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The problems at investigation of state of stress of thick orthotropic plate

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Abstract

The determination of state of stress of thick orthotropic plate is outlined here. The analytical formulations of dispersion curves for arbitrary direction of wave propagation in orthotropic plate, which we defined earlier, are used to obtain results. The false roots appearing in the numerical computation of thick plate dispersion curves from orthotropic materials are mentioned. The displacements and the stresses of the thick orthotropic plate are presented too.

Keywords: wave propagation, orthotropy, thick plate, dispersion curves

1. Introduction

The determination of the dispersion curves of orthotropic plates is very important for ultrasonic nondestructive method based on the guided waves. The measured dispersion curves are compared to the computed dispersion curves of thick infinitely extended plate with free boundary conditions at this method. The distributions of displacements and stresses across the plate thickness play important role in the location of ultrasonic transducers on plate surface.

The analytical formulations of dispersion curves for arbitrary direction of wave propagation in orthotropic plate of thickness $2d$ are resumed in the following chapter. The direction of wave propagation is defined by the angle ϕ, the angle between the [100] axis and the wave vector, see fig. 1. The computed dispersion curves are presented for the unidirectional composite. The problems with false roots of dispersion curves are mentioned here too.

Another two chapters describe the displacements and stresses in the thick orthotropic plate. The results are given for several directions of wave propagation and for the first ten dispersion curves.

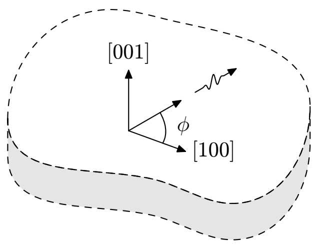

Fig. 1. The scheme of the problem.

2. Dispersion curves

To obtaining analytical dispersion formulae, we have used the method of partial waves, [1]. In this method, the plate wave solutions are constructed from simple exponential-type waves, which reflect back and forth between the boundaries of the plate. Every partial wave have to

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have the same value $k_x = k = \omega/v$, where v is the plate wave phase velocity. The solution has the form $u_j = \alpha_j \exp\left[ik(x + l_z z)\right]$, $j = x, y, z$ and $l_z = k_z/k_x$, for every partial wave solution. u_j are components of displacement, k_j are components of wave vector and α_j are components of polarization of partial waves. By substitution of trial solution into Christoffel equation, we obtain a system of three homogeneous linear equations for α_x, α_y and α_z, [2]. Nontrivial solutions exist only when the determinant of the system equals zero. This gives a sixth order polynomial in l_z with six roots $l_z^{(n)},\ n = 1, \ldots, 6$. The allowed partial wave solutions defined by these roots correspond to the three incident and three reflected waves. The coupling between partial waves at the plate boundaries is given by three boundary conditions for stress $\boldsymbol{T}$

$$T_{xz} = T_{yz} = T_{zz} = 0, \qquad \text{for} \qquad z = \pm d. \tag{1}$$

These conditions are satisfied by taking a linear combination of the six allowed partial waves,

$$u_j = \sum_{n=1}^{6} C_n\, \alpha_j^{(n)} \exp\left[ik(x + l_z^{(n)} z)\right], \quad (j = x, y, z). \tag{2}$$

Substitution of partial waves into the boundary conditions (1) gives a system of six homogeneous linear equations, in which the coefficients, C_n, are now functions of ρ, c_{IJ}, $\omega/k = v$ and kd. Here ρ is density and c_{IJ} are elastic coefficients. Nontrivial solutions exist only when the determinant of the system equals zero, and this defines the dispersion relation between ω and k.

2.1. Orthotropic plate

As was remarked above, it follows for orthotropic thick plate:

1. Direction of wave propagation $\phi = 0°$ and $\phi = 90°$

   The dispersion formula can be written for symmetric modes as

$$\tan\left(l_z^{(1)} kd\right) A - \tan\left(l_z^{(3)} kd\right) B = 0, \tag{3}$$

   and for antisymmetric modes as

$$\cot\left(l_z^{(1)} kd\right) A - \cot\left(l_z^{(3)} kd\right) B = 0. \tag{4}$$

   For $\phi = 0°$

$$A = \left(c_{13}\,\alpha_x^{(1)} + c_{33}\,\alpha_z^{(1)}\,l_z^{(1)}\right)\left(\alpha_x^{(3)}\,l_z^{(3)} + \alpha_z^{(3)}\right),$$
$$B = \left(c_{13}\,\alpha_x^{(3)} + c_{33}\,\alpha_z^{(3)}\,l_z^{(3)}\right)\left(\alpha_x^{(1)}\,l_z^{(1)} + \alpha_z^{(1)}\right),$$

$$\alpha_x^{(n)} = -(c_{13} + c_{55}) l_z^{(n)},\ \alpha_z^{(n)} = c_{11} + c_{55} {l_z^{(n)}}^2 - \rho v^2 \quad \text{for } n = 1, 3.$$

   and for $\phi = 90°$

$$A = \left(c_{23}\,\alpha_x^{(1)} + c_{33}\,\alpha_z^{(1)}\,l_z^{(1)}\right)\left(\alpha_x^{(3)}\,l_z^{(3)} + \alpha_z^{(3)}\right),$$
$$B = \left(c_{23}\,\alpha_x^{(3)} + c_{33}\,\alpha_z^{(3)}\,l_z^{(3)}\right)\left(\alpha_x^{(1)}\,l_z^{(1)} + \alpha_z^{(1)}\right),$$

$$\alpha_x^{(n)} = -(c_{23} + c_{44}) l_z^{(n)},\ \alpha_z^{(n)} = c_{22} + c_{44} {l_z^{(n)}}^2 - \rho v^2 \quad \text{for } n = 1, 3.$$

