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A RETRACT PRINCIPLE ON DISCRETE TIME SCALES

Abstract. In this paper we discuss asymptotic behavior of solutions of a class of scalar discrete equations on discrete real time scales. A powerful tool for the investigation of various qualitative problems in the theory of ordinary differential equations as well as delayed differential equations is the retraction method. The development of this method is discussed in the case of the equation mentioned above. Conditions for the existence of a solution with its graph remaining in a predefined set are formulated. Examples are given to illustrate the results obtained.

Keywords: discrete equation, discrete time scale, asymptotic behavior of solution, retract, retraction.

Mathematics Subject Classification: 39A10, 39A11.

1. INTRODUCTION

We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$, where $s = -\infty$ or $q = \infty$ are admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$ (perhaps with another couple of integers), we always suppose $s \le q$. Moreover, in this paper we suppose that a time scale $\mathbb{T}$ is an arbitrary increasing sequence of real numbers, i.e., $\mathbb{T} := \{t_n\}$ with $t_n \in \mathbb{R}$, $n \in \mathbb{Z}_0^\infty$ and $t_n < t_{n+1}$ for any $n \in \mathbb{Z}_0^\infty$.

A powerful tool for the investigation of various problems in the field of ordinary differential equations as well as delayed differential equations is the retraction method (so-called *Ważewski's method*) described, e.g., in [6, 8] for ordinary differential equations and in [7] for delayed functional differential equations. In this paper we shall give, in the case of one scalar discrete equation, a construction in which the idea of retraction principle is used. The results obtained can be useful in the investigation of asymptotic behavior of solutions of indicated discrete equations.

Let us consider the scalar discrete equation

$$u(t_{n+1}) = f(t_n, u(t_n)), \tag{1}$$

where a mapping $f\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to its second argument. The used scale $\mathbb{T}$ is a special case of a so-called real time scale which is defined as an arbitrary nonempty closed subset of the real line [1, 5].

We consider initial problem (1), (2), where

$$u(t_s)=u^s\in\mathbb{R}, \tag{2}$$

$s\in\mathbb{Z}_0^\infty$ is fixed and $t_s\in\mathbb{T}$. The existence and uniqueness of a solution of initial problem (1), (2) on $\{t_q\}\subset\mathbb{T}$, $q\in\mathbb{Z}_s^\infty$ is obvious. The solution of initial problem (1), (2) is an infinite sequence of numbers

$$\{u(t_s)=u^s, u(t_{s+1}), u(t_{s+2}),\dots,u(t_{s+k}),\dots\}$$

such that equality (1) holds for each $n\in\mathbb{Z}_s^\infty$. Moreover, due to the continuity of the function f with respect to its second argument, initial problem (1), (2) continuously depends on initial data. We define a set $\omega\subset\mathbb{T}\times\mathbb{R}$ as

$$\omega:=\{(t,u):t\in\mathbb{T},\ b(t)<u<c(t)\},$$

where b, c are real functions defined on $\mathbb{T}$ and such that $b(t_n)<c(t_n)$, for each $n\in\mathbb{Z}_0^\infty$. The closure $\overline{\omega}$ is defined as

$$\overline{\omega}:=\{(t,u):t\in\mathbb{T},\ b(t)\le u\le c(t)\}$$

and the boundary $\partial\omega$ as

$$\partial\omega:=\{(t,u):t\in\mathbb{T},\ u=b(t)\quad\text{or}\quad u=c(t)\}.$$

Our aim is to establish sufficient conditions for the right-hand side of equation (1) in order to guarantee the existence of a solution $u=u(t_n)$ defined on the discrete real time scale $\mathbb{T}$ such that $(t_n,u(t_n))\in\omega$ for each $n\in\mathbb{Z}_0^\infty$. Results obtained significantly generalize (in the scalar case) some results given in [3, 4], where investigation of this problem have been performed under supposition that the independent variable varies within the set $N(a):=\{a,a+1,\dots\}$ with a nonnegative integer a. Obviously, $N(a)$ is a partial case of the general discrete time scale $\mathbb{T}$ and therefore these results are not applicable in the case of arbitrary discrete time scale. In Section 4 we apply our results to the investigation of asymptotic behavior of solutions of a discrete equation on the general discrete time scale. Moreover, we give illustration of the result obtained with the aid of two different discrete time scales.

