CORE Metadata, citation and similar papers at core.ac.uk
Provided by Directory of Open Access Journals

ANALELE UNIVERSITĂȚII
"EFTIMIE MURGU" REȘIȚA
ANUL XX, NR. 1, 2013, ISSN 1453 - 7397

Cornel Haţiegan, Gilbert-Rainer Gillich, Eugen Răduca, Marian-Dumitru Nedeloni, Lenuţa Cîndea

Equation of Motion and Determining the Vibration Mode Shapes of a Rectangular Thin Plate Simply Supported on Contour Using MATLAB

This paper presents the differential biharmonic equation of thin plates through which, the vibration mode shapes for a rectangular thin plate simply supported on contour were obtained. Also, the first four vibration mode and the first four natural frequencies of this rectangular thin plate of steel, were obtained. Using MATLAB software, the vibration mode shapes were graphically represented.

Keywords: *Equation of motion, vibration mode shapes, MATLAB*

1. Introduction

Various construction contain in their structure, plates of various shapes and sizes which means that plates have a great importance in mechanical engineering. In the study of plates, there are two well-known manners of scientific approach:

A. The first approach is based on the development in a series of the functions of stress and strain, on the Z coordinate, as we know it from Cauchy. By keeping a minimum possible number of terms into these series, Sophie-Germaine equation was obtained [1]. Using this approach, Navier solved the problems of bending and stability for a rectangular plate with all edges simply supported. The analysis of plate behavior in general, was made by Poisson, who introduced a further condition for imposing the boundary conditions on contour.
B. The second approach that led to a technical theory of plates is owed to Kirchhoff. He introduced the hypothesis of the straight normal. The Mindlin-Reissner theory generalizes Kirchhoff theory, which can obviously be applied to plates of any thickness [2], [3].

In the present paper, for the study of plane plates, in accordance with [4], [5], [6], the following hypotheses are used:

a. the plate is made of a continuous, homogeneous and isotropic material;
b. for moderate loads, the plate is elastic and Hooke law applies;
c. the deformations are negligible compared to the unit;
d. the displacements are small compared to the thickness of the plate and the equilibrium can be written on the undeformed shape;
e. when the plate is deformed, there are no linear deformations and displacements on the mid-surface.

2. Formulation of problem and equation of motion

It is consider a homogeneous and isotropic plate of a constant thickness h, with a uniformly distributed mass and subjected to loads in the direction normal to its plane, with the mid-surface in the XOY plane. The deformed shape of the mid-surface Z(x,y) satisfies the Sophie-Germaine differential biharmonic equation of the plane plate with a constant thickness as follows [5], [7], [8], [9], [10]:

$$\frac{\partial^4 Z}{\partial x^4} + 2\frac{\partial^4 Z}{\partial x^2 \partial y^2} + \frac{\partial^4 Z}{\partial y^4} = \frac{p(x,y)}{D}, \tag{1}$$

where:

$D = \frac{Eh^3}{12(1-\upsilon)}$ is the bending rigidity of the plate, Z=Z(x,y) is the mid-surface displacement of the plate on the normal direction, h is the thickness of the plate and υ is the Poisson coefficient.

When the plate undergoes vibrations on the Z direction, the $W(x,y,t)$ displacements are added to the static arrows Z(x,y), whereas the momentum per surface unit $-\rho h \frac{\partial^2 W}{\partial t^2}$ and the perturbation forces $p_0(x,y,t)$ are added to the static load.

These displacements verify the differential equation of the 4th order:

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} + \frac{\rho h}{D}\frac{\partial^2 W}{\partial t^2} = \frac{p_0(x,y,t)}{D}, \tag{2}$$

where:

$$\lambda^4 = \omega^2 \frac{\rho \cdot h}{D}. \tag{2'}$$

In the case of a rectangular plate simply supported on two parallel edges: x=0 and x=a, the function $Z_{mn}(x,y)$ has the form:

$$Z_{mn}(x,y) = Y_m(y)\sin\left(\frac{n\cdot\pi\cdot x}{a}\right), \tag{3}$$

where: $n \in N^*$.

