doi: 10.32426/engresv2n2-001

Engineering Research

Technical Reports

Volume 2 – Issue 2 – Article 1

ISSN 2179-7625 (online)

INFLUENCE OF THIRD BODY LONG PERIOS PERTURBATIONS ON THE SECULAR RATES DUE TO EARTH OBLATENESS

Bob E. Schutz[1]

JULY / 2011

Taubaté, São Paulo, Brazil

[1] Professor, FSX Professor of Space Applications and Exploration, Joe J. King Chair in Engineering, Center for Space Research and Department of Aerospace Engineering and Engineering Mechanics, R1000, Austin, Texas, 78759-5321.

Engineering Research: Technical Reports

The “Engineering Research” is a publication with purpose of technical and academic knowledge dissemination.

ABSTRACT

Equations for long period and secular perturbations due to the Moon, Sun and the Earth potential field are applied to equations expressing the influence of both the Moon and Sun long period and secular perturbations on the secular rates due to the Earth potential field. Usual Keplerian elements are used. No singularities are present in the equations neither in the final integrated variables.

INTRODUCTION

Lagrange Equations for the non singular variables have been given in several papers by Giacaglia (Reference 1) and by these authors (References 2, 3, 4, 5, 6). The variables used in these works were: the semi-major axis (a) and the mean longitude (λ); the products (ξ and η) of the eccentricity (e) times the cosine and sine of the longitude of perigee ($\tilde{\omega}$); the products (P and Q) of sine of half the inclination (I) times the cosine and sine of the longitude of the ascending node (Ω). Complete equations for all perturbations have been developed in a paper to be presented at DINAME 2011 (Reference 5)

The following notations are used through out this work: R is disturbing function due to any gravitational influence and n is the mean motion in longitude of the satellite. Also

$$\gamma^2 = 1 - e^2 \tag{1}$$

$$s = \sin(I/2) \qquad c = \cos(I/2) \tag{2}$$

MOON DISTURBING FUNCTION

The main part of the disturbing function in a primitive form is

$$R' = \beta' n'^2 r^2 \left(\frac{a'}{r'} \right)^3 P_2(\cos\psi') \tag{3}$$

In a related work (Reference 7) Kaula represented the disturbing function by considering the Moon moving on the ecliptic. Here, considering the Moon in its real orbit, a transformation to orbital elements and rotation of the lunar coordinates to an ecliptic frame of reference gives

$$\begin{aligned} R' = n'^2 a^2 \left(\frac{a_M}{r_M} \right)^3 \sum_{m,m',p,p',q} H_{2pq}(e) \frac{(-1)^m \in_m \in_{m'} (2-m')!}{(\ell + m)} F_{2mp}(\mathrm{I}) F_{2m'p'}(\mathrm{I}_M) \times \\ \times \left[(-1)^{2+m-m'} E_{2,m,-m'} \cos\left(\theta_{2mpq} + \theta'_{2p'm'}\right) + E_{2,m,m'} \cos\left(\theta_{2mpq} - \theta'_{2p'm'}\right) \right] \end{aligned} \tag{4}$$

where summations in m, m′, p, p′ are from 0 to 2 and the value of q gives the maximum power of the eccentricity. Moreover $H_{2pq}(e) = X^{2,2-2p}_{2-2p+q}(e)$ are Hansen Coefficients given by Plummer (Reference 8) and

$$\begin{aligned} \theta_{2mpq} &= (2-2p+q)\lambda - q\tilde{\omega} + (m+2p-2)\Omega \\ \theta'_{2pm'} &= (2-2p')(f_M + \omega_M) + m'(\Omega_M + \pi/2) \end{aligned} \tag{5}$$

$F_{2mp}(\mathrm{I})$ are the usual inclination functions given by Kaula (Reference 9) and for $m + m' \geq 0$,

$$\mathrm{E}_{2,m,m'} = (-1)^{2-m'} \binom{2+m}{2-m'} \left(\cos\frac{\varepsilon}{2}\right)^{m+m'} \left(\sin\frac{\varepsilon}{2}\right)^{m'-m} \times$$
$$\times F\left(-2+m', 2+m'+1, m+m'+1; \cos^2\frac{\varepsilon}{2}\right) \tag{6}$$

and, for $m+m' \le 0$,

$$\mathrm{E}_{2,m,m'} = (-1)^{2-m'} \binom{2-m}{2+m'} \left(\cos\frac{\varepsilon}{2}\right)^{-m-m'} \left(\sin\frac{\varepsilon}{2}\right)^{m-m'} \times$$
$$\times F\left(-2-m', 2-m'+1, -m-m'+1; \cos^2\frac{\varepsilon}{2}\right) \tag{7}$$

In the above relations, F is the hyper geometric series ${}_1F_2$, defined by

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{8}$$

where

$$(a)_n = a(a+1)(a+2)...(a+n-1), (a)_0 = 1 \tag{9}$$

In both cases, $\mathrm{E}_{2,m,m'}$ is a polynomial in $\sin(\varepsilon/2)$, $\cos(\varepsilon/2)$ since at least one of the parameters a, b is negative, and the above series terminate, i.e., when $n = 1-a = 1+|a|$ or $n = 1-b = 1+|b|$.

