Opuscula Math. 33, no. 4 (2013), 631–640
http://dx.doi.org/10.7494/OpMath.2013.33.4.631 Opuscula Mathematica

ON THE LONGEST PATH IN A RECURSIVELY PARTITIONABLE GRAPH

Julien Bensmail

Communicated by Mariusz Meszka

Abstract. A connected graph G with order $n \geq 1$ is said to be recursively arbitrarily partitionable (R-AP for short) if either it is isomorphic to K_1, or for every sequence $(n_1, \ldots, n_p)$ of positive integers summing up to n there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that each V_i induces a connected R-AP subgraph of G on n_i vertices. Since previous investigations, it is believed that a R-AP graph should be "almost traceable" somehow. We first show that the longest path of a R-AP graph on n vertices is not constantly lower than n for every n. This is done by exhibiting a graph family $\mathcal{C}$ such that, for every positive constant $c \geq 1$, there is a R-AP graph in $\mathcal{C}$ that has arbitrary order n and whose longest path has order $n - c$. We then investigate the largest positive constant $c' < 1$ such that every R-AP graph on n vertices has its longest path passing through $n \cdot c'$ vertices. In particular, we show that $c' \leq \frac{2}{3}$. This result holds for R-AP graphs with arbitrary connectivity.

Keywords: recursively partitionable graph, longest path.

Mathematics Subject Classification: 05C99, 68R10.

1. INTRODUCTION

Let $n \geq 1$ be a positive integer. A *n-graph* is a graph whose *order*, i.e. its number of vertices, is n. Throughout this paper, we denote by $LP(G)$ the order of the longest path in a given connected graph G. We say that G is *recursively arbitrarily partitionable* (R-AP for short) if and only if one of the following two conditions hold.

— The graph G is an isolated vertex.
— For every sequence $(n_1, \ldots, n_p)$ of positive integers that performs a partition of n, there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that $G[V_i]$ is a connected R-AP subgraph of G on n_i vertices for all $i \in \{1, \ldots, p\}$.

The property of being R-AP was introduced in [7] as a strengthened version of the property of being *arbitrarily partitionable*. The property of being AP was itself introduced to deal with a problem of resource sharing among an arbitrary number of users (see [1,2,5,8] for further details).

R-AP graphs have been mainly studied in the context of some simple classes of graphs like trees [7], a family of unicyclic 1-connected graphs called *suns* [6], and a class of 2-connected graphs called *balloons* [4,7]. Although these works did not lead to numerous general properties of R-AP graphs, they however suggest that the property of being R-AP is even closer to *traceability*[1)] than the one of being AP. For instance, we know that if T is a R-AP n-tree, then $LP(T) \geq n-2$. It was also empirically observed[2)] that if B is a R-AP n-balloon, then $LP(B) \geq n-4$. Such bounds do not exist regarding AP trees and AP balloons since the structure of these graphs is much less predictable (see [3] and [4], respectively).

Regarding these observations, one could naively think that there should exist a small positive constant $c \geq 1$ such that $LP(G) \geq n-c$ for every R-AP n-graph G. In this work, we first show, in Section 3, that such a constant does not exist by exhibiting a class $\mathcal{C}$ of R-AP graphs such that for every c there exists a n-graph C in $\mathcal{C}$ such that $LP(C) = n-c$ for some n. The graphs of $\mathcal{C}$ are 1-connected, but an equivalent result regarding 2-connected graphs is derived by slightly modifying our construction. We then investigate, in concluding Section 4, the greatest constant $c' \leq 1$ such that every R-AP n-graph has its longest path passing through $n \cdot c'$ of its vertices. In particular, we exhibit another family of graphs showing that $c' \leq \frac{2}{3}$. This upper bound also holds regarding ℓ-connected R-AP graphs, no matter what is the value of ℓ.

2. DEFINITIONS AND PRELIMINARY RESULTS

First observe that adding edges to a R-AP graph does not make it loose its property of being R-AP.

