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CONSTRAINT INTERFACE PRECONDITIONING FOR THE INCOMPRESSIBLE STOKES EQUATIONS*

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Abstract. We introduce a novel substructuring approach for solving the incompressible Stokes equations for the case of enclosed flows. We employ a simple distribution of the global pressure constraint to subdomains which allows for a natural decomposition into Stokes subdomain problems with Dirichlet data which are well-posed and inf-sup stable. This approach yields a saddle-point problem on the interface Γ involving an operator which is continuous and coercive on $H^{1/2}(\Gamma)$ and which is restricted to the interface trace space of functions satisfying the incompressibility constraint. We derive the form of the constraints explicitly, both for the continuous and for the discrete case. This allows us to design directly a class of interface preconditioners of constraint type, thus avoiding the need to formulate a coarse level problem. Our analysis indicates that the resulting solution method has performance independent of the mesh-size, while numerical results point to a mild dependence on the number of subdomains. We illustrate the technique on some standard test problems and for a range of domains, meshes, and decompositions.

Key words. domain decomposition, incompressible Stokes flow, iterative substructuring, interface preconditioners, constraint preconditioners, discrete fractional Sobolev norms

AMS subject classifications. 65F10, 65N55, 65F08, 65F30

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1. Introduction.

Domain decomposition methods (DDMs) form an established methodology for the parallel solution of large systems of equations arising from the discretization of PDE problems. In the case of incompressible flow models, however, DDMs face a number of challenges inherent in their saddle-point formulation such as indefiniteness, coarse grid problem definition (including inf-sup stability), and well-posedness of subdomain problems.

Over the past decade, it has become evident that for constrained problems a black-box approach is generally not possible and that special treatment is required in order to achieve optimality with respect to the geometric and physical parameters involved. In particular, generalizations to the saddle-point case of existing successful methods for scalar elliptic problems require additional effort as well as analysis. A notable example for the case of nonoverlapping DDMs is the extension of the popular BDDC and FETI-DP methods to the indefinite case [26], [25] (see also the recent modified approaches in [22], [19], [20], [38], [21], [42], [43]). Another example is the reformulation of the Stokes problem in [24] where the pressure is eliminated in order to bring the problem into positive-definite form. This strategy is applied to *hp* discretizations of the Stokes problem. For this type of discretization, see also the approach in [1]. Spectral methods are considered in a DDM context in [31], [9]. Overlapping approaches have also been considered for Stokes [23] and Navier-Stokes problems [12], [8], [15], [16], [19].

In general, iterative substructuring approaches target the Schur complement (or the discrete Steklov–Poincaré operator) associated with the interface generated by a decomposition of the computational domain. Typically, this yields a problem involving

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the unknowns corresponding to the interface degrees of freedom. For the Stokes problem, these may include both velocity and pressure unknowns (as is the case for Taylor–Hood elements), or only velocity unknowns (as is the case for discontinuous approximations of the pressure). In the former situation, a substructuring approach will arise from subdomain problems where Dirichlet pressure boundary conditions are imposed on subdomains. Moreover, the corresponding Steklov–Poincaré operator will involve both velocity and pressure variables. On the other hand, the latter case may be seen to represent a more natural approach in the sense that only velocity boundary conditions are needed on subdomains and only velocity variables occur in the interface problem (for details, see [33, Chap. 5]). This also motivates our approach in this paper.

In this work we introduce a novel substructuring approach for the incompressible Stokes problem with Dirichlet boundary conditions. Specifically, we choose a formulation which distributes the pressure constraint to subdomains, so that the resulting nonoverlapping DDM yields well-posed Dirichlet subdomain problems. For the case of discontinuous pressure approximations, the resulting coupling is an interface problem, which involves a constrained velocity $H^{1/2}(\Gamma)$-equivalent operator, with no pressure variables present. This is related in some sense to formulations involving the trace space of functions satisfying the incompressibility constraint introduced in [7], [32]; see also [33, Chap. 5, p. 183]. Moreover, our modified formulation allows for the explicit derivation of the interface constraints. This allows us to avoid the need for a coarse level problem and instead directly design interface preconditioners of constraint type which can be implemented via sparse Krylov methods. The indefiniteness of the problem is thus catered for, both at the subdomain and at the interface level. We provide an analysis which indicates that performance is independent of the mesh size. Numerical experiments on structured and unstructured meshes and decompositions indicate that there is only a mild dependence on the number of subdomains.

The paper is organized as follows. In the next section we introduce the Stokes problem together with auxiliary results. We also present the problem reformulation which allows for an explicit description of the interface problem (or Schur complement). In section 3 we discuss the domain decomposition approach and derive the constrained interface operator. The matrix representations of our substructuring approach are included in section 4, while section 5 discusses constraint preconditioners and the optimal choices available for the Stokes problem; Krylov approximations of constraint preconditioners are also described and analyzed. Finally, in section 6 we present numerical results on a range of decompositions and meshes and for different types of domains.

2. Problem formulation. Let Ω be an open bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We consider the following Stokes problem

$$-\Delta u + \operatorname{grad} p = f \quad \text{in } \Omega, \tag{2.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2.1c}$$

together with the constraint

$$\int_\Omega p = \mu, \tag{2.2}$$

which ensures the uniqueness of the solution (u, p). We also assume that the following compatibility condition holds,

$$\int_{\partial\Omega} n \cdot g = 0. \tag{2.3}$$

In the following, we will consider solving (2.1) for the case of zero pressure mean, i.e., $\mu = 0$ in (2.2). However, the nonzero case will also arise naturally as a result of our domain decomposition approach. Moreover, for simplicity of exposition, we take $g = 0$, although our experiments will consider nonzero Dirichlet data.

Let $V := [H_0^1(\Omega)]^d$, $Q := L^2(\Omega)$. We will also work with the subspace $Q_\mu =: L^2_\mu(\Omega) := \{q \in Q : (q, 1) = \mu\}$, where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-inner product. We associate with problem (2.1) subject to the constraint (2.2) the Lagrangian $\mathcal{L}_1(v, q, \rho) : V \times Q \times \mathbb{R}$ given by

$$\mathcal{L}_1(v, q, \rho) = \mathcal{L}_0(v, q) + \rho\left[(q, 1) - \mu\right],$$

where

$$\mathcal{L}_0(v, q) = \frac{1}{2}(\nabla v, \nabla v) - (\operatorname{div} v, q) - (f, v). \tag{2.4}$$

A critical point (u, p, λ) of $\mathcal{L}_1$ satisfies the variational problem corresponding to the Stokes equations (2.1) subject to the constraint (2.2):

$$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v), \tag{2.5a}$$

$$-(\operatorname{div} u, q) + \lambda(q, 1) = 0, \tag{2.5b}$$

$$\rho(p, 1) = \rho\mu \tag{2.5c}$$

for all $(v, q, \rho) \in V \times Q \times \mathbb{R}$. The pair (u, p) is also a critical point of $\mathcal{L}_0(v, q) : V \times Q_\mu$ with corresponding variational formulation

$$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v), \tag{2.6a}$$

$$-(\operatorname{div} u, q) = 0 \tag{2.6b}$$

for all $(v, q) \in V \times Q_\mu$. In this sense, formulations (2.5), (2.6) are said to be equivalent. Moreover, they are both known to be LBB stable [6].

Consider a nonoverlapping partition of Ω into open bounded subdomains Ω_i satisfying

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i, \qquad \Omega_i \cap \Omega_j \equiv \emptyset \quad (i \neq j).$$

We also let $\Gamma \subset \mathbb{R}^{d-1}$ denote the set of internal boundaries associated with the above partition of Ω,

$$\Gamma = \bigcup_{i=1}^{N} \Gamma_i, \qquad \Gamma_i := \partial\Omega_i \setminus \partial\Omega.$$

One of the issues with formulating problem (2.1) on subdomains is the global constraint (2.2). Recall that we are interested in the zero pressure mean case ($\mu = 0$): however, this relation cannot be distributed in a natural way to each subdomain as a

global zero pressure mean condition does not imply zero local pressure means. A suitable way to "decompose" the constraint (2.2) is provided by the following equivalent formulation

$$\sum_{i=1}^{N}\left(\int_{\Omega_i} p - c_i\right) = 0 \quad \text{subject to } \sum_{i=1}^{N} c_i = 0. \tag{2.7}$$

While the global constraint (2.2) is replaced with another global constraint on the coefficients c_i, the above reformulation allows for a natural decomposition of the Stokes problem into subdomain problems, which are locally constrained by pressure integrals over Ω_i. The corresponding Lagrangian is

$$\mathcal{L}_2(v,q,\rho,d,\omega) = \mathcal{L}_0(v,q) + \sum_{i=1}^{N}\rho_i((q,1)-d_i) + \omega\sum_{i=1}^{N} d_i, \tag{2.8}$$

where $\mathcal{L}_0$ is defined in (2.4). The critical points (u,p,λ,c,τ) of $\mathcal{L}$ satisfy the following variational formulation

$$\text{(WF)}: \begin{cases} \displaystyle\sum_{i=1}^{N}\left[(\nabla u_i,\nabla v_i) - (\operatorname{div} v_i, p_i)\right] = \sum_{i=1}^{N}(f_i,v_i), \\ \displaystyle\sum_{i=1}^{N}\left[-(\operatorname{div} u_i, q_i) + \lambda_i\,(q_i,1)\right] = 0, \\ \rho_i\,((p_i,1)-c_i) = 0 \quad (i=1,\dots,N), \\ d_i(\tau-\lambda_i) = 0 \quad (i=1,\dots,N), \\ \omega\displaystyle\sum_{i=1}^{N} c_i = 0 \end{cases}$$

for all $(v,q,\rho,d,\omega)\in X := V\times Q\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}$. The above formulation is more suitable than (2.5) for domain decomposition formulations; in particular, we note that the first three equations yield a decoupled set of well-posed subdomain problems of the form (2.5) with modified data

$$\begin{cases} (\nabla u_i,\nabla v_i) - (\operatorname{div} v_i, p_i) = (f_i,v_i), \\ -(\operatorname{div} u_i, q_i) + \lambda_i\,(q_i,1) = 0, \qquad\qquad (i=1,\dots,N). \\ \rho_i\,(p_i,1) = \rho_i c_i \end{cases}$$

This approach yields linear systems that can be parallelized directly, without introducing an awkward or nonphysical treatment of the global constraint (2.2). We consider the full formulation in section 3.