DISTURBING FUNCTION FOR SECULAR AND LONG-PERIOD PERTURBATIONS

In case of no resonances between the motion of the satellite and that of the Moon, the elimination of short period terms (depending on the mean anomaly M of the satellite), uses the following definite integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \sin(2-2p) f \, dM = 0 \tag{10}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \cos(2-2p) f \, dM = \left(1+\beta^2\right)^{-3} X_{0,0}^{2,2-2p}(\beta) =$$
$$= \left(1+\beta^2\right)^{-2-1} H_{2,p,(2p-2)}(\beta) \tag{11}$$

$$\beta = e\left(1+\sqrt{1-e^2}\right)^{-1} \tag{12}$$

The H functions may be represented, for $p > 1$, by

$$H_{2,p(2p-2)} = (-\beta)^{2p-2} \binom{2p+1}{2p-2} F\left(-3, 2p-4, 2p-2; \beta^2\right) \tag{13}$$

and for $p \le 1$, by

$$H_{2,p,(2p-2)} = (-\beta)^{2-2p} \binom{5-2p}{2-2p} F\left(-3, 3-2p, 3-2p; \beta^2\right) \tag{14}$$

In both cases, they are polynomials in β. Long period and secular part of the function R' is

$$
\overline{R'} = a^2 \left(\frac{a_M}{r_M} \right)^3 \sum_{m,m',p,p'} \frac{(-1)^m \varepsilon_m \ \varepsilon_{m'} (2-m')!}{(2+m)!} F_{2mp}(I) F_{2m'p'}(I_M) L_{2,p,(2p-2)}(\beta) \times \\
\times \left[(-1)^{2+m-m'} E_{2,m,-m'} \cos\left(\overline{\theta}_{2pm} + \theta'_{2p'm'}\right) + E_{2,m,m'} \cos\left(\overline{\theta}_{2pm} - \theta'_{2p'm'}\right) \right] \tag{15}
$$

where

$$
\overline{\theta}_{2pm} = (2-2p)\tilde{\omega} + \alpha\Omega \tag{16}
$$

$$
L_{2,p,(2p-2)}(\beta) = \left(1+\beta^2\right)^{-3} H_{2;p(2p-2)}(\beta) \tag{17}
$$

where $\alpha = m + 2p - 2$. D´Alembert Characteristics imply that the function $F_{2mp}(I)$ is factored by $sin^{|\alpha|}(I/2)$ and the functions $L_{2,p,(2p-\ell)}(\beta)$ are factored by $e^{|2p-2|}$.

LONG-PERIOD PERTURBATIONS

Moon perturbations are factored by:

$$
Gm_M = \frac{m_M}{m_M + m_\oplus} n_M^2 a_M^3 = \mathrm{N}_M^2 a_M^3 \tag{18}
$$

$$
\mathrm{N}_M^2 \simeq 1.59\mathrm{X}10^{-5} rev^2 day^{-2} \tag{19}
$$

We note that for the solar perturbations these quantities are:

$$
Gm_\odot = \frac{m_\odot}{m_\odot + m_\oplus} n_\odot^2 a_\odot^3 = N_\odot^2 a_\odot^3 \tag{20}
$$

$$
N_\odot^2 \cong 0.75 \times 15^{-5} rev^2 day^{-2} \tag{21}
$$

which is about half the value for the Moon

The satellite Keplerian negative energy is

$$
F_0 = n^2 a^2 / 2 \tag{22}
$$

The relative size of the perturbing force function wrt the main Keplerian central attraction is given by

$$
v = R' / F_0 = 2\,\mathrm{N}_M^2 / n^2 \tag{23}
$$

For low satellites ($\mathrm{T} \cong$ 90 min), $v \cong 1.2\mathrm{X}10^{-7}$.

For high satellites ($\mathrm{T} \cong 24$ h), $v \cong 3.18\mathrm{X}10^{-5}$.

In the above range of periods, with low values of the eccentricity, the dominant part of the disturbing function of a satellite is due to the Earth oblateness (C_{20}), and lunar (and solar) perturbations are about second order with respect to this. In cases of higher satellites, depending on the values of semi-major axis and eccentricity, the situation might even be reverted, so that for a full evaluation of the lunar perturbations, truncation of the corresponding disturbing function may not be advisable.

The integration of the pertinent equations can be performed numerically by using as input lunar ecliptic coordinates – or, for that matter, equatorial coordinates, in which case the theory is greatly simplified – stored in memory. This will produce precise evaluation of the true lunar motion. However, such a method can be very expensive in time. A good approximation can be obtained by considering I_M, e_M, a_M, and ε fixed values and M_M, ω_M and Ω_M linear functions of time, neglecting accelerations of these elements. Also, an expansion in power series of e_M will converge rapidly owing to the small value of the eccentricity of the Moon orbit.