Remark 2.1. If G is spanned by a R-AP subgraph, then G is R-AP.

Because every path is clearly R-AP, the next result follows by Remark 2.1.

Remark 2.2. Every traceable graph is R-AP.

Determining whether a n-graph G is R-AP is laborious since, according to the original definition, one has to check whether G can be partitioned following every partition of n. We thus usually prefer to check the following equivalent condition which derives from the fact that a R-AP graph is partitionable into R-AP subgraphs at will.

Remark 2.3 ([7])**.** A connected n-graph G is R-AP if and only if for every $\lambda \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ there exists a partition $(V_\lambda, V_{n-\lambda})$ of $V(G)$ such that $G[V_\lambda]$ and $G[V_{n-\lambda}]$ are connected R-AP subgraphs of G on λ and $n-\lambda$ vertices, respectively.

Let us now introduce the following subclass of *caterpillar graphs*.

---

1) A *traceable* graph is a graph that has a Hamiltonian path.

2) Private communication.

Definition 2.4. Let $a, b \geq 2$ be two positive integers and consider three vertex-disjoint paths u_1u_2, $v_1, \ldots, v_a$ and $w_1, \ldots, w_b$ of order 2, a and b, respectively. The *caterpillar* $Cat(a, b)$ is the tree obtained by identifying the vertices u_1, v_1 and w_1.

Throughout this paper, every mention to caterpillar graphs actually refers to caterpillars of the form $Cat(a, b)$. Two examples of such caterpillars are given in Figure 1. This family of caterpillars is important regarding R-AP graphs since it was proven in [7] that most of R-AP trees are caterpillars of this kind. The authors of [7] also gave a complete characterization of R-AP caterpillars.

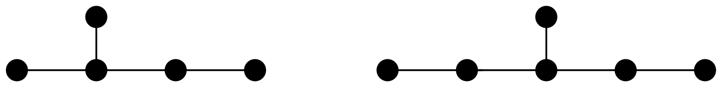

Fig. 1. The caterpillars $Cat(2, 3)$ and $Cat(3, 3)$

Theorem 2.5 ([7]). *A caterpillar $Cat(a, b)$ is R-AP if and only if a and b take values in Table 1.*

Table 1. Values a and b ($a \leq b$) such that $Cat(a, b)$ is R-AP

a	b
$2, 4$	$\equiv 1 \mod 2$
3	$\equiv 1, 2 \mod 3$
5	$6, 7, 9, 11, 14, 19$
6	7
7	$8, 9, 11, 13, 15$

3. LONGEST PATH AND ADDITIVE FACTOR

In this section, we prove the following result.

Theorem 3.1. *There does not exist a positive constant $c \geq 1$ such that we have $LP(G) \geq n - c$ for every R-AP n-graph G.*

This is proved by exhibiting a counterexample for every possible value of c. For this purpose, we introduce the family of *connected cycles* graphs.

Definition 3.2. Let $k \geq 1$ and $x, y \geq 0$ be three positive integers. The *connected cycles* graph $CC_k(x, y)$ is the graph with the following vertices:

— Let $u_1 \ldots u_x$ and $v_1 \ldots v_y$ be paths with order x and y, respectively.
— For every $i \in \{1, \ldots, k\}$, let $a_ib_ie_id_ic_ia_i$ be a cycle with length 5.
— For every $i \in \{1, \ldots, k-1\}$, let $w_{i,i+1}$ be a vertex.

These vertices are linked in $CC_k(x,y)$ in the following way: $u_x a_1, v_y e_k \in E(CC_k(x,y))$ and we have $w_{i,i+1}e_i, w_{i,i+1}a_{i+1} \in E(CC_k(x,y))$ for every $i \in \{1,\ldots,k-1\}$.