2.1. Notation and standard results. Throughout the paper we will use the following notation and standard results. Given an open simply connected domain U in $\mathbb{R}^d$, its boundary will be denoted by ∂U. We denote by $H^m(U)$ the Sobolev space of order m equipped with norm $\|\cdot\|_{m,U}$ and seminorm $|\cdot|_{m,U}$ with the convention $H^0(U) = L^2(U)$. The Sobolev spaces of real index $0\le s\le m$ are defined as interpolation spaces of index $\theta = 1-s/m$ for the pair $(H^m(U),L^2(U))$; we denote this by the following stencil

$$H^s(\Omega) := [H^m(U),L^2(U)]_\theta, \qquad \theta = 1-s/m.$$

For any s, the space $H_0^s(U)$ denotes the completion of $C_0^\infty(U)$ in $H^s(U)$ (see, e.g., [27, p. 60]). In particular, we shall be interested in the interpolation space

$$H^{1/2}(U) = \left[H^1(U), L^2(U)\right]_{1/2}$$

for which there holds $H_0^{1/2}(U) \equiv H^{1/2}(U)$. Another space of interest is $H_{00}^{1/2}(U)$ which is a subspace of $H_0^{1/2}(U)$ and is defined as the interpolation space of index $1/2$ for the pair $[H_0^1(U), L^2(U)]$,

$$H_{00}^{1/2}(U) = \left[H_0^1(U), H^0(U)\right]_{1/2}.$$

Norms on $H^{1/2}(U), H_{00}^{1/2}(U)$ will be denoted by the same notation $|\cdot|_{1/2,U}$ or $\|\cdot\|_{1/2,U}$ with the assumption that it is evident from the context which space is under consideration. The dual of $H_{00}^{1/2}(U)$ is denoted by $(H_{00}^{1/2}(U))' \subset H^{-1/2}(U)$, where $H^{-1/2}(U) := (H^{1/2}(U))' \equiv (H_0^{1/2}(U))'$.

Finally, we will make use of the trace operator $\gamma_0 : H^1(U) \to H^{1/2}(\partial U)$ which is known to be surjective and continuous, i.e., there exists a constant $c_\gamma(U)$ such that

$$\|\gamma_0 v\|_{1/2,\partial U} \le c_\gamma(U)\|v\|_{1,U} \quad \forall v \in H^1(U).$$

A similar inequality holds if we take $\gamma_0 : H_0^1(U) \to H_{00}^{1/2}(\partial U)$:

$$\|\gamma_0 v\|_{1/2,\partial U} \le c_\gamma(U)\|v\|_{1,U} \quad \forall v \in H_0^1(U).$$

We will also assume that the following Poincaré inequality holds,

$$\|v\|_{0,U} \le C_P(U)\, |v|_{1,U}\,. \tag{2.9}$$

Finally, in order to simplify notation, we introduce the generic notation

$$\Lambda := \Lambda(\partial U) := \left[H^{1/2}(\partial U)\right]^d \quad \text{or } \Lambda := \left[H_{00}^{1/2}(\partial U)\right]^d, \tag{2.10}$$

with the choice of product interpolation space and that of ∂U obvious from the context. We write the above trace inequalities for the vector case in the generic form

$$\|\gamma_0 v\|_\Lambda \le c_\gamma(U)\|v\|_{1,U} \quad \forall v \in \left[H_*^1(U)\right]^d. \tag{2.11}$$

2.2. Stokes extensions. The concept of extensions arises naturally in the formulation of nonoverlapping domain decomposition problems [33]. This is also the case for our formulation. Let $U \subset \mathbb{R}^d$ and let $(\xi, \mu) \in [H^{1/2}(\partial U)]^d \times \mathbb{R}$ be given. Let $F : [H^{1/2}(\partial U)]^d \to [H^1(U)]^d$ denote any extension operator from $[H^{1/2}(\partial U)]^d$ to $[H^1(U)]^d$. Consider the problem of finding $(w, r, \nu) \in [H_0^1(U)]^d \times L^2(U) \times \mathbb{R}$ such that for all $(v, q, \rho) \in [H_0^1(U)]^d \times L^2(U) \times \mathbb{R}$

$$(\nabla w, \nabla v) - (\operatorname{div} v, r) = -(\nabla F\xi, \nabla v), \tag{2.12a}$$

$$-(\operatorname{div} w, q) + \nu\,(q, 1) = (\operatorname{div} F\xi, q), \tag{2.12b}$$

$$\rho\,(r, 1) = \rho\mu. \tag{2.12c}$$

The Stokes extension of the data (ξ, μ) is denoted by $E_\mu\xi$ and is defined to be

$$E_\mu\xi := w + F\xi,$$

where $w, F\xi$ are given in (2.12). Equivalently, the Stokes extension satisfies

$$(\nabla E_\mu \xi, \nabla v) - (\operatorname{div} v, r) = 0, \tag{2.13a}$$

$$-(\operatorname{div} E_\mu \xi, q) + \nu\,(q, 1) = 0, \tag{2.13b}$$

$$\rho\,(r, 1) = \rho\mu, \tag{2.13c}$$

where $E_\mu \xi_{|\partial U} = \xi$. We will not include the compatibility condition (2.3) in our definition of Stokes extensions, thus allowing for $\operatorname{div} E_\mu \xi$ to be nonzero in a weak sense. This is perfectly justified, since ultimately we are interested in devising a solver for problem (2.1), rather than an alternative physical model.

The following result provides identities satisfied by Stokes extensions, which will be useful for our later derivation and analysis.

LEMMA 2.1. *Let $q \in L^2(U)$ and let $(E_\mu \xi, r, \nu)$ satisfy* (2.13). *Then*

$$(\operatorname{div} E_\mu \xi, q) = \frac{(q,1)}{|U|}(n \cdot \xi, 1)_{L^2(\partial U)}, \tag{2.14}$$

$$(\operatorname{div} E_\mu \xi, r) = \frac{\mu}{|U|}(n \cdot \xi, 1)_{L^2(\partial U)}, \tag{2.15}$$

$$\nu = \frac{1}{|U|}(n \cdot \xi, 1)_{L^2(\partial U)}. \tag{2.16}$$

Proof. Let $\bar{q} = \frac{1}{|U|}(q,1)$ denote the average of q over the domain U. Then $q - \bar{q} \in L^2_0(U)$ and, by the equivalence between (2.5), (2.6), we have

$$(\operatorname{div} E_\mu \xi, q - \bar{q}) = 0,$$

so that

$$(\operatorname{div} E_\mu \xi, q) = (\operatorname{div} E_\mu \xi, q - \bar{q}) + (\operatorname{div} E_\mu \xi, \bar{q}) = \bar{q} \int_{\partial U} n \cdot \xi,$$

which is (2.14). Using this result, the remaining relations follow from (2.13b) and (2.13c), respectively. ∎

We end this section with the following regularity result which can be found in [39, Prop. 2.3].

PROPOSITION 2.2. *Let Ω be an open set in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. Then there exist a unique function $u \in V$ and a function $p \in Q$ unique up to a constant which are solutions of* (2.1) *and which satisfy*

$$\|u\|_V + \|p\|_{Q/\mathbb{R}} \le C(\Omega)\left(\|f\|_{V'} + \|g\|_\Lambda\right). \tag{2.17}$$

We remark here that the solution of (2.5) also satisfies the above regularity result with p measured in the full space Q.

3. Domain decomposition formulation. Let $V_i := H^1(\Omega_i), V_i^0 = H^1_0(\Omega_i)$ and define $Q_{\mu,i} := L^2_\mu(\Omega_i)$. In order to derive a domain decomposition formulation, we let

$$u_i = u_i^{\{1\}} + u_i^{\{2\}}, \quad p_i = p_i^{\{1\}} + p_i^{\{2\}}, \quad \lambda_i = \lambda_i^{\{1\}} + \lambda_i^{\{2\}},$$

where $u_i^{\{1\}} \in V_i^0, p_i^{\{1\}} \in Q_{0,i}$, and $p_i^{\{2\}} \in Q_{c_i,i}$. We also let $v_i = v_i^0 + E_i w_i$, where $v_i^0 \in V_i^0$, $w_i = v_{i|\Gamma_i} \in V_i$, and $E_i w_i := E_{c_i} w_i$ is the discrete Stokes extension of data (w_i, c_i) to subdomain Ω_i (cf. (2.13)). Finally, we define

$$z := u\,|_\Gamma \in \Lambda, \quad z_i = u_i\,|_{\Gamma_i} = u_i^{\{2\}}\,|_{\Gamma_i} \in \Lambda_i.$$

Replacing these expressions for u_i, p_i, λ_i in (WF), we obtain

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\Big[(\nabla u_i^{\{1\}},\nabla v_i^0)+(\nabla u_i^{\{2\}},\nabla v_i^0)-(\operatorname{div} v_i^0,p_i^{\{1\}})-(\operatorname{div} v_i^0,p_i^{\{2\}})\Big]=\sum_{i=1}^{N}[(f_i,v_i^0)+R_i], \\ \displaystyle\sum_{i=1}^{N}\Big[-(\operatorname{div} u_i^{\{1\}},q_i)-(\operatorname{div} u_i^{\{2\}},q_i)+\lambda_i^{\{1\}}(q_i,1)+\lambda_i^{\{2\}}(q_i,1)\Big]=0, \\ d_i(\tau-\lambda_i^{\{1\}}-\lambda_i^{\{2\}})=0 \quad (i=1,\dots,N), \\ \rho_i\left[\left(p_i^{\{1\}},1\right)+\left(p_i^{\{2\}},1\right)-c_i\right]=0 \quad (i=1,\dots,N), \\ \omega\displaystyle\sum_{i=1}^{N}c_i=0, \end{cases}$$

where

$$R_i=\sum_{i=1}^{N}(f_i,E_iw_i)-\left(\nabla u_i^{\{1\}},\nabla E_iw_i\right)-\left(\nabla u_i^{\{2\}},\nabla E_iw_i\right) +\left(\operatorname{div}E_iw_i,p_i^{\{1\}}\right)+\left(\operatorname{div}E_iw_{ih},p_i^{\{2\}}\right).$$

Using (2.14), (2.15), and the zero-mean assumption on $p_i^{\{1\}}$, we simplify the last two terms in R_i as follows:

$$\left(\operatorname{div}E_iw_i,p_i^{\{1\}}\right)=\frac{(p_i^{\{1\}},1)}{|\Omega_i|}(n_i\cdot w_i,1)_{\partial\Omega_i}=0,\quad \left(\operatorname{div}E_iw_i,p_i^{\{2\}}\right)=\frac{c_i}{|\Omega_i|}(n_i\cdot w_i,1)_{\partial\Omega_i}.$$