The dispersion formula for SH modes is very simple,

$$\sin\left(2\, l_z^{(5)} kd\right) = 0 \quad \Rightarrow \quad l_z^{(5)} = N\pi/2kd \qquad \text{for} \qquad N = 0, 1, 2, \ldots \tag{5}$$

2. Direction of wave propagation $0^\circ < \phi < 90^\circ$
   The dispersion formula can be written for symmetric modes as

$$\cot\left(l_z^{(1)} kd\right)\ A + \cot\left(l_z^{(3)} kd\right)\ B + \cot\left(l_z^{(5)} kd\right)\ C = 0, \tag{6}$$

   and for antisymmetric modes as

$$\tan\left(l_z^{(1)} kd\right)\ A + \tan\left(l_z^{(3)} kd\right)\ B + \tan\left(l_z^{(5)} kd\right)\ C = 0, \tag{7}$$

   where

$$\begin{aligned}
A &= \left(D_x^{(3)} D_y^{(5)} - D_x^{(5)} D_y^{(3)}\right) \left[D_z^{(1)} + \left(E_x^{(1)} - E_y^{(1)}\right) \cos\phi\ (c_{13} - c_{23})\right]\ c_{44} c_{55}, \\
B &= \left(D_x^{(5)} D_y^{(1)} - D_x^{(1)} D_y^{(5)}\right) \left[D_z^{(3)} + \left(E_x^{(3)} - E_y^{(3)}\right) \cos\phi\ (c_{13} - c_{23})\right]\ c_{44} c_{55}, \\
C &= \left(D_x^{(1)} D_y^{(3)} - D_x^{(3)} D_y^{(1)}\right) \left[D_z^{(5)} + \left(E_x^{(5)} - E_y^{(5)}\right) \cos\phi\ (c_{13} - c_{23})\right]\ c_{44} c_{55},
\end{aligned}$$

$$\left.\begin{aligned}
D_x^{(n)} &= \alpha_x^{(n)}\, l_z^{(n)} + \alpha_z^{(n)}, \\
D_y^{(n)} &= \alpha_y^{(n)}\, l_z^{(n)}, \\
D_z^{(n)} &= \alpha_z^{(n)}\, l_z^{(n)}\, c_{33} + \alpha_x^{(n)}\, c_{23}, \\
E_x^{(n)} &= \alpha_x^{(n)}\ \cos\phi, \\
E_y^{(n)} &= \alpha_y^{(n)}\ \sin\phi, \\
\alpha_x^{(n)} &= c_{33}\, g_1\, {l_z^{(n)}}^4 + (g_1\, g_2 + c_{33}\, g_3 - g_4^2)\ {l_z^{(n)}}^2 + g_2\, g_3, \\
\alpha_y^{(n)} &= -c_{33}\, g_5\, {l_z^{(n)}}^4 + (g_4\, g_6 + c_{33}\, g_7 - g_2\, g_5)\ {l_z^{(n)}}^2 + g_2\, g_7, \\
\alpha_z^{(n)} &= \left((g_4\, g_5 - g_1\, g_6)\ {l_z^{(n)}}^2 - (g_3\, g_6 + g_4\, g_7)\right)\ l_z^{(n)},
\end{aligned}\right\} \quad \text{for } n = 1, 3, 5$$

$$\begin{aligned}
g_1 &= \sin^2\phi\, c_{55} + \cos^2\phi\, c_{44}, \\
g_2 &= \sin^2\phi\, c_{44} + \cos^2\phi\, c_{55} - \rho\, v^2, \\
g_3 &= \sin^2\phi \cos^2\phi\, (c_{11} - 2\, c_{12} + c_{22} - 4\, c_{66}) + c_{66} - \rho\, v^2, \\
g_4 &= \sin\phi \cos\phi\, (c_{23} - c_{13} + c_{44} - c_{55})\,, \\
g_5 &= \sin\phi \cos\phi\, (c_{44} - c_{55})\,, \\
g_6 &= \sin^2\phi\ (c_{23} + c_{44}) + \cos^2\phi\ (c_{13} + c_{55})\,, \\
g_7 &= \sin\phi \cos\phi\, (c_{12} - c_{22} + 2\, c_{66} + (c_{11} - 2\, c_{12} + c_{22} - 4\, c_{66})\ \cos^2\phi)\,.
\end{aligned}$$

The planes of symmetry ($\phi = 0^\circ$ and 90°) were investigated at first. In this case, the SH modes are uncoupled from the symmetric and antisymmetric modes. The dispersion relations for direction of wave propagation are shown for $\phi = 0^\circ$ in fig. 2 and for $\phi = 90^\circ$ in fig. 5. The dispersion formulae for direction of wave propagation $\phi = 30^\circ$ are displayed in fig. 3 and for $\phi = 60^\circ$ in fig. 4.

The material parameters for computations were: $c_{11} = 128.2$ GPa, $c_{22} = c_{33} = 14.95$ GPa, $c_{44} = 3.81$ GPa, $c_{55} = c_{66} = 6.73$ GPa, $c_{12} = c_{13} = 6.9$ GPa, $c_{23} = 7.33$ GPa and $\rho = 1580$ kg/m^3 (carbon composite), [3].

