View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Engineering Research: technical reports (E-Journal - Universidade de Taubaté)

doi.org/10.32426/engresv10n1-001

Engineering Research

Technical Reports

Volume 10 – Issue 1 – Article 1

ISSN 2179-7625 (online)

PERTURBATION METHODS IN NON-LINEAR SYSTEMS:

Part I

Giorgio Eugenio Oscare Giacaglia[1]

JANUARY / 2019

Taubaté, São Paulo, Brazil

[1] Department of Mechanical Engineering, University of Taubate, giacaglia@gmail.com.

Engineering Research: Technical Reports

Editor-in-Chief
Wendell de Queiróz Lamas, Universidade de São Paulo at Lorena, Brazil

Executive Editor
Eduardo Hidenori Enari, Universidade de Taubaté, Brazil

Associate Executive Editor
Luís Fernando de Almeida, Universidade de Taubaté, Brazil

Editorial Board
Arcione Ferreira Viagi, Universidade de Taubaté, Brazil
Asfaw Beyene, San Diego State University, USA
Bilal M. Ayyub, University of Maryland, USA
Ciro Morlino, Università degli Studi di Pisa, Italy
Epaminondas Rosa Junior, Illinois State University, USA
Evandro Luís Nohara, Universidade de Taubaté, Brazil
Fernando Manuel Ferreira Lobo Pereira, Universidade do Porto, Portugal
Francisco Carlos Parquet Bizarria, Universidade de Taubaté, Brazil
Francisco José Grandinetti, Universidade de Taubaté, Brazil
Giorgio Eugenio Oscare Giacaglia, Universidade de Taubaté, Brazil
Hubertus F. von Bremen, California State Polytechnic University Pomona, USA
Jorge Muniz Júnior, Universidade Estadual Paulista at Guaratinguetá, Brazil
José Luz Silveira, Universidade Estadual Paulista at Guaratinguetá, Brazil
José Rubens de Camargo, Universidade de Taubaté, Brazil
José Rui Camargo, Universidade de Taubaté, Brazil
José Walter Paquet Bizarria, Universidade de Taubaté, Brazil
María Isabel Sosa, Universidad Nacional de La Plata, Argentina
Miroslava Hamzagic, Universidade de Taubaté, Brazil
Ogbonnaya Inya Okoro, University of Nigeria at Nsukka, Nigeria
Paolo Laranci, Università degli Studi di Perugia, Italy
Rolando A. Zanzi Vigouroux, Kungliga Tekniska högskolan, Sweden
Sanaul Huq Chowdhury, Griffith University, Australia
Tomasz Kapitaniak, Politechnika Lódzka, Poland
Valesca Alves Corrêa, Universidade de Taubaté, Brazil
Valter Bruno Silva, Instituto Politécnico de Portalegre, Portugal

The “Engineering Research” is a publication with purpose of technical and academic knowledge dissemination.

PREFACE

This volume is an on-line reprint of the original book published 1972 by Springer-Verlag which was intended to provide a comprehensive treatment of contemporary developments in methods of perturbation for nonlinear systems of ordinary differential equations. In this respect, it appeared to be a unique work, with hundreds of citations.

Even today is a basic reference in the approximate solution of non-linear differential equations, specially appearing in problems of Celestial Mechanics.

The original goal was to describe perturbation techniques, discuss their advantages and limitations and give some examples. The approach was founded on analytical and numerical methods of nonlinear mechanics.

Attention had been given to the extension of methods to high orders of approximation, required now by the increased accuracy of measurements in all fields of science and technology.

The main theorems relevant to each perturbation technique were outlined, but they only provided a foundation and were not the objective of the original book.

Each chapter concluded with a detailed survey of the contemporary literature, supplemental information and more examples to complement the text, when necessary, for better comprehension.

The references were intended to provide a basic guide for background information and for the reader who wished to analyze any particular point in more detail. The main sources referenced were in the fields of differential equations, nonlinear oscillations and celestial mechanics.

Partial support from the Mathematics Program of the Office of Naval Research is gratefully acknowledged.

July 2016

Giorgio E. O. Giacaglia

Sao Paulo, Brazil

INTRODUCTION

In what follows we are going to describe in short the basic problem of perturbations the way it is to be developed in these notes. We shall here make free and simple statements without entering mathematical details on the functions involved. The necessary hypotheses will be made in the subsequent chapters. Historically we consider Lindstedt's (1882) problem of obtaining a series solution, free from secular and / or mixed secular terms, of the equation.

$$\ddot{x}+\omega_0^2 x = \in \mathrm{f}\left(x,\dot{x},\mathrm{t}\right)$$

Where $0 < \in < 1$ is a parameter. The possibility of obtaining a solution.

$$x = x_0\left(\mathrm{t}\right) + \in x_1\left(\mathrm{t}\right) + \in^2 x_2\left(\mathrm{t}\right) + \ldots$$

$$\dot{x} = \dot{x}_0\left(\mathrm{t}\right) + \in^2 \dot{x}_2\left(\mathrm{t}\right) + \ldots$$

of the above equation, with $x_j\left(t\right), \dot{x}_j\left(t\right)$ bounded functions for all $t \in R$ was found to depend essentially on the nature of f and its derivatives up to some order. The reference solution introduced by Lindstedt, that is, $\left(x_0\left(t\right), \dot{x}_0\left(t\right)\right)$ was given by

$$x_0 = \mathrm{a}\ \cos\ \left(\omega\mathrm{t}+\sigma\right)$$

$$\dot{x}_o = \text{-a}\omega\sin\left(\omega t+\sigma\right)$$

Where ω is a priori unknown but, by assumption, developable in a power series

$$\omega = \omega_o + \in \omega_1 + \in^2 \omega_2 + \in^3 \omega_3 + \ldots$$

Where $\omega_1, \omega_2, \ldots$ are constants depending on ω_o, *a* and *f*. Strictly speaking the very first attempt of dealing with perturbed oscillatory systems had been made by Euler (1772) in his researches on the motion of the Moon. Delaunay was the second in line to recognize that the major difficulty in the avoidance of unbounded terms in the series solution of such systems was the choice of a reference frequency, a fact which lead him to produce perhaps the first systematic series process of determining what are today called Floquet's characteristic exponents (Delaunay, 1860). The transformation of Delaunay's method of successive canonical transformation to a method utilizing a generating function was first foreseen by Tisserand (1868). After some time, the work of Lindstedt was published (1882) and, right after, reduced to a systematic averaging procedure by Poincaré (1886) for Hamiltonian, but not necessarily conservative, systems. In essence, the whole second volume of his "Mécanique Célèste" is devoted to this method and related questions, among the most important, the problem of resonance, in the nonlinear sense. He accomplished a great deal of unification of all previous works including the milestone works of Bohlin and Gyldèn. In chronological order it is again in Celestial Mechanics that new efforts were made on the problem by von Zeipel (1911), by generalizing Poincaré's ideas. We shall not endeavor into details along these works and refer to several surveys on the subject (Cesari, 1959; Giacaglia, 1965; Kyner, 1967). It was only at least a decade later that similar problems and questions arose in nonlinear circuit theory leading to the averaging methods of Krylov and Bogoliubov (1942) made available to the western mathematicians by the efforts of Lefschetz. The work by Brown (1931) on nonlinear resonance came well after Poincaré's dealing with the problem and it is actually based on the examples he produced to illustrate Bohlin's method. Modern literature on perturbation methods and averaging procedures becomes highly dense after about 1950 and specific reference on these will be done along the work, at the proper moment. So far for purely analytic works which aimed the quantitive approach, typical of the classical analysis of last century and beginning of this, of obtaining an explicit time solution for a System of differential equations.

Along different lines, it was Poincaré (1952) who tried to understand, for the first time, the geometry of a differential system. His conjecture on the existence of fixed points for area preserving mapping, associated to the solution of a conservative system, was proved to be right by Birkhoff (1915) whose work is to be considered as one of the deepest changes ever introduced in the concept of solution important concepts like invariant sets, wandering points, etc., all related to the geometric behavior of the integral curves of a system. Along these lines the basic approach is probably best explained by Moser's celebrated work on the area preserving mapping of a circle into itself (1962), by Hale's work on integral manifolds of perturbed systems (1961) and by the work of Krylov and Bogoliubov (1934). Again we shall more specifically to the current literature when dealing with perturbations of invariant sets.

The classical and perhaps the oldest methods of perturbations are of the Euler-Lagrange type, generalized by Poisson. Their conservative analogues are condensed in Jacobi's Theorem on the variation of canonical variables. Since Poisson's method is the most general, it is worth mentioning here, but it will be done in a heuristic manner. We consider a differential system

$$\dot{x} = f(x, t) \tag{1}$$

where x, f are n-vectors. For simplicity we assume f to be analytic in a certain domain D of the vector space x and for $t \in R$. Let, in D,

$$\sigma = \sigma(x,t) \tag{2}$$

be a first uniform integral of (1). It follows that, along any solution of (1) in D, we have

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x}\dot{x} + \frac{\partial \sigma}{\partial t} = 0$$

where σ is an m-vector $(m \leq n)$, so that $\partial\sigma/\partial x$ is a rectangular Jacobian matrix (m x n). We have, for every $x \in D$, the identity

$$\frac{\partial \sigma}{\partial x} f(x,t) + \frac{\partial \sigma}{\partial t} = 0. \tag{3}$$

Consider now the perturbed system

$$\dot{x} = f(x,t) + g(x,t) \tag{4}$$

where, again, g (x, t) is supposed analytical in D X R. We consider the variation of (2) along system (4), that is,

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x}\left[f(x,t) + g(x,t)\right] + \frac{\partial \sigma}{\partial t}$$

or, in view of (3),

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x} g(x,t). \tag{5}$$

Equation (5) is generally credited to Poisson (1956) and contains as particular examples Lagrange's Equations for the variation of arbitrary constants and Jacobi's theorem. In the particular case of a dynamical system

$$\ddot{x} = f(x,x,t) + g(x,\dot{x},t)$$

Poisson's equation becomes

$$\dot{\sigma} = \frac{\partial \sigma}{\partial \dot{x}} g(x,\dot{x},t) \tag{6}$$

where σ is an integral for $g \equiv 0$. Interestingly enough, all basic theorems of Classical Mechanics are immediately derivable from (6). In fact, if σ is the Energy Integral

$$E = \frac{1}{2}\dot{x}^2 + V(x,t)$$

it follows that, $E_{\dot{x}} = \dot{x}$ and

$$\dot{E} = \dot{x}^T g(x,\dot{x},t)$$

which is the basic law of energy and work. If σ is the Angular Momentum Integral

$$L = x \times \dot{x}$$

it follows that $L_{\dot{x}} = x \times$ and

$$\dot{L} = x \times g(x,\dot{x},t)$$

which is the basic law of angular momentum and torque.

If (1) is a Hamiltonian system (x is a 2n-vector), that is,

$$\dot{x} = MH_x^T \tag{7}$$

where $H = H(x,t)$ and M is the canonical matrix $2n \times 2n$,

$$M = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

and we let

$$H = H_0 + H_1$$

if σ is a first integral of (7), in involution with H_0 , for $H = H_0$, it follows that

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x} M \left(\frac{\partial H_1}{\partial x} \right)^T . \quad (8)$$

If, furthermore, the Jacobian matrix $j = \frac{\partial \sigma}{\partial x}$ is symplectic (that is, the transformation is canonical) it follows that

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x} M \left(\frac{\partial \sigma}{\partial x} \right)^T = M \left(\frac{\partial H_1}{\partial \sigma} \right)^T \quad (9)$$

which is Jacobi's theorem. We observe that if $\underline{\sigma}$ is a 2n-vector, (8) are Lagrange's equations for the variation of arbitrary constants in case of conservative forces.

The classical approach to (9) is to assume for σ a power series in a small parameter and reduce the problem to a method of successive approximations. In most cases this procedure leads to secular and mixed secular terms and therefore the series cannot converge for all time. If we limit the time, convergence can eventually be obtained and the earliest reference to this question is probably the work by MacMillan (1910). We refer to this work since it is simple yet quite rigorous.

In the more sophisticated methods of averaging it is generally assumed (Hamiltonian system) that the Hamiltonian function is 2π periodic in every angular variable $Y_{1,} Y_2 ,..., Y_n$ and representable in a convergent Fourier series

$$H = \sum_j A_j(x) \exp i = \sqrt{-1}(j.Y) \quad (10)$$

where $j = (j_1, j_2, \ldots, j_n)$ is an “integer” vector.

The equations generated by (10) are

$$\dot{x} = -\left(\frac{\partial H}{\partial Y}\right)^T \quad (11)$$

$$\dot{Y} = +\left(\frac{\partial H}{\partial x}\right)^T$$

If we consider only the part of H corresponding to j = 0,

$$H_0 = A_0(x)$$

system (11) is obviously integrable and

$$x = x_0$$

$$y = \omega(x_0)t + y_0 \quad (12)$$

where

$$\omega_j(x_0) = \partial H_0 / \partial x_j \big|_x = x_0$$

If in a certain region, the $A_j(x)$ for $j \neq 0$, are such that their derivatives are small (in some sense) with respect to the $\omega_j(x)$, then we can treat $H - H_0$ as a perturbation. Classically it was assumed that if this situation occurs, than the solution of (11) never departs too much from the solution (12). Such supposition is evidently false and seldom verified, even considering “orbital proximity” regard-less of the time. It is actually the “time proximity” of corresponding points which is the most affected by the perturbation, and such phenomenon is well known as the “in-track error”. The analogy with the concept of stability is that it is easier to have orbital than Lyapunov stability.

In any event, using (12) as a reference solution with modified frequency vector $v(\mathrm{x}_0)$ and iterating, we obtain formal series

$$\mathrm{x} = \mathrm{x}_0 + \sum_j \frac{C_j(\mathrm{x}_0)}{\mathrm{j}.v(\mathrm{x}_0)} \exp\left[\mathrm{i}(\mathrm{j}.v)\mathrm{t}\right]$$

$$\mathrm{y} = v(x_0)\mathrm{t} + \mathrm{y}_0 + \sum_j \frac{\mathrm{D}_j(x_0)}{\mathrm{j}.v(x_0)} \exp\left[\mathrm{i}(\mathrm{j.v})\mathrm{t}\right] \tag{13}$$

where $v = \omega_0 + \in \omega_1 + \in^2 \varpi_2 + \ldots$. It is evident that the products $\mathrm{j}.v = \mathrm{j}_1 v_1 + \mathrm{j}_2 v_2 + \ldots + \mathrm{j}_n v_n$ present in the denominators may become arbitrarily small for $\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_n$ covering the all set of integers. In this form, Poincaré concluded that such series were therefore divergent for a set of frequency everywhere dense, which is in fact the case. Nevertheless, as Kolmogorov (1954) suggested, there exists a set of frequencies, of non-zero measure (density as close to one as is close to zero), where the series converge. This is basically due to the fact that it is possible, for all integers $\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_n$ to set a lower bound on the numbers $\mathrm{j}_1 v_1 + \ldots + \mathrm{j}_n v_n$, as shown by the diophantive approximation. The way one can arrive at the series will be shown, in a pure formal fashion, in Chapter II, while the subsequent chapters will be devoted to the problem of convergence of the methods introduced. Chapter I is devoted to the introduction of a basic back-ground and terminology to be used throughout these notes. The last chapter will be devoted to the question of nonlinear resonance.

CHAPTER I

CANONICAL TRANSFORMATION THEORY AND GENERALIZATIONS

1. Introduction.

In this chapter we deal with the terminology and basic well known results, which are necessary to the development of the subsequent chapters. It is not the scope of this chapter to describe Hamiltonian Systems and their general properties. They are found in several books and monographs, among which we wish to mention the classics of Birkhoff (1927), Siegel (1956), Wintner (1947), Abraham (1966), Moser (1968). We avoid any and every sophistication in arriving at intrinsic representations and definitions of Hamiltonian systems on manifolds, not because they are not important, but because they are of no essential necessity in what has to follow.

Initially, we remember the definitions of Lagrange's and Poisson's matrices. They arise naturally from the method of variation of arbitrary constants. We consider the transformation $(\mathrm{y},\mathrm{x}) \to (\eta,\xi)$ to be C^2 and invertible in some domain of a 2n-dimensional space. The vectors y, x are n-dimensional as well as the vectors η,ξ. Also, let $\mathrm{z} = col(\mathrm{y},\mathrm{x})$ and $\zeta = col(\eta,\xi)$ be 2n-dimensional vectors. The Lagrange Matrix is defined as

$$\mathcal{L}(\zeta) = \mathrm{J}^{\mathrm{T}}\mathrm{MJ} \qquad (1.1.1)$$

where M is the 2n x 2n canonical matrix

$$\mathrm{M} = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{-I} & 0 \end{pmatrix}$$

and J the Jacobian matrix of the transformation $\mathrm{Z} \to \zeta$, that is

$$\mathrm{J} = \frac{\partial \mathrm{z}}{\partial \zeta}. \qquad (1.1.2)$$

It is easily verified that

$$\mathcal{L}(\zeta)=\left(\frac{\partial \mathrm{y}}{\partial \zeta}\right)^{\mathrm{T}}\left(\frac{\partial x}{\partial \zeta}\right)-\left(\frac{\partial x}{\partial \zeta}\right)^{\mathrm{T}}\left(\frac{\partial \mathrm{y}}{\partial \zeta}\right) \tag{1.1.3}$$

and, therefore,

$$\mathcal{L}_{ij}(\zeta)=\left[\zeta_1,\zeta_j\right]=\sum_{k=1}^{n}\left(\frac{\partial y_k}{\partial \zeta_1}\frac{\partial x_k}{\partial \zeta_j}-\frac{\partial x_k}{\partial \zeta_1}\frac{\partial y_k}{\partial \zeta_j}\right). \tag{1.1.4}$$

The following properties are obvious

$$\mathcal{L}^{\mathrm{T}}=J^{\mathrm{T}}\mathrm{M}^{\mathrm{T}}J=-J^{\mathrm{T}}\mathrm{M}J=-\mathcal{L},$$

where $|A|\triangleq \det A,$ for any square matrix A.

The Poisson matrix P(z) is defined by

$$\mathrm{P}(\mathrm{z})=\mathrm{JMJ}^{\mathrm{T}} \tag{1.1.5}$$

and one verifies that

$$\mathrm{P}(\mathrm{z})=\left(\frac{\partial \mathrm{z}}{\partial \eta}\right)\left(\frac{\partial \mathrm{z}}{\partial \xi}\right)^{\mathrm{T}}-\left(\frac{\partial \mathrm{z}}{\partial \xi}\right)^{\mathrm{T}}\left(\frac{\partial \mathrm{z}}{\partial \eta}\right) \tag{1.1.6}$$

so that

$$\mathrm{P}_{ij}(\mathrm{z})=\left(z_i,\mathrm{z}_j\right)=\sum_{k=1}^{n}\left(\frac{\partial z_1}{\partial \eta_k}\frac{\partial z_j}{\partial \xi_k}\frac{\partial z_i}{\partial \xi_k}\frac{\partial z_j}{\partial \eta_k}\right). \tag{1.1.7}$$

Also,

$$\mathrm{P}^{\mathrm{T}}=\text{-}\mathrm{P},$$
$$|\mathrm{P}|=\left|\mathrm{J}^{-1}\right|^2=1/|\mathrm{J}|^2$$
$$\mathrm{L}^{-1}(\zeta)=J^{-1}\mathrm{M}^{-1}\left(J^{\mathrm{T}}\right)^{-1}=J^{-1}\mathrm{M}\left(J^{-1}\right)^{\mathrm{T}}=-\mathrm{P}(\zeta).$$

The expressions (1.1.4) and (1.1.7) are called Lagrange's Brackets and Poisson's

Parentheses, respectively.

If one considers the system of n second order ordinary differential equations

$$\ddot{y} = f(y, \dot{y}, t) \qquad (1.1.8)$$

and a solution

$$y = y^0(t; \alpha, \beta)$$

$$\dot{y} = x^0(t; \alpha, \beta) = \frac{\partial y^0}{\partial t} \qquad (1.1.9)$$

corresponding to the initial conditions

$$y^0(0; \alpha, \beta) = y_0$$

$$x^0(0; \alpha, \beta) = \dot{y}_0, \qquad (1.1.10)$$

one verifies

$$\frac{\partial x^0}{\partial t} = f(y, \dot{y}, t).$$

For a perturbed system (1.1.8) one has

$$\ddot{y} = f(y, \dot{y}, t) + g(y, \dot{y}, t) \qquad (1.1.11)$$

and assumes the solutions to be of the same form as (1.1.9), where, of course, α, β are now in general variable. It follows that

$$\frac{dy}{dt} = \frac{\partial y^0}{\partial t} + \frac{\partial y^0}{\partial \alpha}\dot{\alpha} + \frac{\partial y^0}{\partial \beta}\dot{\beta} = x^0(t, \alpha, \beta)$$

and, therefore,

$$\frac{\partial y^0}{\partial \alpha}\dot{\alpha}+\frac{\partial y^0}{\partial \beta}\dot{\beta}=0 \tag{1.1.12}$$

where α, β are, evidently, n-vectors. Moreover

$$\frac{d\dot{y}}{dt}=\frac{\partial x^0}{\partial t}+\frac{\partial x^0}{\partial \alpha}\dot{\alpha}+\frac{\partial x^0}{\partial \beta}\dot{\beta}=f\left(y,\dot{y},t\right)+g\left(y,\dot{y},t\right)$$

and, therefore,

$$\frac{\partial x^0}{\partial \alpha}\dot{\alpha}+\frac{\partial x^0}{\partial \beta}\dot{\beta}=g\left(y^0\left(t;\alpha,\beta\right),x^0\left(t;\alpha,\beta\right),t\right). \tag{1.1.13}$$

The system of 2n first order ordinary differential equations (1.1.12) and (1.1.13) are Lagrange's equations for the variation of arbitrary constants. They can be written in terms of a unique system using, for example, Lagrange's matrix $\mathcal{L}(\gamma)$ where $\gamma=column(\alpha,\beta)$. The result is

$$\mathcal{L}(\gamma)\dot{\gamma}=\left(\frac{\partial x^0}{\partial \gamma}\right)^{\mathrm{T}}g\left(y^0\left(t;\gamma\right),x^0\left(t;\gamma\right),t\right). \tag{1.1.14}$$

Evidently, equation (1.1.14) defines γ under the standard condition

$$\left|\mathcal{L}(\gamma)\right|\neq 0$$

that is,

$$\left|\frac{\partial\left(y^0,x^0\right)}{\partial\left(\alpha,\beta\right)}\right|\neq 0$$

which is met by the fact we assumed $\left(y^0,x^0\right)$ to be the general solution of (1.1.8) under arbitrary initial conditions $\left(y_0,x_0\right)$ or constants of integration $\left(\alpha,\beta\right)$. Moreover, we require that

$$\mathrm{P}(\gamma)\left(\frac{\partial y^0}{\partial \gamma}\right)^{\mathrm{T}} g\left(y^0, x^0, t\right)$$

is Lipschitzian in some domain of the $\gamma-\text{space}$. Strictly speaking all of the above statements have a local character, but it is important, as far as applications are concerned, that they extend to some domain of the variables. Also, the functions we are dealing with are assumed to be continuously differentiable in t, generally for any real t.

Lagrange's and Poisson's matrices satisfy an ordinary differential equation with some remarkable properties. In fact, consider the system of 2n differential equations

$$\dot{z}=\phi(z;t)$$

and a solution $z(\gamma;t) \in \mathrm{C}^2$ in the 2n integration constants γ, and t, in some domain of the γ space and for all $|t|<\mathrm{T}$. Let $J = \partial z / \partial \gamma$ be the non-singular Jacobian matrix of the transformation $\gamma \rightarrow z$, which, by hypothesis, is C^2. Thus

$$\dot{J}=\frac{d}{dt}\frac{\partial z}{\partial \gamma}=\frac{\partial}{\partial t}\frac{\partial z}{\partial \gamma}(\gamma;t)=\frac{\partial}{\partial \gamma}\dot{z}(\gamma;t)$$

$$=\frac{\partial}{\partial \gamma}\phi\left(z(\gamma;t);t\right)=\frac{\partial \phi}{\partial z}J$$

or

$$\dot{\mathrm{J}}=\mathrm{GJ} \qquad (1.1.15)$$

where is a 2n x 2n non-singular matrix. Let us now consider

$$\mathcal{L}(\gamma;t)=J^{\mathrm{T}}\mathrm{M}J$$

so that, making use of (1.1.15), one finds

$$\dot{\mathcal{L}} = J^{T}\left(G^{T}M + MG\right)J. \qquad (1.1.16)$$

<u>Lemma. The Lagrange matrix $\mathcal{L}(\gamma;t)$ of the transformation $\gamma \to z$ is constant if, and only if, the matrix MG is symmetric.</u>

In fact, suppose MG is symmetric, that is

$$MG = (MG)^{T} = -G^{T}M.$$

Then $G^{T}M + MG = 0$ and $\dot{\mathcal{L}} = 0$. Reciprocally let $\dot{\mathcal{L}} = 0$. Under the foregoing hypotheses, it follows that

$$G^{T}M + MG = 0$$

or

$$G^{T}M = -MG = M^{T}G = \left(G^{T}M\right)^{T}$$

which completes the proof. From (1.1.16) and the above Lemma it follows that <u>the flow of a Hamiltonian system is conservative. (Liouville's Theorem).</u> In fact, in this case, if $H = H(z)$ is the Hamiltonian, one has

$$\dot{z} = MH_{z}^{T}$$

so that

$$G = \frac{\partial}{\partial z}\left(MH_{z}^{T}\right) = MH_{zz}$$

and

$$MG = -H_{zz}$$

is therefore symmetric. It follows that $\dot{L}=0$ or

$$\frac{d}{dt}\left(J^T MJ\right) = 0$$

or $J^T MJ$ = constant If γ is the vector of initial conditions $Z_0, J_0 = I$ (the identity matrix), and therefore

$$J^T \mathrm{M} J = \mathrm{M} \qquad (1.1.17)$$

and also, in particular,

$$|\mathrm{J}| = const. = 1$$

which proves the theorem. (The case $|J| = -1$ is discarded for reasons of continuity.) If the 2n-vector z is composed by the n-vectors y and x (coordinates and momenta), one can, more precisely, write

$$J = \begin{pmatrix} \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial x_0} \end{pmatrix}$$

and at $t = 0$,

$$J_0 = \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & \mathrm{I}_0 \end{pmatrix} = \mathrm{I}_{2n}.$$

It follows that the mapping $Z_0 \rightarrow Z$ can be represented by

$$\begin{aligned} y &= y_0 + \tilde{Y}\left(x_0, y_0; t\right) \\ x &= x_0 + \tilde{X}\left(x_0, y_0; t\right) \end{aligned} \qquad (1.1.18)$$

where $Y\left(x_0, y_0; 0\right) = X\left(x_0, y_0; 0\right) = 0$, so that, for t sufficiently small,

$$\tilde{Y}(x_0, y_0; t) = t\, Y(x_0, y_0; t)$$

and

$$\tilde{X}(x_0, y_0; t) = t\, X(x_0, y_0; t).$$

The situation can also be viewed from another point. Since at $t=0$ the mapping $z_0 \rightarrow z$ is the identity, there exists a generating function

$$S = x_0 \cdot y + tW(x_0; y; t) \tag{1.1.19}$$

such that

$$x = S_y^T = x_0 + t\, W_y^T$$

and

$$y_0 = S_{x_0}^T = y + t\, W_{x_0}^T$$

which should be equivalent to (1.1.18).

2. Canonical Transformations.

A transformation $z \rightarrow \zeta$, non-singular and C^2 is canonical if it transforms every Hamiltonian system $\dot{z} = \mathrm{MH}_z^T$ into a Hamiltonian system $\dot{\zeta} = \mathrm{MK}_\zeta^T$. The property is purely local, but, again, the usefulness of such definition and what follows relies on the possible global extension into some domain of the phase space. We consider $z = col(y; x)$, $\zeta = col(\eta; \xi)$, to be 2n-dimensional vector. The invariance of the Hamiltonian Form implies that the transformation is canonical if and only if the form

$$\phi(\mathrm{H}) = \sum_{k=1}^{n} \left(\dot{\eta}_k \delta \xi_k - \dot{\xi}_k \delta \eta_k \right) \tag{1.2.1}$$

is an exact differential, for all H.

From (1.2.1) we shall derive the necessary and sufficient condition for the transformation to be canonical (Breves, 1972). We observe that (1,2, 1) can be written

$$\phi(\mathrm{H}) = \zeta^T M \delta\zeta \qquad (1.2.2)$$

Moreover, given the transformation

$$\zeta = \zeta(z;t) \qquad (1.2.3)$$

we have

$$\dot{\zeta} = J\dot{z} + \zeta_T \qquad (1.2.4)$$

where J is the Jacobian matrix

$$J = \frac{\partial\zeta}{\partial z}.$$

It follows that

$$\dot{\zeta} = J\,\mathrm{M}H_z^T + \zeta_t$$

and, from (1.2.2),

$$\phi(\mathrm{H}) = \left(-\mathrm{H}_z\mathrm{M}J^T + \zeta_t^T\right)\mathrm{M}\,\delta\zeta$$

or, with $\delta\zeta = J\delta z$,

$$\phi(\mathrm{H}) = \left(-\mathrm{H}_z\mathrm{M}J^T + \zeta_t^T\right)\mathrm{M}\,\delta\zeta$$

or

$$\phi(\mathrm{H}) = -\mathrm{H}_z \mathrm{ML}(z)\delta z + \mathrm{L}^*(t,z)\delta z \tag{1.2.5}$$

where

$$\mathrm{L}^*(z) = \left(\frac{\partial \zeta}{\partial z}\right)^T \mathrm{M}\left(\frac{\partial \zeta}{\partial z}\right)$$

and

$$\mathrm{L}^*(t,z) = \left(\frac{\partial \zeta}{\partial t}\right)^T \mathrm{M}\left(\frac{\partial \zeta}{\partial z}\right)$$

The quantity $\mathrm{L}^*(t,z)$ is, evidently, a row vector, whose elements are the Lagrange brackets $[t, z_k]$.

The conditions of integrability of $\phi(\mathrm{H})$, for all H, can be translated into conditions of integrability for

$$\phi(0) = \mathrm{L}^*(t,z)\,\delta z = \sum_k [t, z_k]\delta z_k,$$

$$\phi(y_k) = -\sum_\ell [x_k, z_\ell]\delta z_\ell + \phi(0),$$

$$\phi(x_k) = \sum_\ell [y_k, z_\ell]\delta z_\ell + \phi(0)$$

$$\phi(y_k x_j) = \sum_\ell \left\{[y_j, z_\ell] y_k - [x_k, z_\ell] x_j\right\}\delta z_\ell + \phi(0).$$

It follows that

$$\frac{\partial}{\partial z_j}[t, z_k] = \frac{\partial}{\partial z_k}[t, z],$$

$$\frac{\partial}{\partial z_j}[x_k, z_\ell] = \frac{\partial}{\partial z_\ell}[x_k, z_j],$$

$$\frac{\partial}{\partial z_j}[y_k, z_\ell] = \frac{\partial}{\partial z_\ell}[y_k, z_j],$$

and

$$\begin{aligned} &[y_j, z_\ell] = 0 \text{ for } z_\ell \neq x_j, \\ &[x_k, z_\ell] = 0 \text{ for } z_\ell \neq y_k, \end{aligned} \qquad (1.2.6)$$

and

$$[y_k, x_k] = -[x_\ell, y_\ell] = const. = \lambda.$$

The last relation is obtained in view of the first three from where we conclude, using Jacobi's identity, that

$$\frac{\partial}{\partial t}[z_k, z_j] = 0, \text{ and}$$

$$\frac{\partial}{\partial z_\ell}[z_k, z_j] = 0.$$

In matrix notation, conditions (1.2.6) can be written as

$$\mathrm{L}(z) = J^T \mathrm{M} J = \lambda \mathrm{M} \qquad (1.2.7)$$

and since, by hypothesis, $|J| \neq 0$, the constant λ cannot be zero. Equation (1.2.7) is the necessary and sufficient condition for a transformation to be canonical. On the other hand, since $\mathrm{P}(z) = -\mathcal{L}(z)$, such condition can also be expressed in terms of Poisson's Matrix

$$\mathrm{P}(z) = JMJ^{T} = \lambda M$$

That the condition is sufficient follows immediately from the substitution of (1.2.7) into (1.2.5) which gives

$$\phi(\mathrm{H}) = \lambda \mathrm{H}_z \delta z + \mathrm{L}^*(t,z)\delta z = \delta(\lambda \mathrm{H} + W) \qquad (1.2.9)$$

where $W(z;t)$ is a function such that

$$W_z \delta z = \mathrm{L}^*(t,z)\,\delta z = \phi(0) \qquad (1.2.10)$$

an exact differential form. Under the circumstance, one can easily conclude the following result.

Theorem (Jacobi–Poincaré). "A necessary and sufficient condition that a transformation and non-singular $z \to \zeta$ be canonical and the new Hamiltonian be

$$\mathrm{K} = \lambda \mathrm{H} + \mathrm{W} \qquad (1.2.11)$$

is that the form

$$\psi = \lambda x^{\mathrm{T}} dy - \xi^{\mathrm{T}} d\eta + W dt \qquad (1.2.12)$$

be an exact differential."

In fact,

$$\psi = \left(\lambda x^{\mathrm{T}} - \xi^{\mathrm{T}} \frac{\partial \eta}{\partial y} \right) dy - \xi^{\mathrm{T}} \frac{\partial \eta}{\partial x} dx + \left(W - \xi^{\mathrm{T}} \frac{\partial \eta}{\partial t} \right) dt$$

and the integrability conditions for ψ are

$$\frac{\partial}{\partial x}\left(\lambda x^{\mathrm{T}} - \xi^{\mathrm{T}}\frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\xi^{\mathrm{T}}\frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial}{\partial t}\left(\lambda x^{\mathrm{T}} - \xi^{\mathrm{T}}\frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial y}\left(W - \xi^{\mathrm{T}}\frac{\partial \eta}{\partial t}\right)$$

$$\frac{\partial}{\partial t}\left(-\xi^{\mathrm{T}}\frac{\partial \eta}{\partial x}\right) = \frac{\partial}{\partial x}\left(W - \xi^{\mathrm{T}}\frac{\partial \eta}{\partial t}\right)$$

or, in component form,

$$[z_k, z_\ell] = 0 \qquad (\mathrm{z_k} \neq \mathrm{x}_\ell, \mathrm{z}_\ell \neq \mathrm{x_k}),$$

$$[y_k, x_k] = \lambda,$$

$$[t, z_k] = \frac{\partial W}{\partial z_k},$$

which completes the proof. We finally arrive at the Jacobi-Poincaré relation. From (1.2. 12),

$$\psi = \lambda x^{\mathrm{T}} dy - \xi^{\mathrm{T}} d\eta + (\mathrm{K} - \lambda \mathrm{H})\, dt$$

and, therefore, "the necessary and sufficient condition for a transformation to be canonical can be expressed by the fact that ψ has to be an exact differential, that is,

$$\lambda x^{\mathrm{T}} dy - \xi^{\mathrm{T}} d\eta + (\mathrm{K} - \lambda \mathrm{H})\, dt = dF \qquad (1.2.13)$$

when expressed in terms of the variables η, ξ ."

The set of all matrices A satisfying the condition

$$\mathrm{A^T M A = M}$$

constitutes a group (with respect to matrix multiplication), which is called the Symplectic Group. The case $\lambda \neq 1$ is generally excluded from the definition.

Canonical (and therefore, Symplectic) transformations with $\lambda \neq 1$ are also usually excluded since they are the product of a canonical transformation $\lambda = 1$ and the trivial canonical transformation $\lambda \neq 1$ given by

$$\xi = -\lambda x$$

$$\eta = y$$

for, in this case,

$$J_0 = \frac{\partial(\eta,\xi)}{\partial(y,x)} = \begin{pmatrix} I & 0 \\ 0 & -\lambda I \end{pmatrix}$$

and it is easily seen that

$$J_0^T \mathrm{M} J_0 = \lambda \mathrm{M}$$

as discussed by Siegel (1956).

Excluded such case, the necessary and sufficient condition for a canonical transformation is

$$\mathrm{L}(z) = J^T \mathrm{M} J = \mathrm{M} \qquad (1.2.14)$$

or

$$\mathrm{P}(z) = J \mathrm{M} J^{\mathrm{T}} = \mathrm{M},$$

where

$$J = \frac{\partial \zeta(z;t)}{\partial z}.$$

The Jacobi-Poincaré condition is reduced to

$$x^T dy - \xi^T d\eta + (\mathrm{K} - \mathrm{H}) dt = dF \qquad (1.2.15)$$

and if the transformation does not depend explicitly of t is called completely canonical and if dF =

0, homogeneous.

From the results obtained in Section 1, we also conclude that the transformation defined by the solution of a Hamiltonian system, mapping the phase space into itself, is canonical. The volume preserving property was already established. In more precise form:

"Let $\dot{z} = \mathrm{MH}_z^T$ be a Hamiltonian system of differential equations and let there exist a unique solution $z = z(\zeta, t)$ going through the point $z = \zeta$ at $t = t_0$, and assume $z(\zeta, t)$ to be C^2 with respect to the 2n + 1 variables (z; t) in a neighborhood of $z = \zeta$ and for $|t - t_0|$ sufficiently small. Then the mapping $\zeta \to z$ defined by $z = z(\zeta, t)$ is volume preserving and canonical."