The frequencies involved are

$$\begin{aligned} f^{\pm}_{2mpp'm'q'} &= (2-2p)\dot{\tilde{\omega}}_{\sec} + (m-2+2p)\dot{\Omega}_{\sec} \pm \\ &\pm (2-2p')\dot{\omega}_M \pm (2-2p'+q')\dot{M}_M \pm m'\dot{\Omega}_M \end{aligned} \tag{24}$$

for m = 0,1,2; m′= 0,1,2; p = 0,1,2; p′= 0,1,2; q′=…-2,-1,0,1,2,…

The denominators (no resonances considered) are given by

$$\begin{aligned} D^{\pm}_{2mpp'm'q'p''q''} &= (2-2p)\dot{\tilde{\omega}}_{\sec} + (m-2+2p)\dot{\Omega}_{\sec} \pm \\ &\pm (2-2p')\dot{\omega}_M \pm (2-2p'+q')\dot{M}_M \pm m'\dot{\Omega}_M \pm \\ &\pm (2-2p'')\dot{\omega}_{\odot} \pm (2-2p''+q'')\dot{M}_{\odot} \end{aligned} \tag{25}$$

Resonances will occur if one can find 7 integers $a_i, i = 1,2,...,7$, not "too large", with at most five of them being zeroes, satisfying the condition

$$\begin{aligned} &2a_1(1-p) + a_2(m-2+2p) \pm 2a_3(1-p') \pm a_4(2-2p'+q') \pm a_5 m' \pm \\ &\pm 2a_6(1-p'') \pm a_7(2-2p''+q'') = 0 \end{aligned} \tag{26}$$

should remember that one of the results in Kolmogorov (Reference 10) celebrated work on quasi periodic motions, is that for large enough integers the denominator above can become smaller than any given quantity, which the basis for showing that perturbations techniques based on successive approximations, as for instance, Canonical Methods or Lie Series Methods, cannot converge to the true solution. In this respect, we must rely on Poincaré (Reference 11) statement about asymptotic series "stop the series at a low degree of approximation".

Approximate values for the Moon are $\dot{M}_M = 13.126\, degree/day$, $\dot{\omega}_M = 0.113\, degree/day$, $\dot{\Omega}_M = -0.053\, degree/day$, $\dot{M}_{\odot} = 0.985609\, degree/day$, $\dot{\omega}_{\odot} = 0.985609 \times 10^{-6}\, degree/day$. For a satellite with eccentricity 0.007, inclination 60o and semi-major axis of 26.750 km we have $\dot{\tilde{\omega}}_{\sec} = -24.85 \times 10^{-3} degree/day$, $\dot{\Omega}_{\sec} = -33.26 \times 10^{-3} degree/day$ so we can estimate the amplitude associated to any given set of integers $p, m, p', m', q', p'', q''$ by adding the solar perturbations. Exact evaluation of the integral leading to the mean eccentricity and mean inclination should be from an initial time of observation up to any given successive time, and this requires a transformation from osculating to mean elements, a task to be developed in a future work.

EARTH GRAVITY PERTURBATIONS

Consider only the lowest degree harmonic of the Earth potential

$$R_{20pq} = \frac{\mu b}{a^3} s^{|\alpha_{20p}|} e^{|q|} J_{20p}(c) K_{2pq}(e)(A_{20} \cos\psi_{20pq} + B_{20} \sin\psi_{20pq}) \tag{27}$$

where the coefficients A and B are related to the harmonic coefficients of the Earth Potential Field and

$$\psi_{20pq} = (2-2p+q)\lambda - q\tilde{\omega} + p\Omega \tag{28}$$

where we have set

$$\alpha_{20p} = p \tag{29}$$

It is noted that the absolute value of the coefficient of Ω is the power of sine of half of the inclination in a particular term of the Disturbing Function, as well as the absolute value of the coefficient of $\tilde{\omega}$ is the least power of the eccentricity. The functions $J_{20p}(c)$ are polynomials in the cosine of half the inclination and the functions $K_{2pq}(\beta)$ are power series in the eccentricity and will be written as $K_{2pq}(e)$. In case of long period terms, corresponding to $2-2p+q=0$, in absence of resonance of any kind, these functions are polynomial in β.

SECULAR PERTURBATIONS DUE TO THE EARTH GRAVITY

For a real analysis of perturbations of an artificial satellite we must consider perturbations due to the Earth gravity field, noting that up to geosynchronous heights the dominant term is due the Earth oblateness. The disturbing function for secular perturbations due to the Earth is given by the well known expression

$$R_{2,0,1,0} = \frac{\mu b}{a^3} J_{2,0,1}(c) K_{2,1,0}(e) C_{2,0} \tag{30}$$

where

$$K_{2,1,0} = \frac{1}{(1-e^2)^{1/2}} \sum_{j=0}^{1} \binom{2-1}{2j} \binom{2j}{j} \left(\frac{e}{2}\right)^{2j} \tag{31}$$

$$G_{21q} = e^{|q|} K_{21q}(e) \tag{32}$$

and for $\ell = 2, q = 0$ it follows that $G_{2,1,0} = K_{2,1,0}$.

The J functions are simply derived from Kaula (1966) inclination functions by setting $\ell = 2, m = 0$ and by considering the condition $\alpha_{201} = 0$ it is found that

$$F_{2,0,1}(I) = J_{2,0,1}(c) = \sum_{j=0}^{2} F_{2,0,1}^{j} c^{2(2-j)} (1-c^2)^j \tag{33}$$

where

$$F_{2,0,1}^{j} = (-1)^{1+j} \frac{1}{2} \binom{2}{j} \binom{2}{2-j}. \tag{34}$$

As a first approximation, the secular rates may be computed by a simple quadrature, by keeping constant the right-hand members of the pertinent differential equations. These secular rates are affected by short and long period perturbations, through a, e and I, a task to be undertaken in a future work. In order to simplify matters and give a preliminary example, we shall consider the Moon to move on the ecliptic and, therefore neglect the longitude of the ascending node of the lunar orbit, keeping only the argument of perigee. Under these hypotheses a direct addition of secular rates due to the Moon, Sun and the geopotential gives