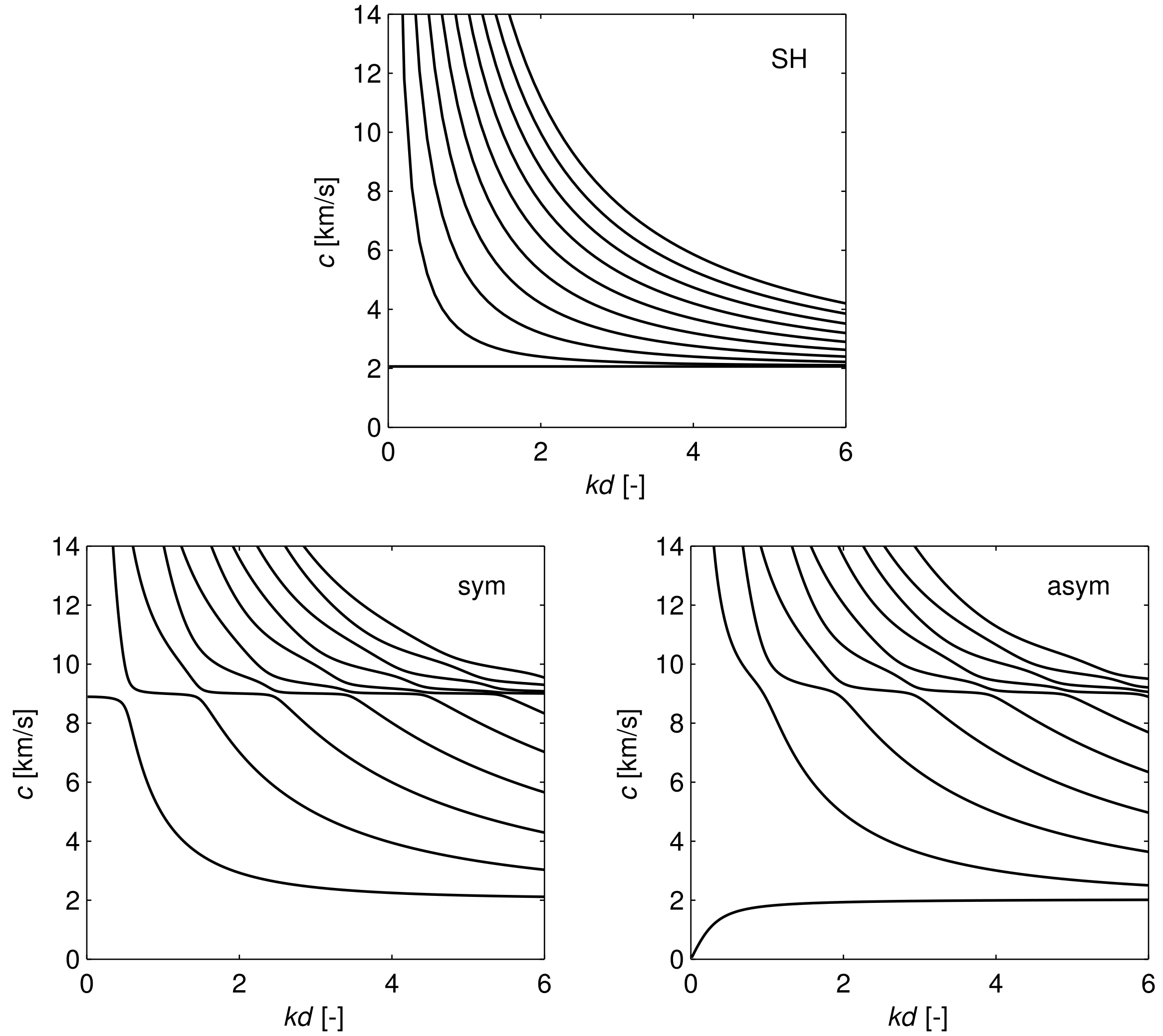

Fig. 2. The dispersion curves for direction of wave propagation 0°.

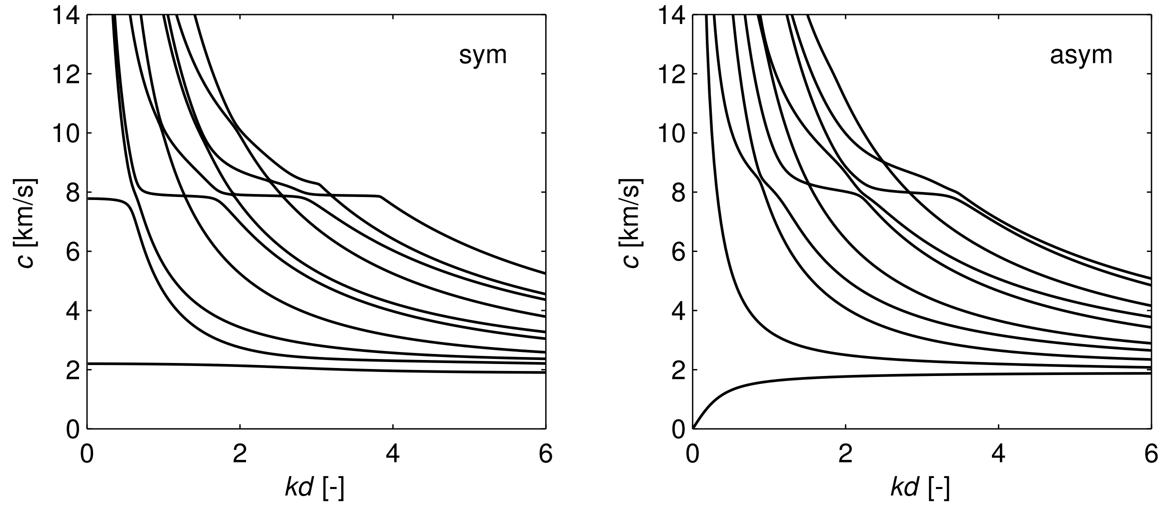

Fig. 3. The dispersion curves for direction of wave propagation 30°.