If the plate is simply supported on the edges: x=0 and x=a, the arrow and the bending moment parallel to the edge are zero and satisfy in this case the limit conditions [5], [7-10]:

$$Z(0,y) = 0; \quad Z(a,y) = Y_m(y)\sin x\pi = 0, \quad (n = 1,2,3,\ldots) \tag{4}$$

$$\left.\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right)\right|_{\substack{x=0\\x=a}} = 0. \tag{5}$$

Introducing the function (3) in the differential equation of the 4th order (2), it is obtain for the function $Y_m(y)$ a linear differential equation of the 4th order with constant coefficients:

$$\frac{d^4Y}{dy^4} - 2\frac{n^2\pi^4}{a^2}\cdot\frac{d^2Y}{dy^2} + \left(\frac{n^4\pi^4}{a^4} - \lambda^4\right)Y = 0. \tag{6}$$

From the characteristic equation:

$$r^4 - 2\frac{n^2\pi^2}{a^2}r^2 + \left(\frac{n^4\pi}{a^4} - \lambda^4\right) = 0, \tag{7}$$

it is obtain the solution:

$$r^2 = \frac{n^2\pi^2}{a^2} \pm \lambda^2. \tag{8}$$

Using the notations:

$$\alpha^2 = \lambda^2 - \frac{n^2\pi^2}{a^2}, \quad \beta^2 = \lambda^2 + \frac{n^2\pi^2}{a^2}, \tag{9}$$

the solutions as the following form are obtained: $r^2 = \beta^2$, with $r_{1,2} = \pm\beta$ and $r^2 = \alpha^2$, with $r_{3,4} = \pm i\alpha$, because: $\lambda^2 < \frac{n^2\pi^2}{a^2}$.

The general solution to the differential equation (6) is a function of the following form:

$$Y(y) = C_1 e^{r_1 \cdot y} + C_2 e^{r_2 \cdot y} + C_3 e^{r_3 \cdot y} + C_4 e^{r_4 \cdot y} = C_1 e^{\beta \cdot y} + C_2 e^{-\beta \cdot y} + C_3 e^{i\alpha \cdot y} + C_4 e^{-i\alpha \cdot y}. \tag{10}$$

The constants C_1, C_2, C_3, C_4 are determined from the boundary conditions that have to be satisfied on the other two edges parallel to the OX axis.

If the plate is simply supported on the edges: y=0 and y=b, for which the arrows and the bending moment parallel to the edge are null, the boundary conditions are:

$$Z(x,0) = Z(x,b) = 0 \Rightarrow Y_m(0) = Y_m(b) = 0 \tag{11}$$

$$\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right)_{\substack{y=0\\y=b}} = 0. \tag{12}$$

Plugging this condition into (3) and using the Euler relations, it will be:

$$\left[-\frac{m^2 \cdot \pi^2}{a^2} \cdot \cos\left(\frac{m \cdot \pi \cdot x}{a}\right) \cdot Y(y) + Y_m''(y) \cdot \sin\left(\frac{m \cdot \pi \cdot x}{a}\right)\right]_{\substack{y=0\\y=b}} = 0. \tag{13}$$

Then:

$$Y_m''(0) = Y_m''(b) = 0, \tag{14}$$

this means that the boundary conditions are satisfied.

Introducing the conditions (11) and (14) into the equation (10), the linear homogeneous system with the unknowns C_1, C_2, C_3, C_4 can be obtained:

$$\begin{cases} C_1 + C_2 + C_3 + C_4 = 0 \\ C_1 e^{\beta \cdot b} + C_2 e^{-\beta \cdot b} + C_3 e^{i\alpha \cdot b} + C_4 e^{-i\alpha \cdot b} = 0 \\ C_1 \beta^2 + C_2 \beta^2 - C_3 \alpha^2 - C_4 \alpha^2 = 0 \\ C_1 \beta^2 e^{\beta \cdot b} + C_2 \beta^2 e^{-\beta \cdot b} - \alpha^2 C_3 e^{i\alpha \cdot b} - \alpha^2 C_4 e^{-i\alpha \cdot b} = 0 \end{cases} \tag{15}$$

We get the constants C_1, C_2, C_3, C_4 from the boundary conditions that have to be satisfied on the other two edges, parallel to OX axis.

