Opuscula Mathematica • Vol. 27 • No. 2 • 2007

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SPECTRA OF SUBNORMAL PAIRS

Abstract. In this short note we present an example related to joint spectra of subnormal pairs of bounded operators. A counterexample to the equality between Taylor's spectrum and the closure of the defect spectrum is given. This example is related to the author's modification of N Sibony's counterexample to Corona Theorem on domains that fail to be strictly pseudoconvex.

Keywords: subnormal operator, corona, joint spectrum.

Mathematics Subject Classification: Primary 47B20; Secondary 46J15.

1. SOME JOINT SPECTRA

It often happens that certain "canonically defined" parts of the spectrum, when considered within a special class of operators, actually fill in the entire set. The advantage is obvious whenever the description of such a part is simpler, since often only one of the possible causes of non-invertibility is then involved. In the case of joint spectra the gain becomes even more essential, especially when there is no practical way of finding the "whole joint spectrum".

Take for example a bounded subnormal operator S on a complex, separable Hilbert space H. Assume also that its normal extension's spectrum: $\sigma(N)$ is "thin", so that the spectrum of S is the closure of $\sigma(S) \setminus \sigma(N)$. (A typical situation is that of S equal to the analytic Toeplitz operator T_u of multiplication by some nonconstant inner function $u \in H^\infty$.) We begin with a simple observation concerning the reflection (in the Real Axis) of of the point spectrum $\sigma_p(S^*)$ of its adjoint, denoted here as $\sigma_p(S^*)^*$.

Lemma 1.1. *Under the above assumption, $\sigma(S)$ is the closure of $\sigma_p(S^*)^*$. In other words, $S - \lambda I$ has no bounded inverse on H if and only if the complex conjugate λ^* of λ is approximable by eigenvalues of S^*.*

Indeed, the remaining part of $\sigma(S)$, contained in the approximate point spectrum, is a subset of $\sigma(N)$ – a nowhere dense subset of $\sigma(S)$. Other arguments are at hand, especially when analytic Toeplitz operators on the Hardy space over a bounded domain $\Omega \subset \mathbb{C}^n$ are considered. Here the symbol u is in $H^\infty(\Omega)$, the algebra of bounded analytic functions on Ω. The reproducing kernel yields eigenvectors of T_u^* corresponding to the eigenvalues $u(\lambda)^*$, as λ runs over Ω. Hence the equality

$$\sigma(T_u) = [\sigma_p(T_u^*)^*]^- \tag{1.1}$$

follows from the invertibility for $u \in H^\infty(\Omega)$ whenever $0 \notin [u(\Omega)]^-$.

The so called *defect spectrum*, $\sigma_\delta(S)$, consisting of those $\lambda \in \mathbb{C}$ for which $S - \lambda I$ is not surjective, clearly contains $\sigma_p(S^*)^*$, since the latter corresponds to the range of $S - \lambda I$ being not dense. Another good candidate for $\sigma(S)$ is the reflected image of the approximate point spectrum of S^*, which has the advantage of being a closed set.

Analogous criterion for non-singularity of multiplication operators by a tuple of elements $u_1, \dots, u_k$ in $H^\infty(\Omega)$ is related to Corona Theorem for Ω (cf.[4], where special classes of operators were considered). In the multi-variable case, the problem has been studied over past 3 decades and still remains open in the most important cases of the unit ball and polydisc.

In this note we present an example of pure subnormal pairs (S,T) for which the multi-variable version of Lemma 1.1 fails. To this end, infinite orthogonal sums of certain multiplication operators are considered. Their Taylor joint spectra are different from the closure of their respective defect spectra.

Notation In this note we use $[A]^-$ for the Euclidean closure of a set $A \subset \mathbb{C}^k, k = 1, 2, \dots,$ and λ^* – for the tuple

$$(\lambda_1^*, \dots, \lambda_k^*)$$

of complex conjugates, if $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$. Similarly, for $E \subset \mathbb{C}^k$, let

$$E^* := \{\lambda^* : \lambda \in E\}.$$

Given a commuting tuple $\tau = (T_1, \dots, T_k)$ of operators, denote by τ^* the tuple of adjoint operators:

$$\tau^* = (T_1^*, \dots, T_k^*).$$

Here "operator" stands for a bounded linear operator on a Hilbert space H over the complex numbers field $\mathbb{C}$ and the adjoint operator means the Hilbert space adjoint. The joint point spectrum, $\sigma_p(\tau)$, is the set of joint eigenvalues $\lambda = (\lambda_1, \dots, \lambda_k)$, so that $T_j x = \lambda_j x$ for some nonzero (joint eigenvector) $x \in H$. This suggests the following generalization of the second term of (1.1), which we propose to call *the c-defect spectrum* and denote $\sigma_*(\tau)$. (Here c stands for „closed")

Definition 1.2. $\sigma_*(T_1, \dots, T_k) := [(\sigma_p(T_1^*, \dots, T_k^*))^*]^-$, i.e., $\sigma_*(\tau) := [(\sigma_p(\tau^*))^*]^-$.

In other words, $\sigma_*(\tau)$ is the closure of the set $(\sigma_p(T_1^*, \dots, T_k^*))^*$ of all complex conjugates of joint eigenvalues of the adjoint tuple. Note that $\sigma_*(\tau)$ is contained in the *Taylor joint spectrum* of τ, denoted usually by $\sigma(\tau, H)$. If no further assumptions are made about the operators, one may have $\sigma_*(\tau) = \emptyset$ and σ_* is not a *joint spectrum* (or even a *subspectrum*) in the terminology of Żelazko (cf. [1, 2]).

