4) and (2.10), the following lemma can be easily verified.

Lemma 8. *The following statements are equivalent:*
(i) $F_k \xrightarrow{stat(w)} \varepsilon_0$,
(ii) for every $\lambda > 0$, $d_L(F_k, \varepsilon_0) < \lambda \quad$ for a.a.k,
(iii)for every $\lambda > 0$, $F_k(\lambda) > 1 - \lambda \quad$ for a.a.k.

Definition 9. *Let (V, ν, τ, τ^*) be a PN space. A sequence $\{p_n\}$ of elements of V is said to be strongly statistically convergent to θ_V in the strong topology if, and only if, for every $\lambda > 0$,*

$$\delta(\{k \in \mathbb{N}\colon d_L(\nu_{p_k}, \varepsilon_0) \geq \lambda\} = 0, \tag{2.11}$$

viz.

$d_L(\nu_{p_k}, \varepsilon_0) < \lambda$ for a.a.k. In this case we write $p_k \xrightarrow{stat} \theta_V$ or $stat - \lim p_k = \theta_V$. In terms of strong neighborhoods, we have

$$p_n \xrightarrow{stat} \theta_V \Leftrightarrow p_k \in \mathcal{N}_{\theta_V}(\lambda) \quad \textit{for a.a.k.} \tag{2.12}$$

Of course, there is nothing special about θ_V as a limit; if one wishes to consider the convergence of sequence $\{p_n\}$ to the vector p in the strong topology, then it suffices to consider the sequence $\{p_n - p\}$ and its convergence to θ_V.

3 Statistical convergence in probability

Let $\{X_k\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ taking values in a separable normed space $(V, \|\cdot\|)$, where $\|\cdot\|$ is the norm. Then we say the sequence X_k *converges in probability* or *converges in measure* to θ_V (the null vector in V) if for every $\lambda > 0$,

$$\lim_{k\to\infty} P(\ \|X_k\|\ > \lambda)) = 0.$$

Equivalently, for any $\lambda > 0$, there is an $n_0 \in \mathbb{N}$ such that

$$P(\ \|X_k\| < \lambda) > 1 - \lambda \quad \text{for all} \quad n \geq n_0.$$

In this case we write $X_k \xrightarrow{P} \theta_V$.

Remark 10. *The need for separability on V is to ensure that the norm, $\|X_k\|$, is a random variable, for all random variables X_k and θ_V. Convergence almost surely implies convergence in probability but not conversely.*

For *statistical convergence in probability*, we give the following definition.

Definition 11. *The sequence $\{X_k\}$ is said to converge statistically in probability to θ_V if for every $\epsilon > 0$,*

$$\delta(\{k \in \mathbb{N} \colon P(\|X_k\| < \lambda) \leq 1 - \lambda\}) = 0,$$

viz. $P(\ \|X_k\| < \lambda) > 1 - \lambda$ *for a.a.k. In this case we write* $X_k \xrightarrow{stat(P)} \theta_V$

Theorem 12. *For a sequence of (equivalence classes of) E-valued random variables $\{f_k\}$, the following statements are equivalent:*
(a) $\{f_k\}$ converges statistically in probability to θ_S, $f_k \xrightarrow{stat(P)} \theta_S$;
(b) the corresponding sequence $\{\nu_{f_k}\}$ of probabilistic norms converges (weakly) statistically to ε_0 for a.a.k, $\nu_{f_k} \xrightarrow{stat(w)} \varepsilon_0$;
(c) $\{f_k\}$ converges statistically to θ_S in the strong topology of the Šerstnev space (L^0, ν, τ_W), $f_k \xrightarrow{stat} \theta_S$.

Proof. Since (b) and (c) are equivalent by definition, it suffices to establish the equivalence of (a) and (b).

Let $K(\lambda) = \{k \in \mathbb{N} \colon P(\|f_k\| < \lambda) \leq 1 - \lambda\}$. We note that $f_k \xrightarrow{stat(P)} \theta_S$ if and only if $\delta(K(\lambda)) = 0$. But $\delta(\mathbb{N} \backslash K(\lambda)) = 1$. Therefore, for every $k \in \mathbb{N} \backslash K(\lambda)$, we have $P(\|f_k\| < \lambda) > 1 - \lambda$. By (2.9), this implies that $\nu_{f_k}(\lambda) > 1 - \lambda$. By the property of strong topology, we observe that

$$\begin{aligned} \{k \in \mathbb{N} \colon d_L(\nu_{p_k}, \varepsilon_0) < \lambda\} &\supseteq \{k \in \mathbb{N} \colon \nu_{f_k}(\lambda) > 1 - \lambda\}\} \\ &\supseteq \{k \in \mathbb{N} \colon P(\|f_k\| < \lambda) > 1 - \lambda\}, \end{aligned}$$

which means,

$$\{k \in \mathbb{N} \colon d_L(\nu_{p_k}, \varepsilon_0) \geq \lambda\}) \subseteq \{k \in \mathbb{N} \colon P(\|f_k\| < \lambda) \leq 1 - \lambda\}.$$