2. PRELIMINARIES

We divide the boundary $\partial\omega$ into $B_1\subset\mathbb{T}\times\mathbb{R}$ and $B_2\subset\mathbb{T}\times\mathbb{R}$, where

$$B_1:=\{(t,u):t\in\mathbb{T},\ u=b(t)\},$$
$$B_2:=\{(t,u):t\in\mathbb{T},\ u=c(t)\}.$$

Farther, on $\mathbb{T} \times \mathbb{R}$, let us define the signed lower and upper distances

$$U_1(t,u) := u - b(t),$$
$$U_2(t,u) := u - c(t).$$

Definition 1. *For $(t,u) = (t_n, b(t_n)) \in B_1$, the full difference*

$$\Delta U_1(t,u) := f(t_n, b(t_n)) - b(t_{n+1})$$

is defined as the signed lower distance to B_1 of the result $f(t,u)$ of the n-th step of (1) *emanating from $u = b(t)$.*

Definition 2. *The point $(t,u) \in B_1$ is called a point of strict egress for the set ω with respect to* (1) *if*

$$\Delta U_1(t,u) < 0.$$

By analogy we define the following notions:

Definition 3. *For $(t,u) = (t_n, c(t_n)) \in B_2$, the full difference*

$$\Delta U_2(t,u) := f(t_n, c(t_n)) - c(t_{n+1})$$

is defined as the signed upper distance to B_2 of the result $f(t,u)$ of the n-th step of (1) *emanating from $u = c(t)$.*

Definition 4. *The point $(t,u) \in B_2$ is called a point strict egress for the set ω with respect to* (1) *if*

$$\Delta U_2(t,u) > 0.$$

Remark 1. *The geometrical sense of the notion of a point of strict egress is evident. Namely, if a point*

$$(t,u) = (t_n, b(t_n)) \in B_1,\ n \in \mathbb{Z}_0^\infty$$

is a point of strict egress for the set ω with respect to (1), *then the consequent point*

$$(t_{n+1}, f(t_n, b(t_n))) \notin \overline{\omega}.$$

Similarly, if

$$(t,u) = (t_n, c(t_n)) \in B_2,\ n \in \mathbb{Z}_0^\infty$$

is a point of strict egress for the set ω with respect to (1), *then*

$$(t_{n+1}, f(t_n, c(t_n))) \notin \overline{\omega}.$$

The point $(t_n, u) \in B_1 \cup B_2$ is a point of strict egress for the set ω with respect to (1) *if and only if*

$$f(t_n, u) - b(t_{n+1}) < 0$$

in the case of $(t_n, u) \in B_1$ *and*

$$f(t_n, u) - c(t_{n+1}) > 0$$

in the case of $(t_n, u) \in B_2$.

Now we recall the notion of a *retraction* and a *retract* (see, e.g., [6]).

Definition 5. *If* $A \subset B$ *are subsets in a topological space and* $\pi : B \to A$ *is a continuous mapping from* B *onto* A *such that* $\pi(p) = p$ *for every* $p \in A$*, then* π *is called a retraction of* B *onto* A*. When there exists a retraction of* B *onto* A, A *is called a retract of* B.

3. EXISTENCE THEOREM

The proof of the following theorem uses the retract idea. Namely, simplifying the matter, supposing that the statement of the theorem is not valid, we prove that there exists a retraction of a segment $[\alpha, \beta]$ with $\alpha < \beta$ onto the two-point set $\{\alpha, \beta\}$. It is well known that such a retraction cannot exist because such a retractive behavior is incompatible with continuity. This statement is a partial case of a more general result – the boundary of k-dimensional ball is not its retract (cf. e.g. [2]).

Theorem 1. *Let us suppose that* $f(w, u)$ *is defined on* $\mathbb{T} \times \mathbb{R}$ *and it is continuous with respect to its second argument. If, moreover,*

$$f(t_n, b(t_n)) - b(t_{n+1}) < 0, \tag{3}$$

$$f(t_n, c(t_n)) - c(t_{n+1}) > 0 \tag{4}$$

for any $n \in \mathbb{Z}_0^\infty$*, then there exists a value* $u^* \in (b(t_0), c(t_0))$ *such that the initial problem*

$$u(t_0) = u^*, \tag{5}$$

defines a solution $u = u^*(t_n)$ *of equation* (1) *satisfying*

$$b(t_n) < u^*(t_n) < c(t_n) \tag{6}$$

for every $n \in \mathbb{Z}_0^\infty$.