$$\dot{\tilde{\omega}}_{\sec}=\frac{3nC_{20}b^2}{4\gamma^4a^2}\left(1+2\cos I-5\cos^2 I\right)+ \\ +\frac{3(\mu_M n_M^2+\mu_\odot n_\odot^2)}{8n\gamma}\left[5\cos^2 I-(3e^2+2)\cos I-(1-e^2)\right] \tag{35}$$

$$\dot{\Omega}_{\sec}=\frac{3nC_{20}b^2}{2\gamma^4a^2}\cos I-\frac{3(\mu_M n_M^2+\mu_\odot n_\odot^2)\cos I}{8n\gamma}(3e^2+2) \tag{36}$$

$$\dot{\lambda}_{\sec}=n+\frac{3nC_{20}b^2}{4\gamma^4a^2}\left[(1+\gamma)-(3\gamma-5)\cos^2 I+2\cos I\right]+\frac{\mu_M n_M^2+\mu_\odot n_\odot^2}{8n\gamma}\times \\ \times\left[-3(5+7\gamma+3\gamma e^2)\cos^2 I-3(3e^2+2)\cos I+\gamma(7+3e^2-2\gamma)\right] \tag{37}$$

where $\mu_M=Gm_M/(m_M+m_\oplus)$ and $\mu_\odot=Gm_\odot/(m_\odot+m_\oplus)$

The long period perturbations due to the Moon and Sun are given by

$$\dot{e}_{lp}=\frac{15(\mu_M n_M^2+\mu_\odot n_\odot^2)\gamma e}{8n}\sin^2 I\sin 2\tilde{\omega} \tag{38}$$

$$\dot{I}_{lp}=-\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)e^2}{16n\gamma}\sin 2I\sin 2\tilde{\omega} \tag{39}$$

$$\dot{\tilde{\omega}}_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)}{8n\gamma}(1-e^2+5e^2\cos I-\cos^2 I)\cos 2\tilde{\omega} \tag{40}$$

$$\dot{\Omega}_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)e^2\cos I}{8n\gamma}\cos 2\tilde{\omega} \tag{41}$$

$$\dot{\lambda}_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)}{8n\gamma}(1-e^2+5e^2\cos I-\cos^2 I-(1-e^4)\sin^2 I)\cos 2\tilde{\omega} \tag{42}$$

Integration , keeping constant the metric variables on the right hand side yields

$$\delta_1 e_{lp}=-\frac{15(\mu_M n_M^2+\mu_\odot n_\odot^2)e\gamma}{16n\dot{\omega}_{\sec}}\sin^2 I\cos 2\tilde{\omega} \tag{43}$$

$$\delta_1 I_{lp}=-\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)e^2}{32n\dot{\tilde{\omega}}_{\sec}\gamma}\sin 2I\cos 2\tilde{\omega} \tag{44}$$

$$\delta_1\tilde{\omega}_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)}{16n\dot{\omega}_{\sec}^2\gamma}(1-e^2+5e^2\cos I-\cos^2 I)\sin 2\tilde{\omega} \tag{45}$$

$$\delta_1\Omega_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)e^2\cos I}{16n\dot{\omega}_{\sec}\gamma}\sin 2\tilde{\omega} \tag{46}$$

$$\delta_1\lambda_{lp}=\frac{15\left(\mu_M n_M^2+\mu_\odot n_\odot^2\right)}{16n\dot{\tilde{\omega}}_{\sec}\gamma}[1-e^2+5e^2\cos I-\cos^2 I-(1-e^4)\sin^2 I]\sin 2\tilde{\omega} \tag{47}$$

The numerator in these perturbations is of the order of 10-5 and will over come the value of the numerator.

INFLUENCE OF LUNI-SOLAR PERTURBATIONS ON SECULAR RATES DUE TO THE EARTH GRAVITY FIELD

The task is to take into consideration the influence of the third body on the secular rates resulting from the Earth potential field. Taking into account variational equations the second order perturbations in secular rates are obtained from

$$\frac{d}{dt}(\delta_2\tilde{\omega}_{\sec}) = \frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial e}\delta_1 e_{\ell p} + \frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial I}\delta_1 I_{\ell p} \tag{48}$$

$$\frac{d}{dt}(\delta_2\Omega_{\sec}) = \frac{\partial\dot{\Omega}_{\sec}}{\partial e}\delta_1 e_{\ell p} + \frac{\partial\dot{\Omega}_{\sec}}{\partial I}\delta_1 I_{\ell p} \tag{49}$$

$$\frac{d}{dt}(\delta_2\lambda_{\sec}) = \frac{\partial\dot{\lambda}_{\sec}}{\partial e}\delta_1 e_{\ell p} + \frac{\partial\dot{\lambda}_{\sec}}{\partial I}\delta_1 I_{\ell p} \tag{50}$$

where $\delta_1 e_{\ell p}, \delta_1 I_{\ell p}$ are perturbations due to a third body while $\delta_2\tilde{\omega}_{\sec}$, $\delta_2\Omega_{\sec}$, $\delta_2\lambda_{\sec}$ are clearly second order perturbations on secular motions induced by the geopotential. We initially consider just third body long period perturbations. In order to do this we consider the above variational equations where the secular variations on the right hand side are given by Eqs. (35), (36) and (37)