3. Hamilton – Jacobi Equation. Generalizations.

Consider the non-singular C^2 transformation

$$\begin{aligned} y &= y(\eta;\xi;t) \\ x &= x(\eta;\xi;t) \end{aligned} \qquad (1.3.1)$$

and suppose the particular situation

$$\left|\frac{\partial y}{\partial \eta}\right| \neq 0, \|\eta - \eta_0\| < \delta, \qquad (1.3.2)$$

so that, locally, one can solve the first system for η,

$$\eta = \eta(y;\xi;t)$$

and, therefore

$$x = x(y;\xi;t).$$

If there exists a function $S(y;\xi;t)$ such that

$$\left|\frac{\partial^2 S}{\partial y \partial \xi}\right| \neq 0,$$

and S is C^2 , the transformation defined by

$$x = S_y^{\mathrm{T}}$$

$$\eta = S_\xi^{\mathrm{T}}$$

is canonical, and the new Hamiltonian is given by

$$\mathrm{K}(\eta;\xi;t) = \mathrm{H}\left(y(\eta;\xi;t);x(\eta;\xi;t);t\right)$$

$$+\frac{\partial S}{\partial t}\left(y(\eta;\xi;t);\xi;t\right)$$

In fact, let us write, in (1.2.15),

$$\xi^{\mathrm{T}} d\eta = d\left(\xi^{\mathrm{T}}\eta\right) - \eta^{\mathrm{T}} d\xi$$

and we have

$$x^{\mathrm{T}} dy + \eta^{\mathrm{T}} d\xi + \left(\mathrm{K} - \mathrm{H}\right) dt = d\left(F - \xi^{\mathrm{T}}\eta\right). \qquad (1.3.3)$$

If we let

$$S = F - \xi^{\mathrm{T}}\eta = S(y;\xi;t)$$

then

$$dS = \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial \xi} d\xi + \frac{\partial S}{\partial t} dt,$$

and from (1.3.3),

$$x^{\mathrm{T}} = \frac{\partial S}{\partial y} = x^{\mathrm{T}}\left(y;\xi;t\right),$$

$$\eta^{\mathrm{T}} = \frac{\partial S}{\partial \xi} = \eta^{\mathrm{T}}\left(y;\xi;t\right), \qquad (1.3.4)$$

$$\mathrm{K} = \mathrm{H} + S_t$$

For the transformation to be written in explicit form we require that

$$\left|\frac{\partial^2 S}{\partial y \partial \xi}\right| \neq 0,$$

in which case one obtains

$$\xi = \xi\left(y;x;t\right)$$

and therefore

$$\eta = \eta\left(y;x;t\right),$$

with the evident condition that $\left|\partial\eta / \partial y\right| \neq 0$. Since S is supposed to be C^2, this implies $\left|\partial y / \partial \eta\right| \neq 0$ and therefore, through (1.3.2), the recovery of (1.3.1).

The important result, to our purposes, is the last of equations (1.3.4), which we write explicitly,

$$\begin{aligned} &\mathrm{K}\left(\eta\left(y;\xi;t\right);\xi;t\right) \\ &= \mathrm{H}\left(\mathrm{y};\mathrm{x}\left(\mathrm{y};\xi;\mathrm{t}\right);t\right) + \frac{\partial S}{\partial t}\left(y;\xi;t\right). \end{aligned} \qquad (1.3.5)$$

If the transformation is time independent, that is, $S_t = 0$, the new Hamiltonian is simply the image of the old one through the mapping $z \rightarrow \zeta$.

<u>The basic problem of Hamiltonian-Jacobi is whether there exists a transformation, generated</u>

by S, and such that the new Hamiltonian reduces to an absolute constant, or, which is equivalent, to a function identically zero. In other words, we seek the solution of the partial differential equation

$$\mathrm{H}\left(y;S_y;t\right)+S_t=0 \qquad (1.3.6)$$

with $S=S\left(y;\xi;t\right)$. As is well known, Jacobi has shown that a general solution is not needed but only a complete solution, in the sense of a function $S\left(y;\xi;t\right)$ depending on *n* arbitrary constants ξ and such that $\left\|\partial S/\partial\xi\right\|\neq 0$. In such case the new variables are constants and the relations

$$\eta=\eta\left(y;x;t\right),$$

$$\xi=\xi\left(y;x;t\right),$$

which are obtained from (1.3.4) are 2π integrals of motion. Obviously, if the original Hamiltonian system is integrable in the sense of existence and uniqueness of solution of the equations

$$z=\mathrm{MH}_z^T$$

a generating function $S\left(y;\xi;t\right)$ must exist (which might not be expressible in terms of elementary functions). In fact, since the solution defines a canonical mapping $z=z\left(\zeta,t\right)$ where ζ is the vector of initial conditions, and since for $t=t_0$, $\partial y/\partial\eta=\mathrm{I}$ (the identity matrix), then for $\left|t-t_0\right|$ sufficiently small $\left|\partial y/\partial\eta\right|\neq 0$, and therefore

$$S=\xi^{\mathrm{T}}y+\left(t-t_0\right)F\left(y;\xi;t\right) \qquad (1.3.7)$$

for $t-t_0$ sufficiently small, in agreement with (1.1.19).

The problem of Hamilton-Jacobi can be generalized by relaxing the condition that the new Hamiltonian be an absolute constant. As far as canonical perturbation methods are concerned the following generalized problem is of great relevance.

We ask if there exists a canonical transformation generated by $S\left(y;\xi;t\right)$, such that the new Hamiltonian has fewer degrees of freedom than the old one. One of that ways to translate this, is to produce a Hamiltonian

$$\mathrm{K}(\eta;\xi;t) = \mathrm{H}(y;x;t) + S_t(y;\xi;t)$$

such that

$$\frac{\partial \mathrm{K}}{\partial \eta_k} = 0 \qquad (1.3.8)$$

for $k = 1, 2, ..., p \leq n.$ The resulting system is obviously reduced to quadratures in the cases p = n or p = n-1. This is the least one require from the transformation, but still it is a much weaker requirement than that proposed by Jacobi. One may also require that the new Hamiltonian does not depend on time proposed by Jacobi. One may also require that the new Hamiltonian does not depend on time explicitly. This process of elimination is generally called an averaging method (Burstein and Solovev, 1961) and is usually applied when H is a periodic function of t. One can also easily generalize the concept for the case of almost period functions of t. If H depends on a small parameter say $\in$, and admits a Taylor series about $\in = 0$, it can be shown that there is a formal series in $\in$ which solves S up to any desired power. The convergence properties of such series are not known in general. The problem of existence of such series and its convergence is strictly related to the theory of periodic surfaces (Diliberto, 1961; Diliberto et. al., 1961) and to the theory of Moser (1962) on invariant curves of area preserving mapping. This last subject will be dealt with in some detail in chapter IV. A qualitative description of these problems are described by Kyner (1964) in relation to the motion of a satellite in the oblate field of a planet. We shall not dealt with Diliberto's theory. Such approach is indeed relevant to the subject, but it is dealt in details elsewhere (e.g. Diliberto, 1961; Hale; 1961).

A new approach to canonical transformations can be viewed by introducing a theory formulated by Lie (1888). Lie Series in problems of dynamics have been used in several occasions and a good reference to the subject, as a general background, is the work by Leimanis (1965). Quite recently they have been introduced as a mean to perturbation methods in non-linear Hamiltonian systems and also have been extended to systems of ordinary differential equations with few restrictions and no requirement for such systems to have a Hamiltonian form. Such applications will be discussed in Chapters II and V. Here, we wish to describe whatever is necessary for understanding of such applications. The motivation for such series is the simple fact that given a system depending on a parameter, one usually knows the solutions when that parameter is set equal to zero. A series solution is then constructed as a power series of the parameter, or, in conservative systems, it can be generated by a canonical transformation which, again, is given by power series on the parameter. Generally speaking, little is known about the convergence of such series, but in many applications they have proved invaluable. Such applicability has been actually checked against precise numerical integrations or observations of the system. At this moment, it is perhaps appropriate to repeat some of the words of Professor Siegel (1941), about the normalization of Hamilton functions. "On account of the small divisors appearing in the coefficients of the transformation, it seemed to be probable that the series would diverge in general, but no single example had hitherto been found. From Poincaré's well known theorem on the analytic integrals of canonical differential equations we can only infer that those series do not uniformly converge… whereas this theorem cannot be applied to a fixed function H." Later, about a specific problem he says "In particular, it would be interesting to decide, whether H is regular or singular (i.e., reducible or not to normal form by convergent series) in the special case… But this seems to be beyond the power of the known methods of analysis." Moser (1955) analyzed similar questions but could not, in essence, prove any general new theorem on denseness of regular Hamiltonians, beside the results of Siegel in 1954 (see Chap. IV, Notes).

4. Lie Series and Lie Transforms.

The subject to be dealt with in this section is related to the following fact (to be proved in the text).

Let $S(y;x;\in)$ and $\mathrm{f}(y;x;\in)$ be functions of the n-vectors y (coordinates) and x (conjugate momenta), and let $\in$ be a dimensionless parameter. We assume S and f to be real analytic functions of the 2n + 1 arguments. Let us define an operator

$$\Delta_w \mathrm{f} = (\mathrm{f}, w) + \frac{\partial f}{\partial \in} \qquad (1.4.1)$$

where (f, W) is Poison parenthesis. Finally, consider the operator

$$E_w \mathrm{f} = \sum_{n \geq 0}^{\infty} \frac{\in^n}{n!} \left(\Delta_w{}^n \mathrm{f} \right) \in = 0 \qquad (1.4.2)$$

where

$$\Delta_w^0 \mathrm{f} = \mathrm{f}$$

$$\Delta_w^1 \mathrm{f} = \Delta_w \mathrm{f}$$

$$\Delta_w^n \mathrm{f} = \Delta_w \Delta_w^{n-1} \mathrm{f} \left(n = 2,3,\ldots\right).$$

The main result is that, under the foregoing conditions, if the series (1.4.2) converges, the transformation

$$\eta_k = \mathrm{E}_w y_k \qquad \xi_k = \mathrm{E}_w x_k \tag{1.4.3}$$

is completely canonical. Moreover, any function g (y; x) real analytic is expressed in the new variables ($\eta;\xi$) by

$$g\left(y(\eta;\xi;\in),x(\eta;\xi;\in)\right) = \mathrm{E}_w g\left(\eta;\xi\right). \tag{1.4.4}$$

Lie's Theorem (1888). The original application of Lie's series to perturbations methods was introduced by Hori (1966). He considered the operator $L_S^n \mathrm{f}$ defined by

$$L^0{}_S \mathrm{f} = \mathrm{f} \qquad L^1{}_S \mathrm{f} = \left(\mathrm{f}, S\right) \qquad L^n{}_S \mathrm{f} = L^1{}_S L_S{}^{n-1} \mathrm{f} \tag{1.4.5}$$

where f, S are real analytic functions of 2n variables ($\eta;\xi$), $\eta = \left(\eta_1,\ldots,\eta_n\right)$, $\xi = \left(\xi_1,\ldots,\xi_n\right)$, canonically conjugate, and wrote Lie's theorem as follows: "A set of 2n variables (y; x) defined by the equation

$$\mathrm{f}\left(y;x\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} D^n{}_S \mathrm{f}\left(\eta;\xi\right) \tag{1.4.6}$$

is canonical if the series converges for $\in$ sufficiently small and independent of ($\eta;\xi$)." The proof of such theorem is quite elementary. One introduces the canonical system of different equations (j = 1, 2, …, n):

$$\frac{d\eta_j}{d\tau} = \frac{\partial S}{\partial \xi_j}, \frac{d\xi_j}{d\tau} = -\frac{\partial S}{\partial \eta_j} \tag{1.4.7}$$

where τ is any parameter, and let $\eta_j(\tau), \xi_j(\tau)$ be the solution of the system which is unique in the region where S is real analytic. It follows that, from (1.4.6)

$$\mathrm{f}(y;x) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \frac{d^n \mathrm{f}}{d\tau^n}\bigg| \in = 0 = \mathrm{f}\left(\eta(\tau + \in); \xi(\tau + \in)\right)$$

or, since f (y; x) is analytic

$$y_j = \eta_j(\tau + \in), x_j = \xi_j(\tau + \in) \tag{1.4.8}$$

for j = 1, 2, …, n and $\in$ sufficiently small. Since (1.4.8) are solutions of the Hamiltonian system (1.4.7), it follows that (y; x) are canonical, because the mapping (1.4.8) is canonical.

If the "generator" S is given, the transformation has the explicit form

$$y_j = \eta_j + \sum_{n=1}^{\infty} \frac{\in^n}{n!} D_S^{\,n-1} \frac{\partial S}{\partial \xi_j}$$

$$x_j = \xi_j + \sum_{n=1}^{\infty} \frac{\in^n}{n!} D_S^{\,n-1} \frac{\partial S}{\partial \xi_j} \tag{1.4.9}$$

which follow from (1.4.6). The apparent incongruence in the application of such theory to a perturbation method is that the functions f and S are to be considered power series in $\in$ and such dependence is not taken care in the formulation. A modified approach to the question was introduced by Deprit (1969) and later was shown to be equivalent to Hori's formulation by several authors (e.g., Mersman, 1970). The equivalence of the generalized Hamilton-Jacobi transformation theory and Lie's transformations as used by Poincaré, Hori and Deprit, will be dealt with at the end of Chapter II. Here, we limit the presentation to the basic theorems involved in Lie's series transformation in the case when f and / or S are functions of $\in$. The main purpose is to establish (1.4.3) and (1.4.4). The exposition follows the lines of Deprit's (1969) work.

Consider f and S to be real analytic functions of 2n canonically conjugate variables (y; x). The Poisson's parenthesis (f, S) may be written

$$\left(\mathrm{f}, S\right) = \frac{\partial \mathrm{f}}{\partial y}\left(\frac{\partial S}{\partial x}\right)^{\mathrm{T}} - \frac{\partial \mathrm{f}}{\partial x}\left(\frac{\partial S}{\partial y}\right)^{\mathrm{T}} \tag{1.4.10}$$

where, as usual, the derivative of a scalar function with respect to a vector is supposed to be a row matrix. One can define the 2n-vector Z = (y; x) and the 2-vector (f, S) and write the Poisson's 2 x 2 matrix

$$\mathrm{P}_z\left(\mathrm{f}, S\right) = J_z \mathrm{M} J_z^{\mathrm{T}} \tag{1.4.11}$$

where $J_z = \dfrac{\partial\left(\mathrm{f}, S\right)}{\partial z}$ is a 2 x 2n matrix and M is the 2n x 2n canonical matrix. Then

$$\mathrm{P}_z\left(\mathrm{f}, S\right) = \left(\mathrm{f}, S\right)_z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a nontrivial canonical transformation z = z (ζ) one has

$$J^{\mathrm{T}} \mathrm{M} J = \mathrm{M}$$

where $J = \partial z / \partial \zeta$. Then

$$J_\zeta = J_z J. \tag{1.4.12}$$

Now one has

$$\mathrm{P}_\zeta\left(\mathrm{f}, S\right) = J_\zeta \mathrm{M} J_\zeta^{\mathrm{T}} = J_z J \mathrm{M} J^{\mathrm{T}} J_z^{\mathrm{T}}$$

$$= J_z \mathrm{M} J_z^{\mathrm{T}} = \mathrm{P}_z\left(\mathrm{f}, S\right)$$

which shows the invariance of P with respect to a canonical transformation.

The Lie Derivative of *f* generated *S* is simply

$$L_S \mathrm{f} = (\mathrm{f}, S), \quad (1.4.13)$$

and the following properties follows from the fact that $L_s \mathrm{f}$ is a bilinear form in f, S (α, β are constants):

$$\begin{aligned}
&a.\ L_S(\alpha \mathrm{f} + \beta g) = \alpha L_S \mathrm{f} + \beta L_S g \\
&b.\ L_S(\mathrm{f}.g) = \mathrm{f}.L_S g + g.L_S \mathrm{f} \\
&\mathrm{c.}\ L_S(\mathrm{f}, g) = (\mathrm{f}, L_S g) + (L_S, \mathrm{f}, g) \\
&d.\ L_S L_{S'} \mathrm{f} = L_{S'} L_S \mathrm{f} + L_{(S,S')} \mathrm{f}.
\end{aligned} \quad (1.4.14)$$

Defining $L_S{}^0 \mathrm{f} = \mathrm{f}$, the n iterate of the Lie Derivative is

$$L_S{}^n \mathrm{f} = L_S L_S{}^{n-1} \mathrm{f}.$$

For this n iterative, the following properties are easily verified:

$$\begin{aligned}
&\mathrm{a.}\ L_S{}^n(\alpha \mathrm{f} + \beta g) = \alpha L_S{}^n g \\
&\mathrm{b.}\ L_S{}^n(\mathrm{f}.g) = \sum_{m=0}^{n} \binom{n}{m} L_S{}^m \mathrm{f}.L_S{}^{n-m} g \\
&\mathrm{c.}\ L_S{}^n(\mathrm{f}, g) = \sum_{m=0}^{n} \binom{n}{m} \left(L_S{}^m \mathrm{f}, L_S{}^{n-m} g \right).
\end{aligned} \quad (1.4.15)$$

If the function S is real analytic one may choose $\in$ sufficiently small so that the series

$$\sum_{n=0}^{\infty} \frac{\in^n}{n!} L_S{}^n \mathrm{f} = \exp(\in L_S) \mathrm{f} \quad (1.4.16)$$

converges when applied to an analytic function f. Again, one can easily verify that

a. exp $\left(\in L_S\right)\left(\alpha \mathrm{f}+\beta g\right)=\alpha \exp\left(\in L_S\right) g$

b. exp $\left(\in L_S\right)\left(\mathrm{f}.g\right)=\exp$ f. exp $\left(\in L_S\right) g$ (1.4.17)

c. exp $\left(\in L_S\right)\left(\mathrm{f}, g\right)=\left(\exp\left(\in L_S\right)\mathrm{f}, \exp\left(\in L_S\right) g\right)$.

From the last of the above relations one concludes the Theorem: "Let $\in$ be a constant parameter and consider the transformation $z=z(\zeta)$ from the 2n-vector z = (y; x) where y, x are canonically conjugate, to the 2n-vector $z=(y;x)$. If there exists a real analytic function S (z) such that the series

$$\zeta=\exp\left(\in L_S\right) z \tag{1.4.18}$$

<u>converges in some domain of the z-space, the transformation is canonical."</u>

Note that is essentially Lie's Theorem as stated before. The proof, under the present approach, follows immediately by considering

$$\zeta_i=\exp\left(\in L_S\right) z_i$$

$$\zeta_j=\exp\left(\in L_S\right) z_j$$

and from (1.4.17) *e*,

$$\left(\zeta_i, \zeta_j\right)=\left(\exp\left(\in L_S\right) z_i, \exp\left(\in L_S\right) z_j\right)$$

$$= \exp\ \left(\in L_S\right) \mathrm{P}(z).$$

or

$$\mathrm{P}(\zeta)=\exp\left(\in L_S\right) \mathrm{P}(z)$$

From the fact the z is a canonical set, P (z) = M and, therefore

$$\mathrm{P}(\zeta) = \mathrm{M}$$

so that ζ is canonical

Another important result gives the transformation law for any function of z into a function of ζ . Theorem: "The image of every real analytic function f (z) under the transformation

$$z = exp(\in L_S)\zeta \quad (1.4.19)$$

is

$$\tilde{\mathrm{f}}(\zeta;\in)\mathrm{f}(\exp(\in L_S)\zeta) = \exp(\in L_S)\mathrm{f}(\zeta)". \quad (1.4.20)$$

In fact,

$$L_S\tilde{\mathrm{f}}(\zeta;\in) = \frac{\partial \mathrm{f}}{\partial z}L_S z \quad (1.4.21)$$

where $\partial \mathrm{f}/\partial z$ is the row matrix $[\partial \mathrm{f}/\partial z_k]$ and $L_S z$ is the column matrix $[(z_k, S)]$.

Differentiating (1.4.19) with respect to $\in$,

$$\frac{\partial z}{\partial \in} = \sum_{m=0}^{\infty}\frac{\in^m}{m!}L_S{}^{m+1}\zeta = L_S z,$$

so that

$$L_S\tilde{\mathrm{f}}(\zeta;\in) = \frac{\partial \mathrm{f}}{\partial z}\frac{\partial z}{\partial \in} = \frac{\partial \tilde{\mathrm{f}}}{\partial \in}.$$

The n-th iterate of such an operation gives

$$L_S{}^{\mathrm{n}}\tilde{\mathrm{f}} = \frac{\partial^{\mathrm{n}}\tilde{\mathrm{f}}}{\partial \in^n}$$

or

$$\left.\frac{\partial^n \tilde{\mathrm{f}}}{\partial \in^{\mathrm{n}}}\right|_{\in=0} = L_S{}^n \tilde{\mathrm{f}}\left(\zeta;0\right) = \mathrm{f}\left(\zeta\right)$$

from (1.4.20). Hence, the Taylor's expansion of $\tilde{\mathrm{f}}\left(\zeta;\in\right)$ is given by

$$\tilde{\mathrm{f}}\left(\zeta;\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \left.\frac{\partial^n \tilde{\mathrm{f}}}{\partial \in^n}\right|_{\in=0}$$

$$= \sum_{n=0}^{\infty} \frac{\in^n}{n!} L_S{}^n \mathrm{f}\left(\zeta\right) = \exp\left(\in L_S\right) \mathrm{f}\left(\zeta\right)$$

which completes the proof.

From this last theorem we conclude a corollary which, ultimately, will establish the validity of Hori's approach who considered S an explicit function of $\in$. Corollary: "If the function $\mathrm{f}\left(z,\in\right)$ admits a Taylor series in the neighborhood of $\in = 0$, that is,

$$\mathrm{f}\left(z;\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{f}_n\left(z\right) \qquad (1.4.23)$$

then, under the canonical mapping (1.4.19),

$$\mathrm{f}\left(z\left(\zeta;\in\right);\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \sum_{m=0}^{\infty} \binom{n}{m} L_S{}^m \mathrm{f}_{n-m}\left(\zeta\right) \qquad (1.4.24)$$

In fact, from (1.4.20),

$$\mathrm{f}_n\left(z\left(\zeta;\in\right)\right) = \sum^{\infty} \frac{1}{m!} \in^m L_S{}^m \mathrm{f}_n\left(\zeta\right)$$

which substituted into (1.4.23) gives the desired result, upon collection of like powers of $\in$.

Finally, we prove the following theorem about the inverse of a canonical transformation defined by Lie's Series:

Theorem: "The inverse of the canonical transformation

$$z = \exp\left(\in L_S\right)\zeta$$

is given by

$$\zeta = \exp\left(\in L_{-S}\right) z." \qquad (1.4.25)$$

In fact

$$\zeta = \exp\left(\in L_{S'}\right) z = \exp\left(\in L_{S'}\right)\left(\exp\left(\in L_S\right)\zeta\right)$$

$$= \exp\left(\in \left(\mathrm{L}_{S'} + L_S\right)\right)\zeta.$$

The operator

$$\exp\left(\in\left(L_{S'} + L_S\right)\right)$$

must reduce to the identity transformation, that is, $L_{S'} + L_S = 0$, and, therefore, S' = -S, necessarily.

5. Lie Transform Depending on a Parameter.

As was stated earlier, canonical transformations associated with perturbation methods are necessarily functions of a parameter, generally small, for the solution is known such parameter is set equal to zero (or any fixed numerical value). In terms of the Lie Transformation Theory presented in the previous section, this means that one should allow for the Generator to depend explicitly on the parameter $\in$. This can be accomplished by defining (Deprit, 1969) the operator

$$\Delta_S = L_S + \frac{\partial}{\partial \in} \qquad (1.5.1)$$

with the obvious properties:

a. $\Delta_S\left(\alpha \mathrm{f}+\beta g\right)=\alpha\Delta_S \mathrm{f}+\beta\Delta_S g$

b. $\Delta_S\left(\mathrm{f}.g\right)=\mathrm{f}.\Delta_S \mathrm{g}+g.\Delta_S \mathrm{f}$

(1.5.2)

c. $\Delta_S\Delta_{S}, \mathrm{f}=\left(\Delta_S \mathrm{f},g\right)+\left(\mathrm{f},\Delta_S g\right)$

d. $\Delta_S\Delta_{S}, \mathrm{f}=\Delta_{S},\Delta_S \mathrm{f}+L\left(S',S\right)^{\mathrm{f}\ +\ \mathrm{L}}\ S'_{\in}-S_{\in}$

where

$$S=S\left(z;\in\right),$$

$$S_{\in}=\frac{\partial S}{\partial \in}.$$

It is also legitimate to define the n iterate of $\Delta_S \mathrm{f}$ by

$$\Delta_S{}^n \mathrm{f}=\Delta_S\left(\Delta_S{}^{n-1}\mathrm{f}\right)$$

$$\Delta_S{}^0 \mathrm{f}=\mathrm{f}$$

and easily obtain the relations corresponding to (1.5.2).

We also define

$$\mathrm{f}_n\left(\zeta;0\right)=\left[\Delta^n_{S\left(\zeta;\in\right)}\mathrm{f}\left(\zeta;\in\right)\right]_{\in=0} \qquad (1.5.3)$$

and the new operator

$$\mathrm{E}_S \mathrm{f}=\sum_{n=0}^{\infty}\frac{\in^n}{n!}\mathrm{f}_n\left(\zeta;0\right). \qquad (1.5.4)$$

Evidently, if there exist a finite quantity A such that

$$\mathrm{f}_n\left(\zeta;0\right)<\mathrm{A}^n$$

for ζ in some neighborhood of a point ζ_0, the series (1.5.4) certainly converges.

The following relations are easily verified:

a. $\mathrm{E}_S\left(\alpha \mathrm{f}+\beta g\right)=\alpha\ \mathrm{E}_S \mathrm{f}+\beta\ \mathrm{E}_S g$

b. $\mathrm{E}_S\left(\mathrm{f}.g\right)=\mathrm{E}_S \mathrm{f}.\ \mathrm{E}_S g$ (1.5.5)

c. $\mathrm{E}_S\left(\mathrm{f},g\right)=\left(\mathrm{E}_S \mathrm{f},\ \mathrm{E}_S g\right)$.

As done previously with operator L_S, one shows that the transformation $\left(\zeta;\in\right)\rightarrow z$ defined by

$$z = \mathrm{E}_S(\zeta) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} z_n(\zeta;0). \qquad (1.5.6)$$

is canonical provided the series converges. In order to establish a Lie Generator for the above transformation, we prove the following theorem.

Theorem: "The transformation $z = \mathrm{E}_S(\zeta)$ is the solution of the Hamiltonian system

$$\frac{dz}{d\in} = \mathrm{M}\left(\frac{\partial S}{\partial z}\right)^T \qquad (1.5.7)$$

corresponding to the initial conditions $z = \zeta$ at $\in = 0$ and where $S(z;\in)$ is related to through (1.5.4) and (1.5.3)."

In fact, considering (1.5.1),

$$\Delta_S z(\zeta;\in) = L_S z(\zeta;\in) + \frac{\partial z}{\partial \in}$$

$$= \frac{\partial z}{\partial y}\left(\frac{\partial S}{\partial x}\right)^T - \frac{\partial z}{\partial x}\left(\frac{\partial S}{\partial y}\right)^T + \frac{\partial z}{\partial \in},$$

where z = col (y; x). From (1.5.7), with $S = S(\zeta;\in)$,

$$\left(\frac{\partial S}{\partial x}\right)^T = \frac{dy}{d\in}$$

and

$$\left(\frac{\partial S}{\partial y}\right)^T = -\frac{dx}{d\in},$$

so that

$$\Delta_S z(\zeta;\in) = \frac{\partial z}{\partial y}\frac{dy}{d\in} + \frac{\partial z}{\partial x}\frac{\partial x}{d\in} + \frac{\partial z}{\partial \in} = \frac{dz}{d\in}.$$

The transformation $z(\zeta;\in)$ being supposed real analytic, we obtain

$$\Delta_S^n z(\zeta;\in) = \frac{d^n z}{d\in^n}, \qquad (1.5.9)$$

and for $\in = 0$ there results

$$\Delta_S^n z(\zeta;\in)\Big|_{\in=0} = \frac{d^n z}{d\in^n}\Big|_{\in=0} = z_n(\zeta;0)$$

so that, using (1.5.6)

$$z = \mathrm{E}_S(\zeta) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \frac{d^n z}{d\in^n}\bigg|_{\in=0} = z(\zeta;0).$$

which completes the proof.

The transformation of a real analytic function $\mathrm{f}(z;\in)$ under the canonical mapping $z = z(\zeta;\in) = \mathrm{E}_S(\zeta)$ defined by (1.5.6), is simply obtained as

$$\mathrm{f}\left(\mathrm{E}_S(\zeta);\in\right) = \mathrm{E}_S \mathrm{f}(\zeta;\in). \qquad (1.5.11)$$

In fact, along the solution $z = z(\zeta;\in)$, going through $z = \zeta$ at $\in = 0$, of system (1.5.7),

$$\mathrm{f}\left(z(\zeta;\in);\in\right)$$

$$= \sum_{n=0}^{\infty} \frac{\in^n}{n!} \left(\frac{d^n \mathrm{f}}{d\in^n}\right)_{\in=0} = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \left(\Delta_S^n \mathrm{f}\right)_{\in=0}$$

as shown by (1.5.10). Therefore, by definition of E_S,

$$\mathrm{f}\left(z(\zeta;\in);\in\right) = \mathrm{E}_S \mathrm{f}(\zeta;\in).$$

which is (1.5.11).

A particular case of interest for the transformation rule (1.5.11) is when both $S(\zeta;\in)$ and $\mathrm{f}(\zeta;\in)$ are power series in $\in$, that is,

$$S(\zeta;\in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} S_{n+1}(\zeta) \qquad (1.5.12)$$

and

$$\mathrm{f}(\zeta;\in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{f}_n(\zeta). \qquad (1.5.13)$$

In this case, let us define

$$L_{S_P} = L_P \left(p \geq 1\right)$$

so that, from the results of the previous section, one finds

$$\frac{\partial}{\partial \in} \mathrm{f}(\zeta;\in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{f}_{n+1}(\zeta)$$

and

$$L_S \mathrm{f}(\zeta;\in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \sum_{m=0}^{n} \binom{n}{m} L_{m+1} \mathrm{f}_{n-m}(\zeta).$$

Thus, representing $\Delta_S \mathrm{f}$ by the series

$$\Delta_S \mathrm{f} = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{f}_n^{(1)}(\zeta),$$

one finds

$$\mathrm{f}_n^{(1)}(\zeta) = \mathrm{f}_{n+1}(\zeta) + \sum_{m=0}^{n} \binom{n}{m} L_{m+1} \mathrm{f}_{n-m}(\zeta)$$

and therefore

$$\mathbf{f}_0^{(1)}(\zeta) = \mathrm{f}_1 + L_1 \mathrm{f}_0 = \mathrm{f}_1 + (\mathrm{f}_{0,} S_1).$$

In the same manner, introducing the series

$$\Delta_S^2 \mathrm{f} = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathbf{f}_n^{(2)}(\zeta)$$

we find

$$\mathbf{f}_0^{(2)}(\zeta) = \mathbf{f}_{n+1}^{(1)}(\zeta) + \sum_{m=0}^{n} \binom{n}{m} L_{m+1} \mathbf{f}_{n-m}^{(1)}(\zeta)$$

and therefore

$$\mathbf{f}_0^{(2)}(\zeta) = \mathbf{f}_1^{(1)} + L_1 \mathbf{f}_0^{(1)},$$

or, using the expression for $\mathbf{f}_0^{(1)}$, $\mathbf{f}_1^{(1)}$, it follows that

$$\mathbf{f}_0^{(2)}(\zeta) = \mathrm{f}_2 + 2(\mathrm{f}_1, S_1) + (\mathrm{f}_0, S_2) + ((\mathrm{f}_0, S_1), S_1).$$

A general recurrence algorithm is thus obtained for the transformation of $\mathrm{f}(z;\in)$ under a Lie Series Transform generated by $S(z;\in)$ when both functions are real analytic in all variables and for $\in$ in the neighborhood of $\in = 0$:

$$\mathbf{f}_n^{(k)}(\zeta) = \mathbf{f}_{\mathrm{n+1}}^{(\mathrm{k-1})} + \sum_{m=0}^{n} \binom{n}{m} L_{m+1} \mathbf{f}_{n-m}^{(k-1)}.$$

This is represented in the following triangular map:

$$
\begin{array}{l}
f_0 \\
\downarrow \\
f_1 \rightarrow f_0^{(1)} \\
\downarrow \\
f_2 \rightarrow f_1^{(1)} \rightarrow f_0^{(2)} \\
\downarrow \quad \downarrow \quad \downarrow \\
f_3 \rightarrow f_2^{(1)} \rightarrow f_1^{(2)} \rightarrow f_0^{(3)} \rightarrow \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
f_4 \rightarrow \\
\downarrow
\end{array}
$$

A particular case of interest is the transformation of the vector z = col (x; y). The canonical transformation is

$$
y = \mathrm{E}_S(\eta) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \eta_0^{(n)}(\zeta;0)
$$

$$
x = \mathrm{E}_S(\xi) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \xi_0^{(n)}(\zeta;0) \tag{1.5.15}
$$

and the recurrence procedure above described gives the coefficients $\eta_0^{(n)} = \eta_0^{n}(\zeta;0)$ and $\xi_0^{(n)} = \xi_0^{(n)}(\zeta;0)$. In (1.5.15) it is worth noting that, obviously, $\eta_0^{(0)} = \eta$ and $\xi_0^{(0)} = \xi$. The all procedure can be extend to the case in which the canonical transformation depends explicitly on time. One way to produce the corresponding result is by simply taking the time as an additional canonical coordinate, the conjugate momentum being the Hamiltonian itself. This leads directly to the algorithm described in detail by Deprit (1969).

6. Equivalence Relations

In previous sections we have described ways of producing canonical transformations as power series of a parameter $\in$. Such transformations are written

$$
y = y(\eta;\xi;\in)
$$

$$
x = x(\eta;\xi;\in) \tag{1.6.1}
$$

or

$$
z = z(\zeta;\in),
$$

where x, y, η, ξ are n-vectors and z, ζ are 2n-vectors, In terms of a generator satisfying Hamilton-Jacobi's equation, that is, the one required to develop Poincaré's method of perturbations, the transformation (1.6.1) is produced by

$$y = \eta + \left(\frac{\partial M}{\partial x}\right)^T = y(\eta; x; \in)$$

$$\xi = x + \left(\frac{\partial W}{\partial \eta}\right)^T = \xi(\eta; x; \in) \quad (1.6.3)$$

Where $W = W(\eta; x; \in)$. The condition

$$W(\eta; x; 0) = 0 \quad (1.6.4)$$

indicates the fact that the transformation (1.6.1) is "near" the identity for $\in$ sufficiently small.

A transformation of the same character is produced, as was seen, by a generator $S = S(y; x; \in)$, through the solution of the Hamiltonian system

$$\frac{dy}{d\in} = \left(\frac{\partial S}{\partial x}\right)^T$$

$$\frac{dx}{d\in} = -\left(\frac{\partial S}{\partial y}\right)^T \quad (1.6.5)$$

with the initial conditions $y = \eta, x = \xi$ at $\in = 0$. We have the following basic equivalence statement: Theorem (Shniad, 1970): "The Generators W and S, satisfying the foregoing conditions, satisfy the relation

$$S(y; x; \in) = \frac{\partial W}{\partial \in}(\eta; x; \in) \quad (1.6.6)$$

where

$$y = \eta + \left(\frac{\partial W}{\partial x}\right)^T = y(\eta; x; \in)." \quad (1.6.7)$$

In fact, applying the canonical transformation (1.6.3) to the system (1.6.5), the new Hamiltonian $S'(\eta; \xi; \in)$ is given, according to Hamilton-Jacobi theory, by

$$S'(\eta; \xi(\eta; x; \in); \in) = S(y(\eta; x; \in); x; \in) - \frac{\partial W}{\partial \in}(\eta; x; \in). \quad (1.6.8)$$

on the other hand, by definition, $\eta$and $\xi$are constants and, therefore, the Hamiltonian $S'(\eta; \xi; \in)$must be identically zero, which proves the theorem.

Now, both W and S are generally defined as power series in $\in$ and (1.6.6) provides the relations among the coefficients of these two series. In fact, since S' is identically zero, the corresponding relation

$$S\left(\eta+\left(\frac{\partial W}{\partial x}\right)^{T};x;\in\right)-\frac{\partial W}{\partial\in}(\eta;x;\in)=0 \tag{1.6.9}$$

must be identically satisfied as a function of the 2n + 1 independent variables $(\eta;x;\in)$.

Let us assume for S and W the series

$$S(y;x;\in)=\sum_{n=0}^{\infty}S_{n+1}S_{n+1}(y;x)\in^{n}$$

$$W(\eta;x;\in)=\sum_{n=1}^{\infty}W_{n}(\eta;x)\in^{n} \tag{1.6.10}$$

where y is defined by (1.6.7).

Substitution of (1.6.10) and (1.6.9) leads to the recurrence relations

$$W_1=S_1$$

$$2W_2=S_2+\left(\frac{\partial S_1}{\partial\eta}\right)\left(\frac{\partial W_1}{\partial x}\right)^{T}$$

$$3W_3=S_3+\left(\frac{\partial S_1}{\partial\eta}\right)\left(\frac{\partial W_2}{\partial x}\right)^{T}+\left(\frac{\partial S_2}{\partial\eta}\right)\left(\frac{\partial W_1}{\partial x}\right)^{T}$$

$$=\frac{1}{2}\frac{\partial W_1}{\partial x}\frac{\partial^2 S_1}{\partial\eta\partial\eta}\left(\frac{\partial W_1}{\partial x}\right)^{T}$$

where $\frac{\partial S_n}{\partial\eta}$ and higher derivatives stand for $\left.\frac{\partial S_n}{\partial y}\right| y=\eta$. In general, Mersman (1971) finds that

$$W_1=S_1$$

$$(n+1)W_{n+1}=S_{n+1}+\sum_{k=1}^{n}\frac{1}{k!}\sum_{p}\frac{\partial^k S_{p_0}}{\partial\eta_{i_1}\partial\eta_{i_2}..\partial\eta_{i_k}}\cdot \tag{1.6.11}$$

$$\cdot\frac{\partial W_{p_i}}{\partial x_{i_1}}\frac{\partial W_{p_2}}{\partial x_{i_2}}....\frac{\partial W_{p_k}}{\partial x_{i_k}}$$

where the second summation is over all sets of k + 1 positive integers $(P_0,P_1,P_2,...,P_k)$ such that $P_0+P_1+P_2+...+P_k=n+1.$ Relation (1.6.11) is totally equivalent to the one originally obtained by Giacaglia (1964) in the development of explicit relations for the von Zeipel (Poincaré) method. The recurrence formula (1.6.11) can now be used to establish explicit relations among the generators defined in Poincaré's method and those given by Hori and Deprit by means of Lie Series. These relations are given in detail by Mersman (1971). The equivalence of Hori's and Deprit's formulations establishes, indirectly, a justification of the the fact that in Hori's original approach the generator S could be considered a function of $\in$, although, apparently the proof of Lie's Theorem falls short in such case. A discussion over the above question was originally presented by Campbell and Jefferys (1970) with respect to some negative remarks by Deprit (1969) about Hori's Theory. Their argument is essentially the one of assuming the generator S imbedded on a one parameter family (parameter $\in_0$), constructing the transformation for a fixed value of the parameter and showing the validity for any value $\in$ of $\in_0$. An analogous reasoning was quite successfully applied by Poincaré (1892) in a problem where the same parameter is fictitiously labeled by two names are identified again.