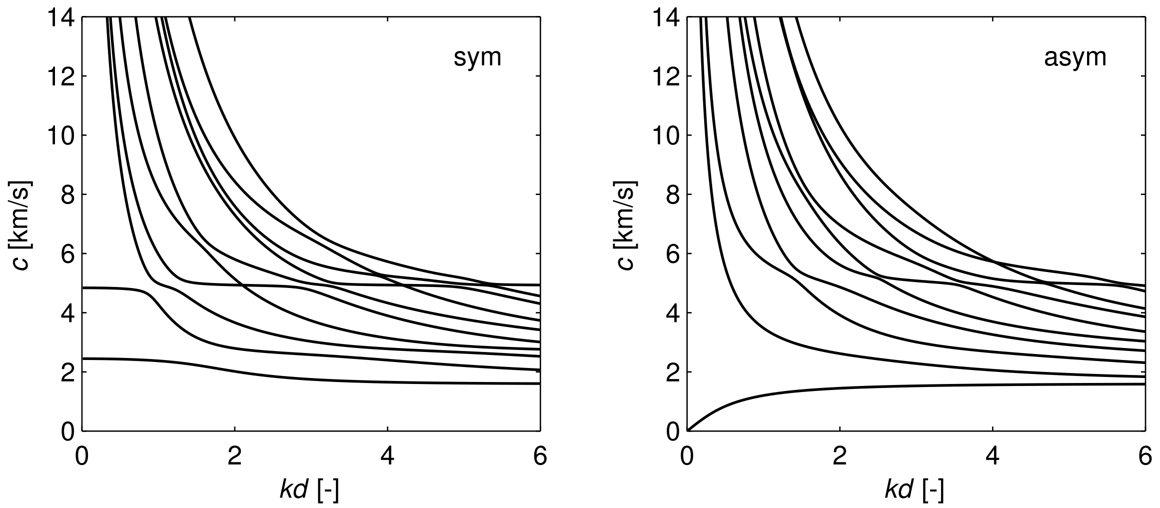

Fig. 4. The dispersion curves for direction of wave propagation $60°$.

