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Unique determination of sound speeds for coupled systems of semi-linear wave equations

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Abstract

We consider coupled systems of semi-linear wave equations with different sound speeds on a finite time interval $[0, T]$ and a bounded domain Ω in $\mathbb{R}^3$ with C^1 boundary $\partial\Omega$. We show the coupled systems are well posed for variable coefficient sound speeds and short times. Under the assumption of small initial data, we prove the source to solution map associated with the nonlinear problem is sufficient to determine the source to solution map for the linear problem. This result is a bit surprising because one does not expect, in general, for the interaction of the waves in the nonlinear problem to always behave in a tractable fashion. As a result, we can reconstruct the sound speeds in Ω for the coupled nonlinear wave equations under certain geometric assumptions. In the case of the full source to solution map in $\Omega \times [0, T]$ this reconstruction could also be accomplished under fewer geometric assumptions.

Keywords: Inverse problems; Coupled systems; Non-linear hyperbolic equations

1. Introduction

We consider coupled systems of semi-linear wave equations with variable sound speeds on an open bounded domain Ω in $\mathbb{R}^3$ with C^1 boundary $\partial\Omega$. In nonlinear problems, when waves are propagated, they interact and the interaction may cause difficulties in building an accurate parametrix and detecting the variable coefficients.

For the problem of elasticity, the stress the material is under going is described by the Lamé parameters, λ and μ. Recently in [35] it was shown that this important linear hyperbolic problem where the solutions are vector valued can be reduced to three variable speed wave

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equations with scalar valued solutions. The authors of [35] are then able to solve the associated inverse boundary value problem for the linear elasticity equation by building solutions to the wave equations. We will eventually consider the fully nonlinear elastic wave equations, which are not considered in [35], but we will report on this in future work. However, even in the simpler model here, for the case of variable sound speeds well posedness estimates are novel.

The general set up is as follows; we consider a coupled system of variable coefficient semi-linear wave equations which has a quadratic non-linearity. In an extended open domain Ω', ($\overline{\Omega} \subset \Omega' \subset \mathbb{R}^3$), which has a smooth boundary so the definitions of the Sobolev spaces make sense, we show that the solution is well posed, so the waves we are studying are meaningful. On any subdomain Ω of Ω' with C^1 boundary, we can measure on the boundary of $\partial\Omega$ and the solution to the non-linear problem determines the behaviour of the linear problem. This conclusion is only possible under the assumption of small Cauchy data ($\mathcal{O}(\epsilon)$, $\epsilon \ll 1$) and an appropriate timescale for the solution to make sense. In this case, the nonlinear problem completely determines the behaviour of the linear waves, which in turn determine the variable coefficient sound speeds. This setup has physical significance because typically when complicated elasticity problems are linearised the linearisation to a hyperbolic system of wave equations only holds up to a quadratic term on a small domain for short times (in [26] Ch6 this is shown for constant coefficients). Moreover, as previously mentioned, the result of [35] also shows that the wave equation with multiple sound speeds model in 3d is the correct one for solving the elastic wave equation to leading order in the sense of pseudo-differential operators. The result in the main theorem here is a bit surprising because it says the waves on the boundary essentially behave as in the linear problem for short timescales. However there is no obvious way to show the source-to-solution map for the non-linear hyperbolic problem is in general Frechet differentiable with Frechet derivative equal to the linear source-to-solution map. The key idea here is the construction of a parametrix which is accurate for small Cauchy data, and has leading order terms in ϵ which are linear, without having to use a tedious Duhamel principle argument. The improvement shown here is a modification of earlier constructions to show the terms are in a bounded hierarchy. This parametrix takes the place of trying to show Frechet differentiability of the map directly from the PDE.

Parametrix construction of solutions to these coupled systems has been done only for the constant coefficient case c.f., [12,13,30,31]. In the case of nonlinear elasticity, constant coefficient equations have been examined in [23–25,32] although many of these references are interested in a different (and challenging!) perspective which is the issue of well-posedness and scattering for long times.

The problem of parameter recovery is well studied for a class of linear hyperbolic problems such as the wave equation $(\partial_t^2 - \Delta_g)u = 0$, for generic Riemannian manifolds (M_0, g) c.f. [4–7,10,11,15,33] for example. One can even recover the metric g for the associated semi-linear problem. The latter problem is handled via a linearisation method, [16]. The authors also apply their linearisation techniques to the case of Einstein's equations in the related article [17]. The difference in these articles and the material presented here is that the coefficients e.g the metric g are time dependent, and ours are not. Time dependence of the metric g adds considerable difficulties. However we are able to handle the case of multiple sound speeds and coupled systems of nonlinear wave equations. Due to the technical difficulties of the problem, such coupled nonlinear wave equations have not been considered before.

Reviewing the literature, the use of only boundary data in the form of the trace of the solutions is new for the nonlinear hyperbolic problem and is one of the main points of the article. Even in the linear case, the pioneering work on parameter recovery in nonlinear inverse problems in [16,17] uses the singularities of their nonlinear hyperbolic problems to determine the metric in their partial differential equations (PDE) in the entirety of the domain on which they are measuring. They use the calculus of cononormal singularities developed in [21] and [22] to recover the metric at every point. In, [16,17] it is necessary to check the interaction of the singularities under the nonlinearity as we know by [27] that waves can interact when a nonlinearity is present and produce more singularities. Moreover in [27,28], they showed that these crossings are the only place where new singularities can form. In their articles [16,17], the authors exploit the singularity crossings to reconstruct the geometry of domains they consider. The main difference is that they have knowledge of the full source to solution map everywhere in the domain where they are measuring. Their argument uses a variant of boundary control (introduced in [3]) which allows for recovery of generic time dependent Lorentzian metrics. The boundary control technique gives limited results in the case of boundary measurements, which is why it is not used here. As we do not use a singularity crossings argument, we can proceed differently than in [16,17]. In [16,17] they have chosen sufficiently regular data for the PDE, we render this approach is unnecessary, in the time independent coefficient case when the data is the *trace* of the source to solution map. The reason the singularity crossing argument disappears is that we are only interested in recovery of the topology from the boundary of where we are measuring. This considerable reduction in the measurements gives much less information about the sound speeds, and we expect different results in this scenario. Indeed, even in the time independent sound speed case there are known results where the trace of the solutions on the boundary coincide but the sound speeds do not [8].

We have to be careful about the type of measurements that we are taking. In particular, it is not known if the coupled nonlinear equations are well posed for generic compact manifolds with boundary. In fact for quadratic nonlinearities, it is likely that they are not, as the simpler case of the scalar semi-linear wave equation is not globally well posed. We could extend our short time well-posedness estimates to generic globally hyperbolic manifolds, but we leave this for future work. In order to avoid difficulties with boundary considerations we examine the solutions on the boundary of $[0, T] \times \Omega$, where T is finite. This scenario is not a traditional boundary value problem. The hyper surface $\partial\Omega$ is not a true boundary for the waves, simply where we are measuring.

Under these same geometric assumptions as in [35], for the nonlinear case, and a small displacement field, we are able to reduce the amount of data required to uniquely determine the vector field to just boundary valued data on the artificial surface $[0, T] \times \partial\Omega$. This result is completely new for nonlinear hyperbolic PDE, even in the case when the solutions are scalar valued. The techniques required for the reduction of data, are new from those in [16,17].

As such, the major contributions of this article are the following:

- A reduction of source-to-solution map output (to co-dimension 1) required to determine the topological structure of the sound speeds.
- Simplification of the parametrix construction for semi-linear wave equations, and an explicit parametrix