Hence,

$$\delta(\{k \in \mathbb{N} \colon d_L(\nu_{p_k}, \varepsilon_0) \geq \lambda\}) \leq \delta(\{k \in \mathbb{N} \colon P(\|f_k\| < \lambda) \leq 1 - \lambda\}).$$

Since, $f_k \xrightarrow{stat(P)} \theta_S$, we have $\delta(\{k \in \mathbb{N} \colon d_L(\nu_{f_k}, \varepsilon_0) \geq \lambda\}) = 0$, hence $\nu_{f_k} \xrightarrow{stat} \varepsilon_0$. □

4 Statistical convergence in L^p

In order to consider statistical convergence in L^p with $p \in [1, +\infty]$, the following result connecting the L^p norms ($\|\cdot\|_p$) with the probabilistic norm (2.9) will be needed (see [12]).

Theorem 13. *Let* $L^p = L^p(\Omega, \mathcal{A}, P) := \{f \in L^0 \colon \int_\Omega |f|^p dP < +\infty\}$ *for* $p \in [1, +\infty[$ *and* $L^\infty := \{f \in L^0 \colon \|f\|_\infty := ess\ sup\, |f| < +\infty\}$. *If the probabilistic norm* $\nu \colon L^0 \to \Delta^+$ *is defined by*

$$\nu_f(\lambda) := P\{\omega \in \Omega \colon |f(\omega)| < \lambda\}, \quad \lambda > 0,$$

then for every $f \in L^p$, $\|f\|_p = \left(\int_{\mathbb{R}_+} \lambda^p d\nu_f(\lambda)\right)^{1/p}$ *and for every* $f \in L^\infty$, $\|f\|_\infty = \sup\{t > 0 \colon \nu_f(\lambda) < 1\}$.

With the help of the previous result one can characterize statistical convergence in L^p. As in the previous section, there is no loss of generality in considering only convergence to θ_V, for, if one wishes to study the statistical convergence of a sequence $\{f_n\}$ to $f \neq \theta_V$, it suffices to replace $\{f_n\}$ by $\{f_n - f\}$.

Theorem 14. *For a sequence of (equivalence classes of) E-valued random variable* $\{f_k\}$ *in* L^p, *the following statements are equivalent:*
if $p \in [1, +\infty)$:
(a) $\{f_k\}$ *statistically converges to* θ_S *in* L^p, $f_k \overset{stat(L^p)}{\longrightarrow} \theta_S$:
(b) the sequence of the p-th moments of the probabilistic norms $\{\nu_{f_n}\}$ *statistically converges to 0, viz.*

$$\int_{\mathbb{R}_+} t^p d\nu_{f_k}(t) \longrightarrow 0 \quad \textit{for a.a.k};$$

if $p = +\infty$:
(c) $\{f_k\}$ *statistically converges to* θ_V *in* L^∞, $f_k \overset{stat(L^\infty)}{\longrightarrow} \theta_S$;
(d) for every $\lambda > 0$, *the sequence* $\{\nu_{f_k}(\lambda)\}$ *is definitely equal to 1 for a.a.k.*

Proof. (a) $\Leftrightarrow$ (b) We note that $f_k \overset{stat(L^p)}{\longrightarrow} \theta_S \Leftrightarrow \delta(\{k \in \mathbb{N} \colon \|f_k\|_p \geq \lambda\} = 0$ for every $\lambda > 0$. But then

$$\delta(\{k \in \mathbb{N} \colon \left(\int_{\mathbb{R}_+} t^p d\nu_{f_k}(t)\right)^{1/p} \geq \lambda\} = \delta(\{k \in \mathbb{N} \colon \|f_k\|_p \geq \lambda\}.$$

Since the right hand side is zero, we have

$$\delta(\{k \in \mathbb{N} \colon \int_{\mathbb{R}_+} t^p d\nu_{f_k}(t) \geq (\lambda)^p\} = 0.$$

This clearly implies that $\int_{\mathbb{R}_+} t^p d\nu_{f_k}(t) \longrightarrow 0$ for a.a.k.