Let the bilinear form $s(\cdot,\cdot):\Lambda\times\Lambda$ be defined via

$$s(z,w):=\sum_{i=1}^{N}(\nabla E_iz_i,\nabla E_iw_i). \tag{3.1}$$

The weak formulation (WF) decouples into the following three sets of problems:

$$\text{I}\begin{cases} (\nabla u_i^{\{1\}},\nabla v_i^0)-(\operatorname{div}v_i^0,p_i^{\{1\}})=(f_i,v_i^0), \\ -(\operatorname{div}u_i^{\{1\}},q_i)+\lambda_i^{\{1\}}(q_i,1)=0, \qquad (i=1,\dots,N), \\ \rho_i\left(p_i^{\{1\}},1\right)=0 \end{cases}$$

$$\text{II}\begin{cases} s(z,w)-\displaystyle\sum_{i=1}^{N}\frac{c_i}{|\Omega_i|}(n_i\cdot w_i,1)_{\Gamma_i}=\sum_{i=1}^{N}\Big[(f_i,E_iw_i)-(\nabla u_i^{\{1\}},\nabla E_iw_i)\Big], \\ d_i\left[\tau-\frac{1}{|\Omega_i|}(n_i\cdot z_i,1)_{\Gamma_i}\right]=0 \quad (i=1,\dots,N), \\ \omega\displaystyle\sum_{i=1}^{N}c_i=0, \end{cases}$$

$$\text{III}\begin{cases} (\nabla u_i^{\{2\}},\nabla v_i^0)-(\operatorname{div}v_i^0,p_i^{\{2\}})=0, \\ -(\operatorname{div}u_i^{\{2\}},q_i)+\lambda_i^{\{2\}}(q_i,1)=0, \qquad (i=1,\dots,N), \\ \rho_i\left(p_i^{\{2\}},1\right)=\rho_ic_i \end{cases}$$

where we used the fact that, by (2.16),

$$\lambda_i^{\{1\}} = 0, \quad \lambda_i^{\{2\}} = \frac{1}{|\Omega_i|}(n_i \cdot z_i, 1)_{\Gamma_i}.$$

DEFINITION 3.1. *We will refer to the above decoupled form of the Stokes problem as the three-step formulation. We will refer to* $s(\cdot,\cdot)$ *defined in* (3.1) *as the bilinear form induced by the three-step formulation of the Stokes problem.*

We remark here that problems I and III are Dirichlet subdomain problems of the form (2.5) and as such they are LBB stable and have a unique solution.

The above formulation does not enforce pressure continuity over the domain Ω. Thus, the subsequent derivation will only be valid for the case of discontinuous finite element spaces for the pressure approximation. We will therefore consider only this choice below. However, we note here that a similar approach is possible if the pressure is required to be continuous. In particular, the resulting subdomain problems will involve Stokes problems with both velocity and pressure essential boundary conditions for which well-posedness will have to be considered specifically. This approach is generally acknowledged to be more technical (cf. [33, Chap. 5], [40, Chap. 9]) and will not be considered here.

Let $\mathcal{T}_h$ denote a conforming subdivision of Ω into disjoint shape-regular simplices T (see, e.g., [10, p. 124]) of diameter at most h. Let $\mathcal{P}_k(U)$ denote the space of polynomials of degree $k \geq 0$ defined on a set $U \subset \mathbb{R}^d$. Let

$$V_h = \mathcal{V}_h^d, \quad \mathcal{V}_h = \left\{\nu_h \in H_0^1(\Omega) \cap C^0(\Omega) : \nu_{h|T} \in \mathcal{P}_2(T), \forall T \in \mathcal{T}_h\right\},$$
$$Q_h = \left\{q_h \in L^2(\Omega) : q_{h|T} \in \mathcal{P}_0(T), \forall T \in \mathcal{T}_h\right\}.$$

We will denote by V_{hi}, Q_{hi} the corresponding subspaces of functions defined on subdomains Ω_i.

Let $\{\phi_j\}_{1\leq j\leq n}, \{\psi_k\}_{1\leq k\leq m}$ denote sets of finite element bases for V_h, Q_h, respectively, so that

$$u_h = \sum_{j=1}^{n} \mathbf{u}_j \phi_j, \quad p_h = \sum_{k=1}^{m} \mathbf{p}_k \psi_k.$$

Define the interface spaces

$$\Lambda_{hi} := \Lambda_{hi}(\Gamma_i) := \operatorname{span}\left\{\chi_j := \gamma_0(\Gamma_i)\phi_j : \operatorname{supp}\phi_j \cap \Gamma_i \neq \emptyset\right\},$$
$$\Lambda_h := \Lambda_h(\Gamma) := \bigcup_{i=1}^{N} \Lambda_{hi}(\Gamma_i).$$

Note $\Lambda_h(\Gamma)$ is a product piecewise polynomial space of degree 2 defined on Γ. Moreover, $\Lambda_h(\Gamma) = \gamma_0(\Gamma)V_h \subset \Lambda(\Gamma)$.

Let $X_h = V_h \times Q_h \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ and consider the following discrete variational formulation: find $(u_h, p_h, \lambda, c, \tau) \in X_h$ such that for all $(v_h, q_h, \rho, d, \omega) \in X_h$ the

following relations hold:

$$(\mathrm{WF})_h : \begin{cases} \displaystyle\sum_{i=1}^{N} [(\nabla u_{hi}, \nabla v_{hi}) - (\operatorname{div} v_{hi}, p_{hi})] = \sum_{i=1}^{N} (f_i, v_{hi}), \\ \displaystyle\sum_{i=1}^{N} [-(\operatorname{div} u_{hi}, q_{hi}) + \lambda_i (q_{hi}, 1)] = 0, \\ \rho_i ((p_{hi}, 1) - c_i) = 0 \quad (i = 1, \dots, N), \\ d_i(\tau - \lambda_i) = 0 \quad (i = 1, \dots, N), \\ \omega \displaystyle\sum_{i=1}^{N} c_i = 0. \end{cases}$$

We note here the slight abuse of notation for the purpose of simplicity: the variables $\lambda, c, \tau, \rho, d, \omega$ also occur in the continuous variational formulation introduced in the previous section. Henceforth, we will work only with the formulation $(\mathrm{WF})_h$.

A similar three-step formulation can be derived for the discrete case as in the previous section; we include it below:

$$\mathrm{I}_h \begin{cases} (\nabla u_{hi}^{\{1\}}, \nabla v_{hi}^0) - (\operatorname{div} v_{hi}^0, p_{hi}^{\{1\}}) = (f_i, v_{hi}^0), \\ -(\operatorname{div} u_{hi}^{\{1\}}, q_{hi}) + \lambda_i^{\{1\}} (q_{hi}, 1) = 0, \qquad (i = 1, \dots, N), \\ \rho_i \left(p_{hi}^{\{1\}}, 1 \right) = 0 \end{cases}$$

$$\mathrm{II}_h \begin{cases} s(z_h, w_h) - \displaystyle\sum_{i=1}^{N} \frac{c_i}{|\Omega_i|} (n_i \cdot w_{hi}, 1)_{\Gamma_i} = \sum_{i=1}^{N} \left[(f_i, E_i w_{hi}) - (\nabla u_{hi}^{\{1\}}, \nabla E_i w_{hi}) \right], \\ d_i \left[\tau - \dfrac{1}{|\Omega_i|} (n_i \cdot z_{hi}, 1)_{\Gamma_i} \right] = 0 \quad (i = 1, \dots, N), \\ \omega \displaystyle\sum_{i=1}^{N} c_i = 0, \end{cases}$$

$$\mathrm{III}_h \begin{cases} (\nabla u_{hi}^{\{2\}}, \nabla v_{hi}^0) - (\operatorname{div} v_{hi}^0, p_{hi}^{\{2\}}) = 0, \\ -(\operatorname{div} u_{hi}^{\{2\}}, q_{hi}) + \lambda_i^{\{2\}} (q_{hi}, 1) = 0, \qquad (i = 1, \dots, N), \\ \rho_i \left(p_{hi}^{\{2\}}, 1 \right) = \rho_i c_i \end{cases}$$

where we recall here that $s(\cdot, \cdot)$ is the bilinear form induced by the three-step formulation of the discrete weak formulation of the Stokes problem

$$s(z_h, w_h) := \sum_{i=1}^{N} (\nabla E_i z_{hi}, \nabla E_i w_{hi})$$

with

$$z_h := u_h \mid_{\Gamma} \in \Lambda_h, \quad z_{hi} = u_{hi} \mid_{\Gamma_i} = u_{hi}^{\{2\}} \mid_{\Gamma_i} \in \Lambda_{hi}.$$

Thus, problems (WF), $(\mathrm{WF})_h$ decouple into a set of independent standard Stokes problems with zero data (problems I, I_h), a set of independent standard Stokes problems with nonzero data (problems III, III_h) and an interface problem which is a globally constrained saddle-point problem (problems II, II_h). This three-step procedure has a matrix representation which can be shown to be equivalent to a Schur complement method, as is the case with other substructuring approaches.

4. Matrix representations. The matrix representation of $(\mathrm{WF})_h$ is

$$K\mathbf{x} := \begin{pmatrix} A & B^T & O & O & \mathbf{0} \\ B & O & R & O & \mathbf{0} \\ O & R^T & O & -I & \mathbf{0} \\ O & O & -I & O & \mathbf{1} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \boldsymbol{\lambda} \\ \mathbf{c} \\ \tau \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \end{pmatrix} =: \mathbf{b},$$

where

$$A_{ij} = (\nabla\phi_j, \nabla\phi_i),\ B_{jk} = -(\operatorname{div}\phi_j, \psi_k),\ R_{k\ell} = (\psi_k, 1)_{\Omega_\ell}$$

for the index ranges $1 \le i, j \le n$, $1 \le k \le m$, $1 \le \ell \le N$, and with $I \in \mathbb{R}^{N\times N}$. We also define $r_i := R_{J_i i}$.