Suppose one is trying to extend (1.1) to pairs, or k-tuples of commuting operators. The problem with interpreting the left-hand term in (1.1) stems from the existence of several different notions of the joint spectrum of a commuting tuple τ. Our primary object will be the joint spectrum

$$\sigma_{\mathcal{R}(\tau)}(\tau)$$

of τ in the unital, inverse-closed Banach algebra $\mathcal{R}(\tau)$ generated by τ. This joint spectrum is also denoted by $\sigma_{\mathcal{R}}(\tau)$ [1].

Equivalently, $\mathcal{R}(\tau)$ is the closed linear span (in the operator norm topology) of finite products of the resolvents $(T_j - \lambda_j)^{-1}$, where λ_j run over $\mathbb{C} \setminus \sigma(T_j)$.

The relation: $\lambda \notin \sigma_{\mathcal{R}(\tau)}(\tau)$ ("Banach algebra nonsingularity of $\tau - \lambda := (T_1 - \lambda_1 I, \dots, T_k - \lambda_k I)$") takes place iff the ideal generated in $\mathcal{R}(\tau)$ by $\tau - \lambda$ is not proper, so that it contains the unit element of $\mathcal{R}(\tau)$ (the identity operator $I : H \to H$). In other words, the equation

$$(T_1 A_1 - \lambda_1 A_1) + \dots + (T_k A_k - \lambda_k A_k) = I \tag{1.2}$$

should have a solution satisfying $A_1 \in \mathcal{R}(\tau), \dots, A_k \in \mathcal{R}(\tau)$.

The inclusions

$$\sigma_*(\tau) \subset \sigma(\tau, H) \subset \sigma_{\mathcal{R}(\tau)}(\tau) \tag{1.3}$$

are well-known even for arbitrary commuting tuples. In certain classes, however, both inclusions in (1.3) become equalities. This uniqueness of joint spectra is well known for, e.g., commuting normal tuples, but also for doubly commuting hyponormal pairs. The uniqueness is also known to hold in certain special cases for more general n-tuples of operators. Therefore, the following result appears to be useful.

Theorem 1.3. *There exists a pair of pure subnormal operators $(S, T) \in \mathcal{B}(H)$ such that $\sigma_*(\tau) \neq \sigma(\tau, H)$.*

Proof. In our example, $S = \bigoplus S_n, T = \bigoplus T_n$, where S_n, T_n are the operators of multiplication by complex coordinate functions z, w on certain function spaces on domains $\Omega_n \subset \mathbb{C}^2$. In [3] the set $G := \{(z, w) \in \mathbb{C}^2 : |w|, |z| < 1, |w| < \exp(V(z))\}$ is introduced with V, a subharmonic function in the unit disc $\mathbb{D}$ whose zero set $V^{-1}\{0\}$ consists of the following countable discrete subset of $\mathbb{D}$:

$$V^{-1}\{0\} = \bigcup_{n=2}^{\infty} C_n,$$

where C_n is the set of n^4 points equidistributed on the circle $|z| = 1 - \frac{1}{n}$. We take an ascending exhaustion of G by a sequence of smoothly bordered domains of holomorphy Ω_n containing the union of $C_n \times \{w : |w| \leq 1 - \frac{1}{n}\}$ with the bidiscs $\{(z, w) : |z| \leq 1 - \frac{1}{2n}, |w| \leq \frac{1}{4}\}$. We also assume that the closure of Ω_n is contained in Ω_{n+1}.

Denote by V_n the restriction of the usual Lebesgue measure to Ω_n. (Its volume element can be written as $dV_n = \frac{-1}{4} d\bar{z} \wedge dz \wedge d\bar{w} \wedge dw$, when $(z, w) \in \Omega_n$, or by $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$, when (z, w) is represented as $(x_1 + iy_1, x_2 + iy_2)$ in the real

coordinates). Let the measure ν_n on Ω_n be the sum $\nu_n = V_n + \mu_n$, where μ_n is the product of planar measure (yielded by $dx_2 \wedge dy_2$) on the disc $\{w : |w| \leq 1 - \frac{1}{n}\}$ with the equidistributed probability measure on C_n.
For example, if $C_n = \{(1 - \frac{1}{n}) \exp(\frac{2k\pi i}{n^4}) : k = 0, 1, \ldots, n^4 - 1\}$, then

$$\int f(z)g(w)d\mu_n = \left(\frac{1}{n^4} \sum_{k=0}^{n^4-1} f\left(\frac{n-1}{n} e^{\frac{2k\pi i}{n^4}} \right) \right) \cdot \int_0^{2\pi} \int_0^{1-\frac{1}{n}} g(re^{i\phi})\, r\, dr\, d\phi.$$

Our Hilbert spaces H_n are defined as the subspaces spanned in $L^2(\nu_n)$ by the complex polynomials $p(z, w)$. These are weighted Bergman spaces and in [3] it is shown that for T_n, S_n defined as the multiplication operators

$$(T_n f)(z, w) := zf(z, w), \ \ (S_n f)(z, w) := wf(z, w) \quad f \in H_n$$

the Taylor spectrum of their orthogonal sums, $S = \bigoplus S_n, T = \bigoplus T_n$ is larger than the closure of the union of the spectra of (S_n, T_n).

Therefore, in view of (1.3) it suffices to show that the union of $\sigma_*(S_n, T_n)$ is dense in $\sigma_*(S, T)$. But for the point spectra of (even infinite) direct sums the following equality holds

$$\sigma_p(\bigoplus_n S_n, \bigoplus_n T_n) = \bigcup_n \sigma_p(S_n, T_n).$$

This fact is as easily established, as for point spectra of single operators. This concludes the proof. □

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Received: October 26, 2006.