By considering the major terms already computed for the above quantities, we find from Eqs. 86 to 88 the following partials

$$\begin{aligned}\frac{\partial}{\partial e}\dot{\tilde{\omega}}_{\sec} &= \frac{3nC_{20}b^2 e}{\gamma^6 a^2}\left(1+2\cos I - 5\cos^2 I\right) + \\ &+ \frac{3e(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{8n\gamma^3}\left[5\cos^2 I + (3e^2-8)\cos I + \gamma^2\right]\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{\partial}{\partial I}\dot{\tilde{\omega}}_{\sec} &= \frac{3nC_{20}b^2}{4\gamma^4 a^2}\left(2\sin I + 5\sin 2I\right) + \\ &+ \frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{8n\gamma}\left[-5\sin 2I + (3e^2+2)\sin I\right]\end{aligned} \tag{52}$$

$$\frac{\partial}{\partial e}\dot{\Omega}_{\sec} = \frac{6nC_{20}b^2 e}{\gamma^6 a^2}\cos I - \frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)e\cos I}{8n\gamma^3}(9e^2-4) \tag{53}$$

$$\frac{\partial}{\partial I}\dot{\Omega}_{\sec} = -\frac{3nC_{20}b^2}{2\gamma^4 a^2}\sin I + \frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)\sin I}{8n\gamma}(3e^2+2) \tag{54}$$

$$\begin{aligned}\frac{\partial}{\partial I}\dot{\lambda}_{\sec} &= \frac{3nC_{20}b^2}{4\gamma^4 a^2}\left[(3\gamma-5)\sin 2I - 2\sin I\right] + \\ &+ \frac{\mu_M n_M^2 + \mu_\odot n_\odot^2}{8n\gamma}\left[3(5+7\gamma+3\gamma e^2)\sin 2I + 3(3e^2+2)\sin I\right]\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial}{\partial e}\dot{\lambda}_{\sec} &= \frac{3neC_{20}b^2}{4\gamma^7 a^2}\left[(e^2+\gamma) + (5-3\gamma+3\gamma^2)\cos^2 I + 2\cos I\right] + \frac{\mu_M n_M^2 + \mu_\odot n_\odot^2}{8n\gamma^3}\times \\ &\times\left[3e\left(5e-6\gamma^3\right)\cos^2 I - \gamma^2\left(18e\gamma^2 + 6\gamma^2 - 9e^3 - 6e^2\right)\cos I + 2e\gamma^2\left(3\gamma - e\right)\right]\end{aligned} \tag{56}$$

Recalling Equations (48), (49) and (50), we find the influence of long period perturbations due to the Moon and Sun on the secular rates due to the Earth gravity field

$$\begin{aligned}&\frac{d}{dt}(\delta_2\tilde{\omega}_{\sec}) = -\frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)e\sin I}{16n\gamma\dot{\tilde{\omega}}_{\sec}}\times \\ &\times\left\{\frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial e}(1-e^2)\sin I + \frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial I}e\cos I\right\}\cos 2\tilde{\omega}\end{aligned} \tag{57}$$

$$\begin{aligned}&\frac{d}{dt}(\delta_2\Omega_{\sec}) = -\frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)e}{16n\gamma\dot{\tilde{\omega}}_{\sec}}\times \\ &\times\left\{\frac{\partial\dot{\Omega}_{\sec}}{\partial e}(1-e^2)\sin I + \frac{\partial\dot{\Omega}_{\sec}}{\partial I}e\cos I\right\}\cos 2\tilde{\omega}\end{aligned} \tag{58}$$

$$\frac{d}{dt}(\delta_2\lambda_{\sec}) = \frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)e\sin I}{16n\dot{\tilde{\omega}}_{\sec}}\left\{-\frac{\partial\dot{\lambda}_{\sec}}{\partial e}\gamma\sin I - \frac{\partial\dot{\lambda}_{\sec}}{\partial I}e\cos I\right\}\cos 2\tilde{\omega} \tag{59}$$

Integration of these equations may be performed by keeping constant all metric elements since they are not affected by secular perturbations. The result is given as follows:

$$\delta_2\tilde{\omega}_{\sec} = -\frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)e\sin I}{32n\gamma\dot{\tilde{\omega}}_{\sec}^2}\left\{\frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial e}(1-e^2)\sin I + \frac{\partial\dot{\tilde{\omega}}_{\sec}}{\partial I}e\cos I\right\}\sin 2\tilde{\omega} \tag{60}$$

$$\delta_2\Omega_{\sec} = -\frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{32n\gamma\dot{\tilde{\omega}}_{\sec}^2}\left\{\frac{\partial\dot{\Omega}_{\sec}}{\partial e}(1-e^2)\sin I + \frac{\partial\dot{\Omega}_{\sec}}{\partial I}e\cos I\right\}\sin 2\tilde{\omega} \tag{61}$$

$$\delta_2\lambda_{\sec} = -\frac{15(\mu_M n_M^2 + \mu_\odot n_\odot^2)\sin I}{32n\dot{\tilde{\omega}}_{\sec}^2}\left\{\frac{\partial\dot{\lambda}_{\sec}}{\partial e}\gamma\sin I + \frac{\partial\dot{\lambda}_{\sec}}{\partial I}e\cos I\right\}\sin 2\tilde{\omega} \tag{62}$$