Let now I_i denote the index set for the velocity basis functions with support contained in Ω_i; with a standard abuse of notation, let also Γ denote the index set for the velocity basis functions with support intersecting Γ. Similarly, let J_i denote the index set for the pressure basis functions with support contained in Ω_i; we note here that $\sup\psi_k \cap \Gamma = \emptyset$, given the choice of Q_h. Thus, we have

$$n = n_I + n_\Gamma, \quad n_I := \sum_{i=1}^{N} n_{I_i}, \quad n_{I_i} := |I_i|, \quad m = \sum_{i=1}^{N} m_i, \quad m_i := |J_i|.$$

Using the numbering given by the index set

$$\left\{ \{I_i, n + J_i, n + m + N + i\}_{1\le i\le N}, \Gamma, \{n + m + i\}_{1\le i\le N}, n + m + 2N + 1 \right\},$$

the above system is permuted to

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}, \tag{4.1}$$

where

$$K_{11} = \bigoplus_{i=1}^{N} \begin{pmatrix} A_{I_iI_i} & B_{J_iI_i}^T & \mathbf{0}_{I_i} \\ B_{J_iI_i} & O_{J_iJ_i} & r_i \\ \mathbf{0}_{I_i}^T & r_i^T & 0 \end{pmatrix}, \quad K_{22} = \begin{pmatrix} A_{\Gamma\Gamma} & O_{N\Gamma}^T & \mathbf{0}_\Gamma \\ O_{N\Gamma} & O_{NN} & \mathbf{1}_N \\ \mathbf{0}_\Gamma^T & \mathbf{1}_N^T & 0 \end{pmatrix},$$

$$K_{12} = \operatorname*{\mathbf{Col}}_{i=1}^{N} \begin{pmatrix} A_{I_i\Gamma} & O_{I_iN} & \mathbf{0}_{I_i} \\ B_{J_i\Gamma} & O_{J_iN} & \mathbf{0}_{J_i} \\ \mathbf{0}_\Gamma^T & -\mathbf{e}_i^T & 0 \end{pmatrix} =: \operatorname*{\mathbf{Col}}_{i=1}^{N} K_{12,i},$$

where **Col** indicates column concatenation of matrices over the indicated range. Correspondingly, we have

$$\mathbf{x}_1 = \operatorname*{\mathbf{Col}}_{i=1}^{N} \begin{pmatrix} \mathbf{u}_{I_i} \\ \mathbf{p}_{J_i} \\ \lambda_i \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \mathbf{u}_\Gamma \\ \mathbf{c} \\ \tau \end{pmatrix}, \quad \mathbf{b}_1 = \operatorname*{\mathbf{Col}}_{i=1}^{N} \begin{pmatrix} \mathbf{f}_{I_i} \\ \mathbf{0}_{J_i} \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} \mathbf{f}_\Gamma \\ \mathbf{0}_N \\ 0 \end{pmatrix}.$$

With this notation in place, a Schur complement method computes a solution $\mathbf{x}$ to (4.1) via the following sequence of problems

$$\text{I. } K_{11}\mathbf{x}_1^{\{1\}} = \mathbf{b}_1,\ \text{II. } \Sigma\mathbf{x}_2 = \mathbf{f}_2 - K_{12}^T\mathbf{x}_1^{\{1\}},\ \text{III. } K_{11}\mathbf{x}_1^{\{2\}} = -K_{12}\mathbf{x}_2,\ \mathbf{x}_1 := \mathbf{x}_1^{\{1\}} + \mathbf{x}_1^{\{2\}},$$

where $\Sigma = K_{22} - K_{12}^T K_{11}^{-1} K_{12}$ is the Schur complement of K_{11} in K.

4.1. The Schur complement. The Schur complement Σ associated with the splitting indicated above will be referred to as the interface Schur complement. For this problem formulation, this matrix has an explicit expression given by the following result.

PROPOSITION 4.1. *The Schur complement Σ has the following explicit structure*

$$\Sigma = \begin{pmatrix} S & -B_{\Gamma m}^T \bar{R}_{mN} & \mathbf{0}_\Gamma \\ -(B_{\Gamma m}^T \bar{R}_{mN})^T & O_{NN} & \mathbf{1}_N \\ \mathbf{0}_\Gamma^T & \mathbf{1}_N^T & 0 \end{pmatrix}, \tag{4.2}$$

where S is an $n_\Gamma \times n_\Gamma$ matrix and where $\bar{R}_{J_i i} = \bar{r}_i := \frac{r_i}{\|r_i\|^2}$.

Proof. We have

$$\Sigma = K_{22} - K_{12}^T K_{11}^{-1} K_{12},$$

where the second term has the form

$$\sum_{i=1}^{N} \begin{pmatrix} A_{I_i\Gamma} & O_{I_iN} & \mathbf{0}_{I_i} \\ B_{J_i\Gamma} & O_{J_iN} & \mathbf{0}_{J_i} \\ \mathbf{0}_\Gamma^T & -\mathbf{e}_i^T & 0 \end{pmatrix}^T \begin{pmatrix} A_{I_iI_i} & B_{J_iI_i}^T & \mathbf{0}_{I_i} \\ B_{J_iI_i} & O_{J_iJ_i} & r_i \\ \mathbf{0}_{I_i}^T & r_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{I_i\Gamma} & O_{I_iN} & \mathbf{0}_{I_i} \\ B_{J_i\Gamma} & O_{J_iN} & \mathbf{0}_{J_i} \\ \mathbf{0}_\Gamma^T & -\mathbf{e}_i^T & 0 \end{pmatrix}.$$

The result follows by direct calculation of the summand, and the expression for the inverse given in Lemma 4.2 below with $\mathbf{q} = r_i$. ∎

LEMMA 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and let $B \in \mathbb{R}^{m\times n}, \mathbf{q} \in \mathbb{R}^n \setminus \{0\}$ with $\ker B^T = \operatorname{span}\{\mathbf{q}\}$. Let*

$$K = \begin{pmatrix} A & B^T & 0 \\ B & O & \mathbf{q} \\ 0 & \mathbf{q}^T & 0 \end{pmatrix}.$$

Then

$$K^{-1} = \begin{pmatrix} M & N^T & 0 \\ N & P & \mathbf{z} \\ 0^T & \mathbf{z}^T & 0 \end{pmatrix},$$

where

$$\mathbf{z} = \frac{\mathbf{q}}{\|\mathbf{q}\|^2}, \quad P = -S_q^{-1}\left[I - \frac{\mathbf{q}\mathbf{q}^T}{\|\mathbf{q}\|^2}\right], \quad N^T = -A^{-1}B^T P,$$
$$M = A^{-1}\left[A - B^T S_q B\right] A^{-1}, \quad S_q = \mathbf{q}\mathbf{q}^T + BA^{-1}B^T.$$

Proof. See the appendix for the proof. ∎

Remark 4.1. The expression for S is also available explicitly, although it is not included here as our convergence analysis in section 5 does not require it. However, we point out that S is the matrix representation of the bilinear form $s(\cdot,\cdot) : \Lambda_h \times \Lambda_h$ induced by our three-step formulation (cf. Definition 3.1).

Remark 4.2. The saddle-point form of the Schur complement Σ is explicit in the constraints. This property does not hold for other domain decomposition formulations for the Stokes problem (e.g., [25], [22]), but only for the reformulation given in (WF). This will represent an important advantage when designing a substructuring preconditioner.

One can show that the decomposition of problem $(\mathrm{WF})_h$ introduced in section 3, corresponds to the Schur complement method described above. More specifically, the three sets of problems in the previous section become three sets of algebraic systems:

$$\text{I.} \left\{ \begin{pmatrix} A_{I_iI_i} & B_{J_iI_i}^T & \mathbf{0}_{I_i} \\ B_{J_iI_i} & O_{J_iJ_i} & r_i \\ \mathbf{0}_{I_i}^T & r_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{I_i}^{\{1\}} \\ \mathbf{p}_{J_i}^{\{1\}} \\ \lambda_i^{\{1\}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{I_i} \\ \mathbf{0}_{J_i} \\ 0 \end{pmatrix} \quad (i = 1, \ldots, N) \,,$$

$$\text{II.} \left\{ \begin{pmatrix} S & -B_{\Gamma m}^T \bar{R}_{mN} & \mathbf{0}_\Gamma \\ -(B_{\Gamma m}^T \bar{R}_{mN})^T & O_{NN} & \mathbf{1}_N \\ \mathbf{0}_\Gamma^T & \mathbf{1}_N^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Gamma \\ \mathbf{c} \\ \tau \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Gamma \\ \mathbf{0}_N \\ 0 \end{pmatrix} - \sum_{i=1}^N K_{12,i}^T \begin{pmatrix} \mathbf{u}_{I_i}^{\{1\}} \\ \mathbf{p}_{J_i}^{\{1\}} \\ \lambda_i^{\{1\}} \end{pmatrix} ,$$

$$\text{III.} \left\{ \begin{pmatrix} A_{I_iI_i} & B_{J_iI_i}^T & \mathbf{0}_{I_i} \\ B_{J_iI_i} & O_{J_iJ_i} & r_i \\ \mathbf{0}_{I_i}^T & r_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{I_i}^{\{2\}} \\ \mathbf{p}_{J_i}^{\{2\}} \\ \lambda_i^{\{2\}} \end{pmatrix} = -K_{12,i} \begin{pmatrix} \mathbf{u}_\Gamma \\ \mathbf{c} \\ \tau \end{pmatrix} \quad (i = 1, \ldots, N) \,.$$

Remark 4.3. The above formulation is the generalization of substructuring techniques for scalar elliptic problems to the case of the Stokes system. In particular, the reformulation (WF) allows one, at the discrete level, to identify the subdomain problems as nonsingular diagonal subblocks of the system matrix K. In this sense, this approach is more natural, without requiring the more involved reformulations implicit in BDDC or FETI methods (e.g., [26]).

5. Constraint interface preconditioners. Given the explicit form of the interface Schur complement derived in the previous section, we aim to design explicitly a class of preconditioners with similar block structure. This will avoid the issue of formulating a coarse level problem, which for the case of the Stokes problem is not a straightforward task (see, for example, [26], [22], [19]).

We start by defining the following generic block matrix, as a permuted version of the interface matrix in system II. above:

$$\Sigma_\pm(P) = \begin{pmatrix} P & \mathbf{0}_\Gamma & -B_{\Gamma m}^T \bar{R}_{mN} \\ \mathbf{0}_\Gamma^T & 0 & \mathbf{1}_N^T \\ \mp(B_{\Gamma m}^T \bar{R}_{mN})^T & \pm\mathbf{1}_N & O_{NN} \end{pmatrix} = \begin{pmatrix} P_0 & C^T \\ \pm C & O_{NN} \end{pmatrix},$$

where P is assumed to be symmetric and nonsingular and P_0 denotes the matrix P bordered on the right and bottom by zero and $C \in \mathbb{R}^{N \times (n_\Gamma + 1)}$ is a constraint matrix with structure given below:

$$P_0 = \begin{pmatrix} P & \mathbf{0}_\Gamma \\ \mathbf{0}_\Gamma^T & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -\left(B_{\Gamma m}^T \bar{R}_{mN}\right)^T & \mathbf{1}_N \end{pmatrix}.$$

The structure of the above matrix is an example of a so-called *constraint preconditioner*. The concept of constraint preconditioners has been employed and analyzed widely (see [5] and the references therein). The following two results describe the eigenvalue distribution of a generic constraint preconditioned system.

LEMMA 5.1. *Let*

$$\mathcal{K} = \begin{pmatrix} K & B^T \\ B & O \end{pmatrix}, \quad \mathcal{G}_\pm = \begin{pmatrix} G & B^T \\ \pm B & O \end{pmatrix},$$

and let $Z_B \in \mathbb{R}^{n \times (n-m)}$ be a matrix whose columns form an orthogonal basis for the null space of B. Assume the following properties hold:

1. $K, G \in \mathbb{R}^{n\times n}$ *are symmetric with* $\operatorname{rank}[K\ G] = n-k\ \ (0<k<n)$;
2. $B \in \mathbb{R}^{m\times n}$ *has full rank and* $\ker K \cap \ker B = \ker G \cap \ker B = \{\mathbf{0}\}$;
3. $K\mathbf{x} \neq \pm G\mathbf{x}$ *for any* $\mathbf{x} \notin \ker K \cap \ker G$.