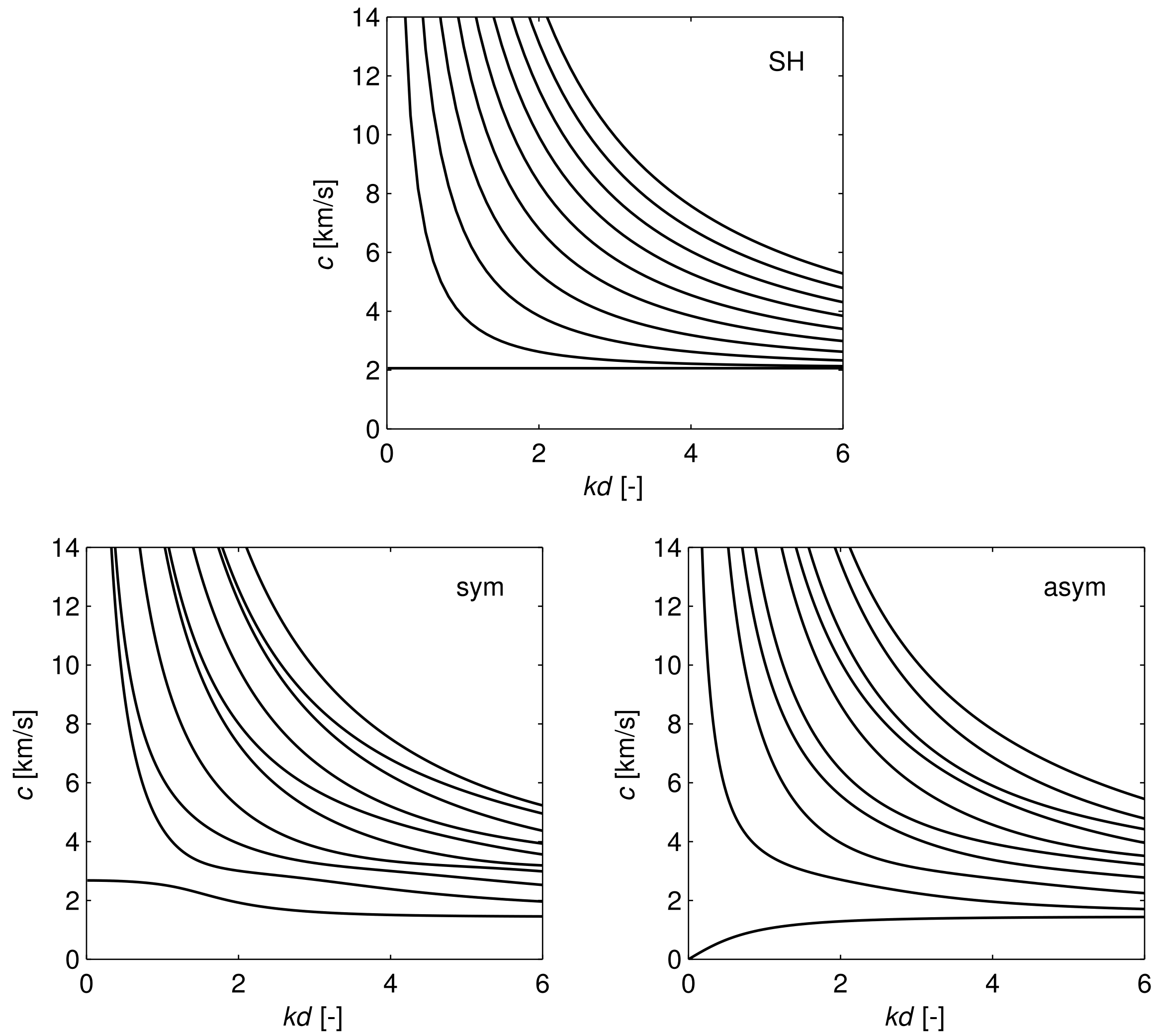

Fig. 5. The dispersion curves for direction of wave propagation $90°$.

2.2. False curves

During computing of the dispersion curves for orthotropic thick plate, we met the occurrence of false dispersion curves. A plot of computed dispersion branches for symmetric modes for direction of wave propagation $\phi = 45°$ is shown in fig. 7 - left. Two false nondispersion curves are displayed by dashed lines in the figure. These curves intersect the other curves, that is in conflict with theory. Since we were interested in the reason of false root occurrence, we plotted the left-hand side of equation (6), see fig. 7 - right. The location of false roots is marked out by arrows. The function intersects zero value as it is seen in the figure. Hence, the numerical programme detected the intersections with zero as valid roots of dispersion curves. We had to find some other method how to remove these false roots. We plotted graphs of parameters A, B, and C from equation (6) in a vicinity of these false roots, see fig. 8 (real part is black, imaginary is grey). It is clear from figure that all of three parameters A, B, and C are equal zero for the false roots. Hence, the equation (6) is valid. The programme was updated for detection of the false roots. The detection is based on finding zero values of all three parameters A, B, and C.

A plot of computed dispersion branches of antisymmetric modes is shown in fig. 9 - left for direction of wave propagation $\phi = 45°$. Only one nondispersion curve appeared in the figure. This curve intersects again the other curves. Left-hand side of equation (7) is plotted in fig. 9 - right. From this figure, it is again clear that function intersects zero value. In the same way as in symmetric case, the false root is caused by zero values of three parameters A, B, and C, see fig. 8. Following analysis shows that displacements for symmetric as well as antisymmetric modes vanish in these false roots.

The distributions of false roots (velocities) on direction of wave propagation for both modes are shown in fig. 6.

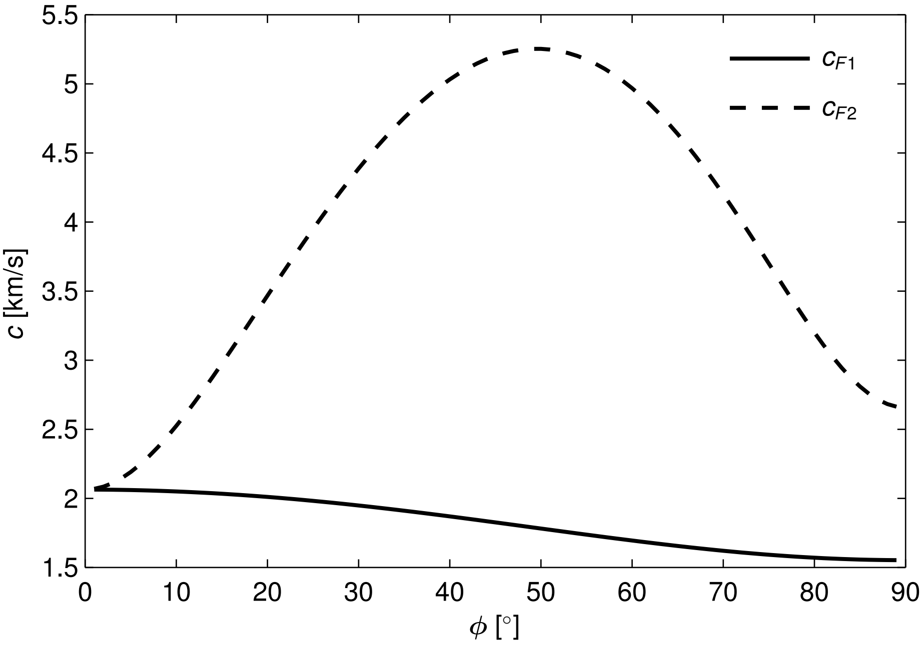

Fig. 6. False velocities for angle ϕ.