An example of a connected cycles graph is depicted in Figure 2. Notice that $LP(CC_k(1,1)) = |V(CC_k(1,1))| - k$. Thus, by showing that all graphs $CC_k(1,1)$ are R-AP, we can contradict the existence of the constant c mentioned in Theorem 3.1.

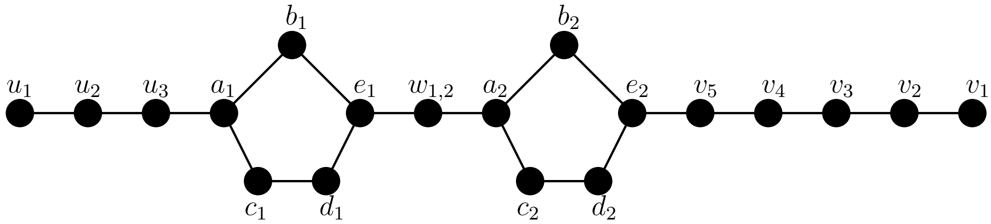

Fig. 2. The connected cycles graph $CC_2(3,5)$

Before proving that $CC_k(1,1)$ is R-AP for every k, we first introduce another graph structure we encounter while partitioning a connected cycles graph.

Definition 3.3. Let $k \geq 1$ and $x \geq 0$ be two positive integers. The *partial connected cycles* graph $PCC_k(x)$ is the graph obtained by removing the vertex e_k from $CC_k(x,0)$.

We are now ready to prove the main result of this section.

Lemma 3.4. *The graph $PCC_k(x)$ is R-AP for every $k \geq 1$ and $x \geq 1$ such that $x \not\equiv 2 \mod 3$. The graph $CC_k(x,y)$ is R-AP for every $k \geq 1$ and $x,y \geq 1$ whenever $x \not\equiv 2 \mod 3$ or $y \not\equiv 2 \mod 3$.*

Proof. The proof is by induction on k and uses the terminology introduced in Definition 3.2. For each value of k, we prove that the result is true for all possible values of x and (possibly) y which satisfy the claim. Recall that, according to Remark 2.3, a connected n-graph G is R-AP if and only if for every $\lambda \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$ we can partition $V(G)$ into two parts V_λ and $V_{n-\lambda}$ inducing connected R-AP subgraphs of G with order λ and $n-\lambda$, respectively.

Case 1. $k=1$.
First, every graph $PCC_1(x)$ is R-AP since it is spanned by $Cat(3,x+1)$, which is R-AP according to the assumption on x.

We now prove that every graph $C = CC_1(x,y)$ is R-AP whenever the conditions of the claim are fulfilled. This is proved by induction on $x+y$ by showing that there is a partition of $V(C)$ into two parts V_λ and $V_{n-\lambda}$ satisfying the conditions above for every $\lambda \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$ where $n = 5+x+y$. For each value of λ, we give a satisfying subset V_λ, and it is understood that $V_{n-\lambda} = V(C) - V_\lambda$. We further assume $x \not\equiv 2 \mod 3$ since the graphs $CC_1(x,y)$ and $CC_1(y,x)$ are isomorphic.

First, when dealing with $\lambda \geq x+5$, we can pick up, as V_λ, the λ first vertices of the ordering $\{u_1, \ldots, u_x, a_1, b_1, c_1, d_1, e_1, v_y, \ldots, v_1\}$ of $V(C)$ to get a partition of C into a traceable graph or $CC_1(x, y-(\lambda-(x+6)))$ which is R-AP by the induction hypothesis, and a path. For $\lambda = x$, one can consider $V_\lambda = \{u_1, \ldots, u_x\}$ so that the two induced graphs are traceable. Now, if $\lambda = x+2$ or $\lambda = x+3$, then we can choose $\{u_1, \ldots, u_x, a_1, b_1\}$ or $\{u_1, \ldots, u_x, a_1, c_1, d_1\}$, respectively, as V_λ, so that the two induced subgraphs are paths. Next, consider $\lambda = x+4$. Then $V_\lambda = \{u_1, \ldots, u_x, a_1, b_1, c_1, d_1\}$ yields a correct partition of C. Indeed, on the one hand, $C[V_\lambda]$ is a caterpillar $Cat(3, x+1)$ which is R-AP since otherwise it would mean that $x \equiv 2 \mod 3$, a contradiction. On the other hand, the graph $C[V_{n-\lambda}]$ is a path.