As an example of Poincaré's remark, consider

$$\mathrm{f}(\in) = \sin\frac{\in}{1-\in}$$

and the Taylor series of $\mathrm{f}(\in)$ about $\in = 0$. One can produce such series as follows. Let

$$\sin\frac{\in}{1-\in} = \sin\frac{\in}{1-\mu}$$

and the Taylor series is

$$\sin\frac{\in}{1-\mu} = \sum_{n=0}^{\infty}\frac{\in^n}{2n!}\frac{1}{(1-\mu)^n}\left[1-(-1)^n\right]$$

$$= \sum_{n=0}^{\infty}\frac{\in^n}{2n!}\left[1-(-1)^n\right]\sum_{m=0}^{\infty}\binom{-n}{m}(-1)^m\,\mu^m.$$

Identification of μ and $\in$ gives

$$\sin\frac{\in}{1-\in} = \sum_{p=0}^{\infty}\left\{\sum_{n=0}^{p}\frac{1}{2n!}\binom{-n}{p-n}(-1)^{p-n}\left[1-(-1)^n\right]\right\}\in^p$$

which in fact is the correct Taylor series of $\mathrm{f}(\in)$ as it is readily verified.

In the case under question, recalling the operator

$$\exp(\in L_S) = \sum_{n=0}^{\infty}\frac{\in^n}{n!}L_S^n,$$

the property

$$\exp(\in L_S)(\mathrm{f}, g) = \left((\exp\in L_S)\mathrm{f}, (\exp\in L_S)g\right)$$

does not depend on the fact that S is dependent or independent of $\in$. Therefore, since the above relation is basically the proof of the transformation

$$z = \exp\left(\in L_S\right)\zeta$$

to be canonical, it can likewise be applied to Hori's development, as a proof independent of the Hamiltonian system of differential equations generated by S.

7. General Transformations induced by Lie Series.

Consider an n-dimensional vector space and a non-singular real analytic transformation from a point x to a point y of this space, defined by

$$y = x + \sum_{m=1}^{\infty} \frac{\in^m}{m!} y_m\left(x\right) \tag{1.7.1}$$

where y_m are n-vectors, and $\in$ a parameter independent of x. For $\in = 0$, (1.7.1) reduces to the identity transformation and for $\in$ small (1.7.1) is "near" the identity if the series converges. We shall however consider (1.7.1) as a formal series and apply the rules of operations with convergent series (e.g. Cartan, 1963).

One of the goals of the following discussion is to construct a simple algorithm for the transformation, under (1.7.1), of a vector function $F\left(y;\in\right)$. We wish the result to be a power series in $\in$, that is, we wish to find the coefficients $F_n\left(x\right)$ in the expansion

$$F\left(y\left(x;\in\right);\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} F_n\left(x\right). \tag{1.7.2}$$

Obviously, the vector function F should be real analytic in $\in$ at $\in$=0 so that the series (1.7.2) exists. We shall also assume it is real analytic in y.

Two different algorithms were developed independently by Hori (1970) and Kamel (1970), having as major goal the solution by formal series of problem in nonlinear oscillations. The description of such applications will be given in the next chapter. Here, we limit ourselves to the description of the formal expansion discussed above.

By hypothesis, one can expand $F\left(x;\in\right)$ as

$$F\left(x;\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \left(\frac{\partial^n F\left(x;\in\right)}{\partial \in^n} \right) \in = 0 = \sum_{n=0}^{\infty} \frac{\in^n}{n!} F_n\left(x\right) \tag{1.7.3}$$

and also,

$$F\left(x\left(y;\in\right);\in\right) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} F^{(n)}\left(y\right) \tag{1.7.4}$$

where

$$F^{(n)} = \frac{d^n}{d\in^n} F(x;\in)\Big|_{\in=0}$$

$$= \left(\frac{\partial}{\partial\in} + \frac{\partial x}{\partial\in}\frac{\partial}{\partial x}\right)^n F(x(y;\in);\in)\Big|_{\in=0}$$

and

$$x = x(y;\in)$$

is the inverse of (1.7.1) which we suppose exists.

We also have, writing the inverse of (1.7.1) as

$$x = y + \sum_{n=1}^{\infty} \frac{\in^n}{n!} \mathrm{X}^{(n)}(y), \qquad (1.7.5)$$

that

$$\frac{\partial x}{\partial\in} = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{X}^{(n+1)}(y). \qquad (1.7.6)$$

The expansion $\partial x/\partial\in$ clearly indicates that y is kept fixed. From (1.7.6) we can write

$$\frac{\partial x}{\partial\in} = \mathrm{T}(x;\in) \qquad (1.7.7)$$

Where

$$\mathrm{T}(x;\in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} \mathrm{X}^{(n+1)}(y)$$

$$= \sum_{n=0} \frac{\in^n}{n!} \mathrm{T}_{n+1}(x). \qquad (1.7.8)$$

We have that

$$\frac{d}{d\in} = \frac{\partial}{\partial\in} + \frac{\partial x}{\partial\in}\frac{\partial}{\partial x} = \frac{\partial}{\partial\in} + \mathrm{T}(x;\in)\frac{\partial}{\partial x} = \frac{\partial}{\partial\in} + L_{\mathrm{T}} \qquad (1.7.9)$$

where the operator L_{T} is defined by

$$L_{\mathrm{T}} = \mathrm{T}(x;\in)\frac{\partial}{\partial x} \qquad (1.7.10)$$

acting on a real analytic function $\mathrm{f}(x;\in)$. In the above relations we have assumed $\mathrm{T}(x;\in)$ to be an n-dimensional row vector and $\frac{\partial}{\partial x}$ an n-dimensional column vector. Now we have

$$\frac{d}{d\in}F(x;\in)=\frac{d}{d\in}\sum_{n=0}^{\infty}\frac{\in^n}{n!}F_n(x)$$

$$=\sum_{n=0}^{\infty}\frac{\in^n}{n!}F_{n+1}(x)+\sum_{n=0}^{\infty}\frac{\in^n}{n!}\sum_{m=0}^{\infty}\frac{\in^m}{m!}\mathrm{T}_{n+1}\frac{\partial F_n(x)}{\partial x} \qquad (1.7.11)$$

$$=\sum_{n=0}\frac{\in^n}{n!}F_n^{(1)}(x)$$

where

$$F_n^{(1)}(x)=F_{n+1}(x)+\sum_{m=0}^{n}\binom{n}{m}\mathrm{T}_{n-m+1}(x)\frac{\partial F_m}{\partial x}$$

$$=\mathrm{F}_{n+1}(x)+\sum_{p=0}^{n}\binom{n}{p}\mathrm{T}_{p+1}(x)\frac{\partial F_{n-p}}{\partial x}. \qquad (1.7.12)$$

In general, we obtain

$$\frac{d^k}{d\in^k}F(x;\in)=\sum_{n=0}^{\infty}\frac{\in^n}{n!}F_n^{(k)}(x), \qquad (1.7.13)$$

where

$$F_n^{(k)}(x)=F_{n+1}^{(k-1)}(x)+\sum_{m=0}^{n}\binom{n}{m}\mathrm{T}_{m+1}(x)\frac{\partial F_{n-m}^{(k-1)}}{\partial x} \qquad (1.7.14)$$

for $k\geq 1$ and $n\geq 0$, where

$$F_n^{(0)}(x)=F_n(x), F_0^{(k)}(x)=F^{(k)}(x)=F^{(k)}(y)\Big|y=x \qquad (1.7.15)$$

The equation (1.7.14) is a recursive algorithm to construct the coefficients $F^{(n)}(x)$ from $F_n(x)$ of the series (1.7.4) and (1.7.3). The variable's name is, obviously, dummy. The corresponding formula to construct the coefficients $F_n(x)$ from $F^{(n)}(x)$ is

$$F_n^{(k)}=F_{n-1}^{(k+1)}-\sum_{m=0}^{n-1}\binom{n-1}{m}\mathrm{T}_{m+1}(x)\frac{\partial F_{n-m-1}^{(k)}}{\partial x} \qquad (1.7.16)$$

Successive substitution of (1.7.16) into itself from n = 1 up, gives

$$F_n^{(k)}=\sum_{j=0}^{n}\binom{n}{j}\mathrm{N}_j\left(F^{(k+n-j)}\right) \qquad (1.7.17)$$

where $n\geq 1,\ k\geq 0$ and $\mathrm{N_j}(j\geq 0)$ is a linear operator given by

$$\mathrm{N}_o = 1$$

$$\mathrm{N}_j = -\sum_{m=1}^{j} \binom{j-1}{m-1} \mathrm{N}_{j-m} \left\{ \mathrm{T}_m(x) \frac{\partial}{\partial x} \right\} = \qquad (1.7.18)$$

$$= -\sum_{m=1}^{j} \binom{j-1}{m-1} \mathrm{N}_{j-m} L_m$$

for $j \geq 1$, and where

$$L_m = \mathrm{T}_m(x) \frac{\partial}{\partial x}. \qquad (1.7.19)$$

For instance, the first few operators N_j are

$$\mathrm{N}_0 = 1$$

$$\mathrm{N}_1 = -L_1$$

$$\mathrm{N}_2 = -\mathrm{N}_1 L_1 - L_2$$

$$\mathrm{N}_3 = -\mathrm{N}_2 L_1 - 2\mathrm{N}_1 L_2 - L_3$$

$$- - - - - - - - - - - - - - -$$

In particular, for k = 0, Eq. (1.7.17) yields

$$F_n = \sum_{j=0}^{n} \binom{n}{j} \mathrm{N}_j \left(F^{(n-j)} \right)$$

which may be written as

$$F_n = -\sum_{j=0}^{n} \binom{n}{j} F_{j,n-j} \qquad (1.7.20)$$

where

$$F_{j,k} = -\sum_{m=1}^{j} \binom{j-1}{m-1} L_m F_{j-m,k} \qquad (1.7.21)$$

and, by definition

$$F_{0,k} = F^{(k)}.$$

Formula (1.7.20) gives the $F^{(n)}$ recursively in terms of the F_n or the F_n recursively in terms of the $F^{(n)}$. This is the simplest possible form, as derived by Kamel.

Vector Transformation.

The coefficients $y_n(x)$ in (1.7.1) are easily obtained now from (1.7.16) for the special case of

(1.7.3) when one takes

$$F^{(0)} = F = y$$

$$F^{(k)} = 0, k > 0$$

$$F_0 = y_0\left(x\right) = x$$

$$F_n^{(0)} = F_n = y_n^{(x)}.$$

In fact, (1.7.16) gives, in this case

$$y_n(x) = -\sum_{m=0}^{n-1}\binom{n-1}{m}T_{m+1}(x)\frac{\partial y^{n-m-1(x)}}{\partial x}$$

or, considering $p \;=\; m+1$,

$$y_n(x) = -\sum_{p=1}^{n}\binom{n-1}{p-1}T_p(x)\frac{y_{n-p}(x)}{\partial x}$$

or

$$y_n(x) = -T_n(n) - \sum_{p-1}^{n-1}\binom{n-1}{p-1}T_p(x)\frac{\partial y_{n-p}(x)}{\partial x} \tag{1.7.22}$$

The inverse transformation follows from (1.7.14) or, more directly, from (1.7.21).

In fact, in notation of (1.7.4),

$$F\left(x\left(y;\in\right);\in\right) = x$$

$$F^{(0)} = y, F^{(n)}\left(y\right) = \mathrm{X}^{(n)}\left(y\right)$$

and we have

$$x = y + \sum_{n=1}^{\infty}\frac{\in^n}{n!}\mathrm{X}^{(n)}\left(y\right).$$

The recurrence relation (1.7.21) gives, together with (1.7.8)

$$\mathrm{X}_{j,k}\left(y\right) = -\sum_{m=1}^{j}\binom{j-1}{m-1}\mathrm{T}_m\left(y\right)\frac{\partial}{\partial y}\mathrm{X}_{j-m,k}\left(y\right)$$

(1.7.23)

$$\mathrm{X}^n\left(y\right) = \mathrm{T}_n\left(y\right) - \sum_{j=1}^{n-1}\binom{n-1}{j}\mathrm{X}_{j,n-j}\left(y\right)$$

with

$$\mathrm{X}_{0,k} = \mathrm{X}^{(k)}\left(y\right).$$

Applications of the above results will be given, explicitly in the next chapter, when the problem of integration of non-linear systems will be dealt with.

NOTES

Lindstedt has been given credit for developing a perturbation method which avoids secular and mixed secular terms in the perturbed harmonic oscillators. He described such a method on several occasions but always thought that the perturbing forces ought to be either odd or even functions of the angle variable involved. Such restriction was shortly after shown not to be necessary by Poincaré. In his celebrated "Méthodes Nouvelles", vol. 2, he developed a canonical analog of Lindstedt method which, even after a superficial look, proves to be a very elaborate generalization. However, it is obvious that the main idea of Poincaré's development comes from Delaunay and some remarks of Tisserand on Delaunay's Lunar Theory. One might in fact go back to Euler's second lunar theory. He obviously had learned a great deal, in between his Lunar Theories, about the development of frequencies of a perturbed system in power series of the small parameter of the problem. Such theories clearly had a great influence in Poincaré's work. The merit of von Zeipel was mainly the application of Poincaré's method to the theory of motion of a well defined system, although the systematic separation of terms of different period, in the development of perturbations, is an important point. Especially when one considers the fact that on several occasions short period phenomena are of no interest, but only long period or secular ones. The Averaging Methods in general, say as discussed by Cesari in his book, had been quite popular in Celestial Mechanics but with no mention to the convergence problem. Perhaps, a big hangover from Poincaré's definite statements on the divergence problem. Perhaps, a big hangover from Poincaré's definite statements on the divergence of Lindstedt's series. Such series were, and still are, used to produce quite accurate prediction of the position of Celestial bodies. Krylov and Bogoliubov did give some bounds in the truncation errors which, as a consequence of new efforts in celestial mechanics, in the sixties, were reviewed by Kyner. Strangely enough there is a great gap in the western literature in problems related to linear and nonlinear oscillations, a field very rich of references in the Soviet literature essentially from Liapunov until about 1950. Celestial Mechanics had been worked down to the bones by means of the available tools of classical analysis by the end of the last century and nonlinear circuit theory and mechanical systems did not seem to be palatable to western mathematicians. The masterpiece work of Cesari in 1940 e was not immediately recognized, but there stood the first proof of convergence of an averaging method for a large variety of problems. Important works followed more than a decade after, by Gambill and Hale. The works of Birkhoff, Siegel and Wintner were more mathematically oriented toward qualitative properties of Dynamical Systems. The simultaneous analysis of Birkhoff's extensive analysis on the restricted problem of there bodies and of Strömgren's numerical experiments was undertaken only recently and summarized in the master work of Szebehely. Moulton and MacMillan should be considered among the scientists who had such capability of analysis of association between theories and numbers. And also Adams and Darwin. The method of Poisson for the variation of integrals of motion is something else that was overlooked for a long time. In the modern literature it is revived again by Kurth in 1959 and mentioned, under different names and aspects, by Danby and Brouwer-Clemence. Lately, stemming from nowhere, it produced a great deal of papers under the name of Universal Variables in the Newtonian problem of two bodies. The use of vector and matrix algebra and calculus is also still very rare, in books written basically more than a century after such tools were given a final form. Siegel's and Abraham's book show the process of evolution from classical to modern mathematical representation of exactly the same things. The definition of Lagrange's and Poisson's matrices is seldom found anywhere, and one has to refer to works on Quantum Mechanics and Field Theory. The proof of the symplectic condition for a canonical transformation is greatly simplified by the use of matrix notation. The connection between nonlinear circuit analysis and nonlinear mechanics methods and the classical averaging methods of Celestial Mechanics was clearly by Cesari in 1959. Equivalence statements between the KBM and von Zeipel's methods were first given in 1961 by Burstein and Sovolev. The efforts for a better theory of artificial satellites were certainly responsible for new researches in analytic and theories. After 1960 it is obvious that a two way flux was established between researches in nonlinear oscillations and in Celestial Mechanics. Milestones were set by Moser, Hale and Diliberto and, in the Soviet Union, by Kolmogorov, Arnol'd and Merman. A true departure from elaborations on works by Poincaré and Birkhoff was introduced by Hori with the application of Lie's Canonical Mappings. Lie's series appeared only one year before in Leimanis' book on Rigid Bodies Motion, but with no reference to perturbation techniques. The extension to non-canonical systems was presented by Hori at a Summer Institute in 1970 and, independently worked out by Kamel. The extension is, nevertheless, not essential since any system can be written in Hamiltonian form as first shown by Dirac. The fact had been long known to researchers in Optimization and Control, although the great majority of Applied Mathematicians in other fields had and have not been aware of this important fact. Earliest references, to our Knowledge, are the works of Miner, Tapley and Powers in 1967 and 1969. Finally, the operations with formal series as it is done with convergent series is well justified, e.g. the work of Cartan. This is not the earliest reference, but is surely one of the best.

REFERENCES

1. Abraham, R., "Foundation of Mechanics", W. A. Benjamin Inc., Philadelphia.

2. Arnol'd, V. I., 1963, "Small Denominators and Problems of Stability of Motion in Classical and

Celestial Mechanics", Uspeki Mat. USSR 18, 91 – 192.

3. Birkhoff, G. D., 1915, "Proof of Poincaré's Geometric Theorem", Trans. Amer. Math. Soc., 14, 14 – 22.

4. ________, 1915, "The Restricted Problem of Three Bodies", Rend. Circ. Mat. Palermo, 39, 1-115.

5. _________, 1927, "Dynamical Systems", Am. Math. Soc. Colloq. Pub., IX, Providence, R. I.

6. Breves, J. A., 1968, "A new proof of the conditions for a canonical transformation", Seminars, Univ. of São Paulo (in Cel. Mech, 6, No. 1, 1972).

7. Brouwer, D. and Clemence, G. M., 1961, "Methods of Celestial Mechanics", Academic Press, New York.

8. Brown, E. W.,1931, "Elements of Theory of Resonance", Rice Inst. Publ. 19, Houston.

9. Burstein, E. L. and Solovev, L. S., 1961, Dokl. Akad. Nauk USSR, 139, 855-858.

10. Campbell, J. A. and Jefferys, W. H., 1970, "equivalence of the Perturbation Theories of Hori and Deprit", Cel. Mech., 2, 467.

11. Cartan, E. 1963, "Elementary Theory of Analytic Functions of One or Several Complex Variables", Addison-Wesley, Reading, Massachusetts.

12. Cesari, L., 1940, "Sulla Stabilità delle Soluzioni dei Sistemi di Equazioni Differenziali Lineari a Coefficienti Periodici", Atti Accad. Ital. Mem. Clãs. Fis. Mat. e Nat., 11, 633-692.

13. _________, 1959, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations", Springer-Verlag, Berlin (Second Ed., 1963, Acad. Press, New York).

14. Danby, J. M. A., 1962, "Fundamentals of Celestial Mechanics", MacMillan, New York.

15. Darwin, G. H., 1911, "Scientific Papers", Cambridge Univ. Press, London.

16. Delaunay, C. E., 1860-1867, Mèm. Acad. Sci. Paris, 28, 29 (entire volumes).

17. Deprit, A., 1969, "Canonical Transformations Depending on a Small Parameter", Cel. Mech., 1, 12-30.

18. Diliberto, S. P. and Hufford, G., 1956, "Perturbation Theorems for Nonlinear Ordinary Differential Equations", Ann. Math. Studies 36, 207-236.

19. Diliberto, S. P., 1960 and 1961, "Perturbation Theorems for Periodic Surfaces. I and II", Rend. Circ. Mat. Palermo, 9, 265-229 and 10, 111.

20. Diliberto, S. P., Kyner, W. T. and Freund, R. F., 1961, "The Application of Periodic Surface Theory to the Study of Satellite Orbits", Astron. J., 66, 118-127.

21. Diliberto, S. P., 1967, "New Results on Periodic Surfaces and the Averaging Principle", U.S.-Japanese Semin. on Diff. Func. Equas., pp. 49-87, W. A. Benjamin, Inc., Philadelphia.

22. Dirac, P. A. M., 1958, "Generalized Hamiltonian Dynamics", Proceed. Roy. Soc. London, A246, 326-332.

23. Euler, L., 1772, “Theoria Motus Lunae, Novo Methodo”, 775 pp., Petrop. (in Brown, E. W., “Lunar Theory”, Dover Public., New York, 1960).

24. Gambill, R. A., 1954, “Criteria for Parametric Instability for Linear Diferential Systems with Periodic Coefficients”, Riv. Mat. Univ. Parma, 5, 169-181.

25. _________, 1955, Ibidem, 6, 37-43.

26. _________, 1956, Ibidem, 6, 311-319.

27. _________ and Hale, J. K., 1956, “Subharmonics and Ultraharmonics Solutions of Weakly Nonlinear Systems”, J. Rat. Mech. Anal., 5, 353-398.

28. Giacaglia, G. E. O., 1964, “Notes on von Zeipel’s Method”, GSFC-NASA Publ. X-547-64-161, Greenbelt, Md.

29. _________, 1965, “Evaluation of Methods of Integration by Series in Celestial Mechanics”, Ph.D. Dissertation, Yale University, New Haven.

30. Hale, J. K., 1954, “On the boundedness of the solution of linear differential systems with periodic coefficients”, Riv. Mat. Univ. Parma, 5, 137-167.

31. _________, 1954, “Periodic Solutions of Nonlinear Systems of Differential Equations”, Riv. Mat. Univ. Parma, 5, 281-311.

32. _________, 1958, “Sufficient Conditions for the Existence of Periodic Solutions of First and Second Order Differential Equations”, J. Math. Mech., 7, 163-172.

33. __________, 1961, “Integral Manifolds of Perturbed Differential Equations”, Ann. Math., 73, 496-531.

34. Hori, G., 1966, “Theory of General Perturbations with Unspecified Canonical Variables”, Publ. Astron. Soc. Japan, 18, 287-296.

35. __________, 1970, “Lie Transformations in Nonhamiltonian Systems”, Lecture Notes, Summer Institute in Orbital Mechanics, The Univ. of Texas at Austin, May 1970.

36. __________, 1971, “Theory of General Perturbations for Noncanonical Systems”, Publ. Astron. Soc. Japan, 23, 567-587.

37. Kamel, A. A., 1970, “Perturbations Methods in the Theory of Nonlinear Oscillations”, Cel. Mech., 3, 90-106.

38. Kolmogorov, N. A., 1954, “General Theory of Dynamical Systems and Classical Mechanics”, Proc. Int. Cong. Math., Amsterdam, 1, 315-333, Noordhoff (1957).

39. Krylov, N. and Bogoliubov, N. N., 1934, “The Applications of Methods of Nonlinear Mechanics to the Theory of Stationary Oscillations”, Publ. Ukranian Acad. Sci., No. 8, kiev.

40. __________, 1947, “Introduction to Nonlinear Mechanics”, Annals. Math. Studies, 11, Princeton Univ. Press, Princeton, New Jersey.

41. Kurth, R., 1959, “Introduction to the Mechanics of the Solar System”, Pergamon Press, New York (Chapt. III, Section 3).

42. Kyner, W. T., 1964, “Qualitative Properties of Orbits about an Oblate Planet” Comm. Pure Appl.

Math., 17, 227-231.

43. __________, 1967, “Averaging Methods in Celestial Mechanics”, Proc. Intern. Astron. Union Symp., No. 25, Thessaloniki, Greece, 1964. Acad. Press, New York.

44. Leimanis, E., 1965, “The General Problem of Motion of Coupled Rigid Bodies about a Fixed Point”, Springer-Verlag, New York (pp. 121-128).

45. Lie, M. S., 1888, “Theorie der Transformationgruppen”, Teubner, Leipzig (vol. 1, I).

46. Lindstedt, A., 1882, “Beitrag zur Integration der Differentialgleichungen der Storungtheorie”, Abh. K. Akad. Wiss. St. Petersburg, 31, No. 4 (See also: 1883, Comptes Rendus Acad. Sci. Paris, 3, and 1884, Bull. Astron., 1, 302).

47. MacMillan, W. D., 1920, “Dynamics of Rigid Bodies”, Dover Publications, New York (pp. 403-413).

48. Merman, G. A., 1961, “Almost Periodic Solutions and the Divergence of Lindstedt’s Series in the Planar Restricted Problem of Three Bodies”, Bull. Inst. Theor. Astron. Leningrad, 8, 5.

49. Mersman, W. A., 1970, “A new algorithm for the Lie Transformation”, Cel. Mech., 3, 81-89.

50. _________, 1970, “A Unified Treatment of Lunar Theory and Artificial Satellite Theory” in “Periodic Orbits, Stability and Resonances”, Ed. G. E. O. Giacaglia, D. Reidel Pub. Co., Dordrecht, Holland.

51. _________, 1971, “Explicit Recursive Algorithm for the Construction of Equivalent Canonical Transformations”, Cel. Mech., 3, 384-389.

52. Miners, W. E., Tapley, B. D. and Powers, W. F., 1967, “The Hamilton-Jacobi Method Applied to the Low-Thrust Trajectory Problem”, Proc. 18th Cong. Intern. Astronaut. Fed., Belgrade, Yugoslavia.

53. Moulton, F. R. et al., 1920, “Periodic Orbits”, Carnegie Inst. Wash. Publ., No. 161, Washington, D. C.

54. Moser, J., 1955, “Nonexistence of Integrals for Canonical Systems of Differential Equations”, Comm. Pure Appl. Math., 8, 409-436.

55. _________, 1962, “On Invariant Curves of Area-Preserving Mappings of an Annulus”, Nachr. Akad. Wiss. Gottingen Math. Phys. KI. II, 1-20.

56. __________, 1968, “Lectures on Hamiltonian Systems”, Mem. Amer. Math. Soc., 81, pp. 1-60.

57. Poincaré, H., 1886, “Sur la Méthode de M. Lindstedt”, Bull. Astron., 3, 57.

58. __________, 1892-93-99, “Les Méthods Nouvelles de la Mécanique Celeste”, (3 vols.) Reprint by Dover Publ., New York, 1957.

59. __________, 1909 – 12, “Lecons de Mécanique Célèste”, Gauthier-Villars, Paris (vol. 2, part 2).

60. __________, 1912, “Sur un Théorème de Géometrie”, Rend. Circ. Mat. Palermo 33, 375 – 407.

61. Poisson, S. D., 1834, “Mémoire sur la Variation dês Constantes Arbitraires”, J. École Polyt., 3, 266 – 344 (See also Ref. 41).

62. Powers, W. F. and Tapley, B. D., 1969, “Canonical Transformation Applications to Optimal

Trajectory Analysis", AIAA Journal, 7, 394 – 399.

63. Shniad, H., 1969, "The Equivalence of von Zeipel Mappings and Lie Transforms", Cel. Mech., 2, 114 – 120. (For the background and earlier proof of the main theorem in Shniad's paper see: Lanczos, C., "The Variational Principles of Mechanics", Univ. of Toronto Press, Toronto, chapt. VII).

64. Siegel, C. L., 1941, "On the Integrals of Canonical Systems", Ann. Math., 42, 806 – 822.

65.__________, 1956, "Vorlesungen über Himmelsmechanik", Springer-Verlag, Berlin.

66. Strömgren, E., (see Ref. 67, pp. 550 – 555).

67. Szebehely, V., 1967, "Theory of Orbits – The Restricted Problem of Three Bodies", Acad. Press, New York.

68. Tisserand, F., 1896, "Traité de Mécanique Célèste", (4 vols.) Gauthier-Villars, Paris, (see also 1868, Thèse de Doctorat, J. de Liouville).

69. Winter, A., 1941, "The Analytical Foundations of Celestial Mechanics", Princeton Univ. Press, Princeton, New Jersey.

CHAPTER II

PERTURBATION METHODS FOR HAMILTONIAN SYSTEMS. GENERALIZATIONS

1. Introduction.

This chapter is devoted to two main goals. First introduce the reader to known methods of canonical perturbations, describe them in a heuristic way and give examples so as to motivate the theorems presented in Chapters III and IV. Second, present some basic results about iterative procedures of fundamental importance on methods of averaging. Major contributors to this area are Lindstedt (1884), Poincaré (1893), Whittaker (1916), Siegel (1941), Krylov (1947), Bogoliubov (1945), Kolmogorov (1953), Arnol'd (1963), Diliberto (1961), Pliss (1966), Kyner (1961), Moser (1962), Hale (1961) with several overlappings in results. Many of these results have been unified and consolidated in celebrated books by Siegel (1956), Wintner (1947), Newytskii-Stepanov (1960), Cesari (1963), Hale (1969), Abraham (1967), Birkhoff (1927), Bogoliubov-Mitropolskii (1961), Lefschetz (1959), Minorsky (1962), Sansone-Conti (1964), Sternberg (1970).

It is a recognized fact, although several times not mentioned, that the averaging methods were introduced by Lindstedt (1882), though it is not clear whether his ideas stemmed from the efforts of Euler (1750) in the solution of the problem of motion of the moon. In linear periodic systems, an averaging method leads directly and essentially to the determination of Floquet's characteristic exponents. In non-linear systems, when they posses a Hamiltonian character, to the separation of the associate Hamilton-Jacobi equation and therefore the specification of the action and angle variables. In general non-linear and non-Hamiltonian systems, an averaging method leads to separability in an extended space, which can be called the contangent space of the original system space. In regard to Hamiltonian systems, it has been an accepted and recognized result the fact that in general, they are not integrable. Nevertheless, such notion should be considered with care, depending on the definition of integrability. In fact, if the Hamiltonian is at least C^2 in a certain open region D of the phase space, there exists and is unique a solution corresponding to any initial point in D. In this respect, the system is certainly integrable. On the other hand, the word integrability is, in Hamiltonian systems, often associated with the idea of separability, so that an integrable system is a Stäckel's (or in particular, Liouville's) system. The two concepts can be associated by recalling the fact that if a solution exists and is unique for a time $0 \leq t < \mathrm{T}$, them the motion in phase space is area preserving (or, the divergent of a Hamiltonian flow is zero). It is also true that such flow is canonical so that any point P(t) of the solution $\left(0 \leq t < \mathrm{T}\right)$ is related to the initial point P(0) by a canonical transformation, which for t sufficiently small is C^2 and invertible. It follows that, in terms of the initial conditions, taken as a particular set of canonical variables, the system must necessarily be separable, for the Hamiltonian is reduced to a constant. Of course, such type of separability can only be achieved after the solution is known explicitly as a function of time and of the initial conditions, so that no help can come from such results. However, it serves to indicate the connection between the two concepts of integrability mentioned above.

As far as periodic linear systems are concerned we know, under quite general conditions, that the solution exists and has a well defined form as given by Floquet's theory.

For non-linear in general, integrability can only be understood as existence and uniqueness of solution. However, a connection with the idea of separability can be established by the "Hamiltonianization" of the system in the cotangent space, as will be shown later.

Most of the results concerning non-integrability are based on the existence of integrals in the vicinity of singular points (Siegel, 1941) or on the reducibility to Birkhoff's normal form by power series or on the convergence of iterative procedures. The negation of the above results does not evidently imply non-integrability. It was proved by Birkhoff that a normal form for Hamiltonian systems obtained by means of a series cannot in general be achieved. If the averaging methods are a translation, into some different language, of Birkhoff's normalization, them we cannot, in general, conclude on the divergence of these since we know that manipulation of a series does change its convergence character. Indeed, we shall formulate, as an example, an averaging method equivalent to a normalization and we shall expect divergence in general. On the other hand, averaging methods can be generalized, redefined, restated, and the perturbations subjected to such conditions that, such methods may converge at least for a certain set of initial conditions. In specific examples, adelphic integrals defined by formal series (Contopoulos, 1966, 1967) have shown remarkable character of true integrals of motion when submitted to a numerical verification for very long periods of time. The method of surface of section (Poincaré, 1893) has served an invaluable service in the search of possible integrals and has shown that integrals (not necessarily uniform or globally valid) may exist for systems notably defined as non-integrable (Bozis, 1970).

2. Convergence of a Classical Method of Iteration.

If one limits the time interval properly, it can be shown that under quite general conditions, the simplest method of successive approximation of solution by series, converges. In fact, we have the following results (MacMillan, 1912).

Let us initially consider a system of n equations in $x_1, x_2, ..., x_n$ depending on a parameter $\in$,

$F_i(x;\in)=0; \qquad i=1,2,...,n,$ (2.2.1)

where x is the set $(x_1, x_2, ..., x_n)$. Further, suppose

a) $F_1(0;0)=0; \quad i=1,2,...,n.$

b) $J=\det\dfrac{\partial(F_1,F_2,...,F_n)}{\partial(x_1,x_2,...,x_n)}\neq 0,$ for $x=0, \in=0.$

c) $\dfrac{\partial F_i}{\partial \in}\neq 0$, for some i, at $x=0$

It follows that the functions F_i can be developed about the point $x=0, \in=0,$ in powers of x and $\in$.

Then, one can easily prove that, if the F_i are analytic in their argument in a certain region $(x;\in)$, the serie

$$x_j=\sum_{s=1}^{\infty}\in^s a_{js} \qquad (2.2.2)$$

obtained by successive approximations converge uniformly (in $\in$). The a_{js} are obtained by substituting the x_j into the expansions of F_i and equating coefficients of the same powers in $\in$. The proof of this can be found in any standard book of Analysis (e. g. Goursat, 1959). For the purpose of later use some details are needed. The expansion of $F_i(x;\in)$ gives

$$F_i(x;\in)=\left(\frac{\partial F_i}{\partial \in}\right)_0 \in+\sum_{j=1}^{n}\left(\frac{\partial F_i}{\partial x_j}\right)_0 x_j$$

$$+\frac{1}{2}+\sum_{j=0}^{n}\sum_{k=0}^{n}\left(\frac{\partial^2 F_i}{\partial x_j \partial x_k}\right)_0 x_j x_k+...=0$$

where, for uniformity of notation $x_0\equiv\in$. The above expansion can be written

$$\sum_{j=1}^{n}\left(\frac{\partial F_i}{\partial x_j}\right)_0 x_j\equiv b_0^{(i)}x_0+\sum_{k,j=0}^{n}b_{jk}^{(i)}x_j x_k$$

$$+\sum_{\ell,k,j=0}^{n}b_{\ell kj}^{(0)}x_j x_k x_\ell+... \qquad (2.2.3)$$

If the series (2.2.2) are substituted into (2.2.3), the comparison of coefficients of same powers in $\in$ (or x_0) gives

$$\sum_{j=1}^{n}\left(\frac{\partial F_i}{\partial x_j}\right)_0 \mathrm{a}_{j1} = b_0^{(\mathrm{i})} \qquad (2.2.4)$$

$$\sum_{j=1}^{n}\left(\frac{\partial F_i}{\partial x_j}\right)_0 \mathrm{a}_{j2} = \sum_{j=0}^{n}\sum_{k=0}^{n} b_{\mathrm{jk}}^{(\mathrm{i})}\mathrm{a}_{j1}\mathrm{a}_{k1} \qquad (2.2.5)$$

$$\sum_{j=1}^{n}\left(\frac{\partial F_1}{\partial x_j}\right)_0 \mathrm{a}_{jp} = \phi_{ip}\left({}^{b}j_0 j_1 \ldots j_k, \mathrm{a}_{rs}\right)$$

Therefore, at every step, the a_{jp} are computed from a given system of n equations whose right-hand sides are known if all previous approximations $\mathrm{a}_{j1}, \mathrm{a}_{j2}, \ldots, \mathrm{a}_{j,p-1}$ are known. The determinant of the system is not zero by hypothesis (b). From a formal point of view, equations (2.2.4) are totally similar to the sequence of linear inhomogeneous partial differential equations one encounters in the averaging method of Lindstedt-Poincaré or else in Lie's series asymptotic solutions.

Now consider the case in which J = 0 and assume at least one of its first minors is not zero. For instance, suppose $\partial F_i / \partial x_1 = 0$. Then, (n – 1) of the equations (2.2.4) can be solved in terms of $x_2, x_3, \ldots, x_n$, as power series in x_0 and x_1. If the results are substituted in the n – th of the equations (2.2.4), an equation in x_0 and x_1 will then result. Since the coefficient of the first power of x_1 will be zero, then the solution of x_1 in terms of power series of x_0 will necessarily contain fractional powers of this parameter. This is a direct consequence of Weierstrass theorem on the factorization of a power series. The use of the "eliminating determinants" defined by Caley (1848) allows the solution in the case where all the first minors are zero. MacMillan (1912) further developed the method. The appearance of fractional powers in these cases has a direct consequence on the appearance of fractional powers in asymptotic series solutions to be developed later in problem of resonance.

Next, consider a system of differential equations

$$\dot{x}_i = \in \mathrm{f}_i\left(x; \in; t\right) \qquad (2.2.6)$$

For $i = 1, 2, \ldots, n,$ where f_i are analytic in $\left(x; \in; t\right)$ for $x \in D$(a given open region of R^n), $0 \leq \in \leq 1, t \in R$, and regular at $x_i = \alpha_1\left(i = 1, 2, \ldots, n\right), \in = 0,$ for all values of $t \in \left[0, \mathrm{T}\right]$. The functions f_i are developable in power series of $\xi_i = x_i - \alpha_i$ and $\in$.