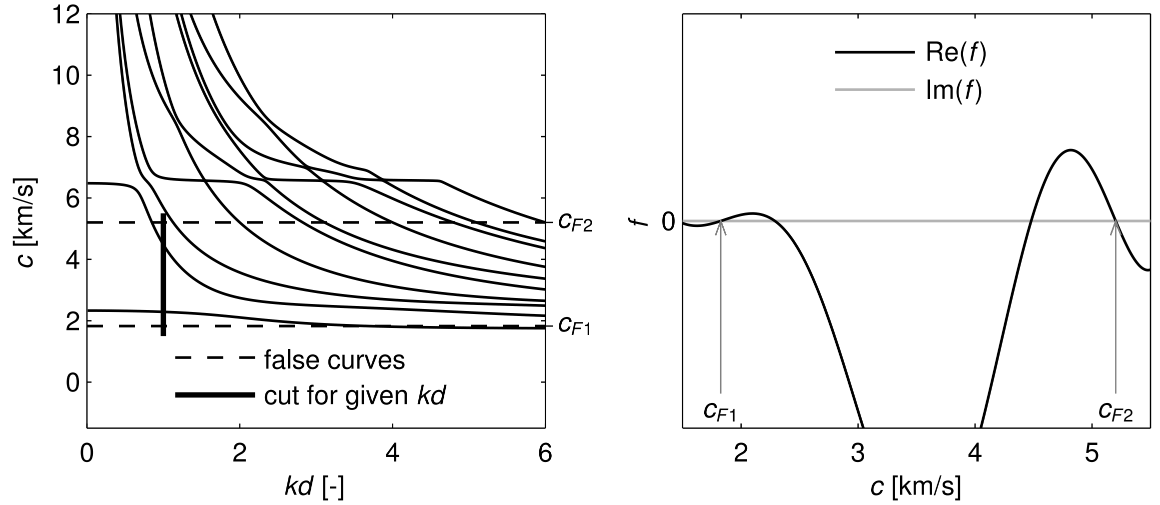

Fig. 7. The dispersion curves for symmetric modes and $\phi = 45°$; plot of function (6).

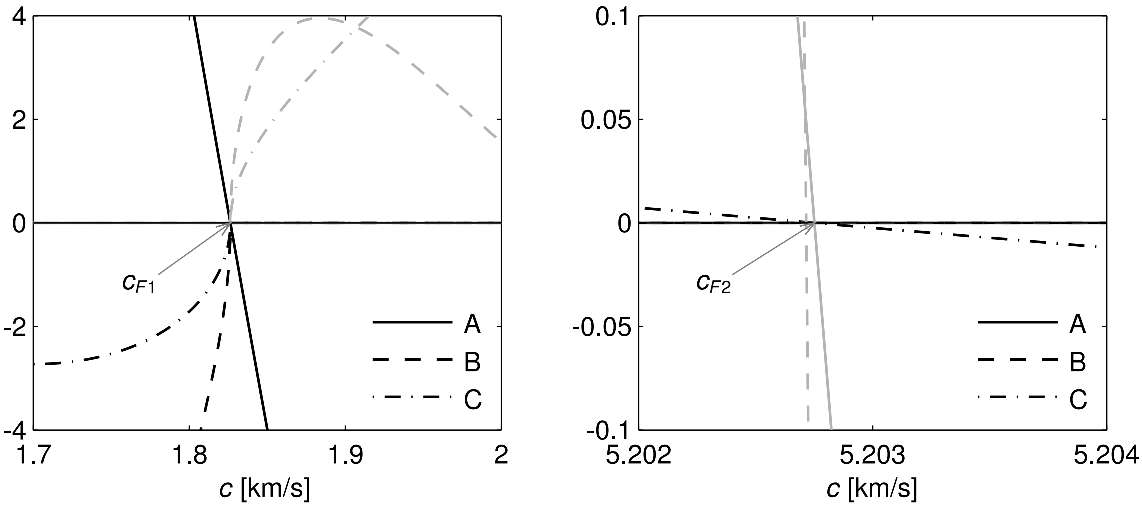

Fig. 8. Plot of parameters A, B, and C for $\phi = 45°$.

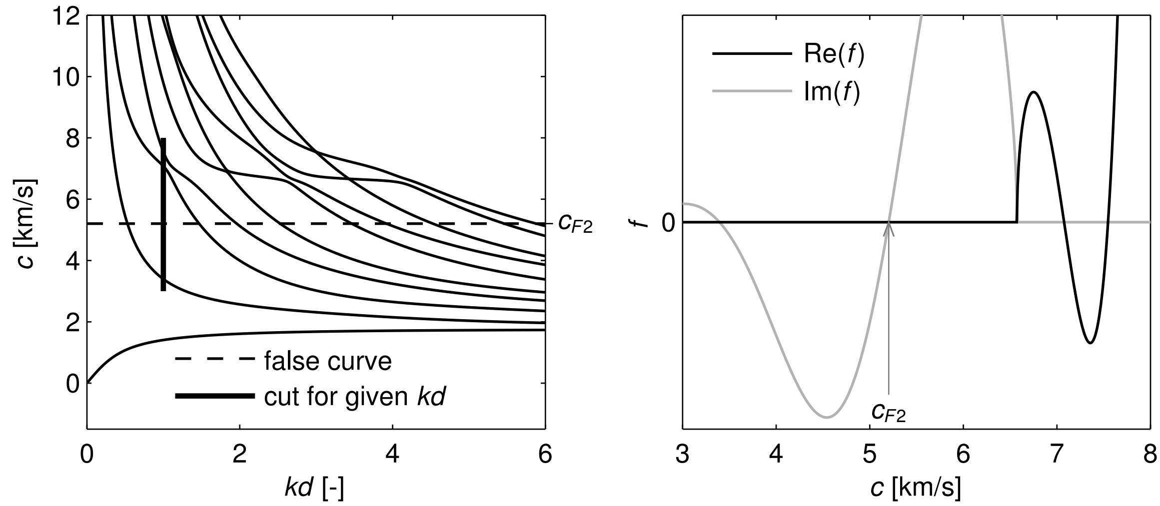

Fig. 9. The dispersion curves for antisymmetric modes and $\phi = 45°$; plot of function (7).