These equations give the major coupling between the secular rates due to the Earth flattening and the long term influence of the Moon and Sun. The period of these perturbations relates to the secular rate of the longitude of perigee, given by

$$\begin{aligned}\dot{\tilde{\omega}}_{\sec} &= \frac{3nC_{20}b^2}{4\gamma^4a^2}\left(1+2\cos I - 5\cos^2 I\right) + \\ &+\frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{8n\gamma}\left[5\cos^2 I - (3e^2+2)\cos I - (1-e^2)\right]\end{aligned} \tag{63}$$

We may consider a good approximation to set the eccentricity equal to zero, so that

$$\dot{\tilde{\omega}}_{\sec} = \frac{3nC_{20}b^2}{4a^2}\left(1+2\cos I - 5\cos^2 I\right) + \frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{8n}(5\cos^2 I - 2\cos I - 1) \tag{64}$$

We assume the values

$$T = 0.498634 day, n = 12.5944 rd/day, a = 26560 km, b = 6378 km, e = 0.006, I = 54^o$$

For the Moon and Sun we have

$$\begin{aligned}\mu_M n_M^2 &= 1.59\times10^{-5} rev^2 day^{-2} = 6.2707\times10^{-4} rd^2 day^{-2} \\ \mu_\odot n_\odot^2 &= 0.75\times15^{-5} rev^2 day^{-2} = 2.9579\times10^{-4} rd^2 day^{-2}\end{aligned} \tag{65}$$

and the Earth flattening coefficient is $C_{20} = -1.083 \times 10^{-3}$.

Introducing these values for the secular rate of the longitude of perigee we find

$$\frac{3nC_{20}b^2}{4\gamma^4 a^2}\left(1 + 2\cos I - 5\cos^2 I\right) = -5.968 \times 10^{-4} \tag{66}$$

$$\frac{3(\mu_M n_M^2 + \mu_\odot n_\odot^2)}{8n}(5\cos^2 I - 2\cos I - 1) = -0.1231 \times 10^{-4} \tag{67}$$

$$\dot{\tilde{\omega}}_{\sec} = -6.0911 \times 10^{-4}\, rd/day \tag{68}$$

The corresponding period is $T_{\varpi} = 10\,310\ days$ which agrees very closely with the observed values as shown in Figure 1 through Figure 4.

DISCUSSION

As given by Schutz (Reference 6) the evolution of the longitude of the perigee and of the longitude of the ascending node, over a decade, of a selected GP satellite are shown in the Fig. 1 and Fig. 2, respectively, while long terms variations in eccentricity and inclination are given in Fig. 3 and Fig.4. We can observe three basic periods in the eccentricity (a period of about 365 days or 0.986 degree/day, a period of 27 days or 13.3 degree per day and a very long period) and three basic periods in the inclination (a periods of about 180 days or 2.0 degree/day, a period of 14 days or 25.7 degree/day and a very long period). The long periods are not well defined but, in any event, they are at least 10,000 days.