These series are convergent, provided in the interval [0, T]

$$\left|x_i-\alpha_i\right| \le m_i\left(i=1,2,...,n\right),$$

and $0<\in\le\in_0\le 1.$

The expansion of (2.2.6) gives

$$\dot{\xi}_i=\in\left[\mathrm{f}_i\left(\alpha;0,t\right)+\sum_{j=0}^{n}\left(\frac{\partial \mathrm{f}_i}{\partial x_j}\right)_0\xi_j+\frac{1}{2}\sum_{j,k=0}^{n}\left(\frac{\partial \mathrm{f}_i}{\partial x_j\partial x_k}\right)_0\xi_j\xi_k+....\right]$$

Where $\xi_0=x_0=\in,\alpha_0=0$ and the subscript zero means that x_i are replaced by the α_i. We can actually write

$$\dot{\xi}_i=\in\left[\phi_0^{(\mathrm{i})}+\sum_{j=0}^{n}\phi_{\mathrm{j}}^{(\mathrm{i})}\xi_j+\sum_{j,k=0}^{n}\phi_{jk}^{(i)}\xi_j\xi_k+...\right] \tag{2.2.7}$$

Where the $\phi\text{'}s$ are functions of the $\alpha\text{'}s$ and t. The goal is to obtain the ξ_i as power series in $\in$, with coefficients functions of time and of the constants α_i, that is

$$\xi_i=\sum_{k=0}^{\infty}\xi_k^{(i)}\in^k \tag{2.2.8}$$

Where

$$\xi_k^{(i)}=\xi_k^{(i)}\left(\alpha;t\right),\ \ i=1,2,...,n.$$

If (2.2.8) are substituted into (2.2.7) and coefficients of the same power of $\in$ compared in both sides, there results a system of differential equations

$$\dot{\xi}_1^{(i)}=\phi_0^{(i)}$$

$$\dot{\xi}_2^{(i)}=\sum_{j=1}^{n}\phi_j^{(i)}\xi_1^{(j)}$$

$$\dot{\xi}_3^{(i)}=\sum_{j=1}^{n}\phi_j^{(i)}\xi_2^{(j)}+\sum_{k,j=1}^{n}\phi_{jk}^{(i)}\xi_1^{(j)}\xi_1^{(k)} \tag{2.2.9}$$

$$-----------$$

$$\dot{\xi}_p^{(i)}=\sum_{j=1}^{n}\phi_j^{(1)}\xi_{p-1}^{(i)}+F_p^{(i)}$$

for p = 1, 2, 3, ... The functions $F_p^{(i)}$ depend on the solution of all approximations up to stage p - 1, so that at every stage of such sequence of approximations

$$\dot{\xi}_p^{(i)} = \mathrm{I}_p^{(i)}\left(\xi_k^{(j)}, t, \beta_\ell\right) \text{ for } i = 1,2,...,n;\ j = 1,2,...,n;$$

$$k = 1,2,..., p-1;\ \ell = 1,2,...,n(p-1).$$

The $\xi_p^{(i)}$ are obtained by quadratures. The constants of integration β are not arbitrary. In fact, if one stops at the p - th stage included, the solutions will depend on the initial constants $\alpha_1, \alpha_2, ..., \alpha_n$ and on np constants β. One might prefer the choice of setting all the $\beta's$ equal to zero or the choice of defining them in a convenient way. In the second case they will be functions of the $\alpha's$. If the constants α_k are <u>initial conditions</u>, that is, $x_i^{(0)}$, then the $\beta's$ constants should be chosen so as to make all the $\xi_p^{(i)}$ vanish at $t = 0$. We now show that the series ξ_i obtained in this way are convergent in [0, T] provided $\in$ is sufficiently small.

Without loss of generality one can assume that the right members of (2.2.7) are convergent for $|\xi_i| \leq 1, \in \leq 1$ in [0, T]. If this is not true, a change of scale for ξ_i and $\in$ will always make this assumption possible. It follows that all coefficients ϕ in the right members of (2.2.7) are bounded and less than a positive number M, i.e..

$$\left|\phi_{j_1 j_2}^{(i)} \cdots j_k\right| \leq \mathrm{M}$$

for $i = 1, 2,..., n$ and $k = 1, 2, 3, ...$. The concept of majorant series can now be used. In fact, consider the equations.

$$\dot{\eta}_i = \frac{\mathrm{M} \in}{(1-\in)\left(1-\sum_{j=1}^{n} \eta_j\right)}, (i = 1,2,...,n). \qquad (2.2.10)$$

The right-hand members can be expanded in power series of $\in$ and η_j, with $|\in| < 1, \left|\sum \eta_j\right|$. Every coefficient is positive and greater than the corresponding coefficient in (2.2.7), in view of the foregoing hypotheses. Equations (2.2.10) can be solved by the method of successive approximations just described. It follows that the right- hand members of the equations (2.2.9) will be less than the corresponding ones for (2.2.10). Thus, if the solution of (2.2.10) converges, the solution of (2.2.9) also converges. But (2.2.10) can be integrated in closed form. If the initial values are all zero (as for the ξ_i if the α_i are initial conditions), it must result that

$$\eta_1 = \eta_2 = ... = \eta_n = \eta$$

or

$$\dot{\eta} = \frac{M \in}{(1 - \in)(1 - n\eta)}$$

and, therefore,

$$\eta = \frac{1}{n}\left[1 - \left(1 - \frac{2Mn \in t}{1 - \in}\right)^{1/2}\right] \qquad (2.2.11)$$

which satisfies the condition $\eta = 0$ for both $t = 0$ and $\in = 0$. The expansion of (2.2.11) in power series of $\in$ is convergent provided

$$\left|\frac{2Mn \in t}{1 - \in}\right| < 1$$

in [0, T], that is, provided

$$|\in| < \frac{1}{1 + 2nMT} = \in_0 . \qquad (2.2.12)$$

Since the method of successive approximations given is unique, it must coincide with the expansion of (2.2.11). Thus, the series for ξ_1 are convergent in [0, T] if $|\in| < \in_0$, where M is the upper bound for the coefficients of (2.2.7). It is seen that, for T large enough, the series only converges, in general, for $\in \to 0$. The above estimate cannot be considered the best possible, so that the term "in general" is kept in for there are actual situations where the method described converges for $\in$ small enough, but not zero, as $T \to \infty$.

Consider now the system of differential equations

$$\dot{x}_i = g(x;t) + \in f_1(x;\in,t), \quad (i=1,2,...,n). \qquad (2.2.13)$$

Substituting the α_i of the previous method by the solutions $x_{i0}(t)$ of (2.2.13) for $\in = 0$, and defining

$$\xi_i = x_i - x_{i0},$$

in the same way it is found that the coefficients $\xi_p^{(i)}$ satisfy differential equations of the type

$$\dot{\xi}_p^{(i)} = \sum_{j=1}^{n} \phi_j^{(i)} \xi_{p-1}^{(j)} + I_p^{(i)}\left(\xi_k^{(\ell)};t\right) \qquad (2.2.14)$$

where $k = 1, 2, ..., p - 1$; $\ell = 1, 2, ...n$; $i = 1, 2, ..., n$; $p = 1, 2, 3, ...$. It is a remarkable property that the $\phi_j^{(i)}$ are independent of the particular p, as before, so that the homogeneous solutions of (2.2.14) are the same for any p. They are functions of the time explicitly and of $x_{i0}(t)$, these last being functions of a set of n integration constants. As far as the integration constants for (2.2.14) they can be chosen so as to make the $\xi_p^{(i)} = 0$ at $t = 0$, at t = 0, and, using the terminology of Celestial Mechanics, in this case the solutions ξ_i and x_{i0} are osculating at t = 0. There are other ways in which the constants of integration can be chosen, but this requires a modification due to the fact the expansions are not done in the neighborhood of $\xi_i = 0$ $(\text{at } t = 0)$.

Picard's classical method of approximations allows to show that the solution of a system

$$\dot{\xi}_i = \sum_{j=1}^{n} \phi_{ij}(t)\xi_j + k_i(t), \quad (i = 1, 2, ..., n),$$

is dominated in the interval [0, T] by the solution of the system

$$\dot{\eta}_i = \mathrm{M}\sum_{j=1}^{n} \eta_j + \mathrm{M}, \quad (i = 1, 2, ..., n),$$

where M is the upper bound of the $\phi_{ij}(\in)$ and $k_i(t)$ in the interval [0, T]. Then, in a similar way as was done before, one proves that, by using the majorant functions defined by

$$\dot{\eta}_i = \mathrm{M}\frac{(\in + \eta_1 + ... + \eta_n)}{1 - (\in + \eta_1 + ... + \eta_n)},$$

the series ξ_i are convergent in [0, T] if

$$|\in| < \exp[-\mathrm{M}n\mathrm{T}] \qquad (2.2.15)$$

where M is the upper bound for the coefficients of (2.2.13) as power series of ξ_j and $\in$. The limitation one obtains in this case is much stronger, as T becomes large, than in the previous case. These cases have been discussed in details by Moulton and others (1920). However, as we shall see later, (2.2.15) may not be the best estimate for this case.

3. Secular Terms. Lindstedt's Device.

The above described methods have the classical characteristic of leading to secular terms, that is, series solutions where the $\xi_p^{(i)}$ contain terms which are linear (at least) in t. If such phenomenon could be avoided, and, more specifically, one could get $\xi_p^{(i)}(t)$ bounded for all t (say, almost periodic or periodic) the rate of convergence would certainly be improved and in special situations, as will be seen in the next chapter, actual convergences for all t can be obtained for sufficiently small $\in$.

At this moment we apply the method described in the previous section to the simple pendulum, show the appearance of secular terms and introduce Lindstedt's device in this particular application. For simplicity we shall assume that the initial conditions correspond to the libration case of the pendular motion, that is, oscillations of finite amplitude around the stable equilibrium solution. The equation of motion can be written as

$$\ddot{\theta}=-\omega_o^2\sin\theta \tag{2.3.1}$$

where $\omega_o^2=g/\ell$. Consider the convergent expansion of $\sin\theta$ in powers of θ and the change of variable $\theta = \sqrt{\in}\ x$, so that (2.3.1) becomes

$$\ddot{x}+\omega_o^2 x=-\omega_o^2\sum_{n=1}^{\infty}(-1)^n\frac{\in^n x^{2n+1}}{(2n+1)!}. \tag{2.3.2}$$

For $\in = 0$ (infinitesimal oscillations), the solution is

$$x_o(t)=\mathrm{A}\sin(\omega_o t+\alpha) \tag{2.3.3}$$

and let us consider the series

$$\xi=x-x_o=\sum_{m=1}^{\infty}\in^m\xi_m(t)$$

or, the solution in the vicinity of $x_o(t)$, as given by

$$x=x_o(t)+\sum_{m=1}^{\infty}\in^m\xi_m(t). \tag{2.3.4}$$

The method just described, substitutes (2.3.4) for x into (2.3.2) and equate coefficients of same powers of $\in$. As the first few approximations we find

$$\ddot{\xi}_1+\omega_o^2\xi_1=\frac{1}{3!}\omega_o^2 x_o^3$$

$$\ddot{\xi}_2+\omega_o^2\xi_2=\frac{1}{2!}\omega^2 x_o^2\xi_1-\frac{1}{5!}\omega_o^2 x_o^5$$

$$\ddot{\xi}_3+\omega_o^2\xi_3=\frac{1}{2!}\omega_o^2\left(x_o^2\xi_2+x_o\xi_1^2\right)-\frac{1}{4!}\omega_o^2 x_o^4\xi_1+\frac{1}{7!}\omega_o^2 x_o^7$$

$$\ddot{\xi}_4+\omega_o^2\xi_4=\frac{1}{3!}\omega_o^2\left(3x_o^2\xi_3+6x_o\xi_1\xi_2+\xi_1^3\right)-\frac{1}{4!}\omega_o^2\left(x_o^4\xi_2+2x_o^3\xi_1^2\right)$$

$$+\frac{1}{6!}\omega_o^2 x_o^6\xi_1-\frac{1}{9!}\omega_o^2 x_o^9$$

(2.3.5)

Let us analyze the solution for ξ_1 which makes $\xi_1=\dot{\xi}_1=0$ at $t=0$. With a proper choice for the unit of time we shall consider $\omega_o=1$ without loss in generality. We also have that the particular solution of

$$\ddot{z}+z=a\sin p\,(t+\alpha)$$

is

$$z=\frac{a}{1-p^2}\sin\ p(t+\alpha), p\neq 1,$$

$$z=-\frac{1}{2}\ a\ \ t\ \ \sin\ (t+\alpha)p\neq 1,$$

and of

$$\ddot{z}+z=a\cos p(t+\alpha)$$

is

$$z=\frac{a}{1-p^2}\ \cos p(t+\alpha), p\neq 1,$$

$$z=\frac{1}{2}\ a\, t\sin\ (t+\alpha),\ p=1.$$

It easily follows that

$$\xi_1=B\ \sin\ (t+\beta)-\frac{1}{16}A^3\ t\sin(t+\alpha)+\frac{1}{192}A^3\ \sin\ 3(t+\alpha) \qquad (2.3.6)$$

where B, β are given by

$$B\sin\beta=\frac{A^3}{192}\ \sin\ 3\alpha,$$

$$B\cos\beta=-\frac{A^3}{64}\ \cos\ 3\alpha+\frac{A^3}{16}\ \sin\ \alpha.$$

In this particular example it is seen that a secular term appears in (2.3.6), that is, in the first approximation. (Actually, that is more often called a mixed secular term.) Evidently, the appearance of t outside trigonometric functions makes it quite difficult to have convergence of the above process for $t\to\infty$. The constants of integration B,β cannot in any way be used to cancel the troublesome term. The solution proposed by Lindstedt (1882) is to assume a reference solution, that is, a function $x_o(t)$ which is a modification of the zero order solution as far as the frequency is concerned. In fact, we consider

$$x_o(t)=A\ \sin(\omega t+\alpha) \qquad (2.3.7)$$

where we assume

$$\omega^2=\omega_o^{(2)}+\in\omega_1+\in^2\omega_2+...$$

or, taking $\omega_o=1$ as before,

$$\omega^2 = 1 \in \omega_1 + \in^2 \omega_2 + ... \quad (2.3.8)$$

where $\omega_1, \omega_2, ...$ are constants (depending on A, α) to be conveniently chosen. By writing the equation (2.3.2) as

$$\ddot{x} + \omega^2 x - \in \omega_1 x - \in^2 \omega_2 x - ... = -\sum_{n=1}^{\infty} (-1)^n \in^n \frac{x^{2n+1}}{(2n+1)!}$$

with the “zero” order solution

$$x_o(t) \text{A}\sin(\omega t + \alpha)$$

where ω is given by (2.3.8), and unknown a priori, we obtain, as before

$$\ddot{\xi}_1 + \omega^2 \xi_1 - \omega_1 x_o = \frac{1}{3!} x_o^3$$

$$\ddot{\xi}_2 + \omega^2 \xi_2 - \omega_1 x_1 - \omega_2 x_o = \frac{1}{2!} x_o^2 \xi_1 - \frac{1}{5!} x_o^5$$

or

$$\ddot{\xi}_1 + \omega^2 \xi_1 = \omega_1 x_o + \frac{1}{3!} x_o^3$$

$$\ddot{\xi}_2 + \omega^2 \xi_2 = \omega_1 x_1 + \omega_2 x_o + \frac{1}{2!} x_o^2 \xi_1 - \frac{1}{5!} x_o^5$$

and the right-hand members are evidently odd functions of $(\omega t + \alpha)$, that is, sine series in $(\omega t + \alpha)$. In the equation for ξ_p the corresponding unknown approximation ω_p has to be determined so that secular (or mixed secular, in this case) terms should be avoided. The first order equation is

$$\ddot{\xi}_1 + \omega^2 \xi_1 = \omega_1 \text{A} \sin(\omega t + \alpha) + \frac{1}{8} \text{A}^3 \sin(\omega t + \alpha)$$

$$-\frac{1}{24} \text{A}^3 \sin(3\omega t + 3\alpha)$$

so that , defining

$$\omega = -\frac{1}{8} \text{A}^2$$

the resonant forcing term is eliminated and the solution is

$$\xi_1 = B \sin\ (\omega \text{t} + \beta) + \frac{1}{192} \text{A}^3 \sin (3\omega \text{t} + 3\alpha)$$

where B, β can be defined by

$$B \sin \beta + \frac{1}{192} \text{A}^3 \sin 3\alpha = 0$$

$$B \cos \beta + \frac{1}{64} A^3 \cos 3\alpha = 0$$

that is,

$$\xi_1 = \dot{\xi}_1 = 0 \quad \text{at} \quad t = 0$$

It is easily seen that to any order of approximation the equation to be integrated is

$$\ddot{\xi}_p + \omega^2 \xi_p = \omega_p x_0 + A_1^p \left(A, \omega_1, \omega_2, ..., \omega_{p-1} \right) \sin\left(\omega t + \alpha\right)$$

$$+ \sum_{j=1}^{n_p} A_j^p \left(A, \omega_1, \omega_2, ..., \omega_{p-1} \right) \sin\left[(2j+1)(\omega t + \alpha) \right]$$

and the solution is found by setting

$$\omega_p = -A_1^p \left(A, \omega_1, \omega_2, ..., \omega_{p-1} \right)$$

$$\xi_p = \sum_{j=1}^{n_p} \frac{A_j^p}{\omega^2 \left[1 - (2j+1)^2 \right]} \sin\left[(2j+1)(\omega t + \alpha) \right]$$

It follows that the frequency ω is determined step by step and the solution is expressed as a purely periodic function of t, that is,

$$x = x_o(t) + \sum_{p=1}^{\infty} \in^p \sum_{j=1}^{n_p} \frac{A_j^p}{\omega^2 \left[1 - (2j+1)^2 \right]} \sin\left[(2j+1)(\omega t + \alpha) \right].$$

In this specific example, since the original equation can be integrated exactly, the convergence of the above procedure can be proved directly as long as the initial conditions are such that an oscillatory motion is verified. The series above diverges in case where the actual motion is a circulation. The case of asymptotic motion cannot, as far as we know, be dealt with an approximation of series. The circulation case can be made convergent by assuming a different change or variable. In fact, in this case, the angle θ increases steadily with time beside undergoing fluctuations. The steady increase with time must be taken care of by assuming

$$\theta = \alpha t + \sqrt{\in} x$$

where

$$\alpha = \alpha_o + \in \alpha_1 + \in^2 \alpha_2 + ...$$

$$x = x_o(t) + \in \xi_1 + \in^2 \xi_2 + ...$$

$$x_o = A \sin\left(\omega t + \beta\right)$$

and

$$\omega^2 = \omega_o^2 + \in \omega_1 + \in^2 \omega_2 + ...$$

When dealing with the canonical equivalent of Lindstedt's method we shall indicate as both cases (libration and circulation) can be treated in a unique fashion. This is possible by introducing elliptic functions with modulus of any value. The asymptotic case, then, will be given by a limiting case of the global solution. The possibility of such global solutions has been studied in details by Garfinkel and others (1971).

4. Poincaré's Method (Lindstedt's Method).

The method of successive adjustment of the frequencies of the system, for which we gave an example in the last section, applies to any system of ordinary differential equations which can be written in normal form and satisfies certain conditions of regularity at least locally. It is however desirable if such regularity extends over a certain domain. In this case we may assume the system to have the form

$$\dot{x}_i = \mathrm{f}_i\left(x;t;\in\right),\left(i=1,2,3,\ldots,n\right)$$

or, in vector form,

$$\dot{x} = \mathrm{f}\left(x;t;\in\right). \qquad (2.4.1)$$

By the well know transformation to Dirac's cotangent space, system (2.4.1) can be brought into a canonical form, by defining the associate generalized momentum vector $y\left(y_i;i=1,2,\ldots,n\right)$ and the Hamiltonian

$$\mathrm{H} = \mathrm{f}^{\mathrm{T}}\left(x;t;\in\right)y. \qquad (2.4.2)$$

The equation of motion are

$$\dot{x} = \mathrm{H}_y^T = \mathrm{f}\left(x;t;\in\right)$$

$$\dot{y} = \mathrm{H}_x^T = -\mathrm{f}_x\left(x;t;\in\right)y$$

and we assume that the system

$$\dot{\xi} = \mathrm{f}\left(\xi;t;0\right)$$

$$\dot{\eta} = -\mathrm{f}_{\xi}\left(\xi;t;0\right)\eta$$

is integrable in some domain D of the 2n- dimensional phase space ($\xi;\eta$) and for $0 \le t \le \mathrm{T}$. We assume $\mathrm{f}\left(x;t;\in\right)$ to be at least C^2 in D, continuous with respect to t in $\left[0,\mathrm{T}\right]$ and analytic in $\in$ for $0 \le \in \le 1$. The same properties are, therefore, also verified by the function H.

By these means, very little restriction is set by assuming a given system of equations to be Hamiltonian and, henceforth, the importance of perturbation methods in Hamiltonian systems.

With the above considerations it seems logical to ask whether a better estimate than (2.2.15) can be obtained. The Hamiltonian of the system under consideration (2.2.13) is

$$\mathrm{H}=\sum_{i=1}^{n} y_i g_i(x;t)+\in\sum_{i=1}^{n} y_i \mathrm{f}_i(x;\in,t)=\mathrm{H}_o+\in \mathrm{H}_1 \tag{2.4.3}$$

and the canonical conjugate equations

$$\begin{aligned}\dot{x}_i &= \frac{\partial \mathrm{H}}{\partial y_i}=g_i+\in \mathrm{f}_i \\ \dot{y}_i &= -\frac{\partial \mathrm{H}}{\partial x_i}=-\sum_j y_j \frac{\partial g_j}{\partial x_i}-\in\sum_j y_j \frac{\partial \mathrm{f}_j}{\partial x_i}.\end{aligned} \tag{2.4.4}$$

For $\in=0$, we assume that the system

$$\begin{aligned}\dot{x}_i &= g_i(x;t) \\ \dot{y}_i &= -\sum_j \frac{\partial g_j}{\partial x_i} y_j\end{aligned} \tag{2.4.5}$$

is integrable. In fact, the first set is integrable by hypothesis and the solution is $x_i=x_{io}(t)$. Substitution of this solution into the second set gives a linear system

$$\dot{y}_i=\sum_j \mathrm{a}_{ij}(t)\, y_j$$

which is, evidently, integrable, for t in the interval of definition of $x_{io}(t)$. Let the solution of (2.4.5) be written as

$$x_i=x_{io}(\alpha;\beta;t)$$

$$y_i=y_{io}(\alpha;\beta;t)$$

with

$$x_{io}(\alpha;\beta;0)=\alpha_i$$

$$y_{io}(\alpha;\beta;0)=\beta_i$$

for $i=1,2,\ldots,n$. It follows from Jacobi's theorem that the solution of system (2.4.4) can be written as

$$x_i=x_{io}(\alpha;\beta;t)$$

$$y_i=y_{io}(\alpha;\beta;t)$$

if α,β are functions of t satisfying the equations

$$\dot{\alpha}_i = \in \frac{\partial \mathrm{H}_1}{\partial \beta_i}$$

$$\dot{\beta}_i = - \in \frac{\partial \mathrm{H}_1}{\partial \alpha_i} \tag{2.4.6}$$

for $i = 1, 2, ..., n$. But system (2.4.6) is of the type studied earlier [Equation (2.2.6)] and the application of the method of successive approximation will give the convergence criterion

$$|\in| < \frac{1}{1 + 4n\mathrm{M}'\mathrm{T}}$$

which, if $\mathrm{M}' \simeq \mathrm{M}$, is a better estimate than (2.2.15) for the system (2.2.13).

We now return to the main purpose of this section and outline the general principle and rationale of Lindstedt's device as explained by Poincaré in canonical language. Let us consider a conservative dynamical system defined by the Hamiltonian

$$\mathrm{H} = \mathrm{H}(y; x; \in) \tag{2.4.7}$$

where y, x are n-dimensional vectors defined in phase space of dimension 2n, $\in$ is a dimensionless constant parameter and H is real analytic in some domain D of the phase space and for $\in$ in [0,1]. We stress the fact that any analytic system $\dot{z} = \mathrm{f}(z; \in)$ can be reduced to the Hamiltonian form above, by introducing the cotangent phase space. Hamilton's principal function $W(y; \mathrm{X}; \in)$ is defined by the partial differential equation

$$\mathrm{H}\left(y; \frac{\partial W}{\partial y}; \in\right) = \mathrm{K}(\mathrm{X}; \in) \tag{2.4.8}$$

where $\mathrm{K}(\mathrm{X}; \in)$ is obviously the Hamiltonian of the system written in terms of the new variables (Y; X) defined by

$$Y_k = \frac{\partial W}{\partial \mathrm{X}_k} = Y_k(y; \mathrm{X}; \in),$$

$$x_k = \frac{\partial W}{\partial W_k} = x_k(y; \mathrm{X}; \in), \tag{2.4.9}$$

for $k = 1, 2, ..., n$. Under the conditions specified for H, a function W satisfying Equation (2.4.8) certainly exists (in the Jacobi sense) since the system of differential equations generated by (2.4.7) has a unique solution in D. The solution is evidently an analytic function of $\in$ and the n constants of integration $\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_n$, in D. We assume that the system of differential equations generated by $\mathrm{H}(y; x; 0) = \mathrm{H}_o(y; x)$ is integrable in the Liouville sense, that is, there exist n first integrals of motion in D, uniform and independent. If $x'_1, x'_2, ..., x'_n$ are such integrals, that is,

$$x'_k(y;x) = \alpha_k$$

along the solutions of (2.4.7) for $\in = 0$ and in D, in general, the angular variables canonically associated to the action variables y'_k have frequencies (in time) which are linearly independent over the set of integers and, therefore, the motion is quasiperiodic (almost periodicity would, this case, correspond to a system with an infinite number of basic frequencies). In terms of these action-angle variables the Hamiltonian (2.4.7) can be written as $\mathrm{H}'(y';x';\in)$ with the obvious condition

$$\mathrm{H}'(y';x';0) = \mathrm{H}'_0(x')$$

It is therefore with no loss of generality that, under the assumption that $\mathrm{H}_o(y;x)$ leads to integrability (in the above specified sense), it can be trought as being a function of the momenta (x) only. It is also logical to expect that almost t everywhere in D the frequencies $\omega_k^o = \partial \mathrm{H}_o / \partial x_k$ are linearly independent over the integers. This implies, in particular, that none of these frequencies are zero in D, or, more precisely, none of the momenta are ignorable. The problem is now reduced to one for which $\mathrm{H}(y;x;0)$ is independent of y and therefore Hamilton's principal function $W(y;\mathrm{X};0)$ is a generator for the identity transformation, that is,

$$W(y;\mathrm{X};0) = y.\mathrm{X}.$$

We assume that W is analytic with respect to $\in$ at $\in = 0$, and therefore, for sufficiently small,

$$W(y;\mathrm{X};\in) = y.\mathrm{X} + \in S(y;\mathrm{X};\in), \qquad (2.4.10)$$

with

$$S(y,\mathrm{X};\in) = S_1(y;\mathrm{X}) + \in S_2(y;\mathrm{X}) + ... \qquad (2.4.11)$$

a convergent power series in $\in$.

It follows that (2.4.9) can be written as

$$Y_k = y_k + \in \frac{\partial S}{\partial \mathrm{X}_k} = y_k + \in F_k(y;\mathrm{X};\in)$$

and

$$x_k = \mathrm{X}_k + \in \frac{\partial S}{\partial y_k} = \mathrm{X}_k + \in G_k(y;\mathrm{X};\in) \qquad (2.4.12)$$

for $k = 1, 2, ..., n$ and $\in$ sufficiently small. Mappings of the sort (2.4.12) have been extensively studied principally by Moser (1955, 1961, 1962, 1967).

Under the above conditions, it is possible to show that there exists a formal series (2.4.11) which solves (2.4.8) up to any order (power) of $\in$. We introduce the "average" value < f > of a quasi-periodic function $f(y_1, y_2, ..., y_n)$, with $y_k = \omega_k t + y_k^o, \omega_k$ constant and linearly independent over the integers, by

$$< f > \;=\; \lim \frac{1}{T}\int_0^T f\, dt. \qquad (2.4.13)$$

In a generalized sense, a quasi-periodic function f with the property < f > = 0, will be said to be said to be purely quasi-periodic. Obviously, if f is a Fourier Series in the n angular variables $y_1, y_2, ..., y_n$, < f > is the constant term of the Fourier's series. On the other hand, in general, if < f > = 0 then

$$\lim_{T\to\infty}\int_0^T f\, dt = \text{finite} \qquad (2.4.14)$$

which is an obvious consequence of (2.4.13) for f quasi-periodic and L_2 for $t \in R$. A function $F(t)$ satisfying the condition

$$\lim_{T\to\infty} F(t) = \text{finite} \qquad (2.4.15)$$

will be said to be free from secular terms. Any primitive of an L_2 purely quasi-periodic function satisfies this properly. Under the integrability assumption of H_o, it follows that, in terms of the action-angle variables (y; x) the Hamiltonian $H(y; x; \in)$ is quasi-periodic if, for example, it has a convergent multi-dimensional Fourier series in $y_1, y_2, ..., y_n$, for $\in$ in $[0,1]$ and $(x; y)$ in D.

The formal series S and K are now obtained by direct substitution of (2.4.10) and (2.4.11) into (2.4.8), that is,

$$H\left(y; \frac{\partial W}{\partial y}; \in\right) = H\left(y; X + \in \frac{\partial S_1}{\partial y} + \in^2 \frac{\partial S_2}{\partial y} + ...; \in\right)$$

$$K(X; \in) = K_0(X) + \in K_1(X) + \in^2 K_2(X) + ...$$

Expansion of the first of these by Taylor series (which by hypothesis converges) gives, symbolically,

$$\mathrm{H}\left(y;\frac{\partial W}{\partial y};\in\right)=\sum_{\mathrm{K}=0}^{\infty}\frac{1}{\mathrm{K}!}\frac{\partial^{\mathrm{K}}\mathrm{H}}{\partial x^{\mathrm{K}}}\bigg| x=\mathrm{X}\left(\in\frac{\partial S_1}{\partial y}+\in^2\frac{\partial S_2}{\partial y}+\ldots\right)^{\mathrm{K}}$$

$$=\sum_{\mathrm{K}=0}^{\infty}\frac{1}{\mathrm{K}!}\sum_{p=0}^{\infty}\in^p\frac{\partial^{\mathrm{K}}\mathrm{H}_p}{\partial x^{\mathrm{K}}}\bigg| x=\mathrm{X}\left(\in\frac{\partial S_1}{\partial y}+\in^2\frac{\partial S_2}{\partial y}+\ldots\right)^{\mathrm{K}}$$

$$=\mathrm{H}_0\left(\mathrm{X}\right)+\in\frac{\partial\mathrm{H}_0}{\partial\mathrm{X}}\left(\frac{\partial S_1}{\partial y}\right)^{\mathrm{T}}+\frac{\in^2}{2!}\left(\frac{\partial S_1}{\partial y}\right)\frac{\partial^2\mathrm{H}_0}{\partial\mathrm{X}\partial\mathrm{X}}\left(\frac{\partial S_1}{\partial y}\right)^{\mathrm{T}}+\ldots \qquad (2.4.16)$$

$$+\in^2\frac{\partial\mathrm{H}_0}{\partial\mathrm{X}}\left(\frac{\partial S_2}{\partial y}\right)^{\mathrm{T}}+\ldots+\in\mathrm{H}_1\left(y;\mathrm{X}\right)+\in^2\frac{\partial\mathrm{H}_1}{\partial\mathrm{X}}\left(y;\mathrm{X}\right)\left(\frac{\partial S_1}{\partial y}\right)^{\mathrm{T}}$$

$$+\ldots+\in^2\mathrm{H}_2\left(y;\mathrm{X}\right)+\ldots \quad .$$

Expressions up to any order of approximation were first obtained by Giacaglia (1963). Equating coefficients of same powers in $\in$, one gets, to any order of approximation, an equation of the type

$$\sum_{\mathrm{K}=1}^{n}\frac{\partial\mathrm{H}_0}{\partial\mathrm{X}_{\mathrm{K}}}\frac{\partial S_p}{\partial y_{\mathrm{K}}}+\phi_p\left(y;\mathrm{X}\right)+\mathrm{H}_p\left(y;\mathrm{X}\right)=\mathrm{K}_p\left(\mathrm{X}\right) \qquad (2.4.17)$$

where $\partial\mathrm{H}_{\mathrm{K}}/\partial\mathrm{X}_{\ell}$ stands for $\partial\mathrm{H}_{\mathrm{K}}/\partial\mathrm{X}_{\ell}\big|_{x=\mathrm{X}}$. For example,

$$\phi_1\left(y;\mathrm{X}\right)=0$$

$$\phi_2\left(y;\mathrm{X}\right)=\frac{1}{2!}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\partial^2\mathrm{H}_0}{\partial\mathrm{X}_{\mathrm{K}}\partial\mathrm{X}_{\ell}}\frac{\partial S_1}{\partial y_{\mathrm{K}}}\frac{\partial S_1}{\partial y_{\ell}}+\sum_{\mathrm{K}=1}^{n}\frac{\partial\mathrm{H}_1}{\partial\mathrm{X}_{\mathrm{K}}}\frac{\partial S_1}{\partial y_{\mathrm{K}}}$$

and so on. In general, $\phi_p\left(y;\mathrm{X}\right)$ is a function of $S_1,S_2,\ldots,S_{p-1},\mathrm{K}_1,\mathrm{K}_2,\ldots,\mathrm{K}_{p-1}$, so that the solutions of equations (2.4.17) can only be obtained in succession. One way of defining $\mathrm{K}_p\left(\mathrm{X}\right)$ is by the use of an averaging procedure

$$\mathrm{K}_p\left(\mathrm{X}\right)=<\phi_p\left(y;\mathrm{X}\right)+\mathrm{H}_p\left(y;\mathrm{X}\right)> \qquad (2.4.18)$$

where y_{K} is supposed to be given by a linear function of time $y_{\mathrm{K}}=\omega_{\mathrm{K}}^0 t+y_{\mathrm{K}}^0$, and all the ω_{K}^0 linearly independent over the set of integers (that is, rationally independent). The resulting $\mathrm{K}_p\left(\mathrm{X}\right)$ is certainly independent of t. It follows that the function

$$F_p=\phi_p+\mathrm{H}_p-\mathrm{K}_p=F_p\left(y;\mathrm{X}\right) \qquad (2.4.19)$$

is purely quasi-periodic in view of the hypotheses on $\mathrm{H}\left(y;x;\in\right)$. The $p-th$ approximate to the generating function, S_p, is obtained from the linear equation

$$\sum_{K=1}^{n} \omega_K^0 \frac{\partial S_p}{\partial y_K} + F_p(y;X) = 0$$

where $\omega_K^0 = \partial H_0 / \partial X_K$. It is now obvious that if every $\omega_K^0 \neq 0, S_p$ results to be a quasi-periodic function in $y_1, y_2, ..., y_n \left(y_K = \omega_K^0 t + y_K^0 \right)$ free from secular terms, that is, for linearly independent ω_K^0 over the integers,

$$S_p(y;X) = \sum_{K=1}^{n} \frac{1}{\omega_K^0} \int F_p(y;X) dy_K + G_p(X) \tag{2.4.20}$$

where $G_p(X)$ is arbitrary. Obviously if one of the ω_K^0 is zero the formula does not apply, unless $F_p(y;X)$ is such that

$$\frac{\partial F_p}{\partial y_K} = 0 \tag{2.4.21}$$

for that particular y_K. It is easily seen that

$$\lim_{t \to \infty} S_p(y;X) = \text{finite} \tag{2.4.22}$$

for $y_K = \omega_K^0 t + y_K^0$. All these relations are easily shown and, by recurrence, it follows that one can determine the formal series

$$y.X + \in S_1 + \in^2 S_2 + \in^3 S_3 + ...$$

and

$$K_0 + \in K_1 + \in^2 K_2 + ...,$$

where $\in S_1, \in^2 S_2 + \in^3 S_3 + ...$ satisfies the property of being quasi-periodic and free from secular terms. The case where some $\omega_K^0 = 0$ or it is small, in some sense, will be studied in chapter 5, under the general problem of resonance. The system is formally solved up to any desired degree of approximation and the "solution", in the new variables (Y; X) is

$$\begin{aligned} Y_K &= \omega_K t + Y_K^0 \\ X_K &= X_K^0 \end{aligned} \tag{2.4.23}$$

where

$$Y_K^0 = \text{const.}$$

$$X_K^0 = \text{const.} + 0\left(\in^{p+1}\right) \qquad (2.4.24)$$

$$\omega_K = \frac{\partial K_0}{\partial X_K} + \in \frac{\partial K_1}{\partial X_K} + ... + \in^p \frac{\partial K_p}{\partial X_K} = \text{const.} + 0\left(\in^{p+1}\right)$$

where $0\left(\in^{p+1}\right)$ is the factor of the first term neglected after the last approximates S_p and K_p have been obtained. By no means, it should be interpreted as the error or an approximation to the error bound of the solution. This might be so, eventually, only in the case of convergences of the method. The problem will be dealt with in the next two chapters. A rough estimate by Kyner (1963) shows that the error bound is equivalent to that obtained by Bogoliubov and Mitropolsky (1951) for the canonical averaging method of Krylov-Bogoliubov-Mitropolsky (KBM), and in fact, Poincaré's Method was shown to be equivalent to that of KBM by Burstein and Solovev (1961). Such error bound is proportional to $\in$ for $t \sim 1/\in$, at worst. Convergence of the method, under particular circumstances, will be given in Chapter 3.

From a purely formal point of view we obtain, from (2.4.21),

$$y_K = \omega_K t + Y_K^0 + \in N_K\left(Y_1, Y_2, ..., Y_n; X_1^0, X_2^0, ..., X_n^0; \in\right)$$

$$x_K = X_K^0 + \in W_K\left(Y_1, Y_2, ..., Y_n; X_1^0, X_2^0, ..., X_n^0; \in\right) \qquad (2.4.25)$$

where N_K, W_K are quasi-periodic in $Y_1, Y_2, ..., Y_n$ and free from secular terms. It is obvious that one of the major causes of error is in the frequency ω_K, since any error is linearly multiplied by time. In practical applications, the best way out, in lack of exact solution, is to use a numerical observed value of ω_K from the average $< y_K >$ with respect to t. Such average, if relations (2.4.25) hold, is obviously ω_K. This use of observational evidence eliminates the in-track error due to a miscalculation of the frequency ω_K.

