Janusz Matkowski

ON EXTENSION OF SOLUTIONS
OF A SIMULTANEOUS SYSTEM
OF ITERATIVE FUNCTIONAL EQUATIONS

Abstract. Some sufficient conditions which allow to extend every local solution of a simultaneous system of equations in a single variable of the form

$$\varphi(x) = h\left(x, \varphi\left[f_1(x)\right], \ldots, \varphi\left[f_m(x)\right]\right),$$
$$\varphi(x) = H\left(x, \varphi\left[F_1(x)\right], \ldots, \varphi\left[F_n(x)\right]\right),$$

to a global one are presented. Extensions of solutions of functional equations, both in single and in several variables, play important role (cf. for instance [1–3]).

Keywords: functional equation, simultaneous system of equations, local solution, extension theorem.

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1. INTRODUCTION

Let $I \subseteq \mathbb{R}$ be an open interval with an endpoint $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $I_\xi \subset I$ be a nonempty open interval with the same endpoint ξ. Let $f : I \to I$ be a continuous function such that one of the following three conditions is fulfilled:
   if ξ is finite then

$$0 < \frac{f(x) - \xi}{x - \xi} < 1, \qquad x \in I;$$

if $\xi = -\infty$ then

$$f(x) < x, \qquad F(x) < x, \qquad x \in I;$$

if $\xi = +\infty$ then

$$f(x) > x, \qquad F(x) > x, \qquad x \in I.$$

In particular, ξ can be treated as a fixed point of f. Let $h : I \times \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Assume that a function $\varphi_\xi : I_\xi \to \mathbb{R}$ satisfies the equation

$$\varphi_\xi(x) = h\left(x, \varphi_\xi[f(x)]\right), \qquad x \in I_\xi.$$

It is well known (cf. for instance [1] and [2]) that then there exists a unique function $\varphi : I \to \mathbb{R}$ such that $\varphi\big|_{I_\xi} = \varphi_\xi$ and

$$\varphi(x) = h\left(x, \varphi[f(x)]\right), \qquad x \in I.$$

Since the unique global extension φ of φ_ξ depends only on the given functions f and h, we denote it by $\varphi_{f,h}$.

Assume that f and $F : I \to I$ satisfy one of the above three conditions, and let $H : I \times \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Suppose that the function $\varphi_\xi : I_\xi \to \mathbb{R}$ satisfies also the equation

$$\varphi_\xi(x) = H\left(x, \varphi_\xi[F(x)]\right), \qquad x \in I_\xi,$$

that is φ_ξ satisfies simultaneously a system of two functional equations in I_ξ. Now a natural question arises under what conditions on functions f, h, F and H we have $\varphi_{f,h} = \varphi_{F,H}$?

Let us mention that study of iteration groups and locally commuting functions leads to the following question (cf. [3]). Assume that a function $\gamma : (-\infty, c) \to \mathbb{R}$ satisfies the simultaneous system of functional equations

$$\gamma(x + a) = \gamma(x) + a, \quad \gamma(x + b) = \gamma(x) + b, \quad x \in (-\infty, c),$$

for some $a, b, c \in \mathbb{R}$. Does there exist a function $\phi : \mathbb{R} \to \mathbb{R}$ extending γ and satisfying this system of equations in $\mathbb{R}$, i.e. such that $\phi|_{(-\infty,c)} = \gamma$ and

$$\phi(x + a) = \phi(x) + a, \quad \phi(x + b) = \phi(x) + b, \quad x \in \mathbb{R}?$$

The extension problem was considered in [3] for the following simultaneous system of two linear functional equations of the first order

$$\gamma\left[f_i\left(x\right)\right] = g_i\left(x\right)\gamma(x) + h_i\left(x\right), \quad i = 1, 2.$$

In the second section we consider this problem for a nonlinear system of functional equations of higher orders. In the third section, applying the result of section two, we formulate the suitable results for linear equations.

Finally we remark that each of the presented results remains true if the simultaneous system of two functional equations is replaced by any finite simultaneous system of functional equations.

2. SIMULTANEOUS SYSTEMS OF NONLINEAR FUNCTIONAL EQUATIONS

In this section we prove the following extension theorem.