3. Displacements

The plate displacements were computed according to eq. (2). The displacements $u_x(z)$ and $u_z(z)$ for direction of wave propagation $\phi = 30°$, $\phi = 45°$, and $\phi = 60°$ are shown for the first ten dispersion curves in fig. 10. The displacements for symmetric modes are shown on the left side of the figure and for antisymmetric modes on the right side of the figure.

Note the displacements $u_x(z)$ for symmetric modes and the displacements $u_z(z)$ for antisymmetric modes are even functions of z. On the other hand, the displacements $u_z(z)$ for symmetric modes and the displacements $u_x(z)$ for antisymmetric modes are odd functions of z. It is in agreement with the theory.

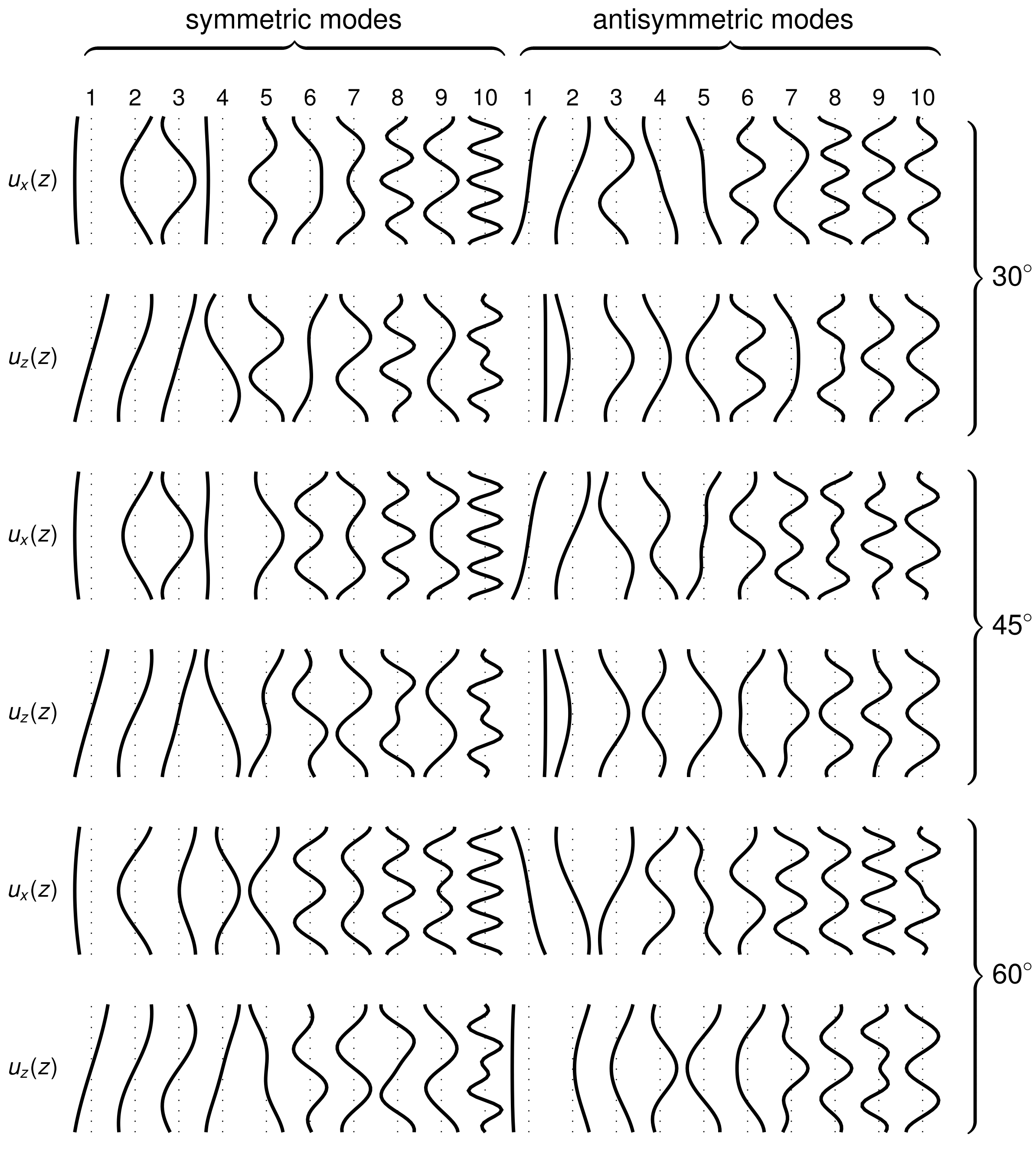

Fig. 10. Displacements u_x and u_z for $\phi = 30°$, $45°$, and $60°$.

4. Stresses

The stresses T_{xz}, T_{yz}, and T_{zz} are shown for direction of wave propagation $\phi = 30°$, $\phi = 45°$, and $\phi = 60°$ and the first ten dispersion curves in fig. 11. The stresses for symmetric modes are shown on the left side of the figure and for antisymmetric modes on the right side of the figure.