5. Fast and Slow Variables.

The case of proper degeneracy (Arnold, 1963) is quite common in perturbation theory. Generally speaking, the problem is defined by non-independent frequencies of the unperturbed system. That is, given the Hamiltonian

$$H_0 = H_0(x)$$

and the frequencies

$$\omega_j = \frac{\partial \mathrm{H}_0}{\partial x_j}; \; j = 1, 2, ..., n$$

one has degeneracy if the matrix

$$\left(\frac{\partial \omega_j}{\partial x_{\mathrm{K}}} \right); \quad j, k = 1, 2, ..., n \tag{2.5.1}$$

is singular. This definition includes cases of rational dependence and when some of the action variables are not present in $\mathrm{H}_0(x)$, that is, at least one of the ω_j is identically zero. It also includes linear systems, that is, cases in which

$$\mathrm{H}_0 = \omega_1 x_1 + \omega_2, x_2 + ... + \omega_n x_n. \tag{2.5.2}$$

Let us consider, here, the case where the matrix (2.5.1) has at least a minor of order $m(0 < m \leq n)$ which is not zero. The unperturbed system is nonlinear, integrable and defined by m independent frequencies, corresponding to a set of m independent angular variables $y_{\mathrm{K}} = \omega_{\mathrm{K}}(x)t + y_{\mathrm{K}}^0, \mathrm{K} = 1, 2, ..., m.$ There exists, in this case, a canonical transformation $(x, y) \to (x', y')$ such that, at least locally, the Hamiltonian H_0 is a function of only m momenta x' and the corresponding matrix (2.5.1) is non-singular. It may be worth noting, however, that if none of the x are absent in H_0, one may perform a transformation to a new Hamiltonian whose Hessian matrix (2.5.1) is non-singular. In fact, consider in general the Hamiltonian

$$\mathrm{H} = \mathrm{H}(y; x; \in) = \mathrm{H}_0(x) + \in \mathrm{H}_1(y; x) + ...$$

and suppose $\omega_j = \partial \mathrm{H}_0 / \partial x_j \neq 0, j = 1, 2, ..., n.$ If a function $F = \phi(\mathrm{H})$ can be found such that

$$F = F_0(x) + \in F_1(y; x) + ...$$

and such that, being $\Omega_j = \partial F_0 / \partial x_j$, the matrix

$$\left\{ \frac{\partial \Omega_j}{\partial x_k} \right\}$$

is non-singular, the apparent degeneracy is eliminated. The equations of motion are now

$$\dot{y}_j = \frac{1}{\alpha} \frac{\partial F}{\partial x_j}$$

$$\dot{x}_j = -\frac{1}{\alpha} \frac{\partial F}{\partial y_j}$$

where α is the constant defined in terms of the initial conditions by

$$\dot{\phi}(\mathrm{H}) = \dot{\phi}(\mathrm{H}(y_0; x_0; \in)) = \dot{\phi}(h) = \alpha$$

and h is the energy integral corresponding to the initial conditions $(y_0; x_0)$. Evidently can be developed in a power series of $\in$ [we suppose H real analytic in all arguments] and if ϕ is analytic, the power series

$$\phi(\mathrm{H}) = F_0(x) + \in F_1(y;x) + \in^2 F_2(y;x) + \ldots \quad .$$

converges. This process does not apply in the linear case (2.5.2) since, as it is easily verified, whatever $\phi(\mathrm{H})$ is, the Hessian of $F_0(x)$ is zero. It does apply, however, in other cases. An important example is, for instance when

$$\mathrm{H}_0 = \frac{1}{x_1^2} + x_2$$

a case of many applications in celestial mechanics (two-body problem in rotating coordinates, restricted three-body problem in rotating coordinates, etc.). Although the Hessian of H_0 is zero (H_0 is linear in x_2) one sees that there are several functions of H_0 leading to an F_0 for which the Hessian is not zero (e.g.; Poincaré, 1893). Excluded the linear case we are therefore left with the case in which some of the momenta are not present in H_0. Let $(x_{p+1}, x_{p+2}, \ldots, x_n)$ be the ignorable momenta and consider the equations generated by

$$\mathrm{H} = \mathrm{H}_0(x_1, x_2, \ldots, x_p) + \in \mathrm{H}_1(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) + \ldots$$

that is,

$$\dot{y}_{\mathrm{K}} = \partial \mathrm{H} / \partial x_{\mathrm{K}}$$

$$\dot{x}_{\mathrm{K}} = -\partial \mathrm{H} / \partial y_{\mathrm{K}} = -\in \partial \mathrm{H}_1 / \partial y_{\mathrm{K}}.$$

It follows that, as a "zero approximation", the X_{K} are constant and the y_{K} are linear functions of time $(k = 1, 2, \ldots, p)$ or are constant $(k = p+1, p+2, \ldots, n)$. If these results are put back into the equations of motion and the average with respect to $y_1, y_2, \ldots, y_p$ is considered, to "first order" one obtains

$$y_k = \omega_k(x)t + y_k^0$$

$$x_k = x_k^0$$

with

$$\omega_k = \omega_k^0 + \in \omega_k^1, \left(k = 1, 2, ..., p\right),$$

$$\omega_j = \in \omega_j^1 \qquad , \left(j = p+1, ..., n\right).$$

This crude description motivates the fact the angular variables $y_1, y_2, ..., y_p$ (whose associate momenta $x_1, x_2, ..., x_p$ are present in H_0) are called fast and the angular variables $y_{p+1}, y_{p+2}, ..., y_n$ (whose associate momenta are absent in H) are called slow. As a consequence, any function containing at least a fast variable is said to be short periodic and any function containing none of the fast variable is said to be short periodic and any function containing none of the fast variable, long periodic. Obviously, we are not seeking here precise definitions, but only a traditional explanation of a terminology.

The problem now is to see whether there are formal series, in this case, which solve the generating function of Poincaré's method. In general the answer is negative, unless a unique situation occurs. This is the subject of the present section.

The elimination of fast variables is accomplished by a generalization of Hamiltonian's problem, where we require the new Hamiltonian to contain only slow variables. More precisely, we construct a generating function, as a formal series

$$W\left(y; \mathrm{X}; \in\right) = y.\mathrm{X} + \in S_1\left(y; \mathrm{X}\right) + ...$$

as in (2.4.10), (2.4.11) and (2.4.12), and require the energy conservation law in the form

$$\mathrm{H}\left(y; x; \in\right) = \mathrm{K}\left(Y_{p+1}, Y_{p+2}, Y_n; \mathrm{X}; \in\right) \tag{2.5.3}$$

so that the system reduces to one with a number of degrees of freedom equal to n-p. This is always possible since at any stage m of approximation the equation to be integrated is

$$\sum_{k=1}^{p} \omega_k\left(\mathrm{X}\right) \frac{\partial S_m}{\partial y_k} + F_m\left(y; \mathrm{X}\right) + \mathrm{H}_m\left(y; \mathrm{X}\right)$$

$$= \mathrm{K}_m\left(y_{p+1}, y_{p+2}, ..., y_n; \mathrm{X}; \in\right)$$

and K_m is defined by the average of $F_m + \mathrm{H}_m$ over the fast variables. The new Hamiltonian is obtained as a formal series. Admitting such series to be convergent (at least over a finite interval of time) the problem is now reduced to the equations generated by the Hamiltonian

$$\mathrm{K} = \mathrm{K}_0\left(\mathrm{X}\right) + \in \mathrm{K}_1\left(Y_{p+1}, Y_{p+2}, \ldots, Y_n; \mathrm{X}\right)$$

$$+ \in^2 \mathrm{K}_2\left(Y_{p+1}, Y_{p+2}, \ldots, Y_n; \mathrm{X}\right) + \ldots \tag{2.5.4}$$

$$= \mathrm{K}\left(Y_{p+1}, Y_{p+2}, \ldots, Y_n; \mathrm{X}; \in\right)$$

while the constant momenta $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_p$ play the role of parameters. In case of convergence, the relations

$$x_k = \mathrm{X}_k + \in \frac{\partial S\left(y; \mathrm{X}; \in\right)}{\partial y_k} \tag{2.5.5}$$

for $k = 1, 2, 3, \ldots, p$ represent first integrals of the original system, depending on p parameters $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_p$ which can be given arbitrary values.

The elimination of the slow variables reduces now to a simple condition. In fact, in (2.5.4), $\mathrm{K}_0\left(\mathrm{X}\right)$ depends only on $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_p$ and is therefore a constant of motion. The Hamiltonian can now be written as

$$\in F = \in F_1\left(q; p\right) + \in^2 F_2\left(q; p\right) + \ldots$$

$$= \in F\left(q; p; \in\right) \tag{2.5.6}$$

where $q = \left(Y_{p+1}, Y_{p+2}, \ldots, Y_n\right)$, $p = \left(\mathrm{X}_{p+1}, \mathrm{X}_{p+2}, \ldots, \mathrm{X}_n\right)$ and the parameters $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_p$ have been omitted. The equations of motion are simply

$$\dot{q}_k = \in \frac{\partial F}{\partial p_k}$$

$$\dot{p}_k = - \in \frac{\partial F}{\partial q_k} \tag{2.5.7}$$

for $k = 1, 2, \ldots, n - p$. If $n - p = 1$, the system has a single degree of freedom and the problem is theoretically solved. If $n - p \geq 2$ the integration by a method of successive approximations of the type under discussion can only be performed, obviously, if the dominant part of $\in F$, that is, $\in F_1\left(q; p\right)$ corresponds to an integrable system. From this point on, we have a repetition of the process of Poincaré described in the previous section. Useless to say, the problem can formally be completely reduced if $F_1\left(q; p\right)$ does not depend on any q and contains all of the p variables, that is, $\mathrm{X}_{p+1}, \mathrm{X}_{p+2}, \ldots, \mathrm{X}_n$. The actual contribution of von Zeipel (1916) was to rec ognize the fact that, although the complete reduction of the system may not be possible, partial reduction is a certain step toward the solution of the problem.

Error estimates of the method have been obtained by Kyner (1966) and, in case of convergence, accelerated process of convergence have been introduced by Moser (1966) based on a Newton-type iterative process. This process, actually first suggested by Kolmogorov (1954), has been widely used by Arnol'd (1963) in several papers. In this respect, much will be said in the next chapter. Evidently, there are several situations where the error estimate $0\left(\in^2\right)$ obtained by Kyner can be improved a lot. For instance, in the proof of convergence in the Twist Mapping of Moser (1962) better than quadratic convergence may be obtained so that the error decreases with a power of $\in$ which is increasing as the iterations are accumulated. For this to be true, the mapping involved does not even have to be analytic but only finitely many times differentiable.

6. Generalization of the Averaging Procedure, Birkhoff's Normalization and Adelphic Integrals.

In most cases, when the averaging method is applied, it is a basic hypothesis to assume that the Hamiltonian be multi-periodic in the angle variables, say $y_1, y_2, ..., y_n$. As seen in section 4 of this chapter, quasi-periodicity can be assumed as a slight generalization of the assumption of multi-periodicity, when a proper definition of average is introduced. Such hypotheses are a reminiscence of the special fields where the methods have been developed: celestial mechanics and oscillations in mechanical and electrical systems.

In order to introduce a more general approach to the problem, where the above mentioned hypotheses are not verified we initially consider a simple example. Let the Hamiltonian be given and such that

$$\mathrm{H}\left(y_1, y_2, x_1, x_2\right) = \mathrm{H}_0 + \mathrm{H}_1 + \mathrm{H}_2 + ...$$

where

$$\mathrm{H}_0 = \frac{1}{2}\mathrm{A}_{11}\left(x_1^2 + y_1^2\right) + \frac{1}{2}\mathrm{A}_{22}\left(x_2^2 + y_2^2\right) + \mathrm{A}_{12}\left(x_1 x_2 + y_1 y_2\right)$$

$$\mathrm{H}_p = \mathrm{H}_p\left(y_1, y_2, x_1, x_2\right),\ p = 1, 2, ...$$

where H_p are homogeneous polynomials of degree $p+2$. The solution of the "dominant" part of the problem is immediate if one can eliminate the part $\left(x_1 x_2 + y_1 y_2\right)$. This can, in general, be accomplished quite easily by a linear canonical transformation

$\left(y;x\right)\rightarrow\left(\eta;\xi\right)$

$$x_j = \sum_{k=1}^{2} \mathrm{a}_{jk}\xi_k$$

$$\eta_j = \sum_{k=1}^{2} \mathrm{a}_{kj}y_k$$

where, for example, one can take

$$\mathrm{a}_{12} = \mathrm{A}_{12}$$

$$\mathrm{a}_{22} = \mathrm{A}_{22} - \mathrm{A}_{11}$$

$$\mathrm{a}_{11} = \left(1+\mathrm{a}_{12}\mathrm{a}_{21}\right)/\,\mathrm{a}_{22}$$

$$\mathrm{a}_{11} = \left(1+\mathrm{a}_{12}\mathrm{a}_{21}\right)/\,\mathrm{a}_{22}$$

$$\mathrm{a}_{21} = \left(\mathrm{A}_{12}\mathrm{a}_{22} - \mathrm{A}_{22}\mathrm{a}_{12}\right)/\left(\mathrm{A}_{22}\mathrm{a}_{12}^2 + \mathrm{A}_{11}\mathrm{a}_{22}^2 - 2\mathrm{A}_{12}\mathrm{a}_{12}\mathrm{a}_{22}\right)$$

excluded the case $\mathrm{A}_{11}=\mathrm{A}_{22}$, where the above transformation is singular. This particular case is, of course, much more easily solved. The Hamiltonian is brought to the form

$$\mathrm{H}=\mathrm{H}_0+\mathrm{H}_1+\mathrm{H}_2+...$$

where $\mathrm{H}_0 = \mathrm{A}_1\left(\xi_1^2+\eta_1^2\right)+\mathrm{A}_2\left(\xi_2^2+\eta_2^2\right)$ and $\mathrm{H}_1,\mathrm{H}_2,...$ are again homogeneous polynomials of degree $3,4,...$ in $\xi_1,\xi_2,\eta_1,\eta_2$. Also

$$\mathrm{A}_1 = \frac{1}{2}\left(\mathrm{A}_{11}\mathrm{a}_{11}^2 + \mathrm{A}_{22}\mathrm{a}_{21}^2 + 2\mathrm{a}_{11}\mathrm{a}_{21}\mathrm{A}_{12}\right),$$

$$\mathrm{A}_2 = \frac{1}{2}\left(\mathrm{A}_{11}\mathrm{a}_{12}^2 + \mathrm{A}_{22}\mathrm{a}_{22}^2 + 2\mathrm{a}_{12}\mathrm{a}_{22}\mathrm{A}_{12}\right).$$

The solution of Hamilton's equation

$$\mathrm{A}_1\left[\left(\frac{\partial W}{\partial\eta_1}\right)^2+\eta_1^2\right]+\mathrm{A}_2\left[\left(\frac{\partial W}{\partial\eta_2}\right)^2\right]=F_0\left(\alpha_1,\alpha_2\right)$$

is immediate. With the "natural" choice

$$F_0=\mathrm{A}_1\alpha_1^2+\mathrm{A}_2\alpha_2^2,$$

we find $S=S_1+S_2$, where

$$\left(\frac{\partial S_k}{\partial \eta_k}\right)^2 + \eta_k^2 = \alpha_k^2, k = 1,2$$

and therefore

$$\alpha_k^2 = \xi_k^2 + \eta_k^2,$$

$$\beta_k = \left(\xi_k^2 + \eta_k^2\right)^{1/2} \text{ arcsin } \left(\eta_k / \alpha_k\right)$$

for $k = 1,2.$ The inverse transformation is

$$\eta_k = \alpha_k \sin\left(\beta_k / \alpha_k\right) \quad ,$$

$$\xi_k = \alpha_k \cos\left(\beta_k / \alpha_k\right) \quad , k = 1,2.$$

The dominant part of the Hamiltonian is reduced to

$$\mathrm{H}_0 = F_0 = \mathrm{A}_1\alpha_1^2 + \mathrm{A}_2\alpha_2^2$$

while the complete Hamiltonian will in general be made up by terms

$$\alpha_1^p \alpha_2^q \cos\left(m\frac{\beta_1}{\alpha_1} + n\frac{\beta_2}{\alpha_2}\right) \tag{2.6.1}$$

The zero-th order solution

$$\alpha_k = const. \quad ,$$

$$\eta_k = \left(-\mathrm{A}_k\alpha_k\right)t + \eta_k^0, \left(k = 1,2\right)$$

shows that Poincaré's method will produce mixed secular terms due to differentiations with respect to α_1 or α_2 in the generating function of the method (containing necessary terms of the form (2.6.1)). The solution to the question is actually simpler, at least in the formal sense. In fact, suppose the Hamiltonian contains the variables (x; y) in the combinations $x_1^2 + y_1^2$ and $x_2^2 + y_2^2$ only, i. e.,

$$\mathrm{H} = \mathrm{H}\left(x_1^2 + y_1^2, x_2^2 + y_2^2\right).$$

In this case, since

$$\dot{x}_j = \frac{\partial \mathrm{H}}{\partial\left(x_j^2 + y_j^2\right)} 2y_j,$$

$$\dot{y}_j = -\frac{\partial \mathrm{H}}{\partial\left(x_j^2 + y_j^2\right)} 2x_j,$$

it follows that

$$x_j^2 + y_j^2 = const. = c_j^2$$

and therefore

$$x_j = c_j \cos\left(\omega_j t + \sigma_j\right)$$

$$y_j = c_j \sin\left(\omega_j t + \sigma_j\right)$$

where

$$\omega_j = -2\frac{\partial \mathrm{H}}{\partial\left(x_j^2 + y_j^2\right)} = const.$$

and c_j, σ_j are arbitrary. This is analogous to Whittaker's (1937) remark that if the Hamiltonian is function of the variables $\omega_j = x_j y_j$ only, then the ω_j are constant. The same remark applies, of course, to any combinations of the associate coordinate and momenta. These considerations lead naturally to the question whether, assuming H_o say to have the form

$$\mathrm{A}_1\left(x_1^2 + y_1^2\right) + \mathrm{A}_2\left(x_2^2 + y_2^2\right),$$

it is possible to reduce all the Hamiltonian to a function of the combinations $x_1^2 + y_1^2$ and $x_2^2 + y_2^2$. The answer to this question is affirmative in the sense that, at least formally, the reduction can in general be obtained by a series of homogeneous polynomials in the variable involved, although the convergences of these series, as such, has never been investigated. The equivalence to the problem of Birkhoff's normalization is, nevertheless, evident.

Consider, then, the dominant part H_o of the Hamiltonian to be a function only of $x_1^2 + y_2^2$ and $x_2^2 + y_2^2$. The higher order parts of the Hamiltonian are functions of the variables (x; y) say in the combinations

$$w_1^p w_2^q u_1^m u_2^n$$

where

$$w_1 = x_1 x_2 + y_1 y_2$$

$$w_2 = x_1 y_2 + x_2 y_1$$

$$u_1 = x_1^2 + y_1^2$$

$$u_2 = x_2^2 + y_2^2$$

This is, for instance, the case of Celestial Mechanics when Poincaré's variables are used (e.g. Brouwer and Clemence, 1961). Generally, one can assume the higher order parts of H to be homogeneous polynomials of increasing degree in x_1, y_1, x_2, y_2. The elimination of all terms except the combinations u_1, u_2, from H_1 can be accomplished by means of a generating function

$$S = x_1^{'} y_1 + x_2^{'} y_2 + S_1 + S_2 + \ldots$$

so that one finds

$$\mathrm{H}_1\left(x';y\right) + \sum_{k=1}^{2} \frac{\partial S_1}{\partial y_k} \frac{\partial \mathrm{H}_0}{\partial x_k^{'}} = \mathrm{H}_1^{'}\left(x';y\right) + \sum_{k=1}^{2} \frac{\partial S_1}{\partial x_k^{'}} \frac{\partial \mathrm{H}_0^{'}}{\partial y_k}$$

where primes indicate new variables and new Hamiltonian. Now, since $\mathrm{H}_0 = \mathrm{A}_1 u_1 + \mathrm{A}_2 u_2,$ the function $\mathrm{H}_1^{'}$ is defined by that part of $\mathrm{H}_1,$ if any, containing purely the combinations u_1 and u_2, which we call H_{1s}. The remaining terms, called H_{1p}, will allow for the determinations of S_1. It follows that, since

$$\mathrm{H}_0^{'}\left(x',y\right) = \mathrm{H}_0\left(x',y\right),$$

$$\sum_{k=1}^{2} \left(\frac{\partial \mathrm{H}_0}{\partial x_k^{'}} \frac{\partial S_1}{\partial y_k} - \frac{\partial \mathrm{H}_0}{\partial y_k} \frac{\partial S_1}{\partial x_k^{'}} \right) = -\mathrm{H}_{1p}\left(x';y\right)$$

where, in H_{1p}, any terms in u_1 and/or u_2 is necessarily factored by a term in w_1 or w_2. Now, considering the form of H_0, one has

$$\frac{\partial \mathrm{H}_0}{\partial x_i^{'}} = 2\alpha_i x_i^{'}, \frac{\partial \mathrm{H}_0}{\partial y_i} = 2\alpha_i y_i$$

where

$$\alpha_i = \left(\frac{\partial \mathrm{H}_0}{\partial u_i} \right) x = x'.$$

Therefore, the equation for S_1 becomes

$$2\sum_i \alpha_i \left(x_i^{'} \frac{\partial S_1}{\partial y_i} - y_i \frac{\partial S_1}{\partial x_i^{'}} \right) = -\mathrm{H}_{1p}\left(x';y\right).$$

On the other hand, considering the definition of w_k and $u_k\left(k=1,2\right)$, it follows that

$$x_1^{'} \frac{\partial \mathrm{S}_1}{\partial \mathrm{y}_1} - y_1 \frac{\partial S_1}{\partial x_1^{'}} = w_2^{'} \frac{\partial \mathrm{S}_1}{\partial \mathrm{w}_1^{'}} - w_1^{'} \frac{\partial S_1}{\partial w_2^{'}},$$

$$x_2^{'} \frac{\partial \mathrm{S}_1}{\partial \mathrm{y}_2} - y_2 \frac{\partial S_1}{\partial x_2^{'}} = w_1^{'} \frac{\partial \mathrm{S}_1}{\partial \mathrm{w}_2^{'}} - w_2^{'} \frac{\partial S_1}{\partial w_2^{'}},$$

where

$$w_1' = x_1' x_2' + y_1 y_2,$$

$$w_2' = x_1' y_2 - x_2' y_1.$$

The equation for S_1 becomes

$$2(\alpha_1 - \alpha_2)\left(w_2' \frac{\partial S_1}{\partial w_1'} - w_1' \frac{\partial S_1}{\partial \mathrm{w}_2'} \right) = -\mathrm{H}_{1p}\left(w_p', w_2', u_1', u_2' \right)$$

$$= \phi_{1p}\left(w_1', w_2' \right),$$

where the dependence on u_1', u_2' is omitted and is not relevant to the subsequent discussion (as long as isolated dependence of u_1', u_2' does not occur). The solution of this last equation is obtained by introducing the auxiliary variables of integration

$$z_1 = w'^2_1 - w'^2_2, z_2 = w'^2_1 + w'^2_2.$$

With this substitution, one finds

$$S_1 = \frac{1}{4(\alpha_1 - \alpha_2)} \int z_1 \frac{\phi^*_{1p}(z_1, z_2)}{(z_2^2 - z_1^2)^{1/2}} dz_1 + \phi_1(z_2)$$

where ϕ_1 is an arbitrary function of z_2, u_1', u_2'. The method does not apply when $\alpha_1 = \alpha_2$, that is

$$\frac{\partial \mathrm{H}_0}{\partial u_1} = \frac{\partial \mathrm{H}_0}{\partial u_2}.$$

This is, obviously, a case of internal resonance of the linear approximation, which is exceptional. A similar treatment and overall discussion holds to any order of approximation. The Hamiltonian is, at least formally, reduced to

$$\mathrm{H}' = \mathrm{H}_o' + \mathrm{H}_1' + \ldots = \mathrm{H}'\left(u_1', u_2' \right)$$

so that

$$u_1' = x'^2_1 + y^2_1 = const.,$$

$$u_2' = x'^2_2 + y^2_2 = const. \quad .$$

The relations between primed and unprimed variables are obtained by

$$y_k = y_k' - \frac{\partial S_1}{\partial x_k'} - \frac{\partial S_2}{\partial x_k'} - \ldots \quad ,$$

$$x_k = x_k' + \frac{\partial S_1}{\partial y_k} + \frac{\partial S_2}{\partial y_k} + \ldots \quad ,$$

The foregoing considerations establish a clear connection between Poincaré’s problem and

Birkhoff's normalisation. (Birkhoff, 1927; Siegel, 1956). The problems are actually identical in scope and such identity has been shown in specific applications quite recently, by Deprit (1969, 1971). It is a well known fact that the series introduced by Birkhoff are generally divergent, although exceptional cases exist. New results connected with such problems are rare and the theorems of Kolmogorov and Moser could apply due to the non-linearity of the equations generated by H_0. The next connection of importance is with the concept of Adelphic Integrals introduced by Whittaker (1937). Recently, the definition and series approximation given by Whittaker, have been explored in specific examples by Contopoulos (1963) who, by the way, has shown that such integrals, supposed only formal results, do hold, in practice, for a very long interval of time, specifically as long as a computer could handle the integration with reasonable confidence in the accuracy of the results. The motivation for the question is: can we find, for a conservative system, some other integral which is independent from the energy integral? Evidently there are systems where such is the case and, in fact, by definition, an integrable system with n degrees of freedom has n such integrals, Although a well known result of Poincaré indicates that dynamical systems are non-integrable, such result relies on the existence of uniform (with respect to a certain parameter) integral. In the vicinity of singular points, Siegel (1941) has also shown the non-existence of analytic integrals and Moser (1955) the non-existence of differentiate integrals. Boneless, integrals may exist for specific values of parameters appearing in the equations, for specific values of initial conditions, or other exceptional cases as, for Instance, just continuous integrals. We shall give, at the end of this section, an example of such exceptions.

Let F(y; x; t) be a (differentiable in D) integral of a conservative system defined by the Hamiltonian H(y; x), supposed to be C^2 in a certain domain D of the 2n-dimensional phase space $y=(y_1,y_2,...,y_n), x=(x_1,x_2,...,x_n)$. It is well known that F being independent of H (here, not a function of H alone) the following condition

$$(F,\mathrm{H})+\frac{\partial F}{\partial t}=0 \tag{2.6.2}$$

is necessary and sufficient. In explicit form, the Poisson parenthesis is, here

$$(F,\mathrm{H})=\sum_{k=1}^{n}\left(\frac{\partial F}{\partial y_k}\frac{\partial \mathrm{H}}{\partial x_k}-\frac{\partial F}{\partial x_k}\frac{\partial \mathrm{H}}{\partial y_k}\right).$$

If F is explicitly time independent, the condition is simply $(F,\ \mathrm{H}) = 0$.

Now consider the case in which H depends on a dimensionless parameter $\in, |\in| \in [0,1]$, and such that it is developable in Taylor series in the vincinity of $\in = 0$, for $|\in| < \in_0$, and also, such that

$$\mathrm{H}(y;x;\in)=\mathrm{H}_0(x)+\in\mathrm{H}_1(y;x)+\in^2\mathrm{H}_2(y;x)+\ldots \quad . \tag{2.6.3}$$

Finally, suppose F to be time independent and an analytic function (in the real sense) of $\in$, for $|\in|<\in_0$. Then

$$F(y;x;\in)=F_0(y;x)+\in F_1(y;x)+\in^2 F_2(y;x)+\ldots \tag{2.6.4}$$

and we require $F_k(y;x)$, $k=0,1,2,\ldots,$ to be differentiable in D. If F is an integral for all $\in, |\in|<\in_0$ say, then one must have $(F_0,\mathrm{H}_0)=0,$ = 0, or, more explicitly

$$\sum_{k=1}^{n}\frac{\partial F_0}{\partial y_k}\frac{\partial \mathrm{H}_0}{\partial x_k}=0. \tag{2.6.5}$$

It is .evident that any $F_0(x)$ satisfies (2.6.5) and also, being $F^{*}_{0}(x,y)$ a solution of (2.6.5) then $F^{*}_{0}(x,y)+F^{**}_{0}(x)$ also is, whatever $F^{**}_{0}(x)$ may be. We shall exclude cases of resonance, in this case, situations where the functions $\omega^{0}_{k}=\partial\mathrm{H}_0/\partial x_k$ are dependent, or, in particular, linearly dependent over the set of integers, for $x\in D$. Actually we shall assume the infinitely many conditions

$$\left|\sum_{k=1}^{n} j_k\omega_k^0\right|>\mathrm{K}(\omega^0)\left[\sum_{k=1}^{n}|j_k|\right]^{-n-1} \tag{2.6.6}$$

for all integers not all zero j_k and a convenient constant K. Cases of resonance or near resonance have been discussed, in details, in the problem of Adelphic tegrals, by Contopoulos (1968, 1970). For systems with n > 2 degrees of free-not even a heuristic solution of the problem is available in the literature, although it can be produced with no major difficulties. The above conditions exclude particular solutions (or "near" solutions) of the type

$$F_0=\sum_{k=1}^{n}p_k y_k \quad ,$$

Where p_k are integers such that

$$\sum_{k=1}^{n} j_k p_k=0.$$

Then one has the following lemma. <u>Lemma 1</u>. "<u>The function</u> F_0 <u>is an arbitrary function of</u> $x_1,x_2,\ldots,x_n$ <u>and of any linear form</u> $\alpha_1y_1+\alpha_2y_2+\ldots+\alpha_ny_n$ <u>where</u> α_k <u>are real non-rational numbers such that</u>

$$\alpha_1\omega_1^0+\alpha_2\omega_2^0+\ldots+\alpha_n\omega_n^0=0."$$

We note that since the solution of the system generated by H_0 is

$$x_k = const.$$

$$y_k = \omega_k^0 (x) t + y_k^0 ,$$

any function of $\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n$ reduces to an absolute constant. For this reason, we shall consider the solution $F_0 = F_0(x)$. This is, in fact, obvious, since F_0 has to be an integral of the system generated by H_0 and therefore, a function of the n integrals $x_1, x_2, ..., x_n$ of that system.

Lemma 2. "If $F_0 = F_0(x)$ and $F_1(y,x)$ is $2\pi-$periodic with respect to $y_1, y_2, ..., y_n$, with zero average, then (H_1, F_1) is $2\pi-$periodic in $y_1, y_2, ..., y_n$ and has zero average, provided $H_1(y,x)$ is $2\pi-$periodic in $y_1, y_2, ..., y_n$."

In fact, the condition $(F,H) = 0$ leads to the sequence of conditions

$$\left(F_p, H_0\right) + \left(F_{p-1}, H_1\right) + \left(F_{p-2}, H_2\right) + ... + \left(F_0, H_p\right) = 0 \qquad (2.6.7)$$

For $p = 1,2,3,...$. For $p = 1$, we have

$$\left(F_1 H_0\right) + \left(F_0, H_1\right) = 0$$

or

$$\sum_{j=1}^{n} \omega_j^0 \frac{\partial F_1}{\partial y_j} = \sum_{j=1}^{n} p_j(x) \frac{\partial H_1}{\partial y_j}$$

where

$$P_j(x) = \frac{\partial F_0}{\partial x_j}.$$

The right-hand member of (2.6.8) certainly is $2\pi-$periodic in each y_k and has zero average. The same will be true for $F_1(y;x)$ provided one disregards any arbitrary function of x in the solution, and the ω_j^0 satisfy (2.6.6). Now let $\theta = p_1 y_1 + p_2 y_2 + ... + p_n y_n$ be any argument in the Fourier series of $H_1(y;x)$, with $p_1, p_2, ..., p_n$ integers, not all zero. In view of the linearity (2.6.8) one can reason with that single argument. Thus, eliminating arbitrary functions of x, we have, for that argument

$$F_1 = \frac{\sum_{j=1}^{n} p_j p_j}{\sum_{j=1}^{n} p_j \omega_j^0} \left(-A \sin\theta + B \cos\theta\right) \qquad (2.6.9)$$

Where we have defined $H_1 = A\cos\theta + B\sin\theta + ...$. The factor of the right-hand member of (2.6.9) is a function of x, let say, $C(x)$, and in view of (2.6.6), is not large (obviously we need the constant $K(\omega^0)$ to be $0(1)$ with respect to $\in$).

It follows that

$$(F_2, H_1)_\theta = \left\{ \sum_{j=1}^{n} p_j \frac{\partial C}{\partial x_j} \right\} \left\{ \frac{B^2 - A^2}{2} \sin 2\theta + AB \cos 2\theta \right\}$$

wich proves the lemma since terms independent of θ can only be produced by trigonometric functions of the same argument.

Consider (2.6.7) for $p = 2$. The function F_2 is defined by

$$(F_2, H_0) + (F_1, H_1) + (F_0, H_2) = 0$$

or

$$\sum_{k=1}^{n} \omega_k^0 \frac{\partial F_2}{\partial y_k} = \sum_{k=1}^{n} P_k(x) \frac{\partial H_2}{\partial y_k} - (F_1, H_1)$$

and it follows that, disregarding arbitrary functions of x, F_2 is also a $2\pi-$periodic function of $y_1, y_2, ..., y_n$.

In general, however, it is not true that F_p will be $2\pi-$periodic in the angular variables $y_1, y_2, ..., y_n$. This is verified only under very special conditions. The most important example is when H is a cosine series in the angles $y_1, y_2, ..., y_n$. In this case, it is easily seen that F is also a cosine series. Therefore, any function obtained from a Poisson's Parenthesis is a sine series, and cannot contain any constant term. This is easily seen by writing

$$(F, H) = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial y_k} \frac{\partial H}{\partial x_k} - \frac{\partial F}{\partial x_k} \frac{\partial H}{\partial y_k} \right)$$

and observing that, in each binomial, one has the product of a sine series by a cosine series.

The same is true also when H is a sine series. In problems of Celestial Mechanics, when Newtonian forces are considered, these conditions are satisfied.

The convergence of such method of approximation has been proved by Whittaker (1916) for some special classes of problems with two degrees of freedom, namely, in the vicinity of an equilibrium point of the general elliptic type and as long as the normal frequencies ω_1, ω_2 are irrational one to the other and for deviations sufficiently small from equilibrium. Although Whittaker felt very strongly in favor of the convergence for more general systems, he pointed out the fact that such adelphic integrals could not generally be uniformly convergent for any value of the independent variable and with respect to all values of the constants of integration or the parameters of the problem in any interval. This last consideration follows clearly from the fact that, as the ratio ω_1/ω_2 changes from an irrational to a rational value, the series defining the adelphic integrals take a completely different form. The same situation occurs in the application of averaging methods with respect to the type of motion defined by H_0 (the reference solution). In non-linear oscillations, the normal modes depend on the initial conditions and therefore, it seems natural to conclude that, as far as the initial conditions are concerned, convergence in any domain of the phase space is not possible. This is, in fact, the essential reasoning behind Poincaré's Theorem on the divergence of series in Celestial Mechanics (Poincaré, 1898, Vol. II). There is, of course, one case where convergence is a fact not even in question: the obvious situation where the series terminates. Even if an integral exists, in the form of a polynomial in $\in$, there remains the problem of what should be the zero-th approximation F_0. The difference between obtaining a series (eventually divergent) and a polynomial, may depend on the choice of $F_0(x)$. If a general principle for such choice could be found, we would have a criterion for the existence of integrals which are polynomials of certain physical parameters. For instance, consider the case

$$\mathrm{H}=\mathrm{H}_0(x)+\mathrm{H}_1(y;x)$$

quite common in problems of perturbation. In this case, the equation defining F_p is

$$(F_p,\mathrm{H}_0)+(F_{p-1},\mathrm{H}_1)=0$$

for $p=1,2,3,...$. Evidently, if $F_k(k\geq p)$ are identically zero, it follows that $F=F_0+F_1+...+F_{p-1}$ where

$$(F_0,\mathrm{H}_0)=0$$

$$(F_1,\mathrm{H}_0)+(F_0,\mathrm{H}_1)=0$$

$$(F_2,\mathrm{H}_0)+(F_1,\mathrm{H}_1)=0$$

$$------------$$

$$(F_{p-1},\mathrm{H}_0)+(F_{p-2},\mathrm{H}_1)=0$$

$$(F_{p-1},\mathrm{H}_0)=0.$$

The last condition implies that F_{p-1} is an integral of the system generated by H_1. This is a necessary condition for the integral F to be a polynomial of degree $p-1$ in $\in$. Evidently, for this to happen it is sufficient that F_{p-1} be equal to, or a function of, H_1. This is the case for instance of Kovalevskaya's integral for the motion of a symmetric top under the influence of gravity. For this motion we introduce Andoyer's variables (1926)

$$L = p_{\psi} = G \cos b$$

$$p_{\theta} = G \sin b \sin \left(\ell - \psi\right)$$

$$p_{\theta} = \mathrm{H} = G \cos \mathrm{I}$$

Where ϕ, ψ, θ are the usual Euler angles as defined in Goldstein (1951), G is the magnitude of the angular momentum, I is the inclination of the invariable plane (normal to the angular momentum vector) with respect to the inertial equatorial plane, b is the inclination of the body principal inertial equatorial plane with respect to the invariable plane and ℓ the angle between the body x-axis and the interception of the body (x, y) plane with the invariable plane. Let h be the angle between the inertial X axis and the interception of the invariable and (X, Y) planes and let g the angle between the interceptions of the invariable plane with the planes (X, Y) and (x, y). Then the quantities $(L, G, H; \ell, g, h)$ are canonically conjugate (e.g., Deprit, 1966) and the kinetic energy is

$$\aleph_0 = \frac{1}{2}\left(\frac{1}{A}\sin^2 + \frac{1}{B}\cos^2 \ell\right)\left(G^2 - L^2\right) + \frac{1}{2C}L^2$$

Where A, B, C are the principal moments of inertia (w.r.t. x, y, z respectively). If one assumes A = B,

$$\aleph_0 = \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L^2 + \frac{1}{2A}G^2$$

While the potential, by proper choice of the axes, can be written

$$w\aleph = w\{x_G[\sin \mathrm{I} \sin g \cos \ell + (\sin$$

$$+ \cos b \sin \mathrm{I} \cos g) \sin \ell] + z_G[\cos b \cos \mathrm{I}$$

$$- \sin b \sin \mathrm{I} \cos g]\}$$

Where w is the weight of the top and $x_G, y_G, =0, z_G$ are the coordinates of the center of mass in the body system. We have, of course, the integrals $\aleph_0 + w\aleph_1 = E$ (energy) and $\mathrm{H} = G\cos \mathrm{I} = \mathrm{H}_0$ (since h is ignorable).