Theorem 2.1. *Let $I \subseteq \mathbb{R}$ be an open interval, $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ be one of the endpoints of I, and $I_\xi \subset I$ be a nonempty open interval with the endpoint ξ. Let $m, n \in \mathbb{N}$ be fixed, the functions $f_i, F_j : I \to \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be continuous and such that one of the following conditions is fulfilled:*

(i) *if ξ is finite then*

$$0 < \frac{f_i(x) - \xi}{x - \xi} < 1, \quad i = 1, \ldots, m, \quad 0 < \frac{F_j(x) - \xi}{x - \xi} < 1, \quad j = 1, \ldots, n, \quad (x \in I),$$

(ii) *if $\xi = -\infty$ then*

$$f_i(x) < x, \quad i = 1, \ldots, m, \qquad F_j(x) < x, \quad j = 1, \ldots, n, \qquad x \in I,$$

(iii) *if $\xi = +\infty$ then*

$$f_i(x) > x, \quad i = 1, \ldots, m, \qquad F_j(x) > x, \quad j = 1, \ldots, n, \quad x \in I,$$

and let some functions $h : I \times \mathbb{R}^m \to \mathbb{R}$, $H : I \times \mathbb{R}^n \to \mathbb{R}$ be fixed. Suppose that a function $\varphi : I_\xi \to \mathbb{R}$ satisfies the simultaneous system of functional equations

$$
\begin{aligned}
\varphi(x) &= h\left(x, \varphi\left[f_1(x)\right], \ldots, \varphi\left[f_m(x)\right]\right), \quad x \in I_\xi, \\
\varphi(x) &= H\left(x, \varphi\left[F_1(x)\right], \ldots, \varphi\left[F_n(x)\right]\right), \quad x \in I_\xi.
\end{aligned}
\tag{2.1}
$$

If f_i and F_j commute, i.e.

$$f_i \circ F_j = F_j \circ f_i, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{2.2}$$

and, for all $x \in I$, $y_{i,j} \in \mathbb{R}$, $= 1, \ldots, m$, $j = 1, \ldots, n$,

$$
\begin{aligned}
h\left(x, H\left(f_1(x), y_{11}, y_{12}, \ldots, y_{1n}\right), \ldots, H\left(f_m(x), y_{m1}, y_{m2}, \ldots, y_{mn}\right)\right) = \\
= H\left(x, h\left(F_1(x), y_{11}, y_{21}, \ldots, y_{m1}\right), \ldots, h\left(F_n(x), y_{1n}, y_{2m}, \ldots, y_{mn}\right)\right),
\end{aligned}
\tag{2.3}
$$

then there exists exactly one function $\Phi : I \to \mathbb{R}$ such that

$$\Phi\big|_{I_\xi} = \varphi$$

and, for all $x \in I$,

$$
\begin{aligned}
\Phi(x) &= h\left(x, \Phi\left[f_1(x)\right], \ldots, \Phi\left[f_m(x)\right]\right), \\
\Phi(x) &= H\left(x, \left[\Phi[F_1(x), \ldots, \Phi\left[F_n(x)\right]\right]\right).
\end{aligned}
$$

Proof. Suppose that the functions f_i, F_j ($i = 1, \ldots, m$, $j = 1, \ldots, n$) satisfy condition (i). In this case ξ is finite and, without any loss of generality, we may assume

that $I = (0, \infty)$, $\xi = 0$ and $I_\xi = (0, a)$ for some $a > 0$. By the extension theorem (cf. M. Kuczma [1], p. 246-247) there exist unique functions $\Phi_1, \Phi_2 : I \to R$ such that

$$\Phi_1(x) = h\left(x, \Phi_1\left[f_1(x)\right], \ldots, \Phi_1\left[f_m(x)\right]\right), \tag{2.4}$$

$$\Phi_2(x) = H\left(x, \Phi_2[F_1(x)], \ldots, \Phi_2\left[F_n(x)\right]\right) \tag{2.5}$$

for all $x > 0$, and

$$\Phi_1\big|_{(0,a)} = \varphi = \Phi_2\big|_{(0,a)} .$$

Let

$$c := \sup\left\{b > 0 : \Phi_1(x) = \Phi_2(x) \text{ for all } x \in (0, b)\right\}.$$