The stresses $T_{xz}(z)$ and $T_{yz}(z)$ for symmetric modes and the stresses $T_{zz}(z)$ for antisymmetric modes are odd functions of z, while the stresses $T_{zz}(z)$ for symmetric modes and the stresses $T_{xz}(z)$ and $T_{yz}(z)$ for antisymmetric modes are even functions of z. Note that according to boundary conditions (1) the stresses on the surfaces vanish.

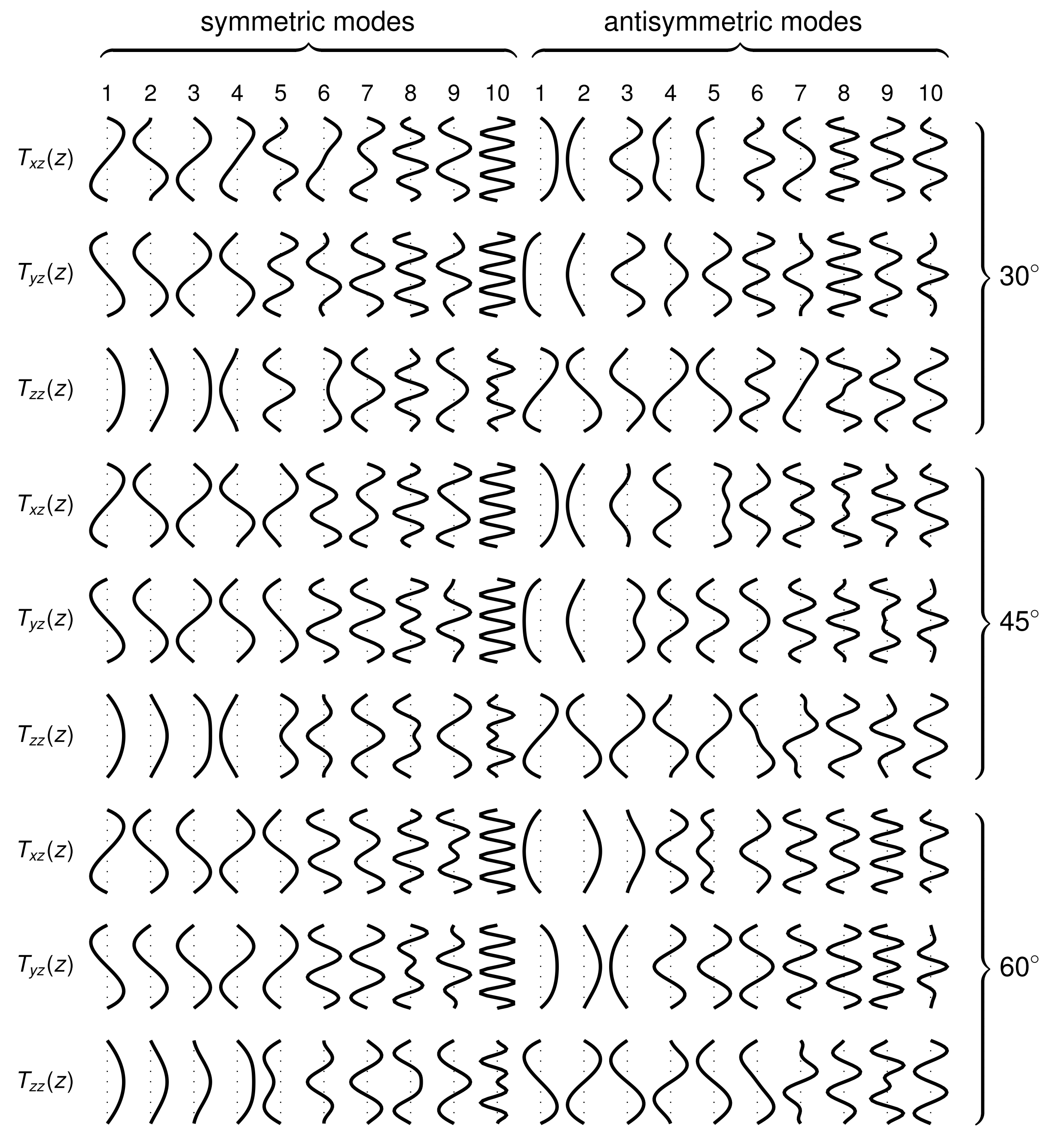

Fig. 11. Stresses T_{xz}, T_{yz}, and T_{zz} for $\phi = 30°$, 45°, and 60°.

5. Conclusion

The knowledge of the dispersion curves of orthotropic plates is very important for advanced ultrasonic nondestructive method, for example the guided wave method. The distributions of displacements and stresses across the plate thickness play important role in the location of ultrasonic transducers on plate surface.

The analytical formulation of dispersion curves was used for obtaining of the state of stress of thick plate for arbitrary direction of wave propagation in orthotropic plate. We used the system for symbolic calculation Maple, [4], for derivation of dispersion curves. The computed dispersion curves are presented for the unidirectional composite.

The false roots, appearing in the numerical computation of thick plate dispersion curves from orthotropic materials, were mentioned and analyzed. The algorithm for removing of false dispersion curves was included into programme for calculation of dispersion curves.

Displacements and stresses are presented for certain directions of wave propagation and for the first ten dispersion curves at the end of the paper.

All calculation were performed in MATLAB, [5].

Acknowledgements

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