Proof. Let us suppose that a value u^* satisfying the inequality $b(t_0) < u^* < c(t_0)$ and generating the solution

$$u = u^*(t_n),\ u(t_0) = u^*$$

which satisfies (6) for any $n \in \mathbb{Z}_0^\infty$ does not exist. This means that for any $\hat{u}$ such that

$$b(t_0) < \hat{u} < c(t_0)$$

there exists $\hat{n} \in \mathbb{Z}_1^\infty$ such that, for the corresponding solution $u = \hat{u}(t_n)$ of initial problem $\hat{u}(t_0) = \hat{u}$, there is $(t_{\hat{n}}, \hat{u}(t_{\hat{n}})) \notin \omega$, while

$$(t_l, \hat{u}(t_l)) \in \omega,\ \ l = 0, 1, \ldots, \hat{n} - 1.$$

Since, in view of inequalities (3), (4) and Remark 1, each point $(w,u) \in B_1 \cup B_2$ is the point of strict egress for the set ω with respect to (1), we can also conclude the following. For any u^0 such that

$$b(t_0) \le u^0 \le c(t_0)$$

there exists a real number $n_0 \in \mathbb{Z}_1^\infty$ such that, for the corresponding solution $u = u^0(t_n)$ of the initial problem $u^0(t_0) = u^0$ there is

$$(t_{n_0}, u^0(t_{n_0})) \notin \overline{\omega}, \tag{7}$$

$$(t_{n_0-1}, u^0(t_{n_0-1})) \in \overline{\omega} \tag{8}$$

and, if $n_0 - 2 \ge 0$,

$$(t_l, u^0(t_l)) \in \omega, \;\; l = 0, 1, \dots, n_0 - 2. \tag{9}$$

This is also a consequence of the above underlying indirect assumption. Obviously, if $u^0 = b(t_0)$ or if $u^0 = c(t_0)$, then $n_0 = 1$.

In this situation we prove that there is a retraction of the set $[b(t_0), c(t_0)]$ onto the two-point set $\{b(t_0), c(t_0)\}$. (See Definition 5 with $B = [b(t_0), c(t_0)]$ and $A = \{b(t_0), c(t_0)\}$.) In other words, in this situation a continuous mapping of a closed interval onto its boundary would exist. This gives a contradiction.

In the following part of the proof, the required retraction is constructed. Let us define auxiliary mappings P_1, P_2 and P_3:

$$P_1 \, : (t_0, u^0) \to (t_{n_0}, u^0(t_{n_0})),$$

where the value n_0 was defined above by (7) – (9);

$$P_2 \, : (t_{n_0}, u^0(t_{n_0})) \to \begin{cases} (t_{n_0}, c(t_{n_0})) & \text{if} \quad u^0(t_{n_0}) > c(t_{n_0}), \\ (t_{n_0}, b(t_{n_0})) & \text{if} \quad u^0(t_{n_0}) < b(t_{n_0}), \end{cases}$$

and for $(t_{n_0}, \tilde{u}) \in \partial\omega$

$$P_3 \, : (t_{n_0}, \tilde{u}) \to \begin{cases} (t_0, c(t_0)) & \text{if} \quad \tilde{u} = c(t_{n_0}), \\ (t_0, b(t_0)) & \text{if} \quad \tilde{u} = b(t_{n_0}). \end{cases}$$

We will show that the composite mapping

$$P \, : (t_0, u^0) \to \{(t_0, b(t_0)), (t_0, c(t_0))\},$$

where

$$P := P_3 \circ P_2 \circ P_1,$$

is continuous with respect to the second coordinate of the point (t_0, u^0). Let us underline that, in view of the construction of mapping P, two result points are possible only, namely, either $P(t_0, u^0) = (t_0, c(t_0))$ or $P(t_0, u^0) = (t_0, b(t_0))$.

We consider the first possibility, i.e., $P(t_0, u^0) = (t_0, c(t_0))$. Then

$$P_1(t_0, u^0) = (t_{n_0}, u^0(t_{n_0})), \; (t_{n_0}, u^0(t_{n_0})) \notin \overline{\omega} \;\text{ and }\; u^0(t_{n_0}) > c(t_{n_0}).$$

We remark that in view of the construction, the value $u^0(t_{n_0})$ continuously depends on u^0. Therefore, the continuity of the mapping P_1 is a consequence of the property of the continuous dependence of an initial problem on its initial data. For small perturbations Δ^0 of u^0. such that $(t_0, u^0 + \Delta^0) \in \overline{\omega}$, in view of the properties of mappings P_2, P_3, we get

$$P(t_0, u^0 + \Delta^0) = (t_0, c(t_0)),$$

i.e., the composite mapping P is continuous in the case considered.

