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SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY RUSCHEWEYH DERIVATIVE II

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Abstract. New class $\Sigma_\lambda(\alpha, \beta, \gamma)$ of univalent meromorphic functions defined by Ruscheweyh derivative in the punctured unit disk U^* is introduced. We study several Hadamard product properties. Some results connected to inclusion relations, neighborhoods of the elements of class $\Sigma_\lambda(\alpha, \beta, \gamma)$ and integral operators are obtained.

1 Introduction

Let Σ denote the class of meromorphic functions in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0,\ n \in I\!N = \{1, 2, 3, \cdots\}). \tag{1}$$

A function $f \in \Sigma$ is meromorphic starlike function of order β $(0 \leq \beta < 1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad (z \in U = U^* \cup \{0\}). \tag{2}$$

The class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma$ is meromorphic convex function of order β $(0 \leq \beta < 1)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \quad (z \in U = U^* \cup \{0\}). \tag{3}$$

Definition 1. *The Ruscheweyh derivative of order λ is denoted by $D^\lambda f$ and is defined as following: If*

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,$$

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then

$$D^{\lambda}f(z)=\frac{1}{z(1-z)^{\lambda+1}}*f(z)=z^{-1}+\sum_{n=1}^{\infty}D_n(\lambda)a_nz^n,\ \lambda>-1,\ z\in U^*, \quad (4)$$

where

$$D_n(\lambda)=\frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n+1)}{(n+1)!}. \quad (5)$$

Definition 2. *The Hadamard product (or convolution) of two functions $f(z)$ given by* (1) *and*

$$g(z)=z^{-1}+\sum_{n=1}^{\infty}b_nz^n \quad (6)$$

is defined by

$$(f*g)(z)=z^{-1}+\sum_{n=1}^{\infty}a_nb_nz^n.$$

Definition 3. *Let $f(z)\in\Sigma$ be given by* (1)*. The new class $\Sigma_\lambda(\alpha,\beta,\gamma)$ is defined by*

$$\begin{aligned}\Sigma_\lambda(\alpha,\beta,\gamma) &= \left\{f\in\Sigma: -Re\left\{\frac{z(D^\lambda f(z))'+\gamma z^2(D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z)+\gamma z(D^\lambda f(z))'}\right\}\right. \\ &\geq \alpha\left|\frac{z(D^\lambda f(z))'+\gamma z^2(D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z)+\gamma z(D^\lambda f(z))'}+1\right|+\beta,\ z\in U,\ 0\leq\beta<1, \\ &\left.\lambda>-1,\ 0\leq\gamma<\frac{1}{2},\quad \alpha\geq 0\right\}.\end{aligned} \quad (7)$$

2 Main Results

We will need the following lemma whose details can be found in [1].

Lemma 4. *The function $f(z)$ defined by* (1) *is in the class $\Sigma_\lambda(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}(n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\beta)]D_n(\lambda)a_n\leq(1-\beta)(1-2\gamma) \quad (8)$$

where $0\leq\beta<1,\alpha\geq0,0\leq\gamma<\frac{1}{2},\lambda>-1,D_n(\lambda)$ given by (5)*.*

Theorem 5. *Let the function $f(z)$ defined by* (1) *and the function $g(z)=z^{-1}+\sum_{n=1}^{\infty}b_nz^n$ $(b_n\geq0,n\in I\!N)$ be in the class $\Sigma_\lambda(\alpha,\beta,\gamma)$. Then the function $k(z)$ defined* *by* $k(z) = z^{-1} + \sum\limits_{n=1}^{\infty} a_n b_n z^n$ *is in the class* $\Sigma_\lambda(\alpha, \sigma, \gamma)$*, where* $(\alpha \geq 0, 0 \leq \gamma < \frac{1}{2},$ $0 \leq \beta < 1, 0 \leq \sigma < 1, n \in I\!N$ *and* $n \geq 1)$ *and* σ *is given by*

$$\sigma = 1 - \frac{(1-\beta)^2(1-2\gamma)(n+1)(\alpha+1)}{(1-\beta)^2(1-2\gamma) + (n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]^2}.$$

Proof. We must find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} a_n b_n \leq 1, \ D_n(\lambda) \text{ is given by } (5).$$

Since $f(z)$ and $g(z)$ are in $\Sigma_\lambda(\alpha, \beta, \gamma)$, then

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} b_n \leq 1.$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} \sqrt{a_n b_n} \leq 1.$$

We want only to show that

$$\frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} a_n b_n \leq \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} \sqrt{a_n b_n}.$$

This equivalently to

$$\sqrt{a_n b_n} \leq \frac{(1-\sigma)[n(\alpha+1) + (\alpha+\beta)]}{(1-\beta)[n(\alpha+1) + (\alpha+\sigma)]}.$$