Consider an integral $F\left(L,G,\ell,g,h\right)$ of the system, such that, $F=F_0+wF_1+...$

$$F_0=\psi\left(L,G\right)$$

$$F_k=F_k\left(L,G,\ell,g\right)\left(k=1,2,...\right)$$

Where we have assumed h to be cyclic, for obvious reasons, and $\mathrm{H}=\mathrm{H}_0$ is simply a parameter not shown explicitly. We shall write

$$\aleph_0=\frac{\mathrm{a}}{2}L^2+\frac{b}{2}G^2$$

$$\aleph_1=A^\circ\sin\left(\ell+g\right)+B^\circ\sin\left(\ell-g\right)+C^\circ\sin\ell+D^\circ\cos g+E^\circ$$

Where

$$\mathrm{A}^o=x_G\left(L+G\right)\left(G^2-\mathrm{H}^2\right)^{1/2}/2G^2\quad,$$

$$\mathrm{B}^o=x_G\left(L-G\right)\left(G^2-\mathrm{H}^2\right)^{1/2}/2G^2\quad,$$

$$C^o=x_G\mathrm{H}\left(G^2-L^2\right)^{1/2}/G^2\qquad,$$

$$D^o=-z_G\left(G^2-L^2\right)^{1/2}\left(G^2-\mathrm{H}^2\right)^{1/2}/G^2\quad,$$

$$\mathrm{E}^o=z_GL\ \mathrm{H}\ G^{-2}$$

The conditions for F to be an integral are

$$\left(\aleph_0,F_k\right)+\left(\aleph_1,F_{k-1}\right)=0\qquad(2.6.10)$$

For $k=1,2,...$. We shall leave $\psi\left(L,G\right)=F_0$ undefined and try to determine under what conditions in ψ and the physical parameters, the series for F terminates. We obtain from (2.6.10)

$$\mathrm{a}L\frac{\partial F_k}{\partial\ell}+bG\frac{\partial F_k}{\partial g}=\left\{A^0\cos\left(\ell+g\right)+B^0\cos\left(\ell-g\right)+C^0\cos\ell\right\}\frac{\partial F_{k-1}}{\partial L}$$

$$+\left\{A^0\cos\left(\ell+g\right)-B^0\cos\left(\ell-g\right)-D^0\sin g\right\}\frac{\partial F_{k-1}}{\partial G}$$

$$-\left\{A_L^0\sin\left(\ell+g\right)+B_L^0\sin\left(\ell-g\right)+C_L^0\sin\ell+D_L^0\cos g+E_L\right\}\frac{\partial F_{k-1}}{\partial\ell}$$

$$-\left\{A_G^0\sin\left(\ell+g\right)+B_G^0\sin\left(\ell-g\right)+C_G^0\sin\ell+D_G^0\cos g+E_G\right\}\frac{\partial F_{k-1}}{\partial g}$$

(2.6.11)

For $k=1$, we find

$$F_1 = \frac{A^0(\psi_L + \psi_G)}{a_L + b_G}\sin(\ell + g) + \frac{B^0(\psi_L - \psi_G)}{a_L - b_G}\sin(\ell - g)$$

$$+\frac{C^0\psi_L}{a_L}\sin\ell + \frac{D^0\psi_G}{b_G}\cos g + E' = A'\sin(\ell + g) \qquad (2.6.12)$$

$$+B'\sin(\ell - g) + C'\sin\ell + D'\cos g + E'$$

Where E' is an arbitrary function of L, G. The function F_1 has the same form as H_1. In fact, this is necessary since, taking $\psi = H_0$, it must result $F_1 = H_1 +$ arbitrary function of L, G. It is also clear that there exists no $\psi \not\equiv 0$ such that $F_1 = 0$. For $k=2$, Eq. (2.6.11) gives (Giacaglia, 1967):

$$F_2 = A_{0,1}\cos g + A_{0,2}\cos 2g + A_{1,-1}\cos(\ell - g)$$

$$+A_{1,0}\cos\ell + A_{1,1}\cos(\ell + g) + A_{1,2}\cos(\ell + 2g)$$

$$+A_{2,-2}\cos(2\ell - 2g) + A_{2,-1}\cos(2\ell - 2g) + A_{2,0}\cos 2\ell$$

$$+A_{2,1}\cos(2\ell + 2g) + A_{2,2}\cos(2\ell + 2g) + B_{1,-2}\sin(\ell - 2g)$$

$$+B_{1,-1}\sin(\ell - g) + B_{1,0}\sin\ell + B_{1,1}\sin(\ell + g)$$

$$+B_{1,2}\sin(\ell + 2g) + E''$$

(2.6.13)

where E'' is an arbitrary function of L, G and $A_{k,j}, B_{k,j}$ are given functions of $\psi_L, \psi_G, A^o, B^o, C^o, E', L, G, a, b$ and their derivatives. It one imposes the condition $F_2 = 0$, all coefficients must be identically zero and we find

$$E'' = E'_L = E'_G = 0$$

and, being k a nonzero constant,

$$A' = kA^o$$

$$B' = kB^o$$

$$C' = kC^o$$

$$D' = kD^o$$

so that $F_0 = k\aleph_0$ and $F_1 = k\aleph_1$. This shows that every differentiable integral (valid for all values of w) and of the form $F_0 + wF_1$ is necessarily proportional to $H = H_0 + wH_1$. From (2.6.11), for $k = 3$, we find

$$F_3 = \sum_{k=-3}^{3} \sum_{j=-3}^{3} \left[A'_{k,j} \cos(k\ell + jg) + B'_{k,j} \sin(k\ell + jg) \right] \quad (2.6.14)$$

with k, j not simultaneously zero and $A'_{k,j}, B'_{k,j}$ functions of $\psi_L, \psi_G, A^o, B^o, C^o, D^0, E', E'', L, G, a, b$ and their derivatives. Setting equal to zero all coefficients of this trigonometric polynomial, we find

$$I) a = b(A = 2C)$$

$$II) D^o = E^o = E' = 0(z_G = 0)$$

$$III) \psi = \left(G^2 - L^2\right)^2 / A^4$$

$$IV) E'' = 2x_G^2 \left[G^2\left(L^2 + H^2\right) - 2L^2H^2 \right] / A^2G^4.$$

(2.6.15)

With these conditions, it follows from (2.6.12) that

$$F_1 = \frac{2xG}{A^3}\left(1 - \frac{1}{G^2}\right)\left(G^2 - H^2\right)^{1/2}\left[(G-L)\sin(\ell+g)\right.$$
$$\left. - (G+L)\sin(\ell - g)\right] + \frac{4xGH^2}{A^3G^2}\left(G^2 - L^2\right)^{3/2}\sin\ell$$

or

$$F_1 = \frac{4xG}{A^3}G^2 \sin^2 b\left[\sin I\,(\sin g\ \cos\ell - \cos b\ \cos g\ \sin\ell)\right]$$
$$- \cos I\ \sin b\ \sin\ \ell\,] \quad (2.6.16)$$

From (2.6.13) we find

$$F_2 = \frac{4x^2G}{A^2}\left\{\frac{1}{2}\left(\frac{L^2+\mathrm{H}^2}{G^4}-2\frac{L^2\mathrm{H}^2}{G^4}\right)-\frac{1}{2}\left(1-\frac{\mathrm{H}^2+L^2}{G^2}+\frac{L^2\mathrm{H}^2}{G^4}\right)\cos 2g\right.$$

$$\left.+2\frac{L\mathrm{H}}{G^2}\left(1-\frac{L^2}{G^2}\right)^{1/2}\left(1-\frac{\mathrm{H}^2}{G^2}\right)^{1/2}\cos g\right\}$$

or

$$F_2 = \frac{4x^2G}{A}\left(1-\cos^2 b\ \cos^2 I - \sin^2 b\ \sin^2 I\ \cos^2 g\right.$$

$$\left.+\ 2\sin\ b\ \cos\ b\ \sin\ I\cos\ I\cos\ g\right)$$

It is easily seen that $F_3, F_4, \ldots$ are all zero, so that we have established the integral

$$F = F_0 + wF_1 + w^2F_2$$

Which is Kowalevskaya's integral (e.g., Leimanis, 1958). In fact, writing F in terms of p, q, r (components x, y, z of the rotation vector) and of Euler's angles, we find

I) $F_0 = A^{-4}G^4\left(1-\frac{L^2}{G^2}\right) = A^{-4}G^4\sin^4 b = \left(p^2+q^2\right)^2$

II) $F_1 = -4xGA^{-1}\left[\left(p^2-q^2\right)\sin\psi\sin\theta + 2pq\cos\psi\sin\theta\right]$

III) $F_2 = 4x^2GA^{-2}\left(1-\cos^2\theta\right) = 4x^2GA^{-2}\sin^2\theta$

Using Leimanis's notation,

$$\mu = wx_G\mathrm{A}^{-1}$$

$$\xi = \sin\psi\sin\theta$$

$$\eta = \cos\psi\sin\theta$$

$$\zeta = \cos\theta$$

it follows

$$F = \left(p^2 - q^2 - 2\mu\xi\right)^2 + \left(2pq - 2\mu\eta\right)^2.$$

We have thus found an integral, valid for any value of w (which here takes the place of $\in$) but under the restriction $\mathrm{A}=\mathrm{B}=2C$. Of course, for more general situations Arnol'd (1963) has shown that the system is integrable for a sufficiently small value of w, i.e., has shown stability of the fast top.

7. The Solution of Poincaré's Problem in Poisson's Parentheses. Elimination of Secular Terms from Adelphic Integrals.

In this section we shall indicate how to solve Poincaré's problem using Poinsson's Parentheses and, at the same time, how to eliminate secular terms in the construction of Adelphic Integrals. We shall deal specifically with a case of degeneracy in which the dominant part of the Hamiltonian depends on a single action variable and the perturbation is $2\pi -$periodic in the angle variables. As we have seen, this situation introduces series difficulties in Poincaré's method, difficulties which led von Zeipel to the already described generalization. Also, as we have seen, Poincaré's method constructs n formal integral whose zero-th order approximations are the action variables, constants of the unperturbed case. The other n formal integrals are essentially the constants of integration for the angle variables when all of these have ultimately been eliminated from the Hamiltonian. The process we are going to discuss is essentially that introduced by Whittaker, although the elimination of secular terms in the procedure was introduced by Giacaglia (1965). With the usual notation, the recurrence relations are

$$\left(\mathrm{H}_0, F_k\right) = -\sum_{j=0}^{k-1}\left(\mathrm{H}_{k-j}, F_j\right) = -\psi_k\left(y; x\right) \qquad (2.7.1)$$

where ψ_k is know when all the $k-1$ preceending approximates are know. For $k=0, \psi_k=0$. Also, assuming $\mathrm{H}_0=\mathrm{H}_0\left(x\right)$,

$$\sum_{j=1}^{n}\omega_j^0\frac{\partial F_k}{\partial y_j} = \psi_k\left(y; x\right) \qquad (2.7.2)$$

With the condition that every $F_k\left(y; x\right)$ should have no secular term, in the sense that the substitution $y_j = \omega_j^0\tau + y_j$ should give

$$\lim_{\tau\to\infty} F_k\left(y\left(\tau\right); x\right) = bounded.$$

Nevertheless, since ψ_k is obtained by multiplication of trigonometric series it will contain terms that are functions only the x-type variable, so that, upon integration of (2.7.2), condition (2.7.3) will not be verified in general. The unwanted "secular behavior' can be eliminated by introducing an averaging procedure to be briefly described hereafter. We shall consider the highly degenerate case where H_0 depends on one of the momenta only, say $\mathrm{H}_0=\mathrm{H}_0\left(x_1\right)$. Also, following Poincaré's results, we try to obtain integrals which, for $\mathrm{H}=\mathrm{H}_0$, reduce to the momenta, that is,

$$F_j = x_j + \in\Delta F_j\left(y; x\right) \qquad (2.7.4)$$

for $j = 1, 2, 3$. The question remains if this choice will lead to the integrals

$$x'_j = x_j + \in W_j\left(y;x\right) \qquad (2.7.5)$$

given by Poincaré's method. Since, by hypothesis, F_j and x'_j are integrals, the function

$$F_j - x'_j = \in\left(\Delta F_j - W_j\right)$$

is also an integral. Now ΔF_j and W_j are not integrals (because x_j is not), so that $\Delta F_j \equiv W_j$. It follows that, if the process converges for $\in$ in some interval, the two methods lead to the same result, although the use of Poisson parentheses gives explicit forms and add extra features to the solution. We let therefore

$$F = x_1 + F_1\left(y;x\right) + F_2\left(y;x\right) + ... \qquad (2.7.6)$$

where H satisfies the foregoing conditions. T he first order equation (k=1) from (2.7.1) gives

$$\omega_1^0 \frac{\partial F_1}{\partial y_1} = -\frac{\partial \mathrm{H}_1}{\partial y_1}$$

so that

$$F_1 = -\frac{1}{\omega_1^0}\mathrm{H}_{1p}\left(y;x\right) + F_{1s}\left(y_2, y_3, ..., y_n; x\right) \qquad (2.7.8)$$

Where H_{1p} is defined by the operation of subtracting from H_1 the average with respect to y_1. In general

$$\mathrm{f}_p\left(y;x\right) = \mathrm{f}\left(y;x\right) - \lim_{T\to\infty}\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{f}\left(t, y_2, ..., y_n; x\right)dt$$

or, for the multiperiodic case under consideration,

$$\mathrm{f}_p\left(y;x\right) = \mathrm{f}\left(y;x\right) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{f}\left(y_1, y_2, ..., y_n; x\right)dy_{1.}$$

The index s indicates absence of y_1 and F_{1s} is, evidently, arbitrary. The second order approximation gives

$$\omega_1^0 \frac{\partial F_2}{\partial y_1} = \sum_i\left(\frac{\partial \mathrm{H}_1}{\partial x_i}\frac{\partial F_1}{\partial y_i} - \frac{\partial \mathrm{H}_1}{\partial y_i}\frac{\partial F_1}{\partial x_i}\right) - \frac{\partial \mathrm{H}_2}{\partial y_1} = \psi_{2p} + \psi_{2s}$$

where and ψ_{2p} are known ψ_{2s} and given by

$$\psi_{2p} = \sum_i\left[\left(\frac{\partial \mathrm{H}_1}{\partial x_i}\frac{\partial F_1}{\partial y_i}\right) - \left(\frac{\partial \mathrm{H}_1}{\partial y_i}\frac{\partial F_1}{\partial x_i}\right)_p\right] - \left(\frac{\partial \mathrm{H}_2}{\partial y_1}\right)$$

$$\psi_{2s} = \sum\left[\left(\frac{\partial \mathrm{H}_1}{\partial x_i}\frac{\partial F_1}{\partial y_i}\right)_s - \left(\frac{\partial \mathrm{H}_1}{\partial y_i}\frac{\partial F_1}{\partial x_i}\right)_s\right].$$

If F_2 has to be free from secular terms, ψ_{2s} must vanish, giving the condition

$$\psi_{2s} = \left(\mathrm{H}_{1s}, F_{1s}\right) - \frac{1}{2\omega_1^{o2}} \frac{\partial \omega_1^o}{\partial x_1} \left[\frac{\partial}{\partial y_1} \mathrm{H}_{1p}^2 \right]_s = 0$$

and since the last term on the right is zero, F_{1s} is defined by

$$\left(\mathrm{H}_{1s}, F_{1s}\right) = 0 \tag{2.7.9}$$

for which we need only a particular solution, the simplest of which, in this case is $F_{1s} = 0$. Consider for simplicity the case

$$\mathrm{H} = \mathrm{H}_0 + \mathrm{H}_1, \tag{2.7.10}$$

so that the second order approximation is given by

$$\omega_1^0 \frac{\partial F_2}{\partial y_1} = \psi_{2p} = \left(\mathrm{H}_{1p}, F_{1s}\right) - \frac{1}{\omega_1^0}\left(\mathrm{H}_{1s}, F_{1p}\right)$$

$$- \frac{1}{2\omega_1^{02}} \frac{\partial \omega_1^0}{\partial x_1} \left[\frac{\partial}{\partial y_1} \mathrm{H}_{1p}^2 \right]_p$$

and since $\left[\frac{\partial}{\partial y_1} \mathrm{H}_{1p}^2 \right]_p = \frac{\partial}{\partial y_1} \mathrm{H}_{1p}^2$ it follows that

$$F_{2p} = -\frac{1}{\omega_1^{o2}} \int \left(\mathrm{H}_1, F_{1p}\right) dy_1 - \frac{1}{2\omega_1^{o3}} \frac{\partial \omega_1^o}{\partial x_1} \left(\mathrm{H}_{1p}^2\right)_p \tag{2.7.11}$$

and

$$F_2 = F_{2p} + F_{2s}$$

where $F_{2s}\left(y_2, y_3, ..., y_n; x\right)$ is arbitrary. Under hypothesis (2.7.10) the third approximation gives

$$\omega_1^o \frac{\partial F_3}{\partial y_1} = \left(\mathrm{H}_1, F_2\right) = \psi_{3p} + \psi_{3s}$$

where

$$\psi_{3s} = \left(\mathrm{H}_{1s}, F_{2s}\right) + \left(\mathrm{H}_{1s}, F_{2p}\right)_s$$

and imposing the condition $\psi_{3s} = 0$, defines the arbitrary function F_{2s} by

$$\left(\mathrm{H}_{1s}, F_{2s}\right) = -\frac{1}{2\pi} \int_0^{2\pi} \left(\mathrm{H}_{1s}, F_{2p}\right) dy_1 = \phi_{3s}\left(y_2, ..., y_n; x\right)$$

where ϕ_{3s} is know. The homogeneous characteristics of $F_{1s}, F_{2s}, ..., F_{ks}$ are the same for any k, and given by where τ is an auxiliary parameter. The solution for $F_{ks}\left(k=1,2,...\right)$ will thus depend on the solution of the system

$$\frac{\partial y_j}{d\tau}=\frac{\partial \mathrm{H}_{1s}}{\partial x_j}$$

$$\left(j=2,3,...,n\right)$$

$$\frac{dx_j}{d\tau}=-\frac{\partial \mathrm{H}_{1s}}{\partial y_j}$$

This corresponds to a dynamical system with $n-1$ degrees of freedom and whose Hamiltonian is H_{1s}. Nevertheless it should be noted that one needs only a particular solution (in the Jacobi sense) of such system. Of course, if for some values of (x; y), one or more of the partials $\partial \mathrm{H}_{1s}/\partial x_k$ is zero or small (say as small as $\in$), the solution will contain singularities or small divisors, and the method cannot proceed. One of the ways to handle this situation is suggested by the considerations pertinent to resonance and to be described in chapter V. Here we limit our discussion to the particular case where the derivative $\partial \mathrm{H}_{1s}/\partial x_2 \simeq 0\left(\in^{1/2}\right)$ and we plainly assume the expansion

$$F=F_o+\in^{1/2}F_1+\in F_2+\in^{3/2}F_3+... \quad .$$

From the fundamental relation $\left(F,\mathrm{H}\right)=0$ it follows that, by equating terms f the same order in $\in$,

$$\left(\mathrm{H}_0,F_0\right)=0 \quad ,$$

$$\left(\mathrm{H}_0,F_1\right)=0 \quad ,$$

$$\left(\mathrm{H}_0,F_2\right)+\left(\frac{\partial \mathrm{H}_1}{\partial x_1}\frac{\partial F_0}{\partial y_1}+\frac{\partial \mathrm{H}_{1p}}{\partial x_2}\frac{\partial F_0}{\partial y_2}+...+\frac{\partial \mathrm{H}_1}{\partial x_n}\frac{\partial F_0}{\partial y_n}\right.$$

$$\left.-\frac{\partial \mathrm{H}_1}{\partial y_1}\frac{\partial F_0}{\partial x_1}-\frac{\partial \mathrm{H}_1}{\partial y_2}\frac{\partial F_0}{\partial x_2}-...-\frac{\partial \mathrm{H}_1}{\partial y_n}\frac{\partial F_0}{\partial x_n}\right)=0$$

$$\left(\mathrm{H}_0,F_3\right)+\frac{\partial \mathrm{H}_{1s}}{\partial x_2}\frac{\partial F_0}{\partial y_2}+\left(\frac{\partial \mathrm{H}_1}{\partial x_1}\frac{\partial F_1}{\partial y_1}\frac{\partial \mathrm{H}_{1p}}{\partial x_2}\frac{\partial F_1}{\partial y_2}+...+\frac{\partial \mathrm{H}_1}{\partial x_n}\frac{\partial F_1}{\partial y_n}\right.$$

$$\left.-\frac{\partial \mathrm{H}_1}{\partial y_1}\frac{\partial F_1}{\partial x_1}-\frac{\partial \mathrm{H}_1}{\partial x_2}\frac{\partial F_1}{\partial x_2}-...-\frac{\partial \mathrm{H}_1}{\partial y_n}\frac{\partial F_1}{\partial x_n}\right)=0,$$

and so forth. If, again, $\mathrm{H}_0=\mathrm{H}_0\left(x_1\right), F_0=x_1$, it follows that

$$\omega_1^0\,\frac{\partial F_1}{\partial y_1}=0\therefore F_1=F_{1s}\left(y_2,y_3,...,y_n;x\right)$$

and

$$\omega_1^0 \frac{\partial F_2}{\partial y_1} = -\frac{\partial \mathrm{H}_1}{\partial y_1} = \psi_2$$

$$\omega_1^0 \frac{\partial F_3}{\partial y_1} = \left(\mathrm{H}_1, F_1\right) - \frac{\partial \mathrm{H}_{1s}}{\partial x_2}\frac{\partial F_1}{\partial y_2} = \psi_3$$

−−−−−−−−−−−−−−−−−−−

$$\omega_1^0 \frac{\partial F_k}{\partial y_1} = \left(\mathrm{H}_1, F_{k-2}\right) + \frac{\partial \mathrm{H}_{1s}}{\partial x_2}\frac{\partial}{\partial y_2}\left(F_{k-3} - F_{k-2}\right) = \psi_k$$

for $k = 4, 5, 6, \ldots$. Now F_{1s} is arbitrary and can be taken equal to zero, so that, automatically, one gets $\psi_3 = 0$ and therefore

$$\omega_1^0 \frac{\partial F_3}{\partial y_1} = 0$$

or

$$F_3 = F_{3s}\left(y_2, y_3, \ldots, y_n; x_1, x_2, \ldots, x_n\right).$$

On the other hand

$$F_2 = -\frac{1}{\omega_1^0}\mathrm{H}_{1p} + F_{2s}$$

so that

$$\psi_{4s} = \left(\mathrm{H}_{1s}, F_{2s}\right) + \left(\mathrm{H}_{1p}, F_{2p}\right)_s - \frac{\partial \mathrm{H}_{1s}}{\partial x_2}\frac{\partial F_{2s}}{\partial y_2}$$

which should be zero. Since

$$\left(\mathrm{H}_{1p}, -\frac{1}{\omega_1^0}\mathrm{H}_{1p}\right)_s = -\frac{1}{\omega_1^{02}}\frac{\partial \omega_1^0}{\partial x_1}\left(\mathrm{H}_{1p}\frac{\partial \mathrm{H}_{1p}}{\partial y_1}\right)_s = 0$$

It follows that

$$\psi_{4s} = \left(\mathrm{H}_{1s}, F_{2s}\right) - \frac{\partial \mathrm{H}_{1s}}{\partial x_2}\frac{\partial F_{2s}}{\partial y_2} = 0$$

and $F_{2s} = 0$satisfies, in particular, the requirements. In any event, the characteristics (up to any order) are

$$\frac{dy_3}{\frac{\partial \mathrm{H}_{1s}}{\partial x_3}} = \ldots = \frac{dy_n}{\frac{\partial \mathrm{H}_{1s}}{\partial x_n}} = \frac{dy_3}{-\frac{\partial \mathrm{H}_{1s}}{\partial y_3}} = \ldots = \frac{dx_n}{-\frac{\partial \mathrm{H}_{1s}}{\partial y_n}} = d\tau$$

with the required disappearance of the small divisor $\partial \mathrm{H}_{1s} / \partial x_2$. If $F_{2s} = 0, F_2$ is completely defined and F_4 is given by

$$F_4 = -\frac{1}{\omega_1^{02}} \int \left[\left(\mathrm{H}_{1s}, \mathrm{H}_{1p} \right) - \frac{\partial \mathrm{H}_{1s}}{\partial x_2} \frac{\partial \mathrm{H}_{1p}}{\partial y_2} \right] dy_1$$

$$-\frac{1}{2\omega_1^{03}} \frac{\partial \omega_1^0}{\partial x_1} \left(\mathrm{H}_{1p}^2 \right)_p$$

and so forth. At every stage of the approximation, the characteristics are the same and do not present any singularity. It is also clear that the method can be applied equivalently to cases in which more than one derivative $\partial \mathrm{H}_{1s} / \partial x_k$ is small.

Suppose now $F_0 = x_2$ so that F will correspond to $x_2^{'}$ of the Poincaré problem. In this case

$$\omega_1^0 \frac{\partial F_1}{\partial y_1} = -\frac{\partial \mathrm{H}_1}{\partial y_2},$$

so that,

$$F_1 = -\frac{1}{\omega_1^0} \int \frac{\partial \mathrm{H}_1}{\partial y_2} dy_1 + \psi_1 \left(y_2, y_3, \ldots, y_n; x \right).$$

However, the integrand $\partial \mathrm{H}_1 / \partial y_2$ may contain terms which are independent from y_1 and, therefore, F_1 will have a secular increase in y_1. Such secular parts will be

$$F_{1s} = -\frac{1}{\omega_1^0} \int \frac{\partial \mathrm{H}_{1s}}{\partial y_2} dy_1 + \psi_1 \left(y_2, y_3, \ldots, y_n; x \right)$$

which cannot be zero unless H_{1s} does not depend on y_2. Therefore, one is forced to deviate from the assumption $F_0 = x_2$ and assume a more general form

$$F_0 = F_0 \left(y_2, y_3, \ldots y_n; x \right).$$

If it is possible to choose F_0 so hat secular terms are not present in the higher approximations, one can at least obtain a formal integral, eventually convergent. The equation for F_1 is obtained from

$$\left(\mathrm{H}_0, F_1 \right) + \left(\mathrm{H}_1, F_0 \right) = 0$$

or

$$\omega_1^0 \frac{\partial F_1}{\partial y_1} = \left(\mathrm{H}_1, F_0 \right).$$

The “secular” part of the right hand member should be zero, that is,

$$\left(\mathrm{H}_{1s}, F_0\right)=0 \qquad (2.7.12)$$

since F_0 does not contain y_1 by hypothesis. This hypotheses is easily justified by the condition $\left(\mathrm{H}_0, F_0\right)=0$, with $\mathrm{H}_0=\mathrm{H}_0\left(x_1\right)$. An immediate solution of (2.7.12) is to assume

$$F_0=k\mathrm{H}_{1s}$$

where k is a constant. From the point of view of Hamilton-Jacobi theory, it is clear that this choice is suggested by the equivalent situation in Poincaré's method (Giacaglia, 1965, p.16).

The interesting physical feature of this process is that the "secular" part of $\overline{\mathrm{H}_1}$ becomes the zero order approximation of an integral of motion. The interpretation of this fact lies in the conservation of the energy of the system under canonical transformations. Also, there is a close connection, at this point, with perturbation methods based on Lie Series Transforms to be discussed later in this chapter.

Now consider the original system

$$\dot{y}_k=\frac{\partial \mathrm{H}}{\partial x_k}, \dot{x}_k=-\frac{\partial \mathrm{H}}{\partial y_k}$$

where $\mathrm{H}=\mathrm{H}(y;x), k=1,2,\ldots,n.$ Let $t=y_{n+1}$, so that

$$\dot{y}_\alpha=\frac{\partial \aleph}{\partial x_\alpha}, \dot{x}_\alpha=-\frac{\partial \aleph}{\partial y_\alpha} \qquad (2.7.13)$$

where $\aleph=\mathrm{H}+x_{n+1}, x_{n+1}=const=\beta, \alpha=1,2,\ldots,n+1.$ The angle variables of the system are, according to Poincaré's method

$$y_\alpha^{'}={}^{\omega}\alpha^t+y_{\alpha o}^{'}$$

where $y_{\alpha o}^{'}$ are absolute constants and

$$\omega_\alpha=-\frac{\partial x_{n+1}}{\partial x_\alpha'}=\omega_\alpha^0+\in\omega_\alpha^1+\in^2\omega_\alpha^2+\ldots.$$

and the ω_α^k are functions of $x_1^{'}, x_2^{'},\ldots,x_n^{'}$. In particular

$$\omega_\alpha^0=\frac{\partial \mathrm{H}_0}{\partial x_\alpha^{'}}.$$

so that

$$y_\alpha^{'}=\omega_\alpha^0 t+\in\nu_\alpha\left(x';\in\right)t+\beta_\alpha.$$

On the other hand

$$y'_\alpha = \frac{\partial W}{\partial x'_\alpha} = y'_\alpha(x';y;\in) = y_\alpha + \in \mu_\alpha(x';y;\in).$$

Comparison of the last two relations gives

$$\beta_\alpha = y_\alpha - \omega_\alpha^0 t + \in(\mu_\alpha - \nu_\alpha t).$$

On the other hand, the β_α are constants of the system (2.7.13), and can be written as

$$\beta_k = y_k - \omega_k^0 y_{n+1} + \in \theta_k(x';y;t;\in) \qquad (2.7.14)$$

and the zero order part of such integrals can be taken as

$$F_{k0} = y_k - \omega_k^0 y_{n+1} \qquad (k = 1,2,...,n). \qquad (2.7.15)$$

Poisson's condition is now written in the form

$$\sum_{\alpha=1}^{n+1}\left(\frac{\partial \aleph}{\partial x_\alpha}\frac{\partial F}{\partial y_\alpha} - \frac{\partial \aleph}{\partial y_\alpha}\frac{\partial F}{\partial x_\alpha}\right) = 0.$$

If $\mathrm{H}_0 = \mathrm{H}_0(x)$, the zero order approximation would be given by

$$\sum_{k=1}^{n}\frac{\partial \mathrm{H}_0}{\partial x_k}\frac{\partial F_0}{\partial y_k} = \frac{\partial F_0}{\partial y_{n+1}}$$

and a particular solution is

$$F_0 = y_1 - \frac{\partial \mathrm{H}_0}{\partial x_1} y_{n+1} = y_1 - \omega_1^0 y_{n+1}$$

which is of the form (2.7.15) for $k = 1$.

The question arises whether the formal series obtained in this form have some meaning, since, in the present case, linear terms in time cannot be eliminated. But the same question is present in Poincaré's method, where the frequencies $\omega_k = \omega_k^0 + \in \omega_k^0 + ...$ are indeed obtained, in practical cases, only up to a certain degree of approximation p. This fact, as mentioned before, is reflected in the conclusion that, even if the series converge, in practical cases the solution cannot be valid for an interval of time which, at best if $0(\in^{-p})$.

Writing the condition as

$$(F, \mathrm{H}) + \frac{\partial F}{\partial t} = 0$$

the "integrals" F corresponding to (2.7.14) are formally obtained as follows. We suppose

$$F_0 = y_1 - \omega_1^0 t$$

and

$$\mathrm{H} = \mathrm{H}_0(x_1) + \mathrm{H}_1(y;x).$$

The recurrence relations for F_k are

$$\left(F_k, \mathrm{H}_0\right) + \frac{\partial F_k}{\partial t} = -\left(F_{k-1}, \mathrm{H}_1\right)$$

or

$$\omega_1^0 \frac{\partial F_k}{\partial y_1} + \frac{\partial F_k}{\partial t} = \left(\mathrm{H}_1, F_{k-1}\right).$$

For $k = 1$,

$$\omega_1^0 \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial t} = \frac{\partial H_1}{\partial x_1} + \frac{\partial \omega_1^0}{\partial x_1} t \frac{\partial H_1}{\partial y_1}$$

for $k = 2$,

$$\omega_1^0 \frac{\partial F_2}{\partial y_1} + \frac{\partial F_1}{\partial t} = \frac{\partial \mathrm{H}_1}{\partial x_1} + \frac{\partial \omega_1^0}{\partial x_1} t \frac{\partial \mathrm{H}_1}{\partial y_1},$$

and so forth. The solution for F_1 is found to be

$$F_1 = \frac{1}{\omega_1^0} \int \frac{\partial \mathrm{H}_1}{\partial x_1} dy_1 + \frac{1}{\omega_1^{02}} \frac{\partial \omega_1^0}{\partial x_1} \int \frac{\partial \mathrm{H}_1}{\partial y_1} y_1 dy_1$$

$$- \frac{1}{\omega_1^{02}} \frac{\partial \omega_1^0}{\partial x_1} y_1 \int \frac{\partial \mathrm{H}_1}{\partial y_1} dy_1 + t \frac{1}{\omega_1^0} \frac{\partial \omega_1^0}{\partial x_1} \int \frac{\partial \omega_1^0}{\partial x_1} dy_1$$

$$+ \psi_1\left(y_2, y_3, \dots, y_n; x\right)$$

with ψ_1 arbitrary. On the other hand, one has

$$\frac{\partial \mathrm{H}_1}{\partial y_1} y_1 = \frac{\partial}{\partial y_1}\left(\mathrm{H}_1 y_1\right) - \mathrm{H}_1,$$

so that, if H_1 is $2\pi -$periodic in every variable $y_1, y_3, \dots, y_n$, we obtain

$$\int \frac{\partial}{\partial y_1}\left(\mathrm{H}_1 y_1\right) dy_1 - \int \mathrm{H}_1 dy_1 = -\int \mathrm{H}_{1p} dy_1.$$

Hence

$$F_1 = \frac{1}{\omega_1^0} \int \frac{\partial \mathrm{H}_1}{\partial x_1} dy_1 - \frac{1}{\omega_1^{02}} \frac{1}{\partial x_1} \int \mathrm{H}_{1p} dy_1 + t \frac{1}{\omega_1^0} \frac{\partial \omega_1^0}{\partial x_1} \mathrm{H}_{1p} + \psi_1.$$

The only undesirable term is the first in the right hand member, from which secular terms in y_1 may arise. They are precisely

$$\frac{1}{\omega_1^0} \int \frac{\partial \mathrm{H}_{1s}}{\partial x_1} dy_1 = \frac{1}{\omega_1^0} \frac{\partial \mathrm{H}_{1s}}{\partial x_1} y_1.$$

The function

$$F_1 - \frac{1}{\omega_1^0}\frac{\partial \mathrm{H}_{1s}}{\partial x_1} y_1 = \frac{1}{\omega_1^0}\left(1 + \frac{\partial \omega_1^0}{\partial x_1} t\right)\mathrm{H}_{1p}$$

$$-\frac{1}{\omega_1^{02}}\frac{\partial \omega_1^0}{\partial x_1}\int \mathrm{H}_{1p} dy_1 + \psi_1$$

is multiperiodic in $y_1, y_2, ..., y_n$ and secular in t, this second characteristic being unavoidable and indeed necessary. The situation suggests therefore a modification of the function F_0 as follows. We consider

$$F_0 = y_1 - \omega_1^0 t + \psi_0 \left(y_2, y_3, ..., y_n; x\right)$$

which, evidently, is a solution for

$$\left(F_0, \mathrm{H}_0\right) + \frac{\partial F_0}{\partial t} = 0.$$

Then, the equation for F_1 becomes

$$\omega_1^0 - \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial t} = \frac{\partial \mathrm{H}_1}{\partial x_1} + t\frac{\partial \omega_1^0}{\partial x_1}\frac{\partial \mathrm{H}_1}{\partial y_1} + \left(\mathrm{H}_1, \psi_0\right)$$

Whose solution is the same as before, with the addition of the term

$$\frac{1}{\omega_1^0}\int \left(\mathrm{H}_1, \psi_0\right) dy_1.$$

The part of this integral which contains secular terms in y_1 will be zero if, and only if,

$$\frac{\partial \mathrm{H}_{1s}}{\partial x_1} + \left(\mathrm{H}_{1s}, \psi_0\right) = 0.$$

The last equation defines the way in which the arbitrary function ψ_0 should be chosen. The solution of this partial differential equation is equivalent to the integration of the characteristics

$$\frac{dy_k}{d\tau} = \frac{\partial \mathrm{H}_{1s}}{\partial x_x}, \frac{dx_k}{d\tau} = -\frac{\partial \mathrm{H}_{1s}}{\partial y_k} \qquad (2.7.16)$$

where τ is any parameter and $k = 2, 3, ..., n,$ whereas x_1 has to be treated as a constant parameter. If y_k, x_k are obtained from these as functions of τ, then H_{1s} is expressed as a function of τ, and ψ_0 is obtained from

$$\psi_0 = -\int \frac{\partial \mathrm{H}_{1s}}{\partial x_1} d\tau.$$

After the integration is performed, ψ_0 is again set in terms of $y_2, y_3, ..., y_n; x_1, x_2, x_3, ..., x_n$. The addition of ψ_0 to F_0 shall have the effect of changing the reference frequency ω_0^1, which, in other terms, is simply Lindstedt's device.

With this, we have established a clear connection between the definition of an Adelphic Integral and the formal integration of a Hamiltonian system by Poincaré's method. Such connection as we shall see next, establishes a fundamental bridge toward the methods using Lie Series Transforms and on Auxiliary System.

8. Perturbation Techniques Based on Lie Transforms.

This section is devoted to a, as brief as possible, view of perturbation methods introduced first by Hori (1966). As we have seen, it is perfectly justified to assume Hori's generator S to depend on the parameter $\in$ and, therefore, define a canonical transformation by

$$y_j = \eta_j + \sum_{n=1}^{\infty} \frac{\in^n}{n!} D^{n-1}_S \frac{\partial S}{\partial \xi_j}$$

(2.8.1)

$$x_j = \xi_j - \sum_{n=1}^{\infty} \frac{\in^n}{n!} D^{n-1}_S \frac{\partial S}{\partial \eta_j}$$

for $j = 1, 2, \ldots, n$, where y_j are coordinates, x_j momenta, and η_j, ξ_j the corresponding new variables, and

$$S = S(\eta; \xi; \in).$$

The image of any function $\mathrm{f}(y; x; \in)$ into the new phase space $(\eta; \xi)$, via the generator S, is given by

$$f(y; x; \in) = \sum_{n=0}^{\infty} \frac{\in^n}{n!} D^n_S \mathrm{f}(\eta; \xi; \in), \qquad (2.8.2)$$

where, we recall the definitions

$$D^0_S \mathrm{f} = \mathrm{f}$$

$$D^0_S \mathrm{f} = (\mathrm{f}, S) = \sum_{k=1}^{n} \left(\frac{\partial \mathrm{f}}{\partial \eta_k} \frac{\partial S}{\partial \xi_k} - \frac{\partial \mathrm{f}}{\partial \xi_k} \frac{\partial S}{\partial \eta_k} \right)$$

$$D^n_S \mathrm{f} = D^1_S \left(D^{n-1}_S \mathrm{f} \right), \quad n = 1, 2, \ldots \; .$$

Obviously, all functions involved f, S must be, at least, infinitely many time differentiable and the series above should converge for $\in$ sufficiently small.