If, having the conditions that the system (15) also admits solutions other than the banal solution, the vibration mode equation can be obtained:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\beta\cdot b} & e^{-\beta\cdot b} & e^{i\alpha\cdot b} & e^{-i\alpha\cdot b} \\ \beta^2 & \beta^2 & -\alpha^2 & -\alpha^2 \\ \beta^2 e^{\beta\cdot b} & \beta^2 e^{-\beta\cdot b} & -\alpha^2 e^{i\alpha\cdot b} & -\alpha^2 e^{-i\alpha\cdot b} \end{vmatrix}$$
$$= (\alpha^2+\beta^2)^2[e^{i\alpha\cdot b}(e^{\beta\cdot b}-e^{-\beta\cdot b})-e^{-i\alpha\cdot b}(e^{\beta\cdot b}-e^{-\beta\cdot b})] =$$
$$= (\alpha^2+\beta^2)^2(e^{\beta\cdot b}-e^{-\beta\cdot b})(e^{i\alpha\cdot b}-e^{-i\alpha\cdot b})$$
$$= (\alpha^2+\beta^2)^2 4\cdot i\cdot \mathrm{sh}(\beta\cdot b)\cdot \sin(\alpha\cdot b) = 0 \quad (16)$$

By solving the vibration mode equation (16) it can be obtained:

$$(\alpha^2+\beta^2)^2 4\cdot i\cdot \mathrm{sh}(\beta\cdot b)\cdot \sin(\alpha\cdot b) = 0. \quad (17)$$

From the (17) equation it can see that:

$\sin(\alpha\cdot b) = 0$, which means that:
$\alpha_m\cdot b = m\cdot\pi,\quad m\in N^*$ and: $\beta = 0.$ (18)

From the expression (18) can be observed that for every natural different from zero value of m, there is a infinity of values of the $\beta_m b$ argument, so:

$$\alpha_m^2 = \frac{m^2\pi^2}{b^2}. \quad (19)$$

Introducing the value of α_m^2 into (9), according to [5], [7], [8], [9], [10], the characteristic equation of the form can be obtained:

$$\lambda_{mn}^2 = \beta_{mn}^2 + \frac{n^2\pi^2}{a^2} = \pi^2\left(\frac{m^2}{b^2}+\frac{n^2}{a^2}\right); m,n\in N^*. \quad (20)$$

Put (2’) into (20), the natural pulsation of motion that has the form can be obtained:

$$\omega_{mn} = \lambda_{mn}^2\sqrt{\frac{D}{\rho\cdot h}} = \pi^2\left(\frac{m^2}{b^2}+\frac{n^2}{a^2}\right)\sqrt{\frac{D}{\rho\cdot h}}. \quad (21)$$

From (21) it can notice that there is a double infinity of natural pulsations with which the plate can perform free harmonic vibrations.

For the integration constants C_1, C_2, C_3, C_4 results that $C_1 = C_2 = C_3 = 0$ and $C_4 \neq 0$. Then the solution to the differential equation (6) has the form:

$$Y_m(y) = C_4 \sin \frac{m \cdot \pi \cdot y}{b}, \tag{22}$$

and the functions of movement Zmn(x,y) have the form:

$$Z_{mn}(x, y) = C_4 \sin \frac{n \cdot \pi \cdot x}{a} \sin \frac{m \cdot \pi \cdot y}{b}, \tag{23}$$

than $C_4 = 1$, because the values are normalized.

Once determined the self pulsations and the self functions, it can be write the expression of displacement in vibration in a self mode:

$$w_{mn}(x, y, t) = Z_{mn}(x, y) \sin(\omega_{mn} t + \theta_{mn}), \tag{24}$$

and the resulting motion is expressed as a superposition of modes:

$$w(x, y, t) = \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} Z_{mn}(x, y) \sin(\omega_{mn} t + \theta_{mn}). \tag{25}$$

