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A NOTE ON VIZING'S GENERALIZED CONJECTURE

Abstract. In this note we give a generalized version of Vizing's conjecture concerning the distance domination number for the cartesian product of two graphs.

Keywords: graph, dominating sets, Vizing's conjecture.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple and finite graph. A set $D \subseteq V$ is a *dominating* set of G if each vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For further details on domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4, 5].

The *cartesian product* $G \times H$ of two graphs G, H is the graph with $V(G \times H) = V(G) \times V(H)$ and two vertices (u, v) and (w, t) being adjacent in $G \times H$ if either $u = w$ and $vt \in E(H)$ or $uw \in E(G)$ and $v = t$. In [11] Vizing conjectured that for any graphs G and H, $\gamma(G \times H) \geq \gamma(G) \times \gamma(H)$. No positive or negative answer to Vizing's conjecture has been given yet. Jacobson and Kinch [6, 7] gave important results to support the conjecture.

The *distance* between two vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path from x to y. If S is a set of vertices of G and v is a vertex of G, then the distance from v to S, denoted by $d_G(v, S)$, is the minimum distance from v to a vertex of S. The *open k-neighborhood* $N_k(x)$ of a vertex x of G is defined as $N_k(x) = \{y \in V : d_G(x, y) \leq k, x \neq y\}$ and the *closed k-neighborhood* $N_k[x]$ of x is $N_k(x) \cup \{x\}$. The *maximum k-degree* of G is given by $\Delta_k(G) = \max\{|N_k(v)| : v \in V\}$.

Meir and Moon [10] introduced distance k concepts, i.e., the k-domination and the k-packing concepts. Let $k \geq 1$ be an integer, a set $D_k \subseteq V$ is said to be *k-dominating* (resp. *k-packing*) set of G if for every vertex v of G, $d_G(v, D_k) \leq k$ (resp. $d_G(x, y) > k$ for all pairs of distinct vertices x and y in D_k). The *k-dominating number* $\gamma_k(G)$ (resp. the *k-packing number* $\rho_k(G)$) is the size of a smallest k-dominating (resp. a largest

k-packing) set of G. Note that $\gamma_1(G) = \gamma(G)$, also every k-packing set of G is a k-dominating set of G, therefore $\rho_k(G) \geq \gamma_k(G)$.

Let $k \geq 1$ be an integer. For any graph G, the *k-th power* of G is the graph denoted G^k with $V(G^k) = V(G)$ and two vertices u and v are adjacent if $1 \leq d_G(u,v) \leq k$.

Remark 1. *It is easy to show that for an arbitrary graph G, $\gamma(G^k) = \gamma_k(G)$ and $\rho_2(G^k) = \rho_{2k}(G)$.*

2. GENERALIZATION OF VIZING'S CONJECTURE

Consider two graphs G and H. Since $G^k \times H^k$ is a spanning graph of $(G \times H)^k$, there is $\gamma(G^k \times H^k) \geq \gamma((G \times H)^k)$.

a) b)

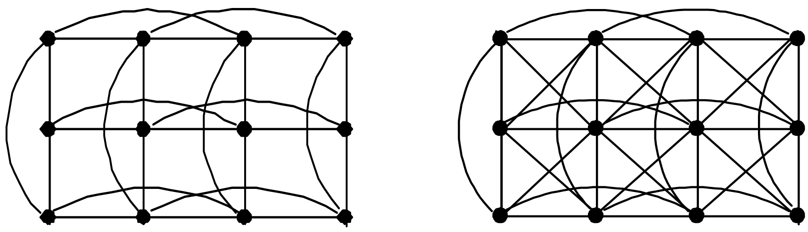

Fig. 1. $P_4^2 \times P_3^2$ (a); $(P_4 \times P_3)^2$ (b)

To see that the above inequality may be strict, consider the graphs shown in Figures 1a and 1b, where the domination numbers are 3 and 2, respectively.

Therefore, even if we suppose that Vizing's conjecture is verified for any graphs G and H, we cannot conclude directly that also the k-domination number of the product of any two graphs is equal to at least the product of the k-domination numbers of these graphs. Hence, we state below a conjecture which generalizes Vizing's.

Vizing's Generalized Conjecture (**VGC**): for any graphs G and H,

$$\gamma_k(G \times H) \geq \gamma_k(G)\gamma_k(H).$$

An immediate consequence of this conjecture is the following result.

Proposition 1. *Let $k \geq 1$ be an integer. If VGC is true for the product of G and H, then Vizing's conjecture is also true for the product of G^k and H^k.*

Proof. Since $G^k \times H^k$ is a spanning graph of $(G \times H)^k$, it follows that $\gamma(G^k \times H^k) \geq \gamma((G \times H)^k) = \gamma_k(G \times H) \geq \gamma_k(G)\gamma_k(H) = \gamma(G^k)\gamma(H^k)$. $\square$

The next proposition is easy to prove.