We proceed in the case when $P(t_0, u^0) = (t_0, b(t_0))$ analogously.

Thus the continuity of P has been proven if $b(t_0) \le u^0 \le c(t_0)$. Hence the required retraction is realized by P, because the mapping

$$[b(t_0), c(t_0)] \quad \xrightarrow{P} \quad \{b(t_0), c(t_0)\}$$

is continuous and

$$\{b(t_0)\} \xrightarrow{P} \{b(t_0)\},$$
$$\{c(t_0)\} \xrightarrow{P} \{c(t_0)\},$$

i.e., the points $\{b(t_0)\}$, $\{c(t_0)\}$ are stationary.

This is by above mentioned fact impossible. Our supposition is false and there exists initial problem (5) such that the corresponding solution $u = u^*(t_n)$ satisfies inequalities (6) for every $n \in \mathbb{Z}_0^\infty$. The theorem is proved. □

Theorem 1 can be generalized in the following way. As it easy follows from its proof, the assumptions with respect to the function $f(t, u)$ were used for values $(t, u) \in \overline{\omega}$ only, although they were supposed to be valid on $\mathbb{T} \times \mathbb{R}$. Therefore, we can reformulate this theorem. In view of the facts mentioned just before, we may omit the proof.

Theorem 2. *Let us suppose that f is defined on $\overline{\omega}$ with values in $\mathbb{R}$ and it is continuous with respect to its second argument. If, moreover, each point $(t, u) \in B_1 \cup B_2$ is a point of strict egress for the set ω with respect to* (1), *then there exists initial problem* (5) *with $u^* \in (b(t_0), c(t_0))$ such that the corresponding solution $u = u^*(t_n)$ satisfies inequalities* (6) *for every $n \in \mathbb{Z}_0^\infty$.*

4. APPLICATIONS

Let $\mathbb{T}$ be the time scale referred to above. We suppose $t_0 > 0$. In this section, we apply Theorem 1 to the investigation of the asymptotic behavior of a particular solution of the equation

$$u(t_{n+1}) = u(t_n) - \frac{a(t_n, u(t_n))}{t_n t_{n+1}} \,, \tag{10}$$

where $a\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a given function. Let us note that equation (10) with

$$a(t_n) := t_{n+1} - t_n$$

has a solution

$$u = u_p^*(t_n) = \frac{1}{t_n}\,, \ \ n \in \mathbb{Z}_0^\infty .$$

Our main purpose is to give sufficient conditions on a for equation (10) to have a solution $u = u_p$ which is in a sense close to u_p^*. In particular, we are interested when

$$u_p(t_n) \sim u_p^*(t_n) = \frac{1}{t_n} \tag{11}$$

as $n\to\infty$. At first we give general conditions for such behavior in the case of the discrete time scale $\mathbb{T}$. Then we will specify these conditions in the case of some concrete time scales.

Theorem 3. *Let there exist functions $b^*\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ and $c^*\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ such that for every $n\in\mathbb{Z}_0^\infty$ inequalities*

$$b^*(t_n) + c^*(t_n) > 0, \tag{12}$$

$$t_{n+2}(t_{n+1} - t_n - a(t_n, b(t_n))) < b^*(t_n)t_{n+2} - b^*(t_{n+1})t_n, \tag{13}$$

$$t_{n+2}(t_{n+1} - t_n - a(t_n, c(t_n))) > c^*(t_{n+1})t_n - c^*(t_n)t_{n+2} \tag{14}$$

hold. Then there exists a solution $u = u_p$ of equation (10) *such that the inequalities*

$$\frac{1}{t_n} - \frac{b^*(t_n)}{t_n t_{n+1}} < u_p(t_n) < \frac{1}{t_n} + \frac{c^*(t_n)}{t_n t_{n+1}} \tag{15}$$

hold for every $n\in\mathbb{Z}_0^\infty$.