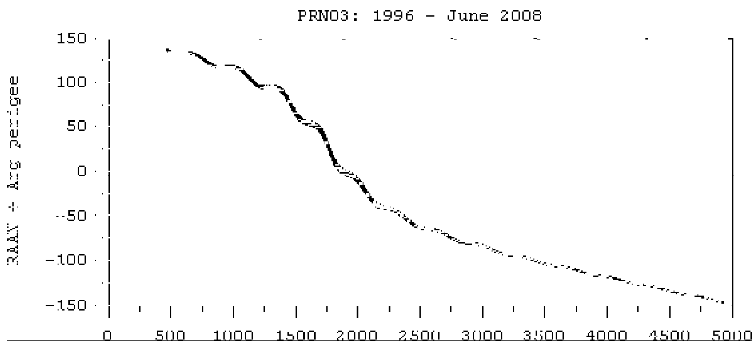

Figure 1. – Long term evolution of the longitude of perigee

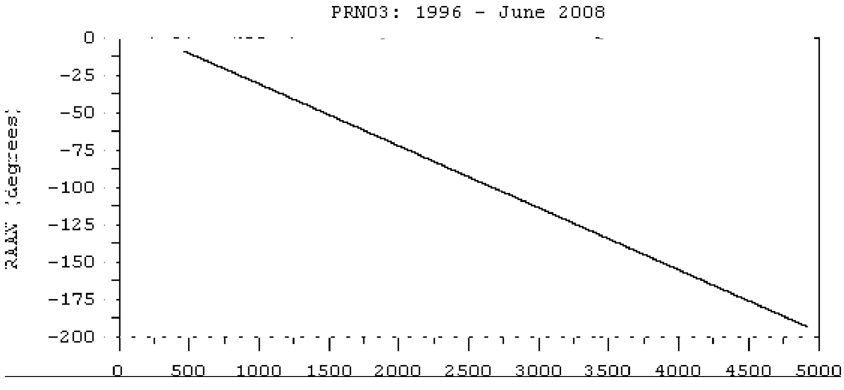

Figure 2. – Long term evolution of the longitude of ascending node

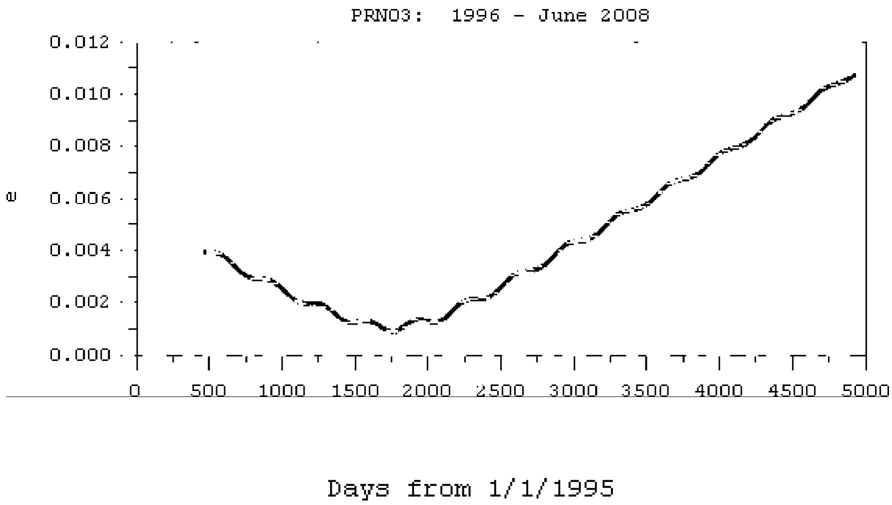

Figure 3. – Long term evolution of the eccentricity

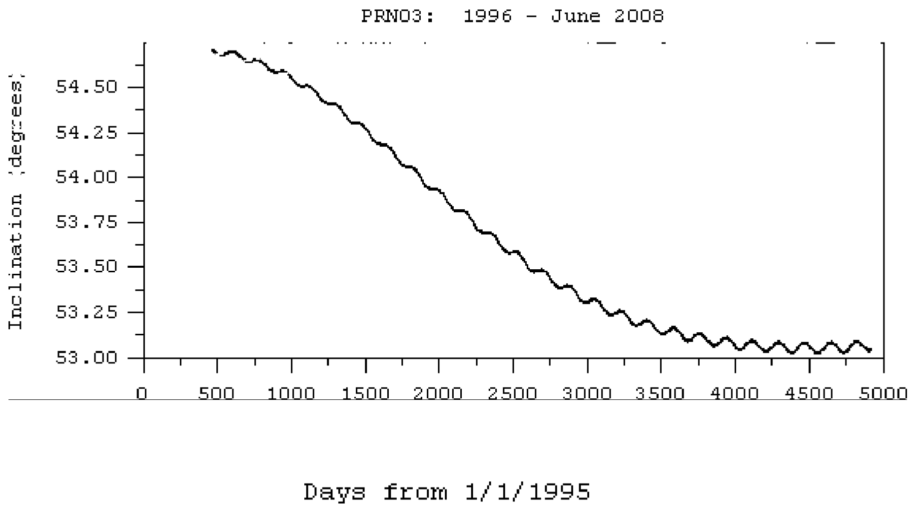

Figure 4. Long term evolution of the Inclination

Now, the possible frequencies due to earth oblateness, lunar and solar perturbations are given by

$$\begin{aligned} f^{\pm}_{\ell mpp'm'q'p''q''} &= \left(\ell - 2p\right)\dot{\bar{\omega}}_{\sec} + (m - \ell + 2p)\dot{\Omega}_{\sec} \pm \left(\ell - 2p'\right)\dot{\omega}_M \\ &\pm \left(\ell - 2p' + q'\right)\dot{M}_M \pm m'\dot{\Omega}_M \pm \left(\ell - 2p''\right)\dot{\omega}_{\odot} \pm \left(\ell - 2p'' + q''\right)\dot{M}_{\odot} \end{aligned} \tag{69}$$

Introducing the approximate values for each secular motion, it follows that all possible frequencies in degree per day are given by

$$\begin{aligned} f^{\pm}_{\ell mpp'm'q'p''q''} &= -\left(\ell - 2p\right)24.85\times10^{-3} - (m - \ell + 2p)33.26\times10^{-3} \pm \\ &\pm \left(\ell - 2p'\right)0.113 \pm \left(\ell - 2p' + q'\right)13.126 \mp m'0.053 \pm \\ &\pm \left(\ell - 2p''\right)0.4707\times10^{-8} \pm \left(\ell - 2p'' + q''\right)0.985627 \end{aligned} \tag{70}$$

It is obvious that one can choose integers involved in Eq. (70) and match very closely the observed frequencies of 0.986, 13.3, 2.0 and 25.7 degree/day. But the point is, for reasonable large amplitudes, as observed in Fig. 1 through Fig.4, only small integers should be considered. It should be remembered that the GPS satellites are in resonance with the following harmonics degrees and orders

Table 1 – Relevant harmonics for the GPS satellites

ℓ	m	p	q
2	2	1	1
2	2	2	1
2	2	0	-1
3	2	1	0

and a deep resonance is observed for the 3-2 harmonic, producing a frozen orbit

If we consider that the eccentricities of the Moon and Sun are very small, then we should start by setting q´ and q” equal zero. The resulting frequencies in degrees per day are

$$f^{\pm}_{\ell mpp'm'0p''0} = -24.85\times10^{-3}\left(\ell-2p\right)-33.26\times10^{-3}(m-\ell+2p)\pm \\ \pm13.239\left(\ell-2p'\right)\mp0.053m'\pm0.985627\left(\ell-2p''\right) \tag{71}$$