Then the preconditioned matrix $\mathcal{P}_\pm := \mathcal{G}_\pm^{-1}\mathcal{K}$ *has the following eigenvalue distribution:*

i. $\lambda = 1$ *with multiplicity* m;
ii. $\lambda = \pm 1$ *with multiplicity* m;
iii. $\lambda = \sigma_j$ $(j = 1, \ldots, n-m)$, *where* σ_j *are the eigenvalues of the pencil* $[Z_B^T K Z_B, Z_B^T G Z_B]$.

Moreover, $\mathcal{P}_+$ *is defective,* $\mathcal{P}_-$ *is diagonalizable, and both have minimal polynomials of degree* $n-m+2$.

Proof. See [28] for the proof. ∎

We will refer to the eigenvalues σ_j as the nonunit eigenvalues of $\mathcal{P}_\pm$.

LEMMA 5.2. *Let the assumptions of Lemma* 5.1 *hold with* $\ker G = \{0\}$. *Assume further that* $G^{-1}K$ *has* n *distinct eigenvalues* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. *Then the nonunit eigenvalues of* $\mathcal{P}_\pm$ *satisfy the following interlacing property*

$$\lambda_k \le \sigma_k \le \lambda_{k+m}, \quad k = 1, \ldots, n-m.$$

Proof. For a proof, see [18]. ∎

A direct application of Lemma 5.1 yields the following result.

PROPOSITION 5.3. *The pencil* $[\Sigma_+(S), \Sigma_-(P)]$ *is diagonalizable with eigenvalues* σ *given below.*

i. $\sigma = 1$ *with multiplicity* N.
ii. $\sigma = -1$ *with multiplicity* N.
iii. *The remaining* $n_\Gamma + 1 - N$ *eigenvalues satisfy*

$$Z_C^T S_0 Z_C \mathbf{v} = \sigma Z_C^T P_0 Z_C \mathbf{v},$$

where the columns of $Z_C \in \mathbb{R}^{(n_\Gamma+1)\times(n_\Gamma+1-N)}$ *form an orthogonal basis for the null space of* C.

Remark 5.1. The extra N eigenvalues arising due to the larger reformulated problem are relocated to ± 1 through the above choice of constraint preconditioner.

Since S_0, P_0 are singular, it is not possible to obtain an interlacing result for the pencil $[\Sigma_+(S), \Sigma_-(P)]$ as provided by Lemma 5.2. However, this issue can be circumvented using a certain reformulation and we obtain the following interlacing result involving the eigenvalues of the pencil $[S, P]$, where we recall that P is a generic symmetric nonsingular matrix.

PROPOSITION 5.4. *Let* $\{\sigma_k, k = 1, \ldots, n_\Gamma + 1 - N\}$ *denote the nonunit eigenvalues of the pencil* $[\Sigma_+(S), \Sigma_-(P)]$. *Then the* σ_k *interlace the eigenvalues* λ_j *of the pencil* $[S, P]$:

$$\lambda_k \le \sigma_k \le \lambda_{k+N-1}, \quad k = 1, \ldots, n_\Gamma + 1 - N.$$

Proof. The nonunit eigenvalues of the pencil $[\Sigma_+(S), \Sigma_-(P)]$ are also the non-unit eigenvalues of the pencil

$$\left[\begin{pmatrix} S & -B_{\Gamma m}^T \bar{R}_{mN} & \mathbf{0}_\Gamma \\ -(B_{\Gamma m}^T \bar{R}_{mN})^T & O_{NN} & \mathbf{1}_N \\ \mathbf{0}_\Gamma^T & \mathbf{1}_N^T & 0 \end{pmatrix}, \begin{pmatrix} P & -B_{\Gamma m}^T \bar{R}_{mN} & \mathbf{0}_\Gamma \\ -(B_{\Gamma m}^T \bar{R}_{mN})^T & O_{NN} & \mathbf{1}_N \\ \mathbf{0}_\Gamma^T & \mathbf{1}_N^T & 0 \end{pmatrix} \right].$$

Letting $B = \begin{pmatrix} \mathbf{0}_\Gamma^T & \mathbf{1}_N^T \end{pmatrix}$ in Lemma 5.1, the corresponding Z_B can be taken to have the form

$$Z_B = \begin{pmatrix} I_\Gamma & O_{\Gamma N-1} \\ O_{N,\Gamma} & Z_{\mathbf{1}_N} \end{pmatrix}.$$

Using Lemma 5.1, with $m = 1$, we conclude that there are 2 eigenvalues at 1, with the remaining $n_\Gamma + N$ given by the eigenvalues of the above matrix pencil projected on the null space spanned by Z_B, which we write taking into account the structure of Z_B as

$$\left[\begin{pmatrix} S & \tilde{B}^T \\ \tilde{B} & O_{N-1,N-1} \end{pmatrix}, \begin{pmatrix} P & \tilde{B}^T \\ \tilde{B} & O_{N-1,N-1} \end{pmatrix} \right].$$

By Lemma 5.1, the above pencil has $2N - 2$ eigenvalues at 1, with the remaining $n_\Gamma - N + 1$ nonunit eigenvalues satisfying by Lemma 5.2 the interlacing property

$$\lambda_k \leq \sigma_k \leq \lambda_{k+N-1}, \quad k = 1, \ldots, n_\Gamma + 1 - N. \qquad \square$$

Remark 5.2. The above results focus on the eigenvalue distribution of the preconditioned system. The classic convergence bound for GMRES [37, p. 216], [14, p. 54], however, requires both the eigenvalues and the condition number of the eigenvector matrix of the preconditioned system (for the nondefective case) be bounded independently of the parameters of interest in the problem. Similarly, a bound that covers also the defective case requires that the contour length of the pseudospectrum of the preconditioned system remains bounded, in addition to the bound on the eigenvalues (see [41], [14, p. 57]).

In light of the above remark, we assume that the convergence of GMRES is governed by the distribution of eigenvalues alone, with the other quantities in various convergence bounds assumed to be benign. In particular, we will say that a preconditioner is optimal if the eigenvalues of the preconditioned system are bounded independently of the size of the problem. Thus, we expect the performance of a Krylov method coupled with an optimal preconditioner to be independent of problem size. The numerical experiments in section 6 confirm that this is a reasonable assumption. If P is an optimal preconditioner for S, then the eigenvalues of the pencil $[\Sigma_+(S), \Sigma_\pm(P)]$ will also be bounded independently of size, i.e., $\Sigma_\pm(P)$ is an optimal preconditioner for $\Sigma_+(S)$. The choice of P is considered next.

5.1. Optimal interface preconditioners. We now turn our attention to suitable candidates P for approximating the matrix S. First, recall that S is the matrix representation of the bilinear form $s(\cdot, \cdot)$ in the basis $\{\chi_j\}$ of Λ_h, where χ_j represents the restriction to Γ of the velocity basis element ϕ_j, whenever this restriction is nonzero (cf. Remark 4.1). Consider again

$$s(z_h, w_h) = \sum_{i=1}^{N} (\nabla E_i z_{hi}, \nabla E_i w_{hi}).$$

In the following we will employ the notation (2.10) introduced in section 1 and will also refer to Definition 3.1.

LEMMA 5.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set with boundary $\partial\Omega$ of class $\mathcal{C}^2$. Let $s(\cdot, \cdot) : \Lambda \times \Lambda$ be the bilinear form induced by a three-step formulation of the Stokes problem. Then there exist constants c_1, c_2 such that for all $\eta, \nu \in \Lambda$,*

$$c_2 \|\eta\|_\Lambda^2 \leq s(\eta, \eta) \leq c_1 \|\eta\|_\Lambda^2.$$

Proof. Note first that

$$s(\eta,\eta)=\sum_{i=1}^{N}(\nabla E_i\eta_i,\nabla E_i\eta_i)=\sum_{i=1}^{N}|E_i\eta_i|^2_{1,\Omega_i}\ .$$

Using the regularity result (2.17), we get

$$s(\eta,\eta)\le\sum_{i=1}^{N}C_i(\Omega_i)\|\eta_i\|^2_{\Lambda(\Gamma_i)}\le c_1\|\eta\|^2_\Lambda$$

with $c_1=\max_i C_i(\Omega_i)$. Using the trace inequality (2.11) we get

$$|E_i\eta_i|^2_{1,\Omega_i}\ge c_i(\Gamma_i)\|\eta_i\|^2_{\Lambda(\Gamma_i)}$$

and the lower bound follows. ∎

As a corollary, we note that if $\eta_h\in\Lambda_h\subset\Lambda$, then

$$c_2\|\eta_h\|^2_\Lambda\le s(\eta_h,\eta_h)\le c_1\|\eta_h\|^2_\Lambda. \tag{5.1}$$

Remark 5.3. The constants $C_i(\Omega_i)$ and $c_i(\Gamma_i)$ arising in the above proof may reflect a dependence of c_1,c_2 on the geometry of the subdivision. In turn, this may translate into a dependence on the number of subdomains, which could affect the scalability of our preconditioners. This issue requires further analysis.

We now turn to the question of defining a norm $\|\cdot\|_{\Lambda_h}$ which is equivalent to $\|\cdot\|_\Lambda$ on Λ_h. This will allow us to derive a suitable preconditioner for S. For simplicity of exposition, we assume that Γ is a planar $(d-1)$-dimensional surface (so that $N=2$). For the general case, see [3]. As before, let $\psi_i=\gamma_0(\Gamma)\phi_i$, where ϕ_i denotes a basis element with support intersecting Γ. Define

$$M_{ij}=(\psi_i,\psi_j)_{L_2(\Gamma)}\,,\qquad L_{ij}=(\gamma_0(\Gamma)\nabla_\Gamma\phi_i,\gamma_0(\Gamma)\nabla_\Gamma\psi_j)_{L_2(\Gamma)}\,, \tag{5.2}$$

where $\nabla_\Gamma v$ denotes the tangential gradient of v with respect to Γ. Let

$$H_{1/2}:=M(M^{-1}L)^{1/2}. \tag{5.3}$$

Then for any $\eta_h=\sum_{i=1}^{n_\Gamma}\boldsymbol{\eta}_i\psi_i\in\Lambda_h$ there exist constants κ_1,κ_2 such that [2]

$$\kappa_2\|\eta_h\|_{\Lambda_h}\le\|\boldsymbol{\eta}\|_{H_{1/2}}\le\kappa_1\|\eta_h\|_{\Lambda_h}. \tag{5.4}$$

We note here that the above result relies on the shape regularity of the subdivision of Γ (in our case inherited from the shape regularity of the subdivision of Ω) and a continuous piecewise polynomial basis set $\{\psi_i\}$. However, there is no explicit restriction with regard to the number of subdomains, or their shape regularity.