To prove the theorem it is enough to show that $c = +\infty$. For an indirect argument assume that $c < +\infty$. Then

$$\Phi_1(x) = \Phi_2(x), \qquad x \in (0, c). \tag{2.6}$$

The assumptions on f_i and F_i imply that there exists a $d > c$ such that

$$f_i(x) < c, \quad F_j(x) < c \qquad x \in (0, d), \ i = 1, \ldots, m, \ j = 1, \ldots, n.$$

Condition (i) easily implies that $f_i \circ F_j(x) \in (0, c)$ and $F_j \circ f_i(x) \in (0, c)$ for all i, j and $x \in (0, d)$. Now for all $x \in (0, d)$, applying in turn (2.4), (2.6), m times (2.5) with x replaced by $f_i(x)$, $i = 1, \ldots, m$, the assumption (2.3), the commutativity assumption (2.2), again (2.6), n times of (2.4) with x replaced by $F_j(x)$, $(j = 1, \ldots, n)$, equality (2.6), and again equation (2.4), we obtain

$$\begin{aligned}
\Phi_1(x) &= h\left(x, \Phi_1\left[f_1(x)\right], \ldots, \Phi_1\left[f_m(x)\right]\right) = \\
&= h\left(x, \Phi_2\left[f_1(x)\right], \ldots, \Phi_2\left[f_m(x)\right]\right) = \\
&= h(x, H\left(f_1(x), \Phi_2[F_1(f_1(x))], \ldots, \Phi_2\left[F_n(f_1(x))\right]\right), \ldots \\
&\qquad \ldots, H\left(f_m(x), \Phi_2[F_1(f_m(x))], \ldots, \Phi_2\left[F_n(f_m(x))\right]\right)) = \\
&= H(x, h\left(F_1(x), \Phi_2[F_1(f_1(x))], \ldots, \Phi_2\left[F_1(f_m(x))\right]\right), \ldots \\
&\qquad \ldots, h\left(F_n(x), \Phi_2[F_n(f_1(x))], \ldots, \Phi_2\left[F_n(f_m(x))\right]\right)) = \\
&= H(x, h\left(F_1(x), \Phi_2[f_1(F_1(x))], \ldots, \Phi_2\left[f_m(F_1(x))\right]\right), \ldots \\
&\qquad \ldots, h\left(F_n(x), \Phi_2[f_1(F_n(x))], \ldots, \Phi_2\left[f_m(F_n(x))\right]\right)) = \\
&= H(x, h\left(F_1(x), \Phi_1[f_1(F_1(x))], \ldots, \Phi_1\left[f_m(F_1(x))\right]\right), \ldots \\
&\qquad \ldots, h\left(F_n(x), \Phi_1[f_1(F_n(x))], \ldots, \Phi_1\left[f_m(F_n(x))\right]\right)) = \\
&= H\left(x, \Phi_1[F_1(x)], \ldots, \Phi_1\left[F_n(x)\right]\right) = \\
&= H\left(x, \Phi_2[F_1(x)], \ldots, \Phi_2\left[F_n(x)\right]\right) = \\
&= \Phi_2(x)
\end{aligned}$$

which contradicts the definition of c.

In the case when condition (ii) or (iii) is fulfilled the argument is similar. This completes the proof. $\qquad\qquad\square$

3. COROLLARIES FOR SIMULTANEOUS SYSTEMS OF LINEAR FUNCTIONAL EQUATIONS

From Theorem 2.1 we obtain the following result for a simultaneous system of linear functional equations of finite order.