Proof. In the case considered we have

$$f(t_n, u) = u - \frac{a(t_n, u)}{t_n t_{n+1}}\,.$$

We put

$$b(t_n) := \frac{1}{t_n} - \frac{b^*(t_n)}{t_n t_{n+1}}\,, \quad c(t_n) := \frac{1}{t_n} + \frac{c^*(t_n)}{t_n t_{n+1}}\,, \tag{16}$$

where $n\in\mathbb{Z}_0^\infty$. Due to (12), the inequality $b(t_n) < c(t_n)$ holds for any $n\in\mathbb{Z}_0^\infty$.

Then, due to (13)

$$\begin{aligned} f(t_n, b(t_n)) - b(t_{n+1}) &= \frac{1}{t_n t_{n+1} t_{n+2}} \times \\ &\times [t_{n+1}t_{n+2} - b^*(t_n)t_{n+2} - a(t_n, b(t_n))t_{n+2} - t_n t_{n+2} + b^*(t_{n+1})t_n] = \\ &= \frac{1}{t_n t_{n+1} t_{n+2}} \times \Big[(t_{n+2}(t_{n+1} - t_n - a(t_n, b(t_n)))) - \\ &\qquad - (b^*(t_n)t_{n+2} - b^*(t_{n+1})t_n) \Big] < 0 \end{aligned}$$

and inequalities (3) hold for every $n \in \mathbb{Z}_0^\infty$. Moreover, due to (14),

$$\begin{aligned} f(t_n, c(t_n)) - c(t_{n+1}) &= \frac{1}{t_n t_{n+1} t_{n+2}} \times \\ &\times [t_{n+1}t_{n+2} + c^*(t_n)t_{n+2} - a(t_n, c(t_n))t_{n+2} - t_n t_{n+2} - c^*(t_{n+1})t_n] = \\ &= \frac{1}{t_n t_{n+1} t_{n+2}} \times \Big[(t_{n+2}(t_{n+1} - t_n - a(t_n, c(t_n)))) + \\ &\qquad + (c^*(t_n)t_{n+2} - c^*(t_{n+1})t_n) \Big] > 0 \end{aligned}$$

and inequalities (4) hold for every $n \in \mathbb{Z}_0^\infty$ as well. All conditions of Theorem 1 are satisfied. Then inequalities (6) turn into inequalities (15). □

Remark 2. *Let the assumptions of Theorem 3 be valid and, moreover,*

$$\lim_{n\to\infty} \frac{b^*(t_n)}{t_{n+1}} = \lim_{n\to\infty} \frac{c^*(t_n)}{t_{n+1}} = 0. \tag{17}$$

Then, obviously, relation (11) *holds.*

Example 1. *We consider equation* (10) *with* $t_n := n + 1$ *and*

$$a(t_n, u(t_n)) := t_{n+1} - t_n - (t_{n+2})^{-1} ,$$

i.e., the equation

$$u(n+2) = u(n+1) - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} , \tag{18}$$

where $n \in \mathbb{Z}_0^\infty$. *Inequalities* (12)–(14) *are valid, e.g., for the choice* $b^* \equiv 1$, $c^* \equiv 0$ *and, in accordance with* (15), *there exists a solution* $u_p(t_n) = u_p(n+1)$ *of equation* (18) *such that the inequalities*

$$\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} < u_p(n+1) < \frac{1}{n+1}$$

hold for every $n \in \mathbb{Z}_0^\infty$. *Moreover, since relations* (17) *hold, there is*

$$u_p(n+1) \sim \frac{1}{n+1}$$

as $n \to \infty$.

As it can be verified easily, the same statements are valid, e.g., for the nonlinear equation

$$u(n+2) = u(n+1) - \frac{1}{(n+1)(n+2)} + \frac{1+|\cos u(n+1)|}{2(n+1)(n+2)(n+3)} \ ,$$

where $n \in \mathbb{Z}_0^\infty$, *which is a special case of equation* (10) *with*

$$a(t_n, u(t_n)) := t_{n+1} - t_n - \frac{(1+|\cos u(t_n)|)}{2t_{n+2}} \ .$$

Example 2. *We consider equation* (10) *with* $t_n := q^n$, *where* $q = \text{const}$, $q > 1$ *and*

$$a(t_n, u(t_n)) := t_{n+1} - t_n + (t_n)^{-1} \ ,$$

i.e., the equation

$$u(q^{n+1}) = u(q^n) - \frac{1}{q^n} + \frac{1}{q^{n+1}} - \frac{1}{q^{3n+1}} \ , \tag{19}$$