The following result follows immediately.

PROPOSITION 5.6. *Let $s(\cdot,\cdot):\Lambda_h\times\Lambda_h$ be the bilinear form induced by a three-step formulation of the Stokes problem. Let $\boldsymbol{\eta},\boldsymbol{\nu}$ denote the coefficients of η_h,ν_h with respect to the basis $\{\psi_i,i=1,\dots,n_\Gamma\}$ of Λ_h. Let S denote the matrix representation of $s(\cdot,\cdot)$ with respect to the same basis. Then there exist constants $\tilde c_1,\tilde c_2$ such that*

$$\tilde c_2\|\boldsymbol{\eta}\|^2_{H_{1/2}}\le\boldsymbol{\eta}^TS\boldsymbol{\eta}\le\tilde c_1\|\boldsymbol{\eta}\|^2_{H_{1/2}}$$

for all $\boldsymbol{\eta}\in\mathbb{R}^{n_\Gamma}$.

Proof. The results follow from (5.1) and equivalence (5.4). ∎

The above spectral equivalence makes $\Sigma_{\pm}(H_{1/2})$ an optimal preconditioner for $\Sigma_{+}(S)$.

PROPOSITION 5.7. *The eigenvalues σ of the matrix pencil $[\Sigma_{+}(S), \Sigma_{\pm}(H_{1/2})]$ satisfy either $|\sigma| = 1$ or*

$$\tilde{c}_2 \leq \sigma \leq \tilde{c}_1,$$

where $\tilde{c}_1, \tilde{c}_2$ are the constants in Proposition 5.6.

Proof. The spectral equivalence in Proposition 5.6 implies

$$\tilde{c}_2 \leq \lambda_k(H_{1/2}^{-1}S) \leq \tilde{c}_1, \quad k = 1, \ldots, n_\Gamma.$$

The result then follows from Lemma 5.1 and the interlacing result of Lemma 5.2. ∎

Remark 5.4. The matrix $H_{1/2}$ is also spectrally equivalent to S_Δ, the Schur complement arising from a DDM applied to a vector Laplacian problem [3]. This means that $\Sigma_{\pm}(S_\Delta)$ is another optimal preconditioner for $\Sigma_{+}(S)$. We will verify this fact in the numerical section.

The preconditioner $\Sigma(H_{1/2})$ is not a practical choice for large values of n_Γ. The matrix $H_{1/2}$ is typically full, and the action of the inverse of $\Sigma(H_{1/2})$ will be expensive, in general. We consider next alternatives that allow a sparse approximation of the action of this preconditioner.

5.2. Krylov approximations of constraint preconditioners. Given the structure of our interface preconditioners, we aim to approximate their action on a given vector via a Krylov subspace approach. In the following, we will refer to the Krylov subspace

$$\mathcal{K}_\ell(A, \mathbf{r}) := \left\{\mathbf{r}, A\mathbf{r}, \ldots, A^{\ell-1}\mathbf{r}\right\}$$

for generic nonsingular square matrix A and vector $\mathbf{r}$. This approach is commonly used to define a sparse approximation of $f(A)\mathbf{z}$ for a range of functions f; e.g., see [35], [13] for $f(t) = \exp(t)$, [44] for $f(t) = t^m$ for integer m, or [11] for $f(t) = t^{1/2}$. More specifically, the method relies on constructing an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ for $\mathcal{K}_\ell(A, \mathbf{r})$, so that the approximation can be taken to be

$$f(A)\mathbf{z} \approx V_\ell f(H_\ell)\mathbf{e}_1\|\mathbf{z}\|, \tag{5.5}$$

where $V_\ell = [\mathbf{v}_1, \ldots, \mathbf{v}_\ell]$, $\mathbf{e}_1 \in \mathbb{R}^{\ell\times\ell}$, and $V_\ell^T A V_\ell = H_\ell$ is an upper Hessenberg matrix; if A is symmetric, H_ℓ is tridiagonal symmetric. It is assumed that ℓ is small, so that the evaluation $f(H_\ell)$ is inexpensive. We consider below modified approximations corresponding to $f(t) = t^{1/2}$ and for the case where both Arnoldi and Lanczos procedures are employed to generate V_ℓ.

Remark 5.5. It is shown in [35] that one can view approximation (5.5) as a polynomial approximation $p(A)\mathbf{z}$ to $f(A)\mathbf{z}$, where p is the Hermite interpolant of degree at most $\ell - 1$ of f at the eigenvalues of H_ℓ. Exactness is achieved when ℓ equals the degree of the minimal polynomial of $\mathbf{z}$ with respect to A. For smaller values of ℓ, the following error bound applies [4]:

$$\|f(A)\mathbf{z} - V_\ell f(H_\ell)\mathbf{e}_1\|\mathbf{z}\|\| \leq C\|\mathbf{z}\| \min_{p\in\mathcal{P}_{\ell-1}} \max_{\zeta\in\mathcal{R}} |f(\zeta) - p(\zeta)|,$$

where $\mathcal{R}$ is a region in the complex plane containing the field of values of A. We note that the bound is small if f can be well-approximated by polynomials of low degree, which is the case for $f(t) = t^{1/2}$.

Consider approximating a typical preconditioning step in a Krylov method:

$$\mathbf{z} = \Sigma_+^{-1}(H_{1/2})\mathbf{r} = \begin{pmatrix} H_{1/2} & \mathbf{0}_\Gamma & -B_{\Gamma m}^T \bar{R}_{mN} \\ \mathbf{0}_\Gamma^T & 0 & \mathbf{1}_N^T \\ -(B_{\Gamma m}^T \bar{R}_{mN})^T & \mathbf{1}_N & O_{NN} \end{pmatrix}^{-1} \mathbf{r}.$$

In order to derive a sparse approximation of the above step, we introduce the equivalent problems

$$\mathbf{z} = \Sigma_-^{-1}(L)\left(\Sigma_-(L)\Sigma_+^{-1}(H_{1/2})\right)\mathbf{r}, \tag{5.6}$$

$$\mathbf{z} = \Sigma_+^{-1}(L)\left(\Sigma_+(L)\Sigma_+^{-1}(H_{1/2})\right)\mathbf{r}. \tag{5.7}$$

Our aim is to replace the action of the matrices $\Sigma_-(L)\Sigma_+^{-1}(H_{1/2})$ and $\Sigma_+(L)\Sigma_+^{-1}(H_{1/2})$ by sparse Krylov approximations. While the first matrix is diagonalizable, the second is not, although they both share the same nonunit eigenvalues. This fact will have an impact on the type of approximation we propose below. The following result describes the relationship between the spectra of $\Sigma_\pm(L)\Sigma_+^{-1}(H_{1/2})$ and $\Sigma_\pm(L)\Sigma_+^{-1}(M)$.

PROPOSITION 5.8. *Let λ_k, σ_k^2 denote the nonunit eigenvalues of the pencils $[\Sigma_\pm(L), \Sigma_+(H_{1/2})]$, $[\Sigma_\pm(L), \Sigma_+(M)]$, respectively. Let α_j denote an eigenvalue of $H_{1/2}$. Then*

$$\alpha_k \le \lambda_k, \sigma_k \le \alpha_{k+N-1}, \quad k = 1 : n_\Gamma + 1 - N.$$

Proof. First, we note that since the matrices M, L are symmetric and positive definite, there exists a matrix Q such that

$$L = Q^T D Q, \quad M = Q^T Q,$$

so that

$$H_{1/2} = Q^T D^{1/2} Q,$$

where D is a diagonal matrix containing the square roots of the eigenvalues of the pencil $[L, M]$. By Lemma 5.2, the nonunit eigenvalues λ_k of the pencil

$$[\Sigma_\pm(L), \Sigma_+(H_{1/2})] = \left[\Sigma_\pm(Q^T D Q), \Sigma_+(Q^T D^{1/2} Q)\right]$$

interlace the eigenvalues of

$$\left(Q^T D^{1/2} Q\right)^{-1} Q^T D Q = Q^{-1} D^{1/2} Q = H_{1/2}.$$

By the same corollary, the nonunit eigenvalues σ_k^2 of the pencil $[\Sigma_\pm(L), \Sigma_+(M)]$ interlace the eigenvalues of $(Q^T Q)^{-1} Q^T D Q = Q^{-1} D Q = H_{1//2}^2$, so that σ_k also interlace the eigenvalues of $H_{1/2}$. ∎

The above result motivates the following definition of a partial square root matrix. Let $\Sigma_-(L)\Sigma_+^{-1}(M) = \mathcal{V} D \mathcal{V}^{-1}$. Define

$$\left[D^{\{1/2\}}\right]_{ii} = \begin{cases} D_{ii} & \text{if } D_{ii} < 0, \\ \sqrt{D_{ii}} & \text{otherwise.} \end{cases}$$

Then the nonunit eigenvalues of

$$\left(\Sigma_-(L)\Sigma_+^{-1}(M)\right)^{\{1/2\}} := \mathcal{V} D^{\{1/2\}} \mathcal{V}^{-1}$$

are the values σ_k in Proposition 5.8.

The above matrix provides an alternative to the impractical choice of preconditioner $\Sigma_+(H_{1/2})$. The action of $\Sigma_-(L)\Sigma_+^{-1}(H_{1/2})$ on $\mathbf{r}$ can be replaced with an approximation to the partial square root of $\Sigma_-(L)\Sigma_+^{-1}(M)$ obtained using a Krylov method. We examine below two possibilities.

5.2.1. An Arnoldi approximation. The Arnoldi method is an iterative procedure which, after k iterations, generates the following factorization

$$V_k^T\Sigma_-(L)\Sigma_+^{-1}(M)V_k = H_k, \tag{5.8}$$

where $H_k \in \mathbb{R}^{k\times k}$ is an upper Hessenberg matrix and V_k is a matrix with orthonormal columns. Since $\Sigma_-(L)\Sigma_+^{-1}(M)$ is diagonalizable, H_k is also diagonalizable, so that the partial square root matrix $H_k^{\{1/2\}}$ exists and therefore we can define the Krylov approximation

$$\left(\Sigma_-(L)\Sigma_+^{-1}(M)\right)^{\{1/2\}}\mathbf{r} \approx V_k H_k^{\{1/2\}}\mathbf{e}_1\|\mathbf{r}\|.$$

This yields the following procedure for constructing an approximation $\tilde{\mathbf{z}}$ to $\mathbf{z}$:

1. Construct V_k, H_k corresponding to Arnoldi factorization (5.8).
2. Compute
$$\tilde{\mathbf{z}}_k = V_k H_k^{\{1/2\}}\mathbf{e}_1\|\mathbf{r}\|.$$
3. Solve for $\tilde{\mathbf{z}}$ the sparse linear system
$$\Sigma_-(L)\tilde{\mathbf{z}} = \tilde{\mathbf{z}}_k.$$

Remark 5.6. The first step in the above procedure constructs the projection H_k of $\Sigma_-(L)\Sigma_+^{-1}(M)$ onto the Krylov space $\mathcal{K}_k(\Sigma_-(L)\Sigma_+^{-1}(M), \mathbf{r})$. The Arnoldi iteration used for this aim requires multiplication by the matrix $\Sigma_-(L)\Sigma_+^{-1}(M)$, which involves at each step one sparse solve with the matrix $\Sigma_+(M)$ and a sparse matrix-vector multiplication with the matrix $\Sigma_-(L)$. Thus, the overall procedure 1–3 can be achieved using sparse operations only. This is crucial for the overall complexity of the resulting algorithm.