The fundamental node of vibration of the rectangular plate corresponds to the values m=1 and n=1, being characterized by the self pulsation ω_{11} and by the self form Z_{11}, expressed by the particular relations:

$$\omega_{11} = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \sqrt{\frac{D}{\rho \cdot h}} \tag{26}$$

and:

$$Z_{11} = \frac{16}{m \cdot n \cdot \pi^2} \sin \frac{\pi \cdot x}{a} \sin \frac{\pi \cdot y}{b}. \tag{26'}$$

3. Simulation and the numerical results obtained

For a rectangular plate made of steel with the edges a=1000 mm and b=500 mm, with the Young module: $E = 2{,}1 \cdot 10^{11}$ N/m^2, the Poisson coefficient $\upsilon = 0{,}3$ and the density $\rho = 7850$ kg/m^3, though Microsoft Excel with the equation:

$$\omega_{mn} = \lambda_{mn}^2 \sqrt{\frac{D}{\rho \cdot h}} = \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{a^2} \right) \sqrt{\frac{D}{\rho \cdot h}}, \tag{27}$$

the first four vibration mode and natural frequencies are calculated in table 1, calculated values according with [11], where n şi m are mode numbers:

Table 1.

n	Angular frequency [rad/s]				Natural frequencies [Hz]			
	m=1	m=2	m=3	m=4	m=1	m=2	m=3	m=4
1	149.607	239.372	388.979	598.43	23.822	38.116	61.939	95.291
2	508.665	598.43	748.037	957.487	80.997	95.291	119.114	152.466
3	1107.09	1196.86	1346.47	1555.92	176.288	190.582	214.406	247.758
4	1944.9	2034.66	2184.27	2393.72	309.697	323.990	347.813	381.165

With MATLAB [12], it was determined their vibration mode shapes for the rectangular plate simply supported on the edges, mode shape presented in Figures 1 ÷ 16, Where it was used the step of 0,005 and the relationship:

$$Z_{mn}(x, y) = C_4 \sin \frac{n \cdot \pi \cdot x}{a} \sin \frac{m \cdot \pi \cdot y}{b}. \tag{28}$$

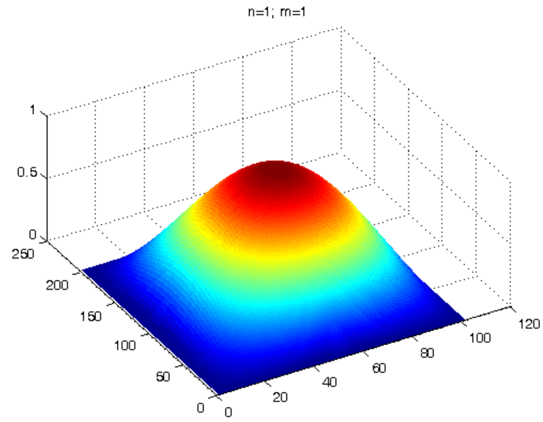


Figure 1. Vibration mode shapes, with n=1 and m=1.

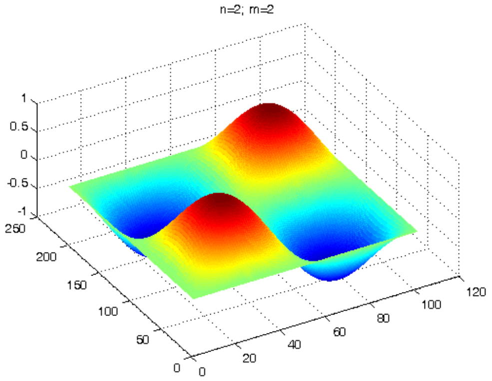


Figure 2. Vibration mode shapes, with n=2 and m=2.

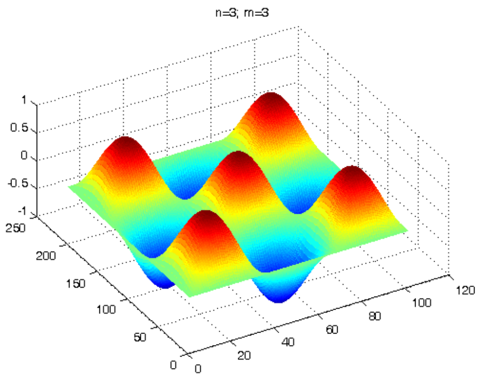


Figure 3. Vibration mode shapes, with n=3 and m=3.