Then

$$\sigma \leq 1 - \frac{(1-\beta)^2(1-2\gamma)(n+1)(\alpha+1)}{(1-\beta)^2(1-2\gamma) + (n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]^2}.$$

□

Theorem 6. *Let the function $f(z)$ defined by (1) and $g(z)$ given by $g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$ be in the class $\Sigma_\lambda(\alpha, \beta, \gamma)$, then the function $k(z)$ defined by $k(z) = z^{-1} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) z^n$ is in the class $\Sigma_\lambda(\alpha, \sigma, \gamma)$, where $0 \le \gamma < \frac{1}{2}$, $\lambda > -1$, $0 \le \sigma < 1$, $0 \le \beta < 1$, $\alpha \ge 0$ and*

$$\sigma = 1 - \frac{8(1-\beta)^2(1-2\gamma)(1+\alpha)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2 + 4(1-\beta)^2(1-2\gamma)}.$$

Proof. We must find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} (a_n^2 + b_n^2) \le 1. \tag{9}$$

Since $f(z)$ and $g(z)$ are in $\Sigma_\lambda(\alpha, \beta, \gamma)$, we get

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \right\}^2 \le 1 \tag{10}$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} b_n \right\}^2 \le 1. \tag{11}$$

Combining the inequalities (10) and (11), gives

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} \right\}^2 (a_n^2 + b_n^2) \le 1.$$

But, $k(z) \in \Sigma_\lambda(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \right\} (a_n^2 + b_n^2) \le 1. \tag{12}$$

The inequality (12) would obviously imply (9) if

$$\begin{aligned} &\frac{(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \\ &\le \frac{[(n\gamma - \gamma + 1)(n(\alpha+1) + (\alpha+\beta))D_n(\lambda)]^2}{2[(1-\beta)(1-2\gamma)]^2} \\ &= \frac{q^2}{2}. \end{aligned}$$

Hence

$$\frac{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \le \frac{q^2}{2}$$

or

$$\frac{(1-\sigma)}{(1+\alpha)} \geq \frac{2(n+1)(n\gamma-\gamma+1)D_n(\lambda)}{(1-2\gamma)q^2+2(n\gamma-\gamma+1)D_n(\lambda)}.$$

The right hand is decreasing function of n and its maximum if $n=1$.

Now

$$\frac{(1-\sigma)}{(1+\alpha)} \tag{13}$$

$$\geq \frac{2(n+1)(n\gamma-\gamma+1)(1-\beta)^2(1-2\gamma)D_n(\lambda)}{[(n\gamma-\gamma+1)(n(\alpha+1)+(\alpha+\beta))D_n(\lambda)]^2+2(n\gamma-\gamma+1)(1-\beta)^2(1-2\gamma)D_n(\lambda)}$$

Simplify (13) we obtain

$$\frac{(1-\sigma)}{(1+\alpha)} \geq \frac{8(1-\beta)^2(1-2\gamma)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2+4(1-\beta)^2(1-2\gamma)}$$

or

$$\sigma \leq 1-\frac{8(1-\beta)^2(1-2\gamma)(1+\alpha)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2+4(1-\beta)^2(1-2\gamma)}.$$

This completes the proof of theorem. □

We get the inclusive properties of the class $\Sigma_\lambda(\alpha,\beta,\gamma)$.

Theorem 7. *Let* $\alpha \geq 0$, $0 \leq \beta < 1$, $0 \leq \gamma < \frac{1}{2}$, $\sigma \geq 0$, *then*

$$\Sigma_\lambda(\alpha,\beta,\gamma) \subseteq \Sigma_\lambda(\alpha,\sigma,0)$$

where

$$\sigma \leq 1-\frac{(n+1)(\alpha+1)(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\gamma)},\ n\in I\!N,\ n\geq 1.$$

Proof. Let $f(z) \in \Sigma_\lambda(\alpha,\beta,\gamma)$, then from Lemma 4, we have

$$\sum_{n=1}^{\infty} \frac{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \leq 1. \tag{14}$$

We want to find the value σ such that

$$\sum_{n=1}^{\infty} \frac{[n(1+\alpha)+(\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)} a_n \leq 1. \tag{15}$$

The inequality (14) would obviously imply (15) if

$$\frac{[n(1+\alpha)+(\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)} \leq \frac{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} = q.$$

Therefore,

$$\frac{[n(1+\alpha)+(\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)} \le q. \tag{16}$$

Hence

$$\frac{(1-\sigma)}{(1+\alpha)} \ge \frac{(n+1)D_n(\lambda)}{(q+D_n(\lambda))}(n \ge 1, n \in I\!N). \tag{17}$$