5.2.2. A Lanczos approximation. By Proposition 5.8, the roots of the eigenvalues of the pencil $[\Sigma_+(L), \Sigma_+(M)]$ satisfy the same interlacing property as the eigenvalues of the pencil $[\Sigma_+(L), \Sigma_+(H_{1/2})$. However, both pencils are defective, and the notion of a partial square root matrix cannot be defined in this case. On the other hand, $[\Sigma_+(L), \Sigma_+(M)]$ is a symmetric pencil, so we can consider the factorization arising from a simplified Lanczos procedure such as that introduced in [30] and related to the bi-Lanczos method [34]. The method is a generalization of the standard Lanczos algorithm but in the indefinite inner product induced by $\Sigma_+^{-1}(M)$. As in the case of the Arnoldi method, after k steps, the following factorization is produced:

$$T_k = V_k^T\Sigma_+^{-1}(M)\Sigma_+(L)\Sigma_+^{-1}(M)V_k, \quad V_k^T\Sigma_+^{-1}(M)V_k = E_k, \tag{5.9}$$

where E_k is a diagonal matrix and T_k is a symmetric tridiagonal matrix. The aim usually associated with such a procedure is to use the pencil $[T_k, E_k]$ to approximate the eigenvalues of the pencil $[\Sigma_+^{-1}(M)\Sigma_+(L)\Sigma_+^{-1}(M), \Sigma_+^{-1}(M)]$, which are also the eigenvalues of the pencil $[\Sigma_+(L), \Sigma_+(M)]$. In our case, we propose to use the above factorizations to approximate the action of $\Sigma_+(L)\Sigma_+(H_{1/2})$ on a vector $\mathbf{r}$ via

$$\Sigma_+(L)\Sigma_+^{-1}(H_{1/2})\mathbf{r} \approx V_k\left(E_k^{-1}T_k\right)^{\{1/2\}}\mathbf{e}_1\|\mathbf{r}\| \tag{5.10}$$

under the assumption that $E_k^{-1}T_k$ is diagonalizable. Our implementation follows that of [45] which uses a normalized version of E_k, i.e., the diagonal entries of E_k are either $+1$ or -1.

Remark 5.7. By Lemma 5.1, the degree of the minimal polynomial of the matrix $\Sigma_+(L)\Sigma_+^{-1}(M)$ is $n_\Gamma - N + 3$, which is the maximum computable number of linearly independent columns in V_k in exact arithmetic. However, in practice, the value of k is much smaller than this maximum admissible value; moreover, computations in the next section indicate that E_k has only positive entries for small values of k (i.e., $E_k = I_k$), so that $(E_k^{-1}T_k)^{\{1/2\}} = T_k^{1/2}$ and the approximation simplifies to

$$\Sigma_+(L)\Sigma_+^{-1}(H_{1/2})\mathbf{r} \approx V_k T_k^{1/2}\mathbf{e}_1\|\mathbf{r}\|.$$

Summary. The preconditioners $\Sigma_\pm(H_{1/2})$ have been shown to yield preconditioned spectra independent of the mesh size h for any fixed number of subdomains N. This is in contrast to most DDMs where dependence on h exists for fixed N. On the other hand, in view of Remark 5.3, a dependence on the number of subdomains may exist for $\Sigma_\pm(H_{1/2})$, although further analysis is needed to clarify this issue.

6. Numerical experiments. The three-step procedure employed and analyzed in the previous sections is aimed at deriving and analyzing the interface Schur complement associated with our domain decomposition formulation. However, our implementation will use this information in order to construct a global (not just interface) preconditioner for the discretization of the incompressible Stokes equations. We illustrate this on two-dimensional test problems; however, three-dimensional problems afford the same treatment. Finally, we note that parallelism is not investigated, although we expect to be able to report on the scalability of our solver, as well as its three-dimensional performance in a future paper. For examples of implementations of discrete fractional Sobolev norms in three dimensions, we refer the reader to [2].

6.1. Implementation. We used a $P_2 - P_0$ discretization of the modified weak formulation (WF) introduced in section 2, which resulted in the linear system (4.1):

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix},$$

where

$$K_{11} = \bigoplus_{i=1}^{N} \begin{pmatrix} A_{I_iI_i} & B_{J_iI_i}^T & \mathbf{0}_{I_i} \\ B_{J_iI_i} & O_{J_iJ_i} & r_i \\ \mathbf{0}_{I_i}^T & r_i^T & 0 \end{pmatrix}, \quad K_{12} = \mathop{\mathbf{Col}}_{i=1}^{N} \begin{pmatrix} A_{I_i\Gamma} & O_{I_iN} & \mathbf{0}_{I_i} \\ B_{J_i\Gamma} & O_{J_iN} & \mathbf{0}_{J_i} \\ \mathbf{0}_\Gamma^T & -I_{iN} & 0 \end{pmatrix}.$$

The inverse of the proposed preconditioner is a block-triangular matrix of the form

$$\begin{pmatrix} K_{11} & K_{12} \\ O & \Sigma_\pm(P) \end{pmatrix}^{-1} = \begin{pmatrix} K_{11}^{-1} & O \\ O & I \end{pmatrix} \begin{pmatrix} I & -K_{12} \\ O & I \end{pmatrix} \begin{pmatrix} I & O \\ O & \Sigma_\pm^{-1}(P) \end{pmatrix},$$

which will be used with a GMRES solver. This is motivated by the fact that the resulting preconditioned system has the form

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ O & \Sigma_\pm(P) \end{pmatrix}^{-1} = \begin{pmatrix} I & O \\ K_{12}^T K_{11}^{-1} & \Sigma_+(S)\Sigma_\pm(P)^{-1} \end{pmatrix},$$

which is known to yield GMRES convergence in 2 iterations if $\Sigma_+(P) = \Sigma_+(S)$ [17], [29]. It is evident from the above factorization that at each GMRES step the implementation requires solution of linear systems with K_{11} and $\Sigma_\pm(P)$, with the former achieved by solving in parallel N subdomain Dirichlet Stokes problems using a sparse direct solver. The choices of P are motivated by our previous discussion. We constructed the constraint matrices $B_{\Gamma m}^T \bar{R}_{mN}$ in $\Sigma_\pm(P)$ (cf. (4.2)) via a finite element assembly, given their explicit form, although we note that this construction can also be achieved cheaply algebraically.

We employed the following choices for P:

1. $P = S_\Delta$—the Schur complement arising from a vector Laplacian problem (see Remark 5.4);
2. $P = H_{1/2} = M(M^{-1}L)^{1/2}$—a discrete vector fractional Sobolev norm;
3. $P = P_{\text{Arn}}$ implicitly defined by the Arnoldi factorization (5.8) and the subsequent three-step procedure indicated;
4. $P = P_{\text{Lan}}$ implicitly defined by the Lanczos factorization (5.9) and the approximation (5.10).

We remark here that the first two choices are expensive in practice, as they involve the inversion of a matrix with a dense (1,1)-block of size n_Γ. On the other hand, the other two are Krylov approximations which involve sparse approximations—indeed, these are the main practical options that we propose.

6.2. Krylov solvers. We employed flexible GMRES (FGMRES) [36] with a standard stopping criterion

$$\|\mathbf{r}^k\| \leq 10^{-6}\|\mathbf{r}\|,$$

where $\mathbf{r}^k = \mathbf{b} - K\mathbf{x}^k$ denotes the residual at the kth iteration. The choice of method is due to the changing nature of the preconditioner for the case where $P_{\text{Arn}}, P_{\text{Lan}}$ are employed. The starting guess $\mathbf{x}^0$ was computed as the solution of the linear system

$$\begin{pmatrix} K_{11} & K_{12} \\ O & \Sigma_\pm(P) \end{pmatrix} \mathbf{x}^0 = \mathbf{b}.$$

This choice ensures that the initial residual has the block form

$$\mathbf{r}^0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{r}_2^0, \end{pmatrix}$$

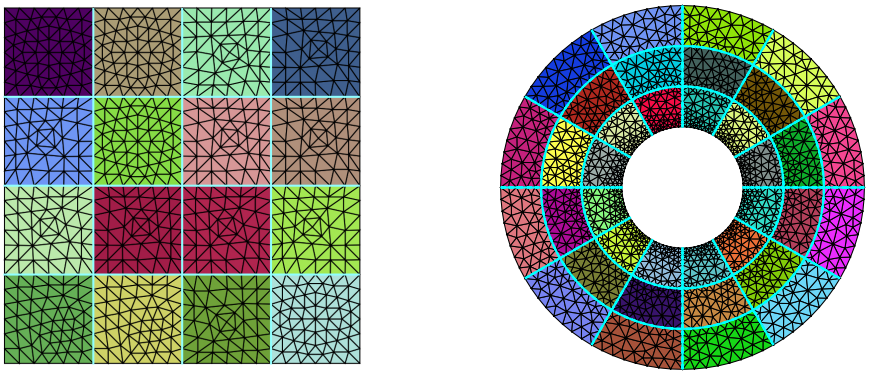
so that the Arnoldi basis generated by GMRES has the same zero pattern. This leads to important savings, as the orthogonal basis generated is nonzero only on the constraint interface space. This was taken into account in our implementation of FGMRES, so that only the nonzero part of the Arnoldi basis was stored. We note that this is mathematically equivalent to using FGMRES on the Schur complement problem, but employing a global stopping criterion.

The implementation of $P_{\text{Arn}}, P_{\text{Lan}}$ used standard Arnoldi and Lanczos algorithms with full reorthogonalization. In all our experiments we worked with bases of size $\ell = 10$.

6.3. Experiments. We employed the above interface preconditioners to solve the Stokes problem (2.1) corresponding to the following domains and boundary conditions.

Problem 1: Regularized lid-driven cavity flow in $\Omega = [0,\ 1]^2$ with boundary data

$$g(x,y) = \begin{cases} (16(x-x^2)^2, 0), & x \in [0,1], y = 1, \\ (0,0) & \text{otherwise.} \end{cases}$$



(a) Problem 1 (b) Problem 3

FIG. 1. *Partitions for test Problems* 1 *and* 3.