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an open interval, $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ be one of the endpoints of I, and $I_\xi \subset I$ be a nonempty open interval with the endpoint ξ. Let $m, n \in \mathbb{N}$, the functions $f_i : I \to \mathbb{R}$, $i = 1, \ldots, m$, $F_j : I \to \mathbb{R}$, $j = 1, \ldots, n$, be continuous and such that one of the conditions* (i)-(iii) *of Theorem 2.1 is fulfilled, and the functions $g_i, h, G_j, H : I \to \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ be fixed. Suppose that a function $\varphi : I_\xi \to \mathbb{R}$ satisfies the simultaneous system of functional equations*

$$\varphi(x) = \sum_{i=1}^{m} g_i(x)\varphi[f_i(x)] + h(x), \qquad x \in I_\xi,$$

$$\varphi(x) = \sum_{j=1}^{n} G_j(x)\varphi[F_j(x)] + H(x), \qquad x \in I_\xi.$$

If f_i and F_j commute, i.e.

$$f_i \circ F_j = F_j \circ f_i, \qquad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

and

$$g_i(x)G_j[f_i(x)] = G_j(x)g_i[F_j(x)], \qquad i = 1, \ldots, m, \ j = 1, \ldots, n, \ x \in I,$$

$$\sum_{i=1}^{m} g_i(x)H(f_i(x)) + h(x) = \sum_{j=1}^{n} G_j(x)h(F_j(x)) + H(x), \qquad x \in I,$$

then there exists exactly one function $\Phi : I \to \mathbb{R}$ such that

$$\Phi\big|_{I_\xi} = \varphi$$

and for all $x \in I$,

$$\Phi(x) = \sum_{i=1}^{m} g_i(x)\Phi[f_i(x)] + h(x),$$

$$\Phi(x) = \sum_{j=1}^{n} G_j(x)\Phi[F_j(x)] + H(x).$$

Taking $m = n = 1$ we hence obtain the following

Corollary 3.2. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ be one of the endpoints of I, and let $I_\xi \subset I$ be a nonempty open interval with the endpoint ξ. Let the functions $f, F : I \to \mathbb{R}$ be continuous and such that*

if ξ is finite then

$$0 < \frac{f(x) - \xi}{x - \xi} < 1, \qquad 0 < \frac{F(x) - \xi}{x - \xi} < 1, \qquad x \in I,$$

if $\xi = -\infty$ then
$$f(x) < x, \qquad F(x) < x, \qquad x \in I,$$

if $\xi = +\infty$ then
$$f(x) > x, \qquad F(x) > x, \qquad x \in I,$$

and let $g, G, h, H : I \to \mathbb{R}$ be given. Suppose that a function $\varphi : I_\xi \to \mathbb{R}$ satisfies the simultaneous system of functional equations

$$\varphi(x) = g(x)\varphi[f(x)] + h(x), \qquad \varphi(x) = G(x)\varphi[F(x)] + H(x), \qquad x \in I_\xi. \qquad (3.1)$$

If f and F commute, i.e.
$$f \circ F = F \circ f,$$

and
$$g(x)G[f(x)] = G(x)g[F(x)], \qquad x \in I,$$
$$g(x)H(f(x)) + h(x) = G(x)h(F(x)) + H(x), \quad x \in I,$$

then there exists exactly one function $\Phi : I \to \mathbb{R}$ such that

$$\Phi\big|_{I_\xi} = \varphi$$

and, for all $x \in I$,

$$\Phi(x) = g(x)\Phi[f(x)] + h(x), \qquad \Phi(x) = G(x)\Phi[F(x)] + H(x).$$

Remark 3.3. The simultaneous systems of functional equations of the type

$$\varphi[f_i(x)] = g_i(x)\varphi(x) + h_i(x), \qquad i = 1, 2,$$

(considered in [3]) in which the value $\varphi[f_i(x)]$ of the unknown function φ is expressed by $\varphi(x)$, reduces to (3.1) if $g_i(x) \neq 0$ for $i = 1, 2$ and for all $x \in I$.

Final remark. Note that each of the above presented results remains true if the simultaneous system of two functional equations is replaced by a simultaneous system of functional equations indexed by the elements of a finite set of the cardinality greater or equal 2.

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Janusz Matkowski
j.matkowski@wmie.im.zgora.pl

University of Zielona Góra
Institute of Mathematics, Computer Science and Econometry,
ul. Podgórna 50, 65-246 Zielona Góra, Poland

Silesian University
Institute of Mathematics
40-007 Katowice, Poland