$n \in \mathbb{Z}_0^\infty$. *Inequalities* (12)–(14) *are valid, e.g., for*

$$b^* \equiv 0, \quad c^* = \text{const}, \quad c^* > \frac{q^2}{q^2-1}.$$

In accordance with (15), *there exists a solution* $u_p(t_n) = u_p(q^n)$ *of equation* (19) *such that the inequalities*

$$\frac{1}{q^n} < u_p(q^n) < \frac{1}{q^n} + \frac{c^*}{q^{2n+1}} \tag{20}$$

hold for every $n \in \mathbb{Z}_0^\infty$. *Moreover, since relations* (17) *hold, there is*

$$u_p(q^n) \sim \frac{1}{q^n}$$

as $n \to \infty$. *As it can be verified easily, the same statements are valid, e.g., for the nonlinear equation*

$$u(q^{n+1}) = u(q^n) - \frac{1}{q^n} + \frac{1}{q^{n+1}} - \frac{1}{2q^{2n+1}(q^{3n+1} + u^4(q^n))} \ ,$$

where $n \in \mathbb{Z}_0^\infty$ *which is a special case of equation* (10) *with*

$$a(t_n, u(t_n)) := t_{n+1} - t_n + \frac{1}{2(t_n^3 \cdot q + u^4(t_n))} \ .$$

We give (Fig. 1) a visualization of the situation described in Example 2. Corresponding functions b and c, defined by (16), are in this case given by the relations

$$b(t_n) = \frac{1}{q^n} \ , \quad c(t_n) = \frac{1}{q^n} + \frac{c^*}{q^{2n+1}} \ .$$

We put $q = 1,5$ and choose $c^* = 2$. Then we have (see (20))

$$\frac{1}{1,5^n} < u_p(1,5^n) < \frac{1}{1,5^n} + \frac{4}{3 \cdot 1,5^{2n}} \tag{21}$$

for $n = 0, 1, \ldots$.

The visualization is performed for $n = 0, 1, \ldots, 5$ on Figure 1. The graph of the solution u_p satisfying (21) lies in the hatched domain.

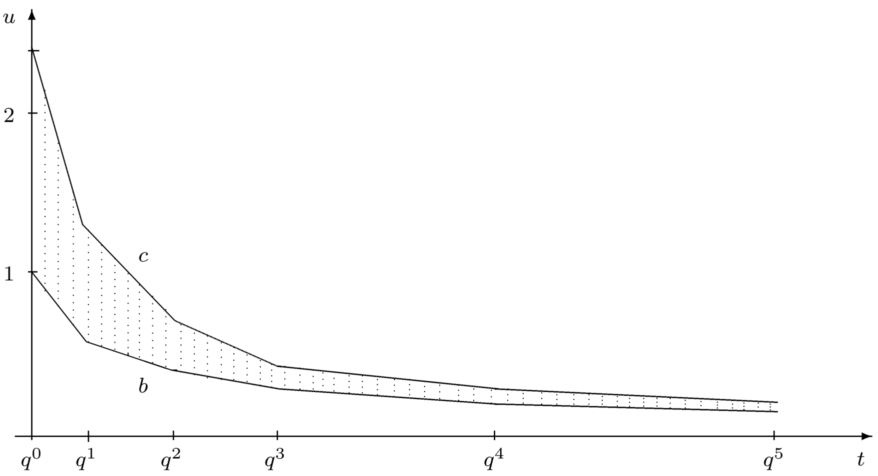

Fig. 1. Asymptotic behavior of solution – Example 2

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REFERENCES

[1] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales,* Birghäuser, 2001.

[2] K. Borsuk, *Theory of Retracts,* PWN, Warsaw, 1967.

[3] J. Diblík, *Asymptotic behavior of solutions of discrete equations,* Funct. Differ. Equ. **11** (2004), 37–48.

[4] J. Diblík, *Anti-Lyapunov method for systems of discrete equations,* Nonlinear Anal. **57** (2004), 1043–1057.

[5] S. Hilger, *Analysis on measure chains – a unified approach to continuous and discrete calculus,* Results Math. **18** (1990), 18–56.

[6] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities, Vol. I – Ordinary Differential Equations,* Academic Press, New York, London, 1969.

[7] K.P. Rybakowski, *Ważewski's principle for retarded functional differential equations,* J. Diff. Equat. **36** (1980), 117–138.

[8] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires,* Ann. Soc. Polon. Math. **20** (1947), 279–313.

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