Problem 2: Poiseuille type flow in $\Omega = [0,8] \times [0,1]$ with boundary data

$$g(x,y) = \begin{cases} (4(y-y^2),0), & x \in \{1,8\}, y \in [0,1], \\ (0,0), & \text{otherwise.} \end{cases}$$

Problem 3: Taylor–Couette-type flow in $\Omega = B_{\mathbf{0},3} \setminus \bar{B}_{\mathbf{0},1}$, where $B_{\mathbf{c},r}$ denotes the open disk centered at $\mathbf{c}$ with radius r and with boundary data

$$g(x,y) = \begin{cases} t(x,y), & x^2+y^2 = (1/2)^2, \\ (0,0) & \text{otherwise,} \end{cases}$$

where $t(x,y)$ denotes the unit tangent vector to $\partial\Omega$ at (x,y).

We chose to experiment with various isotropic partitions for several levels of isotropic mesh refinement with degrees of freedom in the range $O(10^4 - 10^5)$. In particular, we chose partitions into subdomains of diameter H_i and meshes with shape regular simplices such that

$$c_H \leq H_i \leq C_H \tag{6.1}$$

for all i and such that $C_H/c_H = O(1)$. We experimented with straight interfaces, corresponding to the case where the domain is first subdivided and then refined. We did not use uniform meshes as we wanted to avoid any superconvergence effects. Some examples of the meshes and partitions used are included in Figure 1.

Remark 6.1. The regularity requirement (6.1) was not needed in our analysis, but is seen as standard in the domain decomposition literature. However, we found that employing subdomains with large aspect ratio was detrimental to performance. This feature requires further investigation.

We stress here that we did not employ a coarse level in our preconditioning procedure, thus avoiding the complications that arise in defining coarse level operators for constrained problems.

Finally, we remark that the potential for parallelism of the substructuring method proposed in this paper relies on the low complexity of the interface preconditioning procedure. Given the sparse approximation procedure employed, the complexity of

TABLE 1
GMRES iterations for Problem 1: *ideal preconditioners* $S_\Delta, H_{1/2}$.

$P =$	S_Δ				$H_{1/2}$			
$N =$	4	16	64	256	4	16	64	256
$n+m =$ 10,498	13	13	12	13	16	20	22	29
41,474	13	14	13	13	17	20	23	31
164,866	13	14	14	13	16	20	24	32

TABLE 2
GMRES iterations for Problem 1: *Arnoldi and Lanczos approximations.*

$P =$	P_{Arn}				P_{Lan}			
$N =$	4	16	64	256	4	16	64	256
$n+m =$ 10,498	14	17	24	31	14	17	23	30
41,474	16	17	21	30	14	17	23	32
164,866	17	18	18	25	15	17	21	32

TABLE 3
GMRES iterations for Problem 2: *ideal preconditioners* $S_\Delta, H_{1/2}$.

$P =$	S_Δ			$H_{1/2}$		
$N_x \times N_y =$	16×2	32×4	64×8	16×2	32×4	64×8
$n+m =$ 17,890	13	12	10	18	19	20
70,978	14	13	11	19	20	21
282,754	14	13	11	21	21	21

the iterface problem is of order $O(\ell n_B)$. This means that the method should scale well for interface problems of size $n_B = O(n_{I_i})$. This poses balancing constraints on the geometric parameters employed, which is a standard restriction in DDMs.

6.4. Numerical results. We include below the results corresponding to the three test problems and the set of preconditioners indicated.

6.4.1. Problem 1. Table 1 displays the number of iterations corresponding to the ideal choices $P = S_\Delta$ and $P = H_{1/2}$. These results represent the benchmark for the approximations $P_{\text{Arn}}, P_{\text{Lan}}$, shown in Table 2. The approximations were constructed using a Krylov space of dimension $\ell = 10$. First, we note that all the preconditioners exhibit mesh independence. The ideal (and less practical) preconditioner S_Δ appears also to exhibit independence of the number of subdomains, while the other theoretical candidate, $H_{1/2}$ exhibits a mild dependence on N, not dissimilar to the dependence noticed in the scalar case when preconditioning Poisson problems [2]. The two sparse approximations employed also appear to yield performance independent of the refinement level, while inheriting from the exact preconditioner $H_{1/2}$ the dependence on N.

6.4.2. Problem 2. Given the shape of the domain, we chose to work with a subdivision into subdomains that satisfied the isotropy property (6.1). More precisely, we used N_x, N_y subdomains in the x and y directions, respectively, so that each Ω_i was a square. The corresponding results are included in Tables 3 and 4. Unlike the previous example, the dependence on $N = N_x \times N_y$ is insignificant, while the independence of the refinement level is maintained as indicated by the theoretical results from the previous section. Remarkably, this appears to be the case for all the preconditioners employed. We also note that the performance of the Arnoldi and Lanczos approximations appears to be slightly better than that of the exact choice of

TABLE 4
GMRES iterations for Problem 2: *Arnoldi and Lanczos approximations.*

$P =$	P_{Arn}			P_{Lan}		
$N_x \times N_y =$	16×2	32×4	64×8	16×2	32×4	64×8
$n + m = 17{,}890$	15	16	16	16	16	18
70,978	17	15	16	17	16	18
282,754	19	16	16	17	17	16

TABLE 5
GMRES iterations for Problem 3: *ideal preconditioners* $S_\Delta, H_{1/2}$.

$P =$	S_Δ			$H_{1/2}$		
$N_\theta \times N_r =$	4×1	8×2	12×3	4×1	8×2	12×3
$n + m = 10{,}256$	9	12	12	12	16	17
40,512	9	12	13	12	17	17
161,024	9	13	12	11	17	18

TABLE 6
GMRES iterations for Problem 3: *Arnoldi and Lanczos approximations.*

$P =$	P_{Arn}			P_{Lan}		
$N_\theta \times N_r =$	4×1	8×2	12×3	4×1	8×2	12×3
$n + m = 10{,}256$	18	26	28	16	22	23
40,512	18	26	27	15	22	23
161,024	18	27	27	17	22	22

preconditioner that they are designed to approximate. This was also the case for test Problem 1, but is more pronounced for this problem.

6.4.3. Problem 3. Finally, for the case of an annular domain, we chose to work with a subdivision into N_r, N_θ subdomains in the radial and tangential directions, respectively. A typical subdivision and mesh refinement is shown in Figure 1(b). The preconditioned FGMRES performance is included in Tables 5 and 6. We observe that the same behavior is present, with both mesh independence and mild dependence on N clearly displayed. We remark also that the performance of the Lanczos approximation is somewhat better that that of the Arnoldi option, unlike in the other two experiments.

7. Summary. We presented a novel substructuring approach for the parallel solution of the incompressible Stokes equations with Dirichlet boundary conditions. The method involves a simple reformulation of the problem which allows for a natural decomposition into well-posed Stokes subdomain problems, coupled by an interface problem with explicit constraints. In turn, this allows for the design of optimal interface preconditioners which exhibit no mesh dependence and only a mild dependence on the number of subdomains.

Some generalizations are possible and will be investigated in future work. First, we note that while the method was analyzed for the case of discontinuous approximations of the pressure variable, a similar, albeit more technical, approach is possible for the case of continuous pressure approximations. Second, we expect our formulation to arise naturally in a substructuring approach for problems with outflow boundary conditions, for decompositions which involve subdomains with Dirichlet boundary

conditions, as well as for subdomains with mixed (outflow and Dirichlet) boundary conditions. Finally, our approach can be extended directly to the Navier–Stokes equations as well as to the case of generalized Newtonian flow for various standard constitutive laws.

8. Appendix.

Proof of Lemma 4.2. The last column of K^{-1} is the solution of the following linear system:

$$\begin{pmatrix} A & B^T & 0 \\ B & 0 & \mathbf{q} \\ 0 & \mathbf{q}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We find $\mathbf{u} = -A^{-1}B^T\mathbf{p}$ and hence

$$B\mathbf{u} + \lambda\mathbf{q} = -BA^{-1}B^T\mathbf{p} + \lambda\mathbf{q} = 0.$$

Since $B^T\mathbf{q} = 0$, multiplying the above equation by $\mathbf{q}^T$ from the right we obtain

$$0 = -\mathbf{q}^T BA^{-1}B^T\mathbf{p} + \lambda\mathbf{q}^T\mathbf{q} = \lambda\mathbf{q}^T\mathbf{q}$$

and hence $\lambda = 0$. Therefore, $B\mathbf{u} = 0$ and $BA^{-1}B^T\mathbf{p} = 0$. Since A is symmetric and positive definite, we must have $\mathbf{p} \in \ker B^T$ or $\mathbf{p} = a\mathbf{q}$ for some $a \in \mathbb{R}$. But $\mathbf{q}^T\mathbf{p} = 1$ and we find $a = 1/\|\mathbf{q}\|^2$. Finally, $0 = Au + B^T\mathbf{p} = Au$, so that $\mathbf{u} = 0$.

To find the second column of K^{-1} we consider the linear system

$$\begin{pmatrix} A & B^T & 0 \\ B & 0 & \mathbf{q} \\ 0 & \mathbf{q}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \lambda_i \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{e}_i \\ 0 \end{pmatrix},$$

where $\mathbf{e}_i \in \mathbb{R}^m$ is the ith column of the identity matrix I_m. As before,

$$\mathbf{u}_i = -A^{-1}B^T\mathbf{p}_i.$$

We find

$$B\mathbf{u}_i + \lambda_i\mathbf{q} = -BA^{-1}B^T\mathbf{p}_i + \lambda_i\mathbf{q} = \mathbf{e}_i \Leftrightarrow \lambda_i\|\mathbf{q}\|^2 = \mathbf{q}^T\mathbf{e}_i = \mathbf{q}_i \Leftrightarrow \lambda_i = \frac{\mathbf{q}_i}{\|\mathbf{q}\|^2}.$$

Hence, using the last equation $\mathbf{q}^T\mathbf{p}_i = 0$,

$$-BA^{-1}B^T\mathbf{p}_i + \frac{\mathbf{q}\mathbf{q}_i}{\|\mathbf{q}\|^2} = \mathbf{e}_i \Leftrightarrow \left[\mathbf{q}\mathbf{q}^T + BA^{-1}B^T\right]\mathbf{p}_i = -\left[\mathbf{e}_i - \frac{\mathbf{q}\mathbf{q}_i}{\|\mathbf{q}\|^2}\right].$$

Since

$$S_q = \mathbf{q}\mathbf{q}^T + BA^{-1}B^T$$

is symmetric and positive definite, we find the matrix P to be as given. The expression for N^T follows immediately. Finally,

$$AM + B^TN = I_n \Leftrightarrow M = A^{-1}[A - B^TN]A$$

and the expression for M follows.

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