The right hand side of (17) decreases as n increases and so is maximum for $n = 1$. So (17) is satisfied provided

$$\frac{(1-\sigma)}{(1+\alpha)} \ge \frac{(n+1)(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\gamma)} = y.$$

Obviously $y < 1$ and

$$\sigma \le 1 - \frac{(n+1)(\alpha+1)(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\gamma)}.$$

This completes the proof of the theorem. □

3 (n,δ)-Neighborhoods on $\Sigma_\lambda^\sigma(\alpha,\beta,\gamma)$

The next, we determine the inclusion relation involving (n,δ) - neighborhoods. Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruscheweyh [4], but for meromorphic function studied by Liu and Srivastava [3], we define the (n,δ)-neighborhood of a function $f(z) \in \Sigma$ by

$$N_{n,\delta}(f) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \le \delta \right\}. \tag{18}$$

For the identity function $e(z) = z$, we have

$$N_{n,\delta}(e) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|b_n| \le \delta \right\}.$$

Definition 8. *A function $f \in \Sigma$ is said to be in the class $\Sigma_\lambda^\sigma(\alpha,\beta,\gamma)$ if there exists a function $g \in \Sigma_\lambda(\alpha,\beta,\gamma)$ such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \sigma, \quad (z \in U, 0 \le \sigma < 1).$$

Theorem 9. *If $g \in \Sigma_\lambda(\alpha,\beta,\gamma)$ and*

$$\sigma = 1 - \frac{\delta(2\alpha+\beta+1)(\lambda+1)(\lambda+2)}{(2\alpha+\beta+1)(\lambda+1)(\lambda+2) - 2(1-\beta)(1-2\gamma)} \tag{19}$$

then $N_{n,\delta}(g) \subset \Sigma_\lambda^\sigma(\alpha,\beta,\gamma)$.

Proof. Let $f \in N_{n,\delta}(g)$, then we get from (18) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \le \delta$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \le \delta, \quad (n \in I\!N).$$

Since $g \in \Sigma_\lambda(\alpha,\beta,\gamma)$, we have from Lemma 4

$$\sum_{n=1}^{\infty} b_n \le \frac{2(1-\beta)(1-2\gamma)}{(2\alpha+\beta+1)(\lambda+1)(\lambda+2)}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{n=1}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=1}^{\infty} b_n} \le \frac{\delta(2\alpha+\beta+1)(\lambda+1)(\lambda+2)}{(2\alpha+\beta+1)(\lambda+1)(\lambda+2) - 2(1-\beta)(1-2\gamma)} = 1-\sigma.$$

Thus, by Definition 8, $f \in \Sigma_\lambda^\sigma(\alpha,\beta,\gamma)$ for σ given by (19). □

4 Integral Operators

Next, we consider integral transforms of functions in the class $\Sigma_\lambda(\alpha,\beta,\gamma)$.

Theorem 10. *Let the function $f(z)$ given by (1) be in $\Sigma_\lambda(\alpha,\beta,\gamma)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du, \quad (0 < u \le 1,\ 0 < c < \infty) \tag{20}$$

is in $\Sigma_\lambda(\alpha,\delta,\gamma)$, where

$$\delta = \frac{(c+2)(2\alpha+\beta+1) - c(1-\beta)(2\alpha+1)}{(c+2)(2\alpha+\beta+1) + c(1-\beta)}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2(1-\beta)(1-2\gamma)}{(2\alpha+\beta+1)(\lambda+1)(\lambda+2)}z.$$

Proof. Let $f(z) = z^{-1} + \sum\limits_{n=1}^{\infty} a_n z^n$ in $\Sigma_\lambda(\alpha, \beta, \gamma)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^c f(uz)du \\ &= c\int_0^1 \left(\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} a_n u^{n+c} z^n\right) du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\delta)]D_n(\lambda)}{(c+n+1)(1-\delta)(1-2\gamma)} a_n \leq 1. \tag{21}$$

Since $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \leq 1.$$

Note that (21) is satisfied if

$$\frac{c(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\delta)]D_n(\lambda)}{(c+n+1)(1-\delta)(1-2\gamma)} \leq \frac{(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)}.$$

Rewriting the inequality, we have

$$\begin{aligned} & c(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\delta)]D_n(\lambda)(1-\beta)(1-2\gamma) \\ \leq\ & (c+n+1)(1-\delta)(1-2\gamma)(n\gamma - \gamma + 1)[n(1+\alpha) + (\alpha+\beta)]D_n(\lambda). \end{aligned}$$

Solving for δ, we have

$$\delta \leq \frac{(c+n+1)[n(1+\alpha) + (\alpha+\beta)] - c(1-\beta)[n(1+\alpha) + \alpha]}{(c+n+1)[n(1+\alpha) + (\alpha+\beta)] + c(1-\beta)} = F(n).$$