Now consider $\lambda = x+1$. If $V_\lambda = \{u_1, \ldots, u_x, a_1\}$ does not provide a satisfying partition of C, then $y \equiv 2 \mod 3$ since $C[V_{n-\lambda}]$ is $Cat(3, y+1)$ and is not R-AP. Consider now, as V_λ, the λ first vertices of the ordering $(v_1, \ldots, v_y, e_1, b_1, d_1, c_1, a_1, u_x, \ldots, u_1)$ of $V(C)$. If this choice of V_λ does not yield a correct partition of C once again, then it means that either $C[V_\lambda]$ is the caterpillar $Cat(3, y+1)$, or a connected cycles graph $CC_1(x', y)$ with $x' \equiv 2 \mod 3$. But then we get that either $x+1 = y+4$ or $x+1 = y+5+x'$, respectively, which both imply that $x \equiv 2 \mod 3$, a contradiction.

Finally consider every value $\lambda \in \{1, \ldots, x-1\}$. On the one hand, if $x - \lambda \not\equiv 2 \mod 3$, then choose $V_\lambda = \{u_1, \ldots, u_\lambda\}$ so that $C[V_\lambda]$ and $C[V_{n-\lambda}]$ are a path, and $CC_1(x-\lambda, y)$ which is R-AP by the induction hypothesis. On the other hand, i.e. $x - \lambda \equiv 2 \mod 3$, we have $\lambda \not\equiv 0 \mod 3$ since otherwise we would have $x \equiv 2 \mod 3$. We can assume that $\lambda \notin \{y, y+2, y+3\}$, since otherwise we could deduce a correct partition of C as in the cases $\lambda \in \{x, x+2, x+3\}$, respectively. Then consider, as V_λ, the λ first vertices of $(v_1, \ldots, v_y, e_1, b_1, d_1, c_1, a_1, u_x, \ldots, u_1)$. If this choice of V_λ does not yield a correct partition of C, then $C[V_\lambda]$ is either a caterpillar $Cat(3, y+1)$ which is not R-AP, or a graph $CC_1(x', y)$ with $x' \equiv 2 \mod 3$. But note then that the first situation cannot occur because $\lambda \not\equiv 0 \mod 3$. For the second situation, note that, because $\lambda \not\equiv 0 \mod 3$, we have $y \not\equiv 2 \mod 3$. Since we have $x', y < x$, the graph $CC_1(y, x')$ is actually R-AP by the induction hypothesis.

Case 2. Arbitrary k.
Let us now suppose that the result is true for every i up to $k-1$, and let us prove it for k. Consider first $C = PCC_k(x)$ for consecutive values of $x \not\equiv 2 \mod 3$. As we did before, to prove that C is R-AP we show that there exists a partition of $V(C)$ satisfying our conditions for every possible value of λ. One may choose V_λ as follows.

— If $\lambda \equiv 1 \mod 3$, then one may consider, as V_λ, the first λ vertices of the ordering (b_k, d_k, c_k, a_k, $w_{k-1,k}$, e_{k-1}, b_{k-1}, d_{k-1}, c_{k-1}, a_{k-1}, $\ldots, w_{1,2}$, e_1, b_1, d_1, c_1, a_1, u_x, $\ldots$, u_1) of $V(C)$. On the one hand, notice that $C[V_\lambda]$ is either a path, or covered by a R-AP caterpillar or a partial connected cycles graph $PCC_{k'}(x')$ with $k' \leq k-1$ and $x' \not\equiv 2 \mod 3$, which is R-AP by the induction hypothesis. On the other hand, observe that $C[V_{n-\lambda}]$ is either traceable, or spanned by a connected cycles graph $CC_{k''}(x, y)$ for some $k'' \leq k-1$ and y, which is R-AP according to the induction hypothesis.