Consider the small degree harmonics defined in table 149. We shall compute the frequencies associated with the resonance harmonics, that is

$$f^{\pm}_{221p'm'0p''0} = -2\times33.26\times10^{-3}\pm2\times13.239\left(1-p'\right)\pm \\ \pm0.053m'\pm2\times0.985627\left(1-p''\right) \tag{72}$$

$$f^{\pm}_{222p'm'0p''0} = -83.34\times10^{-3}\pm2\times13.239\left(1-p'\right)\pm \\ \pm0.053m'\pm2\times0.985627\left(1-p''\right) \tag{73}$$

$$f^{\pm}_{220p'm'0p''0} = -2\times24.85\times10^{-3}\pm2\times13.239\left(1-p'\right)\mp \\ \mp0.053m'\pm2\times0.985627\left(1-p''\right) \tag{74}$$

$$f^{\pm}_{321p'm'0p''0} = -58.11\times10^{-3}\pm13.239\left(3-2p'\right)\mp \\ \mp0.053m'\pm0.985627\left(3-2p''\right) \tag{75}$$

We shall take m’ = 0, p´= 0 and p” = 0 corresponding to the largest amplitudes.

The resulting frequencies, in degree per day, and the corresponding periods in days are given in Table 2

Table 2- Frequencies, in degree per day, and the corresponding periods in days

ℓ m p p' m' q' p'' q''	f^{+}	f^{-}	PERIOD +	PERIOD -
2 2 1 0 0 0 0 0	28.383	-28.516	10,507	10,556
2 2 2 0 0 0 0 0	28.366	-28.533	10,501	10,562
2 2 0 0 0 0 0 0	28.400	-28.499	10,513	10,550
3 2 1 0 0 0 0 0	42.616	-42.732	15,776	15,819

These values would correspond to the largest amplitudes, due to the chosen harmonics, in the absence of resonances. It is seen that the very long periods are in agreement with those shown in Figures 1 through 4. Periods associated with the period of the Sun and Moon as well half of these periods, as observed in these figures, are easily obtained by proper choice of the degrees and orders involved in Table 1.

CONCLUSIONS

Notwithstanding the necessity of detailed numerical calculations, the theory described in this paper is be able to explain some of the observed behavior in the motion of high altitude satellites with small eccentricity, which is the case of GPS satellites. The drawback is that no resonances have been assumed in the development of the theory, a question to be resolved in a future research paper.

REFERENCES

[1] Giacaglia, G.E.O. (1975). The Equations of motion of an Artificial Satellite in Non Singular Variables. AMRL Report n. 1072, UT Austin

[2] Giacaglia, G.E.O. and Schutz, B. E. (2008a). Third body perturbations on satellite orbits with small eccentricity, Symposium Honoring Byron Tapley's 50 Years of Contributions to Aerospace Education, Research and Service, Austin , February 2008

[3] Giacaglia, G.E.O. and Schutz, B. E. (2008b). Third Body Perturbations on Satellites Orbits with Small Eccentricity, AAS / AIAA Meeting, Galveston

[4] Giacaglia, G.E.O. and Schutz, B. E. (2009). A non singular approach in Satellite Theory, AAS / AIAA Meeting, Savannah

[5] Giacaglia, G. E. O. and Schutz, B. E. (2010). Development of luni-solar and geopotential perturbations on a close satellite with small eccentricity and inclination, (Paper accepted for presentation to the 14th International Symposium on Dynamic Problems of Mechanics – DINAME 2011, March 13 to 18, 2011, São Sebastião, SP, Brazil)

[6] Schutz, B. and Giacaglia, G.E.O., 2009. Decade-Scale GPS Orbit Evolution and Third Body Perturbations. Paper 092407, AIAA / AAS Meeting, Honolulu

[7] Kaula, W. M. (1961). Development of the lunar and solar disturbing functions for a close satellite. Astron. Journ., vol. 67, p. 300.

[8] Plummer, H. C. (1960) "An Introductory Treatise on Dynamical Astronomy", Dover Publications, Inc., New York (a reprint of the original publication by Cambridge University Press, Cambridge, UK, 1918.

[9] Kaula, W. M. (1966) Theory of Satellite Geodesy, Blaisdell Pub. Co., Waltham, Massachussets

[10] Kolmogorov, A. N. (1953) "On the Conservation of Quasiperiodic Motions for a Small Change in the Hamiltonian Function", Dokl. Akad. Nauk USSR, 98, 527-530.

[11] Poincaré, H. (1898) "Les Méthodes Nouvelles de la Mécanique Celèste" (vol. 2) Gauthier-Villars, Paris (Dover Reprint, New York, 1957).

ACKNOWLEDGEMENTS

The first author is grateful to FAPESP, The Sao Paulo Research Foundation, for providing a visiting researcher scholarship at the National Institute for Space Research – INPE, during the year of 2009 and to CNPq, The Brazilian National Council for Research, for providing a Visiting Specialist Scholarship at INPE during the first quarter of 2010. Continuous support from the University of Taubaté is gratefully acknowledged.