A computation shows that

$$\begin{aligned} & F(n+1) - F(n) = \\ = & [((c+n+2)((n+1)(1+\alpha) + (\alpha+\beta)) - c(1-\beta)((n+1)(1+\alpha) + \alpha)) \times \\ & ((c+n+1)(n(1+\alpha) + (\alpha+\beta)) + c(1-\beta)) - ((c+n+1)(n(1+\alpha) + (\alpha+\beta)) \\ & -c(1-\beta)(n(1+\alpha) + \alpha))((c+n+2)((n+1)(1+\alpha) + (\alpha+\beta)) + c(1-\beta))]/ \\ & [((c+n+1)(n(1+\alpha) + (\alpha+\beta)) + c(1-\beta)) \times \\ & ((c+n+2)((n+1)(1+\alpha) + (\alpha+\beta) + c(1-\beta))] > 0 \end{aligned}$$

for all n. This means that $F(n)$ is increasing and $F(n) \geq F(1)$. Using this, the result follows. □

Theorem 11. *If $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$, then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du, \quad (0 < u \leq 1,\ 0 < c < \infty)$$

is in $\Sigma_\lambda(\alpha, \frac{1+\beta c}{2+c}, \gamma)$. The result is sharp for

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)} z^n.$$

Proof. By Definition of F, we get

$$F(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n.$$

By Lemma 4, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)}{(1 - \frac{1+\beta c}{2+c})(1-2\gamma)(c+n+1)} a_n \leq 1. \tag{22}$$

Since if $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$, then (22) satisfies if

$$\frac{c}{(1 - \frac{1+\beta c}{2+c})(1-2\gamma)(c+n+1)} \leq \frac{1}{(1-\beta)(1-2\gamma)}$$

or equivalently, when

$$Y(n, c, \beta) = \frac{c(1-\beta)}{(1 - \frac{1+\beta c}{2+c})(c+n+1)} \leq 1$$

since $Y(n, c, \beta)$ is a decreasing function of n $(n \geq 1)$, then the proof is completed. The result is sharp for

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)[n(\alpha+1) + (\alpha+\beta)]D_n(\lambda)} z^n.$$

□

Theorem 12. *Let the function $f(z)$ given by (1) be in $\Sigma_\lambda(\alpha, \beta, \gamma)$,*

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = z^{-1} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} a_n z^n, \ c > 0. \tag{23}$$

Then $F(z)$ is in $\Sigma_\lambda(\alpha, \beta, \gamma)$ for $|z| \leq r(\alpha, \beta, c, \delta)$, where

$$r(\alpha, \beta, c, \delta) = \inf_n \left(\frac{c(1-\delta)[n(\alpha+1) + (\alpha+\beta)]}{(1-\beta)(c+n+1)[n(\alpha+1) + (\alpha+\delta)]} \right)^{\frac{1}{n+1}},$$

$n = 1, 2, 3, \cdots$.

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\beta)]D_n(\lambda)} z^n, \;\; n = 1, 2, 3, \cdots .$$

Proof. Let

$$w = -\frac{z(D^\lambda f(z))' + \gamma z^2 (D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + z(D^\lambda f(z))'}.$$

Then it is sufficient to show that

$$|w-1| < |w+1-2\delta|.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\delta)]D_n(\lambda)(c+n+1)}{c(1-\delta)(1-2\gamma)} a_n |z|^{n+1} \leq 1 \tag{24}$$

since $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$, then by Lemma 4, we have

$$\sum_{n=1}^{\infty} (n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\beta)]D_n(\lambda) a_n \leq (1-\beta)(1-2\gamma).$$

The equation (24) is satisfied if

$$\begin{aligned} & \frac{(n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\delta)]D_n(\lambda)(c+n+1)}{c(1-\delta)(1-2\gamma)} a_n |z|^{n+1} \\ \leq \; & \frac{(n\gamma-\gamma+1)[n(\alpha+1)+(\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n. \end{aligned}$$

Solving for $|z|$, we obtain the result. □

Corollary 13. *Let the function $f(z)$ given by (1) be in $\Sigma_\lambda(0, \beta, \gamma)$ and $F(z)$ given by (23). Then $F(z)$ is in $\Sigma_\lambda(0, \beta, \gamma)$ for $|z| < r(0, \beta, \gamma, c, \delta)$, where*

$$r(0, \beta, \gamma, c, \delta) = \inf_n \left(\frac{c(1-\delta)(n+\beta)}{(1-\beta)(c+n+1)(n+\delta)} \right)^{\frac{1}{n+1}}, \quad n = 1, 2, 3, \cdots .$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)(n+\beta)D_n(\lambda)} z^n, \;\; n = 1, 2, 3, \cdots .$$

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