— If $\lambda \equiv 2 \mod 3$, then one can obtain similar partitions of C from the ordering $(d_k, c_k, b_k, a_k, w_{k-1,k}, e_{k-1}, d_{k-1}, c_{k-1}, b_{k-1}, a_{k-1}, \ldots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \ldots, u_1)$ of $V(C)$.

— Otherwise, if $\lambda \equiv 0 \mod 3$, then one has to consider as V_λ the first λ vertices of the ordering $(u_1, \ldots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \ldots, a_{k-1}, b_{k-1}, c_{k-1}, d_{k-1}, e_{k-1}, w_{k-1,k}, a_k, b_k, c_k, d_k)$ of $V(C)$ when $x \equiv 1 \mod 3$, or the ordering $(u_1, \ldots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \ldots, a_{k-1}, c_{k-1}, d_{k-1}, b_{k-1}, e_{k-1}, w_{k-1,k}, a_k, c_k, d_k, b_k)$ otherwise, i.e. when $x \equiv 0 \mod 3$. The two induced subgraphs $C[V_\lambda]$ and $C[V_{n-\lambda}]$ are then R-AP. Indeed, on the one hand, $C[V_\lambda]$ is either isomorphic to a path or spanned by a connected cycles graph $CC_{k'}(x, y)$ for $k' \leq k-1$ and some y. On the other hand, the subgraph $C[V_{n-\lambda}]$ is spanned by some $PCC_{k''}(x')$ graph with $k'' \leq k$ and $x' \not\equiv 2 \mod 3$.

To end up proving the claim, we have to show that $CC_k(x, y)$ is R-AP whenever $x \not\equiv 2 \mod 3$ or $y \not\equiv 2 \mod 3$. As for the base case, we show this by induction on $x + y$. Once again, we assume that $x \not\equiv 2 \mod 3$ for a given graph $C = CC_k(x, y)$.

For some $\lambda \in \{1, \ldots, y\}$, one can consider $V_\lambda = \{v_1, \ldots, v_\lambda\}$ so that C is partitioned into a path and $CC_k(x, y - \lambda)$ which is R-AP according to the induction hypothesis. When $\lambda = y + 1$, one can choose $V_\lambda = \{v_1, \ldots, v_y, e_k\}$ so that C is partitioned into a path and a partial connected cycles graph which is R-AP by the induction hypothesis since $x \not\equiv 2 \mod 3$. For other values of λ, one may choose V_λ as follows.

— If $\lambda \equiv 0 \mod 3$, one can consider, as V_λ, the λ first vertices from the ordering $(u_1, \ldots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \ldots, w_{k-1,k}, a_k, b_k, c_k, d_k\ e_k, v_y, \ldots, v_1)$ of $V(C)$ when $x \equiv 1 \mod 3$, from $(u_1, \ldots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \ldots, w_{k-1,k}, a_k, c_k, d_k, b_k, e_k, v_y, \ldots, v_1)$ otherwise, i.e. when $x \equiv 0 \mod 3$. The two induced subgraphs are then R-AP since they are traceable or isomorphic to connected cycles graphs which are R-AP according to the induction hypotheses.