Now consider the original system of differential equations to be defined by the Hamiltonian

$$\mathrm{H} = \mathrm{H}(y; x; \epsilon)$$

which, for simplicity, we assume to be analytic in the $2n+1$ arguments for (x; y) $\in D$ and $0 \leq \epsilon < \epsilon_0$. The equations are

$$\dot{y} = \mathrm{H}_x, \dot{x} = -\mathrm{H}_y, \tag{2.8.3}$$

and we assume that the power series

$$\mathrm{H}(y; x; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \mathrm{H}_k(y; x) \tag{2.8.4}$$

is such that $\mathrm{H}_0(y; x)$ is integrable in D, in the Liouville sense, that is, the system

$$\frac{d\eta_k}{d\tau} = \frac{\partial \mathrm{H}_0}{\partial \xi_k}(\eta; \xi)$$

$$\frac{d\xi_k}{d\tau} = -\frac{\partial \mathrm{H}_0}{\partial \eta_k}(\eta; \xi) \tag{2.8.5}$$

for $k = 1, 2, ..., n$ has the explicit solution

$$\eta_k = \eta^*_k(\alpha_1, \alpha_2, ..., \alpha_n; \beta_1 + \omega_1 \tau, \beta_2, \beta_3, ..., \beta_n)$$

$$\xi_k = \xi^*_k(\alpha_1, \alpha_2, ..., \alpha_n; \beta_1 + \omega_1 \tau, \beta_2, \beta_3, ..., \beta_n) \tag{2.8.6}$$

where $(\alpha; \beta)$ are constants of integration and $\omega_1 = \omega_1(\alpha_1)$, by the usual specific choice of the energy integral dependence on only one of the α's, say α_1. The requirement that the jacobian matrix

$$\frac{\partial(\eta^*; \xi^*)}{\partial(\beta; \alpha)}$$

be non-singular, for sufficiently small τ, allows the inversion of the above relations as

$$\alpha_k = \alpha_k(\eta; \xi), \quad k = 1, 2, ..., n$$

$$\beta_1 + \omega_1 \tau = \beta^*_1(\eta; \xi) \tag{2.8.7}$$

$$\beta_k = \beta_k(\eta; \xi), \quad k = 2, 3, ..., n.$$

Following Hori's definition we shall call (2.8.5) the <u>auxiliary system</u>. It should be kept in mind that, since H_0 is supposed to be integrable in the Liouville sense, there exists a canonical transformation, in particular (2.8.6) if $(\alpha;\beta)$ are action-angle variables, which reduces H_0 to a function only of the new momenta, in this case of α_1 only.

We now consider the problem of producing first integrals of motion of (2.8.3), independent of H. We consider a complete canonical transformation (2.8.1) with a generator

$$\in S(\eta;\xi;\in)=\sum_{k=1}^{\infty}\in^{k} S_k(\eta;\xi). \qquad (2.8.8)$$

The transformation being time independent, if $K(\eta;\xi;\in)$ is the new Hamiltonian, it follows that

$$\mathrm{H}(y;x;\in)=K(\eta;\xi;\in) \qquad (2.8.9)$$

where, in the left hand side the coordinates and momenta (y; x) are supposed functions of $(\eta;\xi;\in)$ through (2.8.1). According to (2.8.2), such transformation is obtained by direct application of S, if this is a known function, so that

$$K(\eta;\xi;\in)=\sum_{n=0}^{\infty}\frac{\in^{n}}{n!}L_S^{\,n}\mathrm{H}(x,y) \qquad (2.8.10)$$

If the series on the right converges, as a power series in $\in$, we must assume that a similar convergent power series exists for K, that is,

$$K(\eta;\xi;\in)=\sum_{n=0}^{\infty}\in^{n} K_n(\eta;\xi). \qquad (2.8.11)$$

Making use of (2.8.4) and (2.8.8), the right hand side of (2.8.10) yields, equating coefficients of the same power in $\in$,

$$K_0(\eta;\xi)=\mathrm{H}_0(\eta;\xi)$$

$$K_p(\eta;\xi)=(\mathrm{H}_0,S_p)+F_p \qquad (2.8.12)$$

$$p=1,2,3,...,$$

where F_p is a function of $\mathrm{H}_0,\mathrm{H}_1,...,\mathrm{H}_{p-1},S_1,S_2,...,S_{p-1}$, and possible to be specified either directly or by recurrence. The specification of F_p is not important as far as the discussion of the method is concerned and the advantage of one or another form is pertinent to the specific problem under study. For $p\geq 1$, equation (2.8.12) represents a partial differential equation in S_p with the typical characteristic of averaging methods, that is, k_p is also unknown. The equation can be written as

$$\sum_{k=1}^{n}\left(\frac{\partial \mathrm{H}_0}{\partial \eta_k}\frac{\partial S_p}{\partial \xi_k}-\frac{\partial \mathrm{H}_0}{\partial \xi_k}\frac{\partial S_p}{\partial \eta_k}\right)+F_p\left(\eta_1,...,\eta_n;\xi_1,...,\xi_n\right)$$

$$=k_p\left(\eta_1,...,\eta_n;\xi_1,...,\xi_n\right) \tag{2.8.13}$$

or, using the auxiliary system

$$-\frac{dS_p}{d\tau}+F'_p\left(\alpha;\beta_1+\tau,\beta_2,...,\beta_n\right)$$

$$=K'_p\left(\alpha;\beta_1+\tau,\beta_2,...,\beta_n\right). \tag{2.8.14}$$

The averaging principle in this method can be interpreted by imposing the condition that K_p should not depend on τ. If, as usual, we assume $\mathrm{H}(y;x;\in)$ to be a $2\pi-$periodic function of each y and because H_0 is Liouville integrable, the y^* and x^* are quasiperiodic, or periodic, functions of τ, a classical result following from the general theory of action and angle variables. We generalize the average to a quasiperiodic function, as was discussed previously, by setting

$$K'_p=\lim_{T\to\infty}\frac{1}{T}\int_0^T F'_p(\alpha;\beta_1+\tau,\beta_2,...,\beta_n)d\tau$$

$$=K'_p(\alpha;-,\beta_2,\beta_3,...,\beta_n)=K_p(\eta;\xi) \tag{2.8.15}$$

the last transformation in (2.8.15) being obtained by means of (2.8.7). It follows that

$$\frac{dS_p}{d\tau}=F'_p(\alpha;\beta_1+\tau,\beta_2,....,\beta_n)-K'_p(\alpha;\beta_2,....,\beta_n)$$

or

$$S_p=\int(F'_p-K'_p)d\tau=S'_p(\alpha;\beta_1+\tau,\beta_2,...,\beta_n)$$

$$=S_p(\eta;\xi), \tag{2.8.16}$$

again making use of (2.8.7) to perform the last transformation. It is also obvious that, in view of the definition of K'_p,

$$\lim_{\tau\to\infty}S'_p(\alpha;\beta_1+\tau,\beta_2,...,\beta_n)=\text{finite}$$

and, under the foregoing hypotheses, S'_p is quasiperiodic (or periodic with no constant term) with respect to τ. By recurrence, or otherwise, one can show that the process can be repeated for any $p=1,2,3,...$ which proves that there exists a formal series

$$S=S_0\left(\eta;\xi\right)+\in S_1\left(\eta;\xi\right)+... \tag{2.8.17}$$

which reduces the Hamiltonian to

$$K = K_0(\eta;\xi) + \in K_1(\eta;\xi) + \ldots \qquad (2.8.18)$$

with the property that, if $(\eta;\xi)$ are substituted by the solution of the auxiliary system, K does not depend explicitly on τ, and therefore,

$$\frac{\partial K'}{\partial \tau} = \frac{dK'}{d\tau} = 0 \qquad (2.8.19)$$

where K' is defined by the formal series

$$K' = K_0^{'}(\alpha;\ -\ ,\beta_2,\ldots,\beta_n) + K_1^{'}(\alpha;\ -\ ,\beta_2,\beta_3,\ldots,\beta_n) + \ldots\ .$$

Obviously, one can write

$$\frac{dK'}{d\tau} = \sum_{k=1}^{n}\left(\frac{\partial K}{\partial \eta_k}\frac{\partial \eta_k}{d\tau} + \frac{\partial K}{\partial \xi_k}\frac{\partial \xi_k}{\partial \tau}\right)$$

$$= \sum_{k=1}^{n}\left(-\dot{\xi}_k\frac{d\eta_k}{d\tau} + \dot{\eta}_k\frac{d\xi_k}{d\tau}\right)$$

$$= \sum_{k=1}^{n}\left(-\frac{\partial K_0}{\partial \xi_k}\dot{\xi}_k - \frac{\partial K_0}{\partial \eta_k}\dot{\eta}_k\right) = -\frac{dK_0}{dt}$$

and in view of (2.8.14),

$$\frac{dK_0}{dt} = 0$$

so that

$$K_0(\eta;\xi) = \text{constant} = J_0. \qquad (2.8.20)$$

We conclude that as a result of a Lie Transform, such that the new Hamiltonian does not depend on the auxiliary time τ, one obtains a new (formal) integral of motion, given by (2.8.20). The validity of this formal result can only be verified by analyzing the convergence of the method. Since it has been shown that Lie's Method and von Ziepel's Method are equivalent (Shniad) and that, if Kolmogorov's Method converges (under variable frequencies) so does von Ziepel's (Moser, 1966), the convergence, under sufficiently small and several time differentiable perturbations, of the Lie Transform Method, can be inferred indirectly. Again, such convergence cannot be uniform with respect to $\in$ or the initial conditions. The advantage of the method outlined here is that only quadratures are involved, in opposition to Poincaré's Method where, in general, one has to deal with partial differential equations. Equivalently important advantages are, of course, the production of the transformation in explicit form (see 2.8.1) ability of writing any function of (x; y) in terms of $(\eta;\xi)$ by the direct use of the generator S (see 2.8.2) and the invariance of the method and resulting quantities with respect to canonical transformations, a fact which follows directly from the invariance of Poinsson's parentheses with respect to such transformations.

We recall that a canonical transformation

$$Q = Q(q; p; \tau)$$
$$P = P(q; p; \tau) \tag{2.8.21}$$

defined by a Lie generator $S(Q; P; \tau)$ can be defined by the solution of the system

$$\frac{dQ}{d\tau} = \left(\frac{\partial S}{\partial P}\right)^{\mathrm{T}}$$
$$\frac{dP}{d\tau} = -\left(\frac{\partial S}{\partial Q}\right)^{\mathrm{T}} \tag{2.8.22}$$

for the initial conditions $(\tau = 0)$

$$Q(q; p; 0) = q,$$
$$P(q; p; 0) = p, \tag{2.8.23}$$

where τ is a parameter. The right hand members of (2.8.21) are supposed C^2 in all the $2n+1$ variables, in some domain of the phase space and τ restricted to some interval, say, $|\tau| \le \tau_0$. For the Poincaré generator $W(q; P; \tau)$ the same canonical transformation is given by

$$Q = q + \frac{\partial W}{\partial P^{\mathrm{T}}}(q; P; \tau)$$
$$p = P + \frac{\partial W}{\partial q^{\mathrm{T}}}(q; P; \tau) \tag{2.8.24}$$

under the condition

$$W(q; P; 0) \equiv 0$$

which is equivalent to the initial conditions (2.8.23). It has been established that

$$S(Q; P; \tau) = \frac{\partial W}{\partial \tau}(q; P; \tau) \tag{2.8.25}$$

where Q is given by the first of (2.8.24). Assuming the expansions

$$W(q;P;\tau)=\sum_{n=1}^{\infty}W_n(Q;P)\tau^n$$

$$S(Q;P;\tau)=\sum_{n=0}^{\infty}S_{n+1}(Q;P)\tau^n, \tag{2.8.26}$$

and

$$S(Q;P;\tau)=\sum_{n=1}^{\infty}S_{n+1}(Q;P)\tau^n,$$

equating coefficients of like powers in τ in (2.8.25), gives the relations among the W_k and the S_j, as obtained earlier.

Mersman (1971) produced Deprit's algorithm by setting $\tau = \in$ in the above formalism. If S corresponds now to Lie's generator S of equation (1.5.7), to keep the notation used there one should substitute $S_{n+1}/n!$ for S_n in the expansion of (2.8.25) and obtain

$$S_2 = 2W_2 - \sum_i \frac{\partial W_1}{\partial Q_i}\frac{\partial W_1}{\partial P_i}$$

$$S_3 = 6W_3 - \sum_i 2\frac{\partial W_1}{\partial Q_i}\frac{\partial W_2}{\partial P_i} + 2\frac{\partial W_2}{\partial Q_i}\frac{\partial W_1}{\partial P_i}$$

$$+\sum_{i,j}\frac{\partial^2 W_1}{\partial Q_i \partial Q_j}\frac{\partial W_1}{\partial P_i}\frac{\partial W_1}{\partial P_j} + 2\frac{\partial^2 W_1}{\partial Q_i \partial P_j}\frac{\partial W_1}{\partial P_i}\frac{\partial W_1}{\partial Q_j}$$

and so on. Hori's formalism is also obtained from (2.8.25) by substituting $S(Q;P)$ for $S(Q;P;\tau)$ and setting $\tau = 1$ thereafter, that is, the expansions of (2.8.25) corresponding to (2.8.26) are

$$W(q;P)=\sum_{n=1}^{\infty}W_n(Q;P)$$

$$S(Q;P)=S_1(Q;P)$$

which, substituted into (2.8.25), or directly into the expansions following (1.6.10) give

$$W = S_1 + \frac{1}{2}\sum_i \frac{\partial S_1}{\partial Q_i}\frac{\partial S_1}{\partial P_j}$$

$$+\frac{1}{6}\sum_{i,j}\left(\frac{\partial^2 S_1}{\partial Q_i \partial Q_j}\frac{\partial S_1}{\partial P_i}\frac{\partial S_1}{\partial P_j} + \frac{\partial^2 S_1}{\partial Q_i \partial P_j}\frac{\partial S_1}{\partial P_i}\frac{\partial S_1}{\partial Q_j}\right. \tag{2.8.27}$$

$$\left.+\frac{\partial^2 S_1}{\partial P_i \partial Q_j}\frac{\partial S_1}{\partial Q_i}\frac{\partial S_1}{\partial P_j}\right) + \ldots \quad .$$

The parameter $\in$ is then introduced into W and S_1, as

$$W = W\left(Q;P;\in\right)$$

$$S_1 = U\left(Q;P;\in\right)$$

and one assumes the formal series

$$W = \sum_{n=1}^{\infty} W_n\left(Q;P\right)\in^n,$$

$$U = \sum_{n=1}^{\infty} U_n\left(Q;P\right)\in^n . \tag{2.8.28}$$

The inverse of (2.8.27) is found to be, by a way or another,

$$S_1 = W - \frac{1}{2}\sum_i \frac{\partial W}{\partial Q_i}\frac{\partial W}{\partial P_i}$$

$$+\frac{1}{12}\sum_{i,j}\left(\frac{\partial^2 W}{\partial Q_i \partial Q_j}\frac{\partial W}{\partial P_i}\frac{\partial W}{\partial P_j} + 4\frac{\partial^2 W}{\partial Q_i \partial P_j}\frac{\partial W}{\partial P_i}\frac{\partial W}{\partial Q_j}\right. \tag{2.8.29}$$

$$\left.+\frac{\partial^2 W}{\partial P_i \partial P_j}\frac{\partial W}{\partial Q_i}\frac{\partial W}{\partial Q_j}\right) + \ldots \quad .$$

Introducing (2.8.28) and equating like powers of $\in$, one finds from (2.8.27)

$$W_1 = U_1,$$

$$W_2 = U_2 + \frac{1}{2}\sum_i \frac{\partial U_1}{\partial Q_i}\frac{\partial U_1}{\partial P_i}, \tag{2.8.30}$$

$$-----------$$

or, from (2.8.29)

$$U_1 = W_1,$$
$$U_2 = W_2 - \frac{1}{2}\sum_i \frac{\partial W_1}{\partial Q_i}\frac{\partial W_1}{\partial P_i}, etc. \qquad (2.8.31)$$

The foregoing relations allow the translation of the perturbation method introduced by Hori (1966) and described at the beginning of this section, into Deprit's formalism.

As an example consider Duffing's Equation without damping, that is,

$$\ddot{u} + u + \in \gamma u^3 = \in B \cos \omega t \qquad (2.8.32)$$

where $\in \geq 0, \gamma \geq 0, B, \omega \neq 0$ are constant parameters. We consider the case when ω is not rational and moreover for $p \neq 0,\ q$ integers, a relation

$$\left|p\omega - q\right| \geq K\left(p\right) \in^{1/2} \qquad (2.8.33)$$

is satisfied for a conveniently chosen $K\left(p\right)$, say $K\left(p\right) = p^{5/2-\sigma}, \sigma \geq 4,$ integer. If (2.8.33) is not satisfied we do have a case of resonance and it will be discussed in the last chapter.

Introducing the homogeneous complete canonical transformation

$$u = \left(2p_1\right)^{1/2} \sin q_1$$

$$\dot{u} = \left(2p_1\right)^{1/2} \cos q_1$$

the equation (2.8.32) can be written

$$\dot{q}_1 = \frac{\partial \mathrm{H}}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial \mathrm{H}}{\partial q_1} \qquad (2.8.34)$$

where

$$\mathrm{H} = p_1 + \in \gamma p_1^2 \sin 4q_1 - \in B\left(2p_1\right)^{1/2} \sin q_1 \cos \omega t.$$

Further, introducing the coordinate

$$q_2 = \omega t$$

with the conjugate momentum p_2, the system takes the form

$$\dot{q}_j = \frac{\partial K}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial K}{\partial q_j} \qquad (2.8.35)$$

for $j = 1, 2,$ and

$$K = p_1 + \omega p_2 + \in \left(\gamma p_1^2 \sin 4q_1 - B\left(2p_1\right)^{1/2} \sin q_1 \cos q_2\right)$$

$$= K_0\left(p_1, p_2\right) + \in K_1\left(q_1, q_2, p_1, -\right)$$

The auxiliary system is defined by K_0 and has the solution

$$q_1^0 = \tau + \beta_1$$

$$q_2^0 = \omega\tau + \beta_2$$

$$p_1^0 = \alpha_1$$

$$p_2^0 = \alpha_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants. Let the new Hamiltonian be

$$K^* = K_0^* + \in K_1^* + \in^2 K_2^* + ...$$

and the Lie generator

$$\in S = \in S_1 + \in^2 S_2 + ... \quad ,$$

with the condition that K^* should not depend on τ and therefore K_0^* is an integral of motion in the new coordinates and momenta $q_1^*, q_2^*, p_1^*, p_2^*$.

The equation

$$-\frac{dS_1}{d\tau} + K_1\left(q_1^0, q_2^0, p_1^0, -\right) = K_1^*$$

gives, under the condition that ω is not an integer

$$K_1^* = \frac{3}{8}\gamma p_1^2$$

$$S_1 = -\frac{1}{4}\gamma p_1^2 \sin 2q_1^* + \frac{1}{32} p_1^* \sin 4q_1^*$$

$$+\frac{B\left(2p_1^*\right)^{1/2}}{2(1+\omega)} \cos\left(q_1^* + q_2^*\right)$$

$$+\frac{B\left(2p_1^*\right)^{1/2}}{2(1-\omega)} \cos\left(q_1^* - q_2^*\right).$$

The second order approximation, using

$$-\frac{dS_2}{d\tau} + \frac{1}{2}\left(K_1 + K_1^*, S_1\right) + K_2 = K_2^*$$

where case, $K_2 = 0$, , in our gives

$$K_2^* = \frac{17}{64}\gamma^2 p_1^{*3} + \frac{B^2}{8\left(1-\omega^2\right)}$$

$$S_2 = -\frac{1}{2}\left(\frac{21}{32}\gamma^2 p_1^{*3} + \frac{B^2}{4\left(1-\omega^2\right)} \sin 2q_1^*\right)$$

$$-\frac{3}{128}\gamma^2 p_1^{*3} \sin 4q_1^* - \frac{7}{192}\gamma^2 p_1^{*3} \sin 6q_1^*$$

$$-\frac{B^2}{8\omega\left(1-\omega^2\right)} \sin 2q_2^* + \frac{B\gamma\left(2p_1^*\right)^{3/2}}{32\left(1-\omega^2\right)\left(1+\omega\right)}\left(13-\omega^2\right) \sin\left(q_1^*+q_2^*\right)$$

$$+\frac{B\gamma\left(2p_1^*\right)^{3/2}}{32\left(1-\omega^2\right)\left(1-\omega\right)}\left(13-\omega^2\right) \sin\left(q_1^*-q_2^*\right)$$

$$-\frac{B\gamma\left(2p_1^*\right)^{3/2}}{128\left(1-\omega^2\right)\left(3+\omega\right)}\left(21-5\omega^2\right) \cos\left(3q_1^*+q_2^*\right)$$

$$-\frac{B\gamma\left(2p_1^*\right)^{3/2}}{128\left(1-\omega^2\right)\left(3-\omega\right)}\left(21-5\omega^2\right) \cos\left(3q_1^*-q_2^*\right)$$

$$+\frac{B\gamma\left(2p_1^*\right)^{3/2}}{128\left(5+\omega\right)} \cos\left(5q_1^*+q_2^*\right) + \frac{B\gamma\left(2p_1^*\right)^{3/2}}{128\left(5-\omega\right)} \cos\left(5q_1^*-q_2^*\right)$$

$$-\frac{B^2}{16\left(1+\omega\right)^2} \sin\left(2q_1^*+2q_2^*\right) - \frac{B^2}{16\left(1-\omega\right)^2} \sin\left(2q_1^*-2q_2^*\right).$$

With the current approximation the new Hamiltonian is given by

$$K^* = p_1^* + \omega p_2^* + \frac{3}{8} \in \gamma p_1^{*2} + \frac{17}{64} \in^2 \gamma p_1^{*3} + 0\left(\in^3\right)$$

where we have neglected absolute constants. On the other hand K_0^* is an integral of motion, that is,

$$p_1^* + \omega p_2^* = const.$$

so that

$$K^* - p_1^* - \omega p_2^* = \frac{3}{8} \in \gamma p_1^{*2} + \frac{17}{64} \in^2 \gamma p_1^{*3} + \ldots$$

is also an integral, so that the problem is, in principle, reduced to quadratures and, except for values of ω rational or "close" to rational, the general solution can be found. The relations between the two sets of variables $(q;p)$ and $(q^*;p^*)$ are given by (2.8.1), or in the present notation

$$q_j = q_j^* + \sum_{n\geq 1} \frac{\in^n}{n!} D_S^{n-1} \frac{\partial S}{\partial p_j^*}$$

$$p_j = p_j^* - \sum_{n\geq 1} \frac{\in^n}{n!} D_S^{n-1} \frac{\partial S}{\partial q_j^*} \qquad (2.8.36)$$

for $j=1,2$. Obviously, since S does not depend on p_2^*, it follows that $q_2 = q_2^*$, that is, the transformation does not change the time $(q_2^* = \omega t)$. Since we have defined

$$\in S = \in S_1 + \in^2 S_2 + ...$$

if one sets

$$W = \in S \qquad (2.8.37)$$

the transformations can be written

$$q_j = q_j^* + \sum_{n\geq 1} \frac{1}{n!} D_W^{n-1} \frac{\partial W}{\partial p_j^*}$$

$$(2.8.38)$$

$$p_j = p_j^* - \sum_{n\geq 1} \frac{1}{n!} D_W^{n-1} \frac{\partial W}{\partial q_j^*}$$

or to second order in $\in$,

$$q_j = q_j^* + \frac{\partial W_1}{\partial p_j^*} + \frac{\partial W_2}{\partial p_j^*} + \frac{1}{2}\left(\frac{\partial W_1}{\partial p_j^*}, W_1\right)$$

$$p_j = p_j^* - \frac{\partial W_1}{\partial q_j^*} - \frac{\partial W_2}{\partial q_j^*} - \frac{1}{2}\left(\frac{\partial W_1}{\partial q_j^*}, W_1\right)$$

where

$$W_1 = \in S_1$$

$$W_2 = \in S_2$$

Clearly, assuming convergence of the method, the p_j^* are reduced to constants and the q_j^* to linear functions of time $(q_2^* = \omega t)$. The frequency of the angle variable q_1^* is a power series in $\in$. To the second order,

$$q_1^* = \left(1 + \frac{3}{4} \in \gamma p_1^* + \frac{51}{64} \in^2 \gamma p_1^{*2} + ...\right) t + \beta_1^*$$

where p_1^*, β_1^* are constants.

9. Perturbation Methods of Non-Hamiltonian Systems Based on Lie Transforms.

Hori (1970, 1971) and Kamel (1970) have developed, independently, methods of perturbations of non-linear systems in general, by generalizing the approach to Hamiltonian systems. Clearly, such generalization is not strictly necessary since, as mentioned before, any system can be reduced to Hamiltonian form by doubling its dimension and introducing Dirac's cotangent space. The price one has to pay by having twice the number of differential equations we started with, is more than compensate by the fact that only two functions are to be solved of the transformation. The direct approach requires the dealing with as many unknowns as there are variables, in fact, by direct application of the results of section 1.7, twice as many, as will be clear in a moment. Here we follow closely the presentation given by Kamel (1970). Consider a system of n first order differential equations

$$\dot{x} = \mathrm{f}\left(x;\in\right) \tag{2.9.1}$$

and assume $\mathrm{f}\left(x;\in\right)$ real analytic in the n+1 variables $\left(x_1, x_2, ..., x_n, \in\right)$ in some domain $\Omega\left\{x \in D \subset R^n, \left|\in\right| < \in_0\right\}$. The right-hand side of (2.9.1) can be expanded for $\in$ sufficiently small in the convergent power series

$$\dot{x} = \sum_{k \geq 0} \frac{\in^k}{k!} \mathrm{f}^{(k)}\left(x\right) \tag{2.9.2}$$

where

$$\mathrm{f}^{(x)}\left(x\right) = \left.\frac{\partial^k \mathrm{f}}{\partial \in^k}\right|_{\in=0} .$$

The functions $\mathrm{f}^{(x)}\left(x\right)$ are obviously real analytic in D. This condition can eventually be relaxed by attaching to the process to follow a smoothing operation at every stage of approximation but, for the general understanding of the method, this is not advisable. We shall not consider nonautonomous systems and the observation that such cases can be treated just as well by treating t as another x-type coordinate is not generally appropriate. Such is the case, for instance, when questions asymptotic behavior, stability and periodic solutions are dealt with.

If equation (2.9.1) or (2.9.2) cannot be integrated in general, one seeks a transformation to a new system of n variables ξ, say

$$x = x\left(\xi;\in\right) \tag{2.9.3}$$

such that the differential equation in ξ

$$\dot{\xi} = g(\xi;\in) \tag{2.9.4}$$

resulting from (2.9.3) and (2.9.1) be more easily treatable. Obviously, stated in this form, the problem is too general to define what should be the properties of that transformation. One way to look at it is, of course, to assume that for $\in = 0$, the equation (2.9.1) has a known general solution, that is, the equation

$$\dot{y} = \mathrm{f}(y;0) = \mathrm{f}^{(0)}(y) \tag{2.9.5}$$

is integrable. We might then ask the question whether there exists a transformation (2.9.3) such that (2.9.1) is brought into the form (2.9.5), that is,

$$\dot{\xi} = \mathrm{f}^{(0)}(\xi). \tag{2.9.6}$$

Since for $\in = 0$ the transformation (2.9.3) is obviously the identity, again we are lead to the search of a near identity transformation

$$x = \xi + \in h(\xi;\in) \tag{2.9.7}$$

and assume $h(\xi;\in)$ to be analytic in some domain of the $n+1$ variables $(\xi;\in)$ containing $\in = 0$. It is obviously invertible, therefore, near $\in = 0$, for $\in$ sufficiently small. So one writes

$$x = \xi + \sum_{k\geq 1} \frac{\in^k}{k!} E_k(\xi) \tag{2.9.8}$$

and the transformed system of differential equations will be, in general,

$$\dot{\xi} = \phi(\xi;\in) = \sum_{k\geq 0} \frac{\in^k}{k!} \phi^{(k)}(\xi),$$

with

$$\phi^{(k)}(\xi) = \left.\frac{\partial^k \phi}{\partial \in^k}\right|_{\in=0}.$$

The problem is now, given the transformation (2.9.8), to obtain the functions $\phi^{(k)}(\xi)$ in (2.9.9) from the functions $\mathrm{f}^{(k)}(x)$ in (2.9.2). Obviously this can be accomplished in several ways but a recursive algorithm like the one discussed in section 1.7 is recommended if high orders and systematic formalism are sought. Differentiation of (2.9.8) with respect to t gives

$$\dot{x} = \dot{\xi} + \sum_{k\geq 1} \frac{\in^k}{k!} \frac{\partial E_k}{\partial \xi} \dot{\xi}$$

and introducing (2.9.2) and (2.9.9), one finds

$$\sum_{k\geq 0}\frac{\in^k}{k!}\mathrm{f}^{(k)}(x)=\sum_{k\geq 0}\frac{\in^k}{k!}\ \phi^{(k)}(\xi)$$

$$+\sum_{k\geq 1}\frac{\in^k}{k!}\frac{\partial E_k}{\partial \xi}\sum_{j\geq 0}\frac{\in^j}{j!}\ \phi^{(j)}(\xi) \tag{2.9.10}$$

From relation (1.7.2) we now see that

$$\mathrm{f}\left(x(\xi;\in);\in\right)=\sum_{n=0}^{\infty}\frac{\in^n}{n!}(\xi)$$

and recursive relations are available for the definition of $\mathrm{f}_n(\xi)$, as for instance, equation (1.7.14) or (1.7.15) or the resulting relations in section (1.7). From (2.9.10) it now follows that

$$\mathrm{f}_n(\xi)=\phi^{(n)}(\xi)+\sum_{m=1}^{\infty}\binom{n}{m}\frac{\partial E_m(\xi)}{\partial \xi}\phi^{(n-m)}(\xi) \tag{2.9.11}$$

If one considers (1.7.22)

$$E_n(\xi)=-\mathrm{T}_n(\xi)-\sum_{m=1}^{n-1}\binom{n-1}{m-1}\mathrm{T}_m(\xi)\frac{\partial E_{n-m}(\xi)}{\partial \xi}$$

or, with notation (1.7.19),

$$E_n(\xi)=-\mathrm{T}_n(\xi)-\sum_{m=1}^{n-1}\binom{n-1}{m-1}\ L_m E_{n-m}(\xi). \tag{2.9.12}$$

We write the inverse of (2.9.8) as

$$\xi=x+\sum_{k\geq 1}\frac{\in^k}{k!}\mathrm{X}^{(k)}(x) \tag{2.9.13}$$

so that

$$\mathrm{X}^n(x)=\mathrm{T}_n(x)-\sum_{m=1}^{n-1}\binom{n-1}{m}\ L_m\mathrm{X}_{m,\,n-m}(x) \tag{2.9.14}$$

using the notation introduced in (1.7.20) and (1.7.21), that is,

$$\mathrm{X}_{p,q}(x)=-\sum_{m=1}^{p}\binom{p-1}{m-1}L_m\ \mathrm{X}_{p-m,\,q}(x),$$

$$\mathrm{X}_{o,q}(x)=\mathrm{X}^{(q)}(x). \tag{2.9.15}$$

Finally, one finds

$$\phi^{(n)}(\xi)=\mathrm{f}^{(n)}(\xi)+\sum_{j=1}^{n}\binom{n}{j}\left[\mathrm{f}_{j,n-j}(\xi)-\frac{\partial E_j}{\partial \xi}\phi^{(n-j)}(\xi)\right] \tag{2.9.16}$$

which is the recurrence relation we have sought. Obviously, Equation (2.9.16) contains the coefficients T_n defining the mapping (2.9.8), that is, the coefficients $\mathrm{X}^{(n+1)}(x)$ of the expansion

$$\frac{\partial \xi}{\partial \in} = \sum_{n \geq 0} \frac{\in^n}{n!} X^{(n+1)}(x)$$

from (2.9.13), and

$$\frac{\partial \xi}{\partial \in} = T(\xi;\in) = \sum_{n \geq 0} \frac{\in^n}{n!} T_{n+1}(\xi)$$

as in (1.7.6), (1.7.7) and (1.7.8). At each stage of the approximation, $T_n(\xi)$ has to be chosen properly so as to meet our special requirements, whenever necessary. Such unknown can be put in direct evidence in (2.9.16), by writing

$$\frac{\partial T_n}{\partial \xi}\phi^{(0)}(\xi) - \frac{\partial \phi^{(0)}(\xi)}{\partial \xi} T_n(\xi) = \phi^{(n)} - f^{(n)} + G_n(\xi) \qquad (2.9.17)$$

where $G_n(\xi)$ depends on all previous approximations. In fact, Kamel finds

$$G_n(\xi) = \frac{\partial E_n^*(\xi)}{\partial \xi} \phi^{(o)}(\xi) - f_{n,o}^*(\xi) + \sum_{m=1}^{n-1} \binom{n}{m} \left[\frac{\partial E_m}{\partial \xi} \phi^{(n-m)}(\xi) - f_{m,n-m}(\xi) \right] \qquad (2.9.18)$$

where

$$E_n^*(\xi) = E_n(\xi) \text{ for } T_n = 0,$$

$$f_{n,o}^*(\xi) = f_{n,o}(\xi) \text{ for } T_n = 0,$$

$$f_{p,q}(\xi) = -\sum_{m=1}^{p} \binom{p-1}{m-1} L_m f_{p-m}, q'$$

$$f_{o,q}(\xi) = \phi^{(q)}(\xi).$$

A thorough development of the method has been given by Kamel (1970) and Henrard (1970) and more recently by Hori (1971). Kamel shows how the generalized Lie Transform approach contains in essence the important methods of two-variable expansions procedures and matching of asymptotic solutions due to Kevorkian (1966). This subject is not dealt with here since it is explored in detail in the work of Cole (1968). It is worth noting that Deprit's presentation of Lie Transforms generated by functions depending on a (small) parameter and applied to Hamiltonians also depending on that parameter as it has been shown earlier in these notes and following Mersman's work (1971). In like manner the foregoing formalism can be simplified by introducing operators and functions which are not functions of a parameter and, later, introduce power series in $\in$ in all results. Consider n variables $(\xi_1, \xi_2, \ldots, \xi_n)$ and an operator $T_k(\xi)$, and le

$$D_\xi = \sum_{k=1}^{n} \mathrm{T}_k(\xi) \frac{\partial}{\partial \xi_k}. \qquad (2.9.19)$$

Consider the mapping

$$x_j = \xi_j + \sum_{p \geq 1} \frac{1}{p!} D_\xi^{p-1} \mathrm{T}_j(\xi) \qquad (2.9.20)$$

where

$$D_\xi^0 = 1,$$

$$D_\xi^1 = D_\xi,$$

$$D_\xi^p = D_\xi D_\xi^{p-1},$$

which are the analog of (2.8.1) and subsequent definitions. In particular $\mathrm{T}_k(\xi)$ plays the role of $\partial S / \partial \xi_k$. We also consider the mapping of a real analytic function f(x) of n variables $(x_1, x_2, ..., x_n)$ into the ξ – space as given by

$$\mathrm{f}(x) = f(\xi) + \sum_{p \geq 1} \frac{1}{p!} D_\xi^p \mathrm{f}(\xi) \qquad (2.9.21)$$

and, actually, (2.9.20) is a consequence of (2.9.21). We define the inverse transformation

$$\mathrm{T}_k^{-1}(x) = \mathrm{T}_k(\xi)\Big|_{\xi = x} \qquad (2.9.22)$$

and

$$D_x = \sum_{k=1}^{n} \mathrm{T}_k^{-1}(x) \frac{\partial}{\partial x_k} \qquad (2.9.23)$$

so that, the inverse mapping of (2.9.21) is

$$\mathrm{f}(\xi) = \mathrm{f}(x) + \sum_{p \geq 1} \frac{(-1)^p}{p!} D_x^p \, \mathrm{f}(x), \qquad (2.9.24)$$

a direct generalization of Lie's Transform. All of the above relations are actually contained in the previous formalism ($\in$ dependent) and their proof is straight-forward.

The equation

$$\dot{x}_k = \mathrm{f}_k(x), \qquad (2.9.25)$$

by means of the transformation (2.9.20) generated by T_k via the mapping (2.9.20), changes into

$$\dot{\xi}_k = \phi_k(\xi). \qquad (2.9.26)$$

Making use of (2.9.24), the inverse of (2.9.20) is given by

$$\xi_j = x_j + \sum_{p\geq 1} \frac{(-1)^p}{p!} D_x^{p-1} \mathrm{T}_j^{-1}(x) \qquad (2.9.27)$$

Since from (2.9.25)

$$\frac{d}{dt} = \sum_{k=1}^{n} \mathrm{f}_k \frac{\partial}{\partial x_k} \qquad (2.9.28)$$

for any function $F(x)$, that is,

$$\frac{d}{dt} F(x) = \dot{F}(x) = \sum_{k=1}^{n} \frac{\partial F}{\partial x_k} \dot{x}_k = \left(\sum_{k=1}^{n} \mathrm{f}_k \frac{\partial}{\partial x_k} \right) F,$$

the computation of $\phi_k(\xi)$ in (2.9.26) is obtained as follows. Differentiate (2.9.27) to get

$$\dot{\xi}_j = \dot{x}_j + \sum_{p\geq 1} \frac{(-1)^p}{p!} \frac{d}{dt} \left\{ D_x^{p-1} \mathrm{T}_j^{-1}(x) \right\}$$

and introduce (2.9.25) and (2.9.23) to find

$$\dot{\xi}_j = \mathrm{f}_j(x) + \sum_{p\geq 1} \frac{(-1)^p}{p!} \sum_{k=1}^{n} \mathrm{f}_k(x) \frac{\partial}{\partial x_k} \left(D_x^{p-1} \mathrm{T}_j^{-1}(x) \right)$$

or, using (2.9.21) and (2.9.20),

$$\dot{\xi}_j = \mathrm{f}_j(\xi) + \sum_{p\geq 1} \frac{1}{p!} D_\xi^p \mathrm{f}_j(\xi) + \sum_{p\geq 1} \frac{(-1)^p}{p!} \sum_{k=1}^{n} \mathrm{f}_k(\xi) \frac{\partial}{\partial \xi_k} \left(D_\xi^{p-1}(\xi) \right)$$

$$+ \sum_{p\leq 1} \frac{(-1)^p}{p!} \sum_{q\geq 1} \frac{1}{q!} D_\xi^q \left\{ \sum_{k=1}^{n} \mathrm{f}_k(\xi) \frac{\partial}{\partial \xi_k} \left(D_\xi^{p-1} \mathrm{T}_j(\xi) \right) \right\} = \phi_j(\xi). \qquad (2.9.29)$$

Now we consider the series

$$\mathrm{f}_j = \mathrm{f}_j^{(0)} + \mathrm{f}_j^{(1)} + \ldots$$

$$\phi_j = \phi_j^{(0)} + \phi_j^{(1)} + \ldots \qquad (2.9.30)$$

$$\mathrm{T}_j = \mathrm{T}_j^{(0)} + \mathrm{T}_j^{(1)} + \ldots$$

and we search for the operators T_j so that the ϕ_j take a desired form. Obviously, the equations

$$\dot{y}_k = \mathrm{f}_k^{(0)}(y) \qquad (2.9.31)$$

are supposed to have a well defined general solution. The decomposition (2.9.30) of f_j is intended not necessarily as a power series in some small parameter $\in$ and also is not necessarily an infinite series. In fact, the normal case of a perturbed integrable system (2.9.31) will be $\mathrm{f}_j^{(k)}$ for $k \geq 2$, that is,

$$f_j = f^{(0)}_j + f^{(1)}_j.$$

By feeding the series (2.9.30) into (2.9.29) one obtains a recursive algorithm for the unknowns $\phi^{(k)}_j$ and $T^{(k)}_j$, by equating terms of same order. In this respect, the explicit use of a parameter $\in$ to represent the orders is quite useful, though not necessary. That is, equating terms of same order can be translated into the easier language of equating coefficients of like powers of $\in$, assuming $f^{(k)}_j = 0\left(\in^k\right)$, $\phi^{(k)}_j = 0\left(\in^k\right)$, $T^{(k)}_j = 0\left(\in^k\right)$.