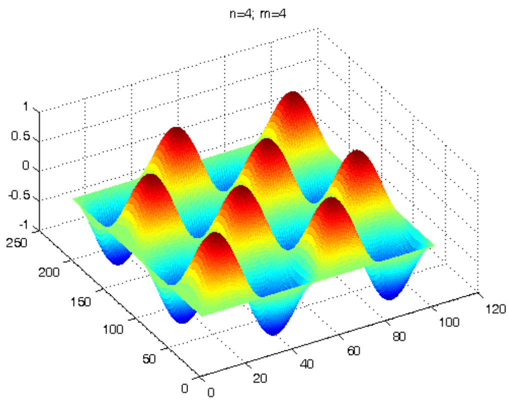
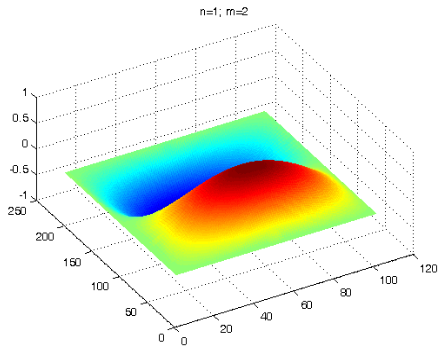


Figure 4. Vibration mode shapes, with n=4 and m=4.



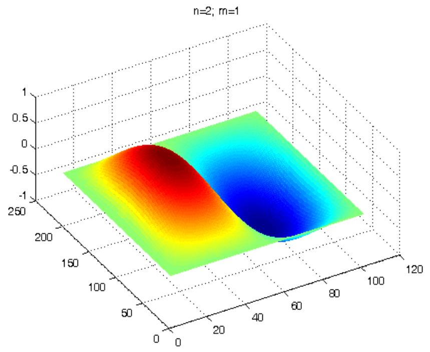
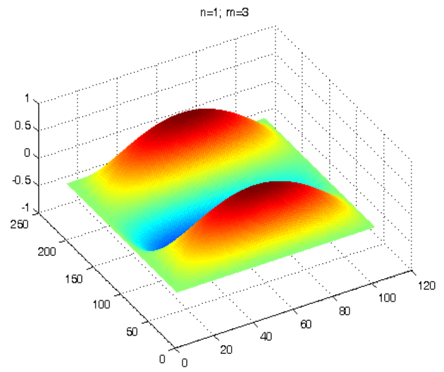


Fig. 5 Vibration mode, with n=1 and m=2. **Fig. 6** Vibration mode, with n=2 and m=1.



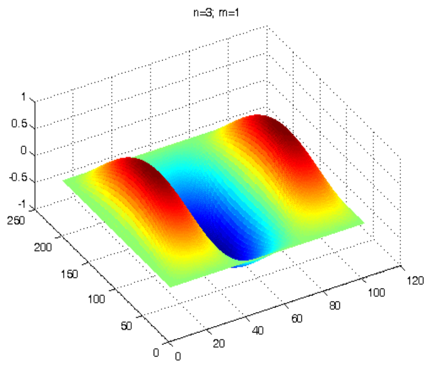
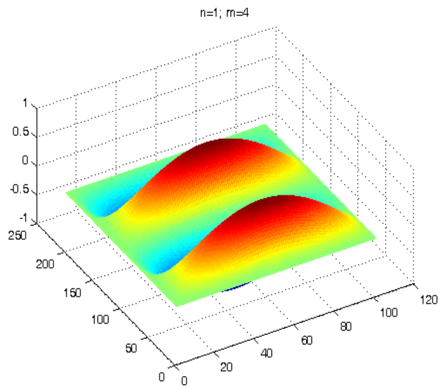


Fig. 7 Vibration mode, with n=1 and m=3. **Fig. 8** Vibration mode, with n=3 and m=1.