— If $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y + 1) \equiv 0 \mod 3$, then one can consider the λ first vertices of the ordering $(v_1, \ldots, v_y, e_k, b_k, d_k, c_k, a_k, w_{k-1,k} \ldots, e_1, b_1, d_1, c_1, a_1, u_x, \ldots, u_1)$ of $V(C)$. For each such partition, we get, on the one hand, that $C[V_\lambda]$ is either a path, a R-AP caterpillar, or a R-AP (partial) connected cycles graph. In particular, note that when $C[V_\lambda]$ is a caterpillar or partial connected cycles graph, then this graph is R-AP since $y \not\equiv 2 \mod 3$ because of the assumptions on λ. On the other hand, the graph $C[V_{n-\lambda}]$ is either a path, or a (partial) connected cycles graph which is R-AP by the induction hypothesis.

— If $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y + 1) \equiv 1 \mod 3$, then one may pick up, as V_λ, the λ first vertices from the ordering given to deal with the previous case. This choice of V_λ makes, on the one hand, $C[V_\lambda]$ being spanned by either a path, or $CC_{k'}(x', y)$ where $k' \leq k - 1$ and $x' \not\equiv 2 \mod 3$ which is R-AP by the induction hypothesis. On the other hand, $C[V_{n-\lambda}]$ is a path, or is spanned by some graph $CC_{k''}(x, y')$ for $k'' \leq k - 1$ and some y' which is R-AP, again by the induction hypothesis.

— Otherwise, if $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y+1) \equiv 2 \mod 3$, then some similar partitions of C may be obtained from the ordering $(v_1, \ldots, v_y, e_k, d_k, c_k, b_k, a_k, w_{k-1,k} \ldots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \ldots, u_1)$ of $V(C)$. □

Note that Lemma 3.4 provides a full characterization of R-AP (partial) connected cycles graphs since every such graph whose parameters do not satisfy this lemma is not R-AP. To be convinced of that fact, one just has to consider successive partitions of such a graph for $\lambda = 3$.

We finally deduce Theorem 3.1 as a corollary of Lemma 3.4.

Proof of Theorem 3.1. We have $LP(CC_{c+1}(1,1)) = |V(CC_{c+1}(1,1))| - (c+1)$ for every $c \geq 1$. Therefore, for every possible value of c, we have a graph showing that c does not contradict the claim. □

Finally notice that by adding the edge u_1v_1 to any connected cycles graph $CC_k(1,1)$, we get a 2-connected graph which is R-AP according to Remark 2.1 and whose longest path has order $LP(CC_k(1,1)) + 1$. Therefore, Theorem 3.1 is also true when restricted to 2-connected graphs.

4. LONGEST PATH AND MULTIPLICATIVE FACTOR

The graph $CC_k(1,1)$ has order $n = 6k + 1$ while its longest path has order $n - k$ for every $k \geq 1$. Thus, even if the connected cycles graphs confirm that the order of the longest path in a R-AP n-graph is not constantly lower than n up to an additive factor, they do not reject the strong relationship between the properties of being R-AP and traceable. We now discuss how to create this relationship thanks to a multiplicative factor.

Question 4.1. *What is the biggest $c < 1$ such that $LP(G) \geq n \cdot c$ for every R-AP n-graph G?*

Regarding the connected cycles graphs, we get that $c \leq \frac{5}{6}$. In this section, we deduce a better upper bound on c thanks to the following graph construction.

Definition 4.2. Let $k, k' \geq 1$ be two positive integers. The *urchin* $W(k,k')$ is the graph obtained as follows.

— Let A, B, C be three sets of k, k and k' distinct vertices, respectively.
— Add a perfect matching between the vertices of A and B.
— Add all possible edges between distinct vertices in $B \cup C$.

This construction is illustrated in Figure 3. Note that the urchin $W(k,k)$ has order $3k$ while its longest path has order $2k + 2$. We then get that $LP(W(k,k))/n$ tends to $\frac{2}{3}$ as k grows to infinity. In what follows, we show that any urchin $W(k,k)$ is R-AP, and thus that the following holds regarding Question 4.1.