$(c) \Rightarrow (d)$ Assume $f_k \overset{stat(L^\infty)}{\longrightarrow} \theta_S$, i.e., $\|f_k\|_\infty \overset{stat}{\longrightarrow} 0$, and let $t > 0$; then for every $\epsilon \in (0,t)$, one has

$$\delta(\{k \in \mathbb{N}\colon \sup\{t > 0\colon \nu_{f_k}(t) < 1\} \geq \epsilon\}) = \delta(\{k \in \mathbb{N}\colon \|f_k\| \geq \epsilon\}) = 0.$$

This means that

$$\sup\{t > 0\colon \nu_{f_k}(t) < 1\} < \epsilon \quad \text{for a.a.k.}$$

But then, for a.a.k, $\nu_{f_k}(t) \geq \nu_{f_k}(\epsilon) = 1$.

$(d) \Leftrightarrow (c)$ For $t > 0$, let $\nu_{f_k}(t) = 1$ for a.a.k; therefore, $\|f_k\|_\infty < t$ for a.a.k, which yields $f_k \overset{stat(L^\infty)}{\longrightarrow} \theta_S$. □

5 Almost sure statistical convergence

We begin with the following definition of almost surely statistical convergence of random variables:

Definition 15. *A sequence $\{f_k\}$ of E-valued random variables is said to be statistical convergent almost surely to θ_E, the null vector of E, provided that for every $t > 0$,*

$$\delta(\{k \in \mathbb{N}\colon P(|f_k| < t) < 1\}) = 0,$$

viz. for every $t > 0$, $P(|f_k| < t) = 1$ for a.a.k. In this case we write $f_k \overset{stat\ a.s.}{\longrightarrow} \theta_E$

Consider the family $V = \{L^0(\mathcal{A})\}^{\mathbb{N}}$ of all sequences of (equivalence classes of) E-valued random variables. The set V is a real vector space with respect to the componentwise operations; specifically, if $s = \{f_k\}$ and $s' = \{g_k\}$ are two sequences in V and if α is a real number, then the sum $s \oplus s'$ of s and s' and the scalar product $\alpha \odot s$ of α and s are defined via

$$s \oplus s' := \{f_k\} \oplus \{g_k\} := \{f_k + g_k\}, \quad \alpha \odot s = \alpha \odot \{f_k\} := \{\alpha f_k\}.$$

A mapping $\phi\colon V \to \Delta^+$ will be defined on V via

$$\phi_s(x) := P\left(\sup_{k \in \mathbb{N}} |f_k| < x\right) = P\left(\bigcap_{k \in \mathbb{N}} \{|f_k| < x\}\right),$$

where $x > 0$ and $s = \{f_k\}$. In [14], it is proved that the triple (V, ϕ, τ_W) is a Šerstnev space.

Given an element s of V, viz. given a sequence $s = \{f_k\colon k \in \mathbb{N}\}$ of E-valued random variables, $f_k \in L^0(\mathcal{A})$ for every $k \in \mathbb{Z}_+ := \{0, 1, \cdots\}$, consider the n-shift s_n of s, $s_n := \{f_{k+n}\colon k \in \mathbb{N}\}$, which again belongs to V.

Theorem 16. *A sequence* $s = \{f_k : k \in \mathbb{N}\}$ *of* E*-valued random variables statistically converges almost surely to* θ_E*, if and only if, the sequence* $\{\phi_{s_n} : n \in \mathbb{N}\}$ *of the probabilistic norms of the* n*-shifts of* s *statistically converges (weakly) to* ε_0*, or, if and only if, the sequence* $\{s_n\}$ *of the* n*-shifts of* s *converges statistically to* $\mathcal{O} := \{\theta_E, \theta_E, \cdots\}$ *in the strong topology of* (V, ϕ, τ_W)*.*

Proof. All statements are equivalent to the assertion, which holds for every $t > 0$,

$$\begin{aligned} 0 = \delta(\{n \in \mathbb{N}\colon \phi_{s_n}(t) < 1\}) &= \delta(\{n \in \mathbb{N}\colon P\left(\bigcap_{k\in\mathbb{N}} \{|f_{k+n}| < t\}\right) < 1\}) \\ &\geq \delta(\{n \in \mathbb{N}\colon P\left(\bigcap_{k\geq n} \{|f_k| < t\}\right) < 1\}) \\ &\geq \delta(\{k \in \mathbb{N}\colon P(|f_k| < t) < 1\}). \end{aligned}$$

This proves the result. ☐

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