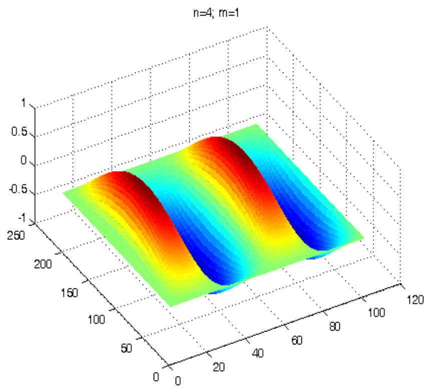


Fig. 9 Vibration mode, with n=1 and m=4. **Fig. 10** Vibration mode, with n=4 and m=1.

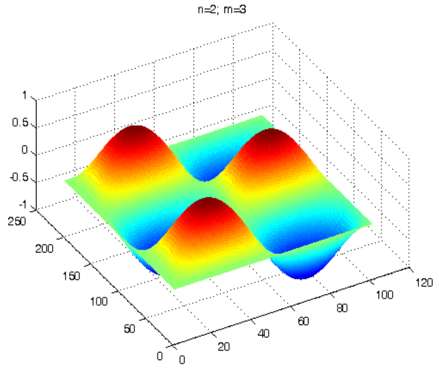


Fig. 11 Vibration mode, with n=2 and m=3.

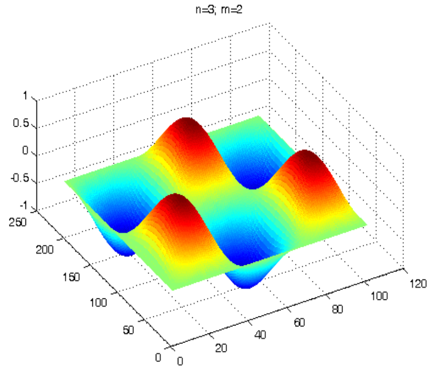


Fig. 12 Vibration mode, with n=3 and m=2.

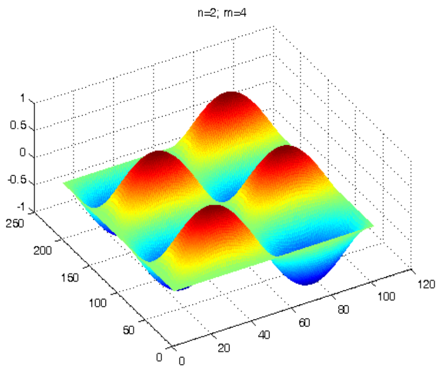


Fig. 13 Vibration mode, with n=2 and m=4.

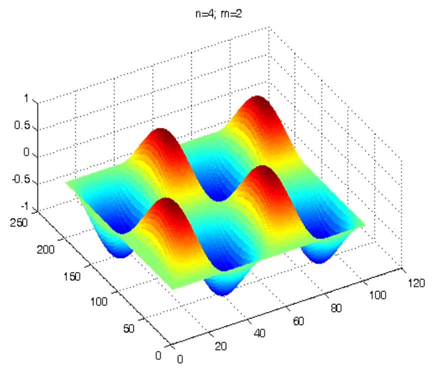


Fig. 14 Vibration mode, with n=4 and m=2.

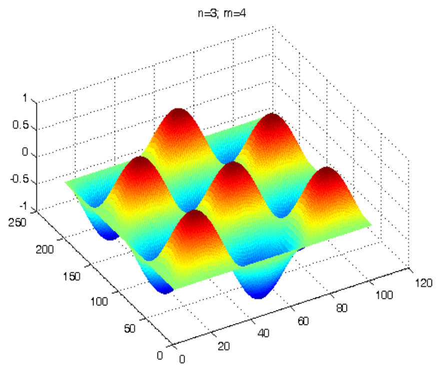


Fig. 15 Vibration mode, with n=3 and m=4.

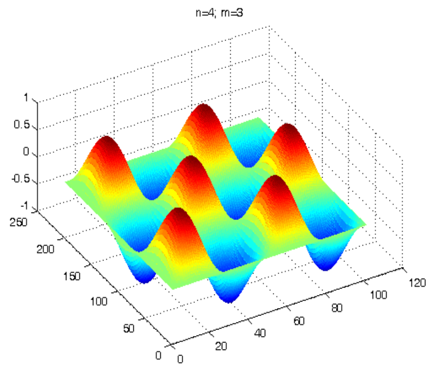


Fig. 16 Vibration mode, with n=4 and m=3.