Theorem 4.3. *Regarding Question 4.1, we have $c \leq \frac{2}{3}$.*

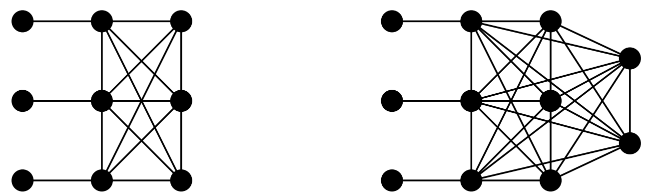

Fig. 3. The urchins $W(3,3)$ and $W(3,5)$

We prove that an urchin $W(k,k')$ is R-AP for some values of k and k'.

Lemma 4.4. *The urchin* $W(k,k')$ *is R-AP for every* $k \geq 2$ *and* $k' \geq k-2$.

Proof. We introduce some terminology to deal with the vertices of any urchin $W(k,k')$. The vertices of A are denoted $u_1,\dots,u_k$, and those of B are denoted $v_1,\dots,v_k$ in such a way that u_iv_i is an edge for every $i \in \{1,\dots,k\}$. The vertices of C are denoted $w_1,\dots,w_{k'}$ arbitrarily.

The claim is proved by induction on both k and k'. As a base case, note that every urchin $W(2,k')$ is traceable, and thus R-AP by Remark 2.2. Suppose now that $W(i,i')$ is R-AP for every i up to $k-1$ and $i' \geq i-2$. We now prove that the urchin n-graph $W = W(k,k')$ is R-AP for every $k' \geq k-2$. For this purpose, we show, for every value of $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$, that $V(W)$ can be partitioned into two parts V_λ and $V_{n-\lambda}$ inducing R-AP graphs on λ and $n-\lambda$ vertices, respectively.

We first deal with the easy cases, i.e. $\lambda \in \{1,2,3\}$. For $\lambda = 1$, consider $V_\lambda = \{u_1\}$ so that the two induced subgraphs are K_1 and $W(k-1,k'+1)$. Since $k' \geq k-2$, this subgraph is R-AP by the induction hypothesis. For $\lambda = 2$, let $V_\lambda = \{u_1,v_1\}$. The two induced subgraphs then are K_2 and $W(k-1,k')$, which is R-AP for the same reason as the previous case. Now, for $\lambda = 3$, choose $V_\lambda = \{u_1,v_1,w_1\}$. The two induced subgraphs then are a path, and the urchin $W(k-1,k'-1)$ which is R-AP, again by the induction hypothesis.

We now deal with the remaining values of λ, i.e. $\lambda \geq 4$. The part V_λ is obtained by choosing two disjoint sets V'_λ and V''_λ, and then considering their union. On the one hand, in the case where $\lambda \equiv 1 \mod 3$, let $x = \lfloor \frac{\lambda-4}{3} \rfloor$. Clearly, x is an integer. First, let $V'_\lambda = \emptyset$ if $x = 0$, or $V'_\lambda = \bigcup_{i=1}^{x}\{u_i,v_i,w_i\}$ otherwise. Then set $V''_\lambda = \{v_{x+1},u_{x+1},v_{x+2},u_{x+2}\}$. The two induced subgraphs then are a path or $W(x+2,x)$, and $W(k-(x+2),k'-(x-2))$, which are R-AP by the induction hypothesis since $k' \geq k-2$.

On the other hand, i.e. $\lambda \not\equiv 1 \mod 3$, let $x = \lfloor \frac{\lambda}{3} \rfloor$ and $y \equiv \lambda \mod 3$ with $y \in \{0,2\}$. Then, let $V'_\lambda = \bigcup_{i=1}^{x}\{u_i,v_i,w_i\}$. The strategy for choosing V''_λ depends on whether $y = 0$ or $y = 2$.

— $y = 0$. Choose $V''_\lambda = \emptyset$. In this situation, the two induced subgraphs are $W(x,x)$ and $W(k-x,k'-x)$ which are R-AP by the induction hypothesis since $k' \geq k-2$.