The first few approximations give, all functions intended to be in terms of the ξ variables,

$$f^{(0)}_j = \phi^{(0)}_j$$

$$\sum_{k=1}^{n}\left\{-f^{(0)}_k \frac{\partial T^{(1)}_j}{\partial \xi_k} + T^{(1)}_k \frac{\partial f^{(0)}_j}{\partial \xi_k}\right\} + f^{(1)}_j = \phi^{(1)}_j$$

$$\sum_{k=1}^{n}\left\{-f^{(0)}_k \frac{\partial T^{(2)}_j}{\partial \xi_k} + T^{(2)}_k \frac{\partial f^{(0)}_j}{\partial \xi_k}\right\} + \frac{1}{2!}\sum_{k=1}^{n} T^{(1)}_k.$$

$$.\frac{\partial}{\partial \xi_k}\left(f^{(1)}_j + \phi^{(1)}_j\right) - \frac{1}{2!}\sum_{k=1}^{n}\frac{\partial T^{(1)}_j}{\partial \xi_k}\left(f^{(1)}_k + \phi^{(1)}_k\right) + f^{(2)}_j = \phi^{(2)}_j$$

and, in general,

$$\sum_{k=1}^{n}\left\{-f^{(0)}_k \frac{\partial T^{(p)}_j}{\partial \xi_k} + T^{(p)}_k \frac{\partial f^{(0)}_j}{\partial \xi_k}\right\} + \ldots + f^{(p)}_j = \phi^{(p)}_j \qquad (2.9.32)$$

which, in fact, is equivalent to the previous relation (2.9.17). Here we introduce the important notion of auxiliary system by defining

$$\frac{d\xi_j}{d\tau} = f^{(0)}_j(\xi) \qquad (2.9.33)$$

with the general solution

$$\xi_j = \xi_j(\tau) \qquad (2.9.34)$$

so that

$$\sum_{k=1}^{n}\left\{-f^{(0)}_k \frac{\partial T^{(p)}_j}{\partial \xi_k} + T^{(p)}_k \frac{\partial f^{(0)}_j}{\partial \xi_k}\right\} = -\frac{dT^{(p)}_j}{d\tau} + \sum_{k=1}^{n} T^{(p)}_k \frac{\partial f^{(0)}_j}{\partial \xi_k}$$

The general equation (2.9.32) reduces to a linear system for the $T^{(p)}_j(\xi)$ at every stage of approximation, that is,

$$-\frac{dT_j^{(p)}}{d\tau}+\sum_{k=1}^{n}\frac{\partial f_j^{(0)}}{\partial \xi_j}\left(\xi(\tau)\right)T_k^{(p)}(\tau)+F_j^{(p)}(\tau)=\phi_j^{(p)} \qquad (2.9.35)$$

where the $\xi\,'s$ are substituted by the solution (2.9.34) of the auxiliary system. It is clear that (2.9.35) is a straight generalization of (2.8.9). It is noted that, as in the usual averaging methods, $\phi_j^{(p)}$ should be chosen so as to avoid secular terms in $T_j^{(p)}(\tau)$, that is,

$$\lim_{\tau\to\infty} T_j^{(p)}(\tau)=\text{finite}$$

The simplest case is when the $f_j^{(0)}$ are linear functions of the $\xi\,'s$, so that (2.9.35) is a linear (non-homogeneous) system with constant coefficients, for any order of approximation. If this is not the case, say the $\partial f_j^{(0)}/\partial \xi_k\Big|_{\xi=\xi(\tau)}$ are periodic or quasiperiodic functions of τ the integration of (2.9.36) is obviously not a trivial task. It is therefore advisable, in general, to produce a decomposition of the $f_j^{(0)}(\xi)$ such that $f_j^{(0)}(\xi)$ are linear.

<u>Van der Pol Equation</u>

As an example consider the equation

$$\ddot{x}+\in\left(1-x^2\right)\dot{x}+x=0$$

which can be written

$$\dot{x}_1=x_2$$

$$\dot{x}_2=-x_1-\in\left(1-x_1^2\right)x_2. \qquad (2.9.36)$$

Here we consider

$$f_1^{(0)}=x_2, \quad f_2^{(0)}=-x_1$$

$$f_1^{(1)}=f_1^{(2)}=\ldots=f_1^{(p)}=\ldots=0$$

$$f_2^{(1)}=-\in\left(1-x_1^2\right)x_2$$

$$f_2^{(2)}=f_2^{(3)}=\ldots=f_2^{(p)}=\ldots=0.$$

The auxiliary system is

$$\frac{d\xi_j}{d\tau}=f_j^{(0)}=\phi_j^{(0)}$$

whose solution we write in the form

$$\xi_1 = \alpha \cos\left(\tau + \beta\right)$$

$$\xi_2 = -\alpha \sin\left(\tau + \beta\right) \qquad (2.9.37)$$

where α, β are scalar constants. The first order equations become

$$-\frac{dT_1^{(1)}}{d\tau} + T_2^{(1)} = \phi_1^{(1)}$$

$$-\frac{dT_2^{(1)}}{d\tau} - T_1^{(1)} + \in\left[1 - \alpha^2 \cos^2\left(\tau + \beta\right)\right]\alpha \sin\left(\tau + \beta\right) = \phi_2^{(1)}$$

or

$$\frac{d^2T_1^{(1)}}{d\tau^2} + T_1^{(1)} = \in\left[\left(1 - \frac{\alpha^2}{4}\right)\alpha \sin\left(\tau + \beta\right) - \frac{\alpha^3}{4}\sin\left(3\tau + 3\beta\right)\right]$$

$$-\frac{d\phi_1^{(1)}}{d\tau} - \phi_2^{(1)}.$$

In order to avoid singular terms, the term in $\sin\left(\tau + \beta\right)$ must be avoided in the equation for $T_1^{(1)}$ and a possible choice is

$$\frac{d^2T_1^{(1)}}{d\tau^2} + T_1^{(1)} = -\in\frac{\alpha^3}{4}\sin\left(3\tau + 3\beta\right),$$

$$\frac{d\phi_1^{(1)}}{d\tau} + \phi_2^{(1)} = \in\left(1 - \frac{\alpha^2}{4}\right)\alpha \sin\left(\tau + \beta\right),$$

$$\phi_2^{(1)} = \frac{d\phi_1^{(1)}}{d\tau} = \frac{\in}{2}\left(1 - \frac{\alpha^2}{4}\right)\alpha \sin\left(\tau + \beta\right)$$

so that

$$\phi_2^{(1)} = -\in\left[1 - \frac{1}{4}\left(\xi_1^2 + \xi_2^2\right)\right]\xi_2$$

$$\phi_1^{(1)} = -\in\left[1 - \frac{1}{4}\left(\xi_1^2 + \xi_2^2\right)\right]\xi_1$$

and therefore

$$T_1^{(1)} = \frac{\in\alpha^3}{32}\sin 3\left(\tau + \beta\right) = \frac{\in}{32}\xi_2\left(\xi_2^2 - 3\xi_1^2\right)$$

$$T_2^{(1)} = \frac{\in}{2}\xi_1\left(-1 + \frac{7}{16}\xi_1^2 - \frac{5}{16}\xi_2^2\right).$$

The first order equations in the new variables are thus

$$\frac{d\xi_1}{dt} = \xi_2 - \in \left[1 - \frac{1}{4}\left(\xi_1^2 + \xi_2^2\right)\right]\xi_1$$

$$\frac{d\xi_2}{dt} = -\xi_1 - \in \left[1 - \frac{1}{4}\left(\xi_1^2 + \xi_2^2\right)\right]\xi_2$$

and, in fact, one easily verifies that the equation for ξ_2 is obtained from that of ξ_1 by the substitutions $\xi_2 \to -\xi_1, \xi_1 \to \xi_2$, provided the choice $\phi_2^{(1)} = d\phi_1^{(1)}/d\tau$ is made. If we let

$$u^2 = \xi_1^2 + \xi_2^2$$

it is found that

$$\frac{du^2}{dt} = -2 \in \left(1 - \frac{u^2}{4}\right)u^2$$

and, therefore,

$$u^2 = \xi_1^2 + \xi_2^2 = \frac{4ke^{-\in t}}{ke^{-\in t} \pm 1}$$

and $\pm$ sign choice depending on the sign of the constant k, that is, on the initial conditions, since u^2 has to be positive.

For $\in > 0$ we obtain the asymptotic behavior

$$u^2 \to 0 \text{ as } t \to \infty$$

which is the well known damped motion toward a focus. If $\in < 0$, $u^2 \to 4$ as $t \to \infty$, which is the limit cycle of the van der Pol equation. The fact that a first order theory (in $\in$) is able to give full information in the asymptotic behavior of the system is that, to any order, the equations for ξ_1, ξ_2 have the same character of the first order equations, that is,

$$\dot{\xi}_1 = \left[1 + \in^2 f_2\left(u^2\right) + \in^4 f_4\left(u^4\right) + \ldots\right]\xi_2$$

$$- \in \left[1 - \frac{u^2}{4} + \in^2 g_3\left(u^2\right) + \in^4 g_5\left(u^2\right) + \ldots\right]\xi_1$$

$$\dot{\xi}_2 = \dot{\xi}_1\left(\xi_2 \to -\xi_1, \xi_1 \to \xi_2\right),$$

so that the above described asymptotic properties are conserved.

NOTES

Integrability of a Dynamical System has been quite a controversial issue. One feels that, as far as Hamiltonian systems are concerned, separability of the Hamilton-Jacobi might be a good definition, although this is not the general opinion. Stackel Theorem, unfortunately, does not give any indication on how to actually construct a coordinates system which separates the equation. The only thing we clearly know is that if there are n independent integrals for an n dimensional system, then, according to Arnol'd, the invariant manifolds are tori and, on these, the motion is generally quasi-periodic. The existence of such manifolds for a certain class of systems is also conjectured by Diliberto under the name of periodic susfaces. The issue for non Hamiltonian systems is more complex, although, as for the example give at the end of Chapter 5, one may think of a generalized Birkhoff normalization, in case of disturbed harmonic oscillators. Many problems can actually be reduced to harmonic oscillators by a proper choice of variables and time.

For instance, the Newtonian problem of two bodies, by making use of the Levi- Civita transformation

$$x = u^2 - v^2$$

$$y = 2uv$$

combined with the time transformation

$$d\tau = dt / r,$$

reduces to a simple harmonic motion. Other force laws have been recently considered by Giacaglia and associates, following methods introduced by Kustaanheimo.

We are also lead to the study of integrability of a system in the vicinity of a stable equilibrium solution, a subject where many efforts have been made by Siegel and Moser, as well as well as many others. Although convergence of normalization methods cannot be established, it is obvious from the results of Contopoulos, Barbanis and Bozis that, under quite general circumstances, other integrals (or quase-integrals) may exist both in normal and resonant systems. Evidence of existence of integrals has also been established by means of the method of Surface of Section by Hènon and associates.

As far as methods of successive approximations are concerned, to produce series solutions of a system, any simple method will do, and convergence in a properly bounded interval of time can be achieved. The question could also be answered by simply applying Picard's method of iterations, which, in fact, has been done by several researchers, especially where numerical techniques are involved.

Given a system depending on a small parameter, the way the solution goes in terms of powers of such parameter, is set by how close one is to a singular (equilibrium) point of the system and on the stability character of such singular point. Properties of this sort were studied originally by Birkhoff for the behavior of area preserving mappings in the vicinity of fixed points. More recent and important results are due to Moser and Gelfand-Lidskii. A typical example of the change in behavior of expansions with respect to a parameter in the vicinity of an equilibrium point can be seen in the Restricted Problem of three bodies at the five Euler-Lagrange solutions. Such expansions can go in powers of $\in^{1/3}, \in^{1/2}$ or $\in$, as recently shown by Szebehely and associates (1970). The method of successive approximations by MacMillan given in section 2 can be changed easily into an averaging method, for the Eq. (2.2.6) or its expanded form (2.2.7). Such a method was given by Cesari and lately by Hale. The appearance of secular terms in a solution, as given in the example at the beginning of section 3, led Lindstedt to the introduction of the averaging methods. In several problems, a poor choice of a reference solution decides on the success of the subsequent approximations, in the same fashion as the wrong choice of coordinates decides on the integrability (separability) of a system. The Hamiltonianization of a system, originally due to Dirac, is only practical in cases where system (2.4.5) has constant coefficients (excluding exceptional cases), that is, the g_i are linear, with constant coefficients, in the components x_j of x. If this is not the case, the definition of the reference solution from (2.4.5) might be a very difficult task. As far as Poincaré's method (which he calls Lindstedt's Method), it has been called von Zeipel's methods mainly because it was through his work on Asteriods that Brouwer obtained a spectacular solution for the problem of artificial satellites of the earth in 1959. The averaging methods entered with full power in Celestial Mechanics, including the Russian Literature, before that time. Also, equation (2.4.8) indicates that, except for the averaging operation, all these methods, in conservative systems, are just a solution of Hamilton-Jacobi's equation by successive approximations. The main disadvantage is that the relations between original and new variables, generated by W (Eq. 2.4.10), are implicit and their inversion has only been recently fully solved by the introduction of Lie's Series. The fact that, if the average of a quasi-periodic function is zero, the integral of such function is bounded, can also be verified if one assumes a certain irrationality condition among the basic frequencies of the corresponding Fourier series $\omega_1, \omega_2, \ldots, \omega_n$; precisely,

$$\left| \sum_{j=1}^{n} p_j \omega_j \right| \geq K \left| \sum_{j=1}^{n} p_j \right|^{-\sigma}$$

for some positive constants K and $\sigma > n-1$. If such conditions are not verified (they are not for a set of $\omega's$ of zero measure), then the integral of a zero average quasi-periodic function may not be bounded due to presence of small divisors, as discussed by Moser, in the theory of quasi-periodic motions. From the purely geometric point of view, Moser has made important steps on the study of area preserving mappings which are "close" to the identity (see 2.4.12). His work has the obvious influence of Birkhoff and Siegel. The expansions involved in actual calculations and decurring from (2.4.16) are actually tedious and incredibly long. The recent introduction of automatic sumbol processors in fast electronic calculators has nevertheless eliminated most of the practical difficulties. Important results have been announced by Kovalevsky, Chapront and Deprit, in typical problems of Celestial Mechanics. We are not aware of analogous developments in Nonlinear Mechnics and Circuit Theory.

Degenerate systems, as defined by Arnol'd, are unfortunately very common in actual problems, thus the importance of the understanding of their behavior under perturbations. The essential geometric difficulty lies in the fact that the Invariant Manifolds of the Unperturbed Problem have a lower dimension than those of the perturbed one. Also, linear perturbed systems are much more sensible to resonance conditions and very difficult to describe. The stress given to the definition of fast and slow variables in justified by the fact that the former correspond usually to small amplitude oscillations and do not affect the latter which are associated with large scale deviations, with respect to the unperturbed system, over a long time. In many instances averaging is understood simply as a process of elimination of time when it appers explicitly in the equations. It is achieved simply by taking the average of the right-hand members of the differential equations. This is, in fact, the first step in the KBM Method. Such a procedure is explored in many ways by Hale (section V.3, pp. 171-208, 1969) who studies the deviation, as time goes to infinity, from a given non-autonomous system

$$\dot{x} = \in \mathrm{f}\left(t, x, \in\right) \qquad \text{(A)}$$

and the average system

$$\dot{x} = \in \mathrm{f}_0\left(x\right) \qquad \text{(B)}$$

where

$$\mathrm{f}_0\left(x\right) = \lim_{\mathrm{T}\to\infty} \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} \mathrm{f}\left(t, x, 0\right) dt.$$

Hale obtains conditions for the existence of periodic solutions, as well as their stability character. The starting proposition is that, under quite general conditions, there exists a transformation

$$x = y + \in u\left(t, y, \in\right) \qquad \text{(C)}$$

such that the equation (A) above is reduced to

$$\dot{y} = \in \mathrm{f}_0\left(y\right) + \in F\left(t, y, \in\right) \qquad \text{(D)}$$

with $F\left(t, y, 0\right) = 0.$ One sees clearly that near identity transformation (C) produces a system (D) which differs from the average system (B) by a quantity $0\left(\in^2\right)$ at least. The error estimate by Kyner is actually derived from this basic result.

From a sophisticated point of view, Moser in 1970 studied the topology of Kepler's Motion, the singularities of the manifold of the state of motion and introduces the concept of averaging on manifolds, avoiding the explicit use of coordinates. His intrinsic representation applies special techniques to the vector field defined by the Keplerian motion. The regularization process used by Moser to study orbits in the vicinity of the origin (r = 0) is due to Levi-Civita and generally known as the inversion transformation. It is not usually applied in global studies since it introduces new singularities at points where the velocity of the particle is zero.

Assuming the right-hand sides of the differential equations to be periodic in time, Laricheva obtains much better error bounds for the averaged equations of Celestial Mechanics than those given by Bogoliubov and Mitropolski. In his work "Theory of Orbits about an Oblate Plant", Kyner in 1963 gives an excellent description of the averaging methods as well as the connection, in that particular example, with Diliberto's theory of periodic surfaces. In the case, they happen to be, as expected, two-dimensional tori, since the field of the Planet is supposed to have rotational symmetry. He also applies a technique developed by Hale in the book "Nonlinear Oscillations", in order to obtain conditions of periodicity and also develop approximate solutions.

As far as the application of Poincaré's Method to Hamiltonian systems, when the Hamiltonian is a power series in both coordinates and momenta, as in the example at the beginning of section 6, it was described by Giacaglia in 1965. The problem arises naturally in small oscillations and, in Celestial Mechanics, in the use of Poincaré's variables and problems of resonance. In this way one provides a certain generalization in the concept of Birkhoff's Normalization, by assuming, in principle, any combination of coordinates and momenta and, second, by giving a more systematic way of producing the Normalization. The application of Lie's series by Deprit is an example, however, on how complex the actual development of the Method might become. The characteristic exponents are better obtained, in this case, by using Cesari's method developed in 1940, as was shown by Giacaglia in the libration cases of the Elliptic Restricted Problem in 1971. Obviously, after the characteristic exponents are obtained to some order as power series in the small parameter of the problem, Lyapunov's transformation easily reduces the problem to the integration of a linear system whose coefficients are constant within that same order. The problem os small divisors in Poincaré's Method is here translated into a problem of parametric resonance.

The construction of integrals of motion via a successive approximation to the Poisson condition, undertaken by Contopoulos in several works, shows very well the change on the form of such integrals (or quasi-integrals) when a region of resonance is crossed. Since, in the limit, the resonance points are at least as dense as the rational numbers on the line, one expects a very wild behavior of the inegrals, changing from one form to another, infinitely many times, in every finite interval of frequencies, defined by the small parameter of the problem and / or by the initial conditions. This fact will not prevent the convergence for a specific value of the frequencies , in fact, over a set of values with non zero measure. Such integrals however cannot be analytic, nor can their series by uniformly convergent or continuous. However, the number of discontinuities is countable and with zero measure. All these considerations and conjectures are intimately connected with Moser's and Kolmogorov's theories.

The construction of Kovalevskaya's Integral we have given in section 6 is a rare example of a series which terminates and, obviously, must be an exceptional situation. It is nevertheless an indication of the danger in defining a system integrable or nonintegrable for all possible situations.

Lie Transform techniques are quite popular at present and they actually represent a real breakthrough from Classical Methods. At least one can say they were not known to Poincare, a thing hard to discover in perturbation theories. The credit for this new method goes to Hori. Later works and modified algorithms should only be considered as refinements or different forms of the same basic idea. One of the best examples of applications of the method has been give by Deprit et al for the main problem of earth's artificial satellites. Also, a recent application to the motion of a rigid body under the influence of central gravitation has been given by Giacaglia et al. Several examples are also treated by Choi and Tapley. The example we have given for the solution of van der Pol equation is extended to third order by Hori in his recent paper on the subject of non Hamiltonian systems, and is the best reference on the actual use of the method for non Hamiltonian systems. The Hamiltonianization of wan der Pol equation

$$\ddot{x} = -\in\left(1-x^2\right)\dot{x} - x$$

is readily obtained by defining $x = y_1, \dot{x} = y_2,$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -\in\left(1-y_1^2\right)y_2 - y_1$$

and the Hamiltonian is

$$\mathrm{H} = x_1\dot{y}_1 + x_2\dot{y}_2 = \mathrm{H}_0 + \mathrm{H}_1,$$

where

$$\mathrm{H}_0 = x_1x_2 - x_2y_1,$$

$$\mathrm{H}_1 = -\in\left(1-y_1^2\right)x_2y_2.$$

The equations of motion are

$$\dot{y}_k = \mathrm{H}_{x_k}, \quad \dot{x}_k = -\mathrm{H}_{y_k}$$

and the auxiliary system is defined by

$$K_0 = \xi_1\eta_2 - \xi_2\eta_1,$$

that is

$$\frac{d\xi_1}{d\tau} = \xi_2, \frac{d\xi_2}{d\tau} = -\xi_1,$$

$$\frac{d\eta_1}{d\tau} = \eta_2, \frac{d\eta_2}{d\tau} = -\eta_1$$

with the obvious solution

$$\xi_1^0 = \alpha_1 \sin(\tau + \beta_1)$$

$$\xi_2^0 = \alpha_1 \cos(\tau + \beta_1)$$

$$\eta_1^0 = \alpha_2 \sin(\tau + \beta_2)$$

$$\eta_2^0 = \alpha_2 \cos(\tau + \beta_2).$$

From the first order equation

$$-\frac{dS_1}{d\tau} + \mathrm{H}_1 = K_1$$

we obtain

$$K_1 = \lim_{\mathrm{T}\to\infty} \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} \mathrm{H}_1(\tau)\, d\tau$$

$$= \lim_{\mathrm{T}\to\infty} \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} \left[- \in \left(1 - \eta_1^{o^2}\right) \eta_2^o - \eta_1^o \right] \mathrm{d}\tau$$

$$= -\frac{\in \alpha_1 \alpha_2}{2} \left(1 - \frac{\alpha_2^2}{4}\right) \cos(\beta_1 - \beta_2),$$

$$S_1 = \int \left[\mathrm{H}_1(\tau) - K_1 \right] d\tau.$$

For a complete solution up to third order the paper by Choi is suggested.

Finally, for a detailed and excellent description of the averaging methods both from the point of view of Krylov-Bogoliubov and of Poincaré, as well as the meaning of neglecting high order terms, we refer to the classical work of Musen, and to the extensive work of Volosov.

REFERENCES

1. Abraham, R., 1967, “Foundations of Mechanics”, W. A. Benjamin, Inc., Philadelphia.
2. Andoyer, H., 1926, “Cours de Mécanique Célèste”, (Vol. I), Gauthier-Villars, Paris.
3. Arnol’d, V. I., 1963, “Proof of A. N. Kolmogorov’s Theorem on the Conservation of Quasiperiodic Motions under Small Perturbations of the Hamiltonian”, Uspekhi Mat. Nauk USSR, 18, 13-40.
4. ____________, 1963, “Small Denominators and Problems of Stability of Motion in Classical and Celestial Mechanics”, Uspekhi Mat. Nauk USSR, 18, 91-192.
5. Barbanis, B., 1966, “The Topology of the Third Integral”, Intern. Astrom. Union Symp. No. 25, pp. 19-25, Academic Press, New York.
6. Birkhoff, G. D., 1927, “Dynamical Systems”, Am. Math. Soc. Colloq. Public. IX, Providence, Rhode Island.
7. Bogoliubov, N. N., 1945, “On some statistica methods in mathematical physics”, (Paper) Izv. Akad. Nauk USSR, Morcow.
8. ____________ and Mitropolskii, Y. A., 1951, “Asymptotic Methods in the Theory of Nonlinear Oscillations”, Gordon and Breach, New York.
9. Bozis, G., 1966, “A New Integral in the Restricted Problem of Three Bodies”, Doctoral Thesis, Univ. of Thessaloniki, Greece.
10. ____________, 1966, “On the Existence of a New Integral in the Restricted Three-Body Problem”, Astron. J., 71, 404-414.
11. Brouwer, D., 1959, “Solution of the Problem of Artifical Satellites without Drag”, Astron. J., 64, 378-390.
12. ____________ and Clemense, G. M., 1961, “Methods of Celestial Mechanics”, Academic Press, New York.
13. Caley, A., 1848, Comb. Dublin Math. J., 3, 116.

14. Cesari, L., 1940, "Sulla Stabilità delle Soluzioni dei Sistemi de Equazioni Differenziali Lineari a Coefficienti Periodic", Atti Accad. Ital. Mem. Classe Fis. Mat. e Nat., 11, 633-692.
15. ____________, 1963, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations", Springer-Verlag New York Inc., New York.
16. Choi, J. S. and Tapley, B. D., 1972, "An Extended Canonical Perturbation Method", Cel. Mech. (to appear).
17. Contopoulos, G., "A Third of Motion in a Galaxy", Zeits. fur Astrophys., 49, 273-291.
18. ____________ and Barbanis, B., 1961, "An Application of the Third Integral of Motion", The Observatory, 82, 80-82.
19. ____________, 1962, "On the Existence of a Third Integral", Astron. J., 68, 1-14.
20. ____________, 1963, "A Classification of the Integrals of Motion", Astrophys. J., 138, 1297-1305.
21. ____________, 1963, "Resonances Cases and Small Dvisors in a Third Integral of Motion. I", Astron. J., 68
22. ____________ and Woltjer, L., 1964, "The Third Integral in Non-Smooth Potentials", Astrophys. J., 140, 1106-1119.
23. ____________, 1965, "The Third Integral in the Restricted Three-Body Problem", Astrophys. J., 142, 802-804.
24. ____________, 1966, "Adiabatic Invariants and the Third Integral", J. Math. Phys., 7, 788-797.
25. ____________ and Hadjidemetrious, J. P., 1968, "Characteristics of Invariant Curves of Plane Orbits", Astron. J., 73, 86-96.
26. ____________, 1970, "Resonance Phenomena in Spiral Galaxies", in "Periodic Orbits, Stability and Resonances" (Ed. G. E. O. Giacaglia), D. Reidel Publ. Co., Dordrecht, Holland.
27. ____________, "Orbits in Highly Perturbed Dynamical Systems", I (1970), Astron. J., 75, 96-107; II (1970), Astron. J., 75, 108-130; III (1971), Astron. J., 76, 147-156.
28. Deprit, A. et. al., 1969, "Birkhoff's Normalization", Cel. Mech., 1, 222-251.
29. ____________ et. al., 1970, "Analytical Lunar Ephermeris: Brouwer's Suggestion", Astron. J., 75, 747-750.
30. ____________ and Rom, A., 1970, "Characteristic Exponents of -------- in the Elliptic Restricted Problem", Astron. Astrophys., 5, 416-428.
31. ____________ and Rom, A., 1970, "The Main Problem of Artificial Sattelite Theory for Small and Moderate Eccemtricities", Cel. Mech., 2, 166-206.

32. Diliberto, S. P., 1967, "New Results on Periodic Surfaces and the Averaging Prounciple", U.S.-Japapenese Sem. on Diff. Funct. Eq., pp. 49-87, Benjamin, Philadelphia.
33. Dirac, P. A. M., 1958, "Generalized Hamiltonian Dynamics", Proc. Roy. Soc. London A246, 326-332.
34. Euler, L., 1753, "Theoria Motus Lunae" and 1772, "Theoria Motus Lunae, Novo Methodo", Petropoli, Typ. Acad. Imp. Scient.
35. Gelfand, I. M. and Lidskii, U. B., 1958, "On the Structure of the Regions of Stability of Linear Canonical Systems of Disserential Equations with Periodic Coefficients", Am. Math. Soc. Transl. (2), 8, 143-182.
36. Giacaglia, G. E. O., 1964, "Notes on von Zeipel's Method", GSFC-NASA Publ. X-547-64-161.
37. ____________, 1965, Evaluation of Methods of Integration by Series in Celestial Mechanics", Doctoral Thesis, Yale Univerity, New Haven.
38. ____________, 1967, "Nonintegrable Dynamical Systems", Chair Thesis, General Mechanics, Uninv. of Sao Paulo, Sao Paulo.
39. ____________, et al., 1970, "A Semi-Analytic Theory for the Motion of a Lunar Satellite", Cel. Mech., 3, 3-66.
40. ____________ and Jefferys, W. H., 1971, "Motion of a Space Station", Cel. Mech. 4, 442-467.
41. ____________, 1971, "Characteristic Exponents at ------- and ------- in the Elliptic Restricted Problem of Three Bodies", Cel. Mech., 4, 468-489.
42. ____________, 1972, "Regularization of Conservative Central Fields", Public. Astron. Soc. Japan, 24, No. 3, July.
43. ____________ and Nuotio, V. I., 1972, "Spinor Regularization of Conservative Central Fields", 3rd Annual Meeting, Div. Dynamical Astron., Am. Astron. Soc., Univ. of Maryland, In "Bull. Amer. Astron. Soc.", (to appear).
44. Goldstein, H., 1951, "Classical Mechanics", Addison-Wesley, Reading, Massachusetts.
45. Goursat, E., 1959, "Cours d'Analyse Mathematique", 7th. Ed., Vol. 2, Gauthier-Villars, Paris. (Reprinted by Dover Publ., New York).
46. Hale, J. K., 1954, "On the Boundedness of the Solution of Linear Differential Systems with Periodic Coefficients", Riv. Mat. Univ. Parma, 5, 137-167.
47. ____________, 1961, "Integral Manifolds f Perturbed Differential Equations", Ann. Math., 73, 496-531.
48. ____________, 1962, "On Differential Equations Containing a Small Parameter", Contrib. Diff. Eq., Vol. 1, J. Wiley, New York (Ed. J. P. LaSalle et. al).

49. Hale, J. K., 1963, “Oscillations in Nonlinear Systems”, MacGraw-Hill, New York.

50. ____________, 1969, “Ordinary Differential Equations”, (Chapt. 5), Wiley-Interscience, New York.

51. Henrand, J., 1970, “On a Perturbation Theory Using Lie Transforms”, Cel. Mech., 3, 107-120.

52. Hènon, M. and Heiles, C., 1964, “The Applicability of the Third Integral of Motion; Some Numerical Experiments”, Astron. J., 69, 73-79.

53. ____________, 1965, “Exploration Numérique du Problème Restreint”, Ann. Astrophys., 28, 499 and 992.

54. Hori, G., 1966, “Theory of General Perturbations with Unspecified Canonical Variables”, Public. Astron. Soc. Japan, 18, 287-296.

55. ____________, 1971, “Theory of General Perturbations for Noncanonical Variables”, Public. Astron. Soc. Japan, 23, 567-587.

56. Kamel, A. A., 1969, “Expansion Formulae in Canonical Transformations Depending on a Small Parameter”, Cel. Mech., 1, 190-199.

57. ____________, 1970, “Perturbation Method in the Theory of Nonlinear Oscillations”, Cel. Mech., 3, 90-106.

58. Kevorkian, J., 1966, “The Two Variable Expansion Procedure for the Approximate Solution of Certain Nonlinear Differential Equations” in “Lectures in Applied Mathematics”, vol. 7, p. 206, Amer. Math. Soc., Providence, R. I.

59. Kolmogorov, A. N., 1953, “On the Conservation of Quasiperiodic Motions for a Small Change in the Hamiltonian Function”, Dokl. Akad. Nauk USSR, 98, 527-530.

60. Kovalevsky, J., 1968, “Review of Some Methods of Programming of Litoral Developments in Celestial Mechanics”, Astron. J., 73, 203-209.

61. Krylov, N. and Bogoliubov, N. N., 1947, “Introduction to Nonlinear Mechanics”, Ann. Math. Stud., 11, Princeton Univ. Press, Princeton, New Jersey.

62. Kustaanheimo, P., 1964, “Spinor Regularization of Kepler Motion”, Ann. Univ. Turkuensis, A73, 3-7.

63. Kyner, W. T., 1961, “Invariant Manifolds”, Rend. Circ. Mat. Palermo, 10, 98-110.

64. ____________, 1963, “A Mathematical Theory of the Orbits about and Oblate Planet”, Tech. Rep. to ONR, Det. Math., Univ. of Southern Calif., Los Angeles (51 pp.)

65. Kyner, W. T., 1968, “Rigorous and Formal Stability of Orbits about an Oblate Planet”, Mem. Amer. Math. Soc., 81, 1-27.

66. Laricheva, V. V., 1966, “On Averaging a Certain Class of Systems of Nonlinear Differential Equations”, Diff. Equa., 2, No. 1, 169-173.

67. Lefschetz, S., 1959, "Differential Equations. Geometric Theory", Interscience, New York.

68. Leimanis, E. and Minorsky, N., 1958, "Dynamical and Nonlinear Mechanics", (Ch. I), J. Wiley, New York.

69. Levi-Civita, T., 1903, "Traiettorie Singolari ed Urti nel Problema Ristretto dei Tre Carpi", Ann. Math., 9, 1-27.

70. Lindstedt, A., 1882, "Beitrag zur Integration der Differentialgleichungen der Störungtheorie", Abh. K. Akad. Wiss. St. Petersburg, 31, No. 4.

71. MacMillan, W. D., 1912, "A Method of Determining Solutions of a System of Analytic Functions in the Neighborhood of a Branch Point", Math. Annalen, 72, 180.

72. MacMillan, W. D., 1920, "Dynamics of Rigid Bodies", Dover Publ., New York (pp. 403-413).

73. Mersman, W. A., 1971, "Explicit Recursive Algorithms for the Construction of Equivalent Canonical Transformations", Cel. Mech., 3, 384-389.

74. Minorsky, N., 1962, "Nonlinear Oscillations", van Nostrand, Princeton, New Jersey.

75. Moser, J., 1955, "Nonexistence of Integrals for Canonical Systems of Differential Equations", Comm. Pure Appl. Math., 8, 409-436.

76. ____________, 1958, "New Aspects in the Theory of Stability of Hamiltonian Systems", Comm. Pure Appl. Math., 2, 81-114.

77. ____________, 1961, "A New Technique for the Construction of Solutions of Nonlinear Differential Equations", Proc. Nat. Acad. Sci., 47, 1824-1831.

78. ____________, 1962, "On Invariant Curves of Area-Preserving Mappings of an Annulus", Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II, 1-20.

79. ____________, 1963, "Perturbation Theory for Almost Periodic Solutions for Undamped Nonliear Differential Equations", Int. Symp. on Nonlinear Diff. Eq. and Nonlinear Mech., Colorado Springs, 1961, Acad. Press, New York (pp. 71-79).

80. ____________, 1964, "Hamiltonian Systems", Lecture Notes, New York Univ., New York.

81. Moser, J., 1966, "A Rapidly Convergent Iteration Method and Nonlinear Partial Differential Equations", I, Ann. Scu. Norm. Sup. Pisa, 20, 265-315; II, Ann. Scu. Norm. Sup. Pisa, 20, 499-533.

82. ____________, 1966, "On the Theory of Quasi-Periodic Motions", SIAM Rev., 8, 145-172.

83. ____________, 1967, "Convergent Series Expansions of Quasi-Periodic Motions", Math. Ann., 169, 136-176.

84. ____________, 1970, "Regularization of Kepler's Problem and the Averaging Method on a Manifold", Comm. Pure Appl. Math., 23, 609-636.

85. Moulton, F. R., 1913, "Periodic Oscillating Satellites", Math. Ann., 73, 441-479.

86. ____________, 1920, “Periodic Orbits”m Carnegie Inst. Washington Publ., 161, Washington, D. C.
87. Musen, P., 1965, “On the high order effects in the Methods of Krylov-Bogoliubov and Poincaré”, J. Astronaut. Sci., 12, 129-134.
88. Nemitskii, V. V. and Stepanov, V. V., 1960, “Qualitative Theory of Differential Equations”, Princeton Univ. Press, Princeton, New Jersey.
89. Pliss, V. A., “On the Theory of Invariant Surfaces”, Diff. Eq., 2, 1139-1150.
90. ____________, 1966, “Nonlocal Problems of the Theory of Oscillations”, Acad. Press, New York.
91. Poincaré, H., 1893, “Les Methodes Nouvelles de la Mécanique Céleste”, Vol. 2, Reprint by Dover Publ. New York (1957).
92. Poincaré, H., 1893, “Les Methodes Nouvelles de la Mécanique Céleste”, (Vol. 3), Gauthier-Villars, Paris (Dover Publ. Reprint, New York).
93. Poincaré, H., 1909, “Lécons de Mécanique Céleste”, (Vol. 2), Gauthier-Villars, Paris.
94. Roels, J. and Louterman, G., 1970, “Normalization des Systèmes Linèaires Canoniques et Application au Problème Restreinte des Trois Corps”, Cel. Mech., 3, 129-140.
95. Sansone, G. and Conti, R., 1964, “Nonlinear Differential Equations”, Pergamon Press, New York.
96. Siegel, C. L., 1941, “On the Integrals of Canonical Systems”, Ann. Math. 42, 806-822.
97. Siegel, C. L., 1954, “Über die Existenz einer Normal form analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung”, Math. Am., 128, 144-170.
98. ____________, 1956, “Vorlesungen uber Himmelsmechanik”, Springer-Verlag, Berlin.
99. Sternberg, S., 1969, “Celestial Mechanics”, (2. Vols.), W. A. Benjamin, New York.
100. Szebehely, V. et al., 1970, “Mean Motions and Characteristic Exponents at the Libration Points”, Astron. J., 75, 92-95.
101. Volosov, V. M., 1962, “Averaging in Systems of Ordinary Differential Equations”, Russian Math. Surv., 17, 1-126.
102. Whittaker, E. T., 1961, “On the Adelphic Integrals of the Differential Equations od Dynamics”, Proc. Roy. Soc. Edinburg, 37, 95.
103. ____________, 1937, “A Treatise on the Analytical Dynamics of Particles and Rigid Bodies”, Cambridge, Univ, Press, London.
104. Wintner, A., 1947, “The Analytical Foundations of Celestial Mechanics”, Princeton Univ. Press, Princeton, New Jersey.

Zeipel, H. von, 1916-17, “Recherches sur le Mouvement des Petits Planets”, Arkiv. Astron. Mat.

Phys., 11, 12, 13.