Proposition 2. *Let $k \geq 1$ be an integer. For any graphs G and H with $\min\{\gamma_k(G), \gamma_k(H)\} = 1, \gamma_k(G \times H) \geq \gamma_k(G)\gamma_k(H)$.*

Proposition 3. *Let $k \geq 1$ be an integer. For any graph G and its complement $\overline{G}$,*

$$\gamma_k(G \times \overline{G}) \geq \gamma_k(G)\gamma_k(\overline{G}).$$

Proof. The case $k = 1$ has been proved by Jaeger and Payan [8] as well as El-Zahar and Pareek [2]. Assume that $k \geq 2$, and let S be a maximum k-packing set of G. If $|S| = 1$, then $\gamma_k(G) = 1$, and by Proposition 2, we obtain the result. If $|S| \geq 2$, then every vertex v of $V(G)$ is at a distance not longer than two from S in $\overline{G}$. Thus, each vertex of S $k-$dominates $\overline{G}$. Hence, $\gamma_k(\overline{G}) = 1$, and by Proposition 2, we obtain the result. □

Our next results extend those of [6] and [7].

Proposition 4. *Let $k \geq 1$ be an integer. For any graphs G and H,*

$$\gamma_k(G \times H) \geq \max\left(\frac{|V(H)|}{1 + \Delta_k(H)}\gamma_k(G), \frac{|V(G)|}{1 + \Delta_k(G)}\gamma_k(H)\right).$$

Proof. Let D be a minimum k-dominating set of $G \times H$ and let i be a vertex of $V(H)$. We put:

$G_i = G \times \{i\}$,

$D_i = D \cap G_i$ for $i = 1, \ldots, |V(H)|$, and

$F_i = \{t \in V(G_i) : d_{G\times H}(t, D_i) > k\}$.

Since the graphs G_i and G are isomorphic, $\gamma_k(G_i) = \gamma_k(G)$. Then for every i, $|F_i| \geq \gamma_k(G) - |D_i|$, as otherwise $F_i \cup D_i$ is a k-dominating set of G_i of a size less than $\gamma_k(G)$, a contradiction. Consequently,

$\Delta_k(H)\,|D| \geq \sum_{i=1}^{|V(H)|} |F_i| \geq \sum_{i=1}^{|V(H)|} (\ \gamma_k(G) - |D_i|) = |V(H)|\ \gamma_k(G) - |D|$, and hence

$$\gamma_k(G \times H) \geq \frac{|V(H)|}{1 + \Delta_k(H)}\gamma_k(G).$$

The second part is obtained similarly. □

Proposition 5. *Let $k \geq 1$ be an integer. For any graphs G and H,*

$$\gamma_k(G \times H) \geq \max\{\gamma_k(G)\rho_{2k}(H), \gamma_k(H)\rho_{2k}(G)\}.$$

Proof. Let D be a minimum k-dominating set of $G \times H$. Clearly, for any vertex $i \in V(H)$, there is $|(V(G) \times N_k[i]) \cap D| \geq \gamma_k(G)$.

Now let W be the maximum $2k$-packing set of H. Then for any pair of vertices i and j in W, $N_k[i] \cap N_k[j] = \emptyset$. It follows that $(V(G) \times N_k[i]) \cap (V(G) \times N_k[j]) = \emptyset$, implying that

$$|D| \geq \sum_{i\in W} |(V(G) \times N_k[i]) \cap D| \geq \sum_{i\in W} \gamma_k(G) = \gamma_k(G)\rho_{2k}(H).$$

Likewise, $\gamma_k(G \times H) \geq \gamma_k(H)\rho_{2k}(G)$. □

In [10], Meir and Moon showed that every tree T satisfies $\gamma_k(T) = \rho_{2k}(T)$. This equality is also true for strongly chordal graphs (which contain trees). Indeed, it is enough to combine the results of Lubiw and Farber with Remark 1. Lubiw [9] has proved that if G is a strongly chordal graph, then for any positive integer k, G^k is strongly chordal graph, while Farber [3] has established that $\gamma(G) = \rho_2(G)$. We note that Domke, Hedetniemi and Laskar [1] have proved the equality between γ_k and ρ_{2k} for block graphs which are contained in strongly chordal class.

According to Proposition 5, the following holds true.

Proposition 6. *Let $k \geq 1$ be an integer. If G is a strongly chordal graph, then for any graph H,*

$$\gamma_k(G \times H) \geq \gamma_k(H)\gamma_k(G).$$

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