— $y = 2$. Let $V''_\lambda = \{v_{x+1},u_{x+1}\}$. The two induced subgraphs then are $W(x+1,x)$ and $W(k-(x+1),k'-x)$, which are R-AP according to the induction hypothesis. □

Theorem 4.3 follows as a corollary of Lemma 4.4. Note that Lemma 4.4 is tight in the sense that urchins $W(k,k-x)$ with $x \geq 3$ are not R-AP since such a graph W

cannot be partitioned as requested for $\lambda = 3$. Indeed, as a set V_λ with size 3 inducing a R-AP subgraph of W, one has to consider, following the terminology introduced in the proof of Lemma 4.4, a part of the form $\{u_i, v_i, w_j\}$ or $\{w_i, w_j, w_\ell\}$. After having successively picked several sets with size 3 off W, one necessarily gets an urchin $W(k', 0)$ with $k' \geq 3$. Such a graph is clearly not partitionable for $\lambda = 3$ once again.

We can strengthen Theorem 4.3 as follows. Let $W = W(k, k')$ be a R-AP urchin. Observe that by adding the edges $u_1u_2, \ldots, u_1u_k$ to W, we get a 2-connected graph W_2 which is R-AP by Remark 2.1. By then adding the edges $u_2u_3, \ldots, u_2u_k$ to W_2, we get another R-AP graph W_3 which is 3-connected. By repeating this procedure as many times as needed, we get an ℓ-connected R-AP graph W_ℓ for any value of ℓ assuming k and k' are big enough. Note that we have $LP(W_i) = LP(W) + 2i$, and thus that $LP(W_i)/LP(W)$ tends to 1 as k grows to infinity. Therefore, the statement of Theorem 4.3 is also true when restricted to ℓ-connected R-AP graphs, no matter what is the value ℓ.

Theorem 4.5. *Theorem 4.3 is also true when Question 4.1 is restricted to R-AP graphs of arbitrary connectivity.*

Acknowledgments

The author would like to thank the anonymous referee for his helpful and constructive comments that greatly improved the overall quality of this paper, as well as the results it contains.

REFERENCES

[1] D. Barth, O. Baudon, J. Puech, *Decomposable trees: a polynomial algorithm for tripodes*, Discret. Appl. Math. **119** (2002) 3, 205–216.

[2] D. Barth, H. Fournier, *A degree bound on decomposable trees*, Discret. Math. **306** (2006) 5, 469–477.

[3] D. Barth, H. Fournier, R. Ravaux, *On the shape of decomposable trees*, Discret. Math. **309** (2009), 3882–3887.

[4] O. Baudon, J. Bensmail, F. Foucaud, M. Pilśniak, *Structural properties of recursively partitionable graphs with connectivity 2*. Preprint available at: http://hal.archives-ouvertes.fr/hal-00672505.

[5] O. Baudon, F. Foucaud, J. Przybyło, M. Woźniak, *Structure of k-connected arbitrarily partitionable graphs*. Preprint available at: http://hal.archives-ouvertes.fr/hal-00690253.

[6] O. Baudon, F. Gilbert, M. Woźniak, *Recursively arbitrarily vertex-decomposable suns*, Opuscula Math. **31** (2011) 4, 533–547.

[7] O. Baudon, F. Gilbert, M. Woźniak, *Recursively arbitrarily vertex-decomposable graphs*, Opuscula Math. **32** (2012) 4, 689–706.

[8] A. Marczyk, *An ore-type condition for arbitrarily vertex decomposable graphs*, Discret. Math. **309** (2009) 11, 3588–3594.

Julien Bensmail
julien.bensmail@labri.fr

Univ. Bordeaux
LaBRI, UMR 5800, F-33400 Talence, France

CNRS
LaBRI, UMR 5800, F-33400 Talence, France

Received: January 7, 2013.
Revised: May 23, 2013.
Accepted: May 24, 2013.