4. Conclusion

The following conclusions can be made:

- it is observed from the mathematical model that the solution of the Y direction, can be achieved only if the solution of the characteristic equation is complex;
- in Table 1 and 2, it can be observed that the fundamental vibration mode, has the lowest value of their mode shapes, respectively the fundamental natural frequencies has the lower of their calculated natural frequencies;
- in Figures 2÷16, it is observed that the vibration mode shapes of the rectangular plate are damped harmonic, periodic harmonic and with sudden variations;
- the results can be used to validate finite element models.

Acknowledgements

The authors gratefully acknowledge the support of the Managing Authority for Sectoral Operational Programme for Human Resources Development (MASOPHRD), within the Romanian Ministry of Labour, Family and Equal Opportunities by co-financing the project "Investment in Research-innovation-development for the future (DocInvest)" ID 76813.

References

[1] Wu J.H., Liu A.Q., Chen H.L., *Exact Solutions for Free-Vibration Analysis of Rectangular Plates Using Bessel Functions*, Journal of Applied Mechanics, vol. 74, 2007, 1247-1251.

[2] Xing Y., Liu B., *Closed form solutions for free vibrations of rectangular Mindlin plates*, Acta Mech Sin, vol. 25, 2009, 689-698.

[3] Hashemi S.H., Arsanjani M., *Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates*, International J. of Solids and Structures, vol. 42, 2005, 819-853.

[4] Alămoreanu E., Buzdugan Gh., Iliescu N., Mincă I., Sandu M., *Îndrumar de calcul în ingineria mecanică (Guidelines for calculating in mechanical engineering)*, Editura Tehnică, Bucureşti, 1996.

[5] Vrabie M., *Calculul plăcilor (Calculus of plates)*, Editura Societăţii Academice "Matei-Teiu Botez", Iaşi, 2011.

[6] Gillich G.R., *Dinamica maşinilor. Vibraţii (Machine dynamics. Vibration)*, Editura AGIR, Bucureşti, 2006.

[7] Amza Gh., Barb D., Constantinescu F., *Sisteme ultraacustice (Ultra-acoustic systems)*, Editura Tehnică, Bucureşti, 1988.
[8] Timoshenko S.P., Gere J.M., *Teoria stabilităţii elastice (Theory of elastic stability)*, Editura Tehnică, Bucureşti, 1967.
[9] Buzdugan Gh., Fetcu L., Radeş M., *Vibraţii mecanice (Mechanical vibration)*, Editura Didactică şi Pedagogică, Bucureşti, 1982.
[10] Bratu P., *Vibraţiile sistemelor elastice (Vibrations of elastic systems)*, Editura Tehnică, Bucureşti, 2000.
[11] Leissa A.W, *Vibration of plates*, Scientific and Technical Information Division. Office of Technology Utilization, National Aeronautics And Space Administration, Washington DC., 1969.
[12] Predoi M.V., *Vibraţii mecanice. Modele şi aplicaţii în MATLAB (Mechanical vibration. Models and Applications in MATLAB)*, Editura Matrix Rom, Bucureşti, 2011.

Addresses:

- PhD. stud. Phys. Cornel Haţiegan, "Eftimie Murgu" University of Reşiţa, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, c.hatiegan@uem.ro
- Prof. PhD. Eng. Gilbert-Rainer Gillich, "Eftimie Murgu" University of Reşiţa, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, gr.gillich@uem.ro
- Prof. PhD. Eng. Eugen Răduca, "Eftimie Murgu" University of Reşiţa, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, e.raduca@uem.ro
- Asist. PhD. Eng. Marian-Dumitru Nedeloni, "Eftimie Murgu" University of Reşiţa, Piaţa Traian Vuia, nr. 1-4, 320085, m.nedeloni@uem.ro
- Lect. PhD. Eng. Lenuţa Cîndea, "Eftimie Murgu" University of Reşiţa, Piaţa Traian Vuia, nr. 1-4, 320085, Reşiţa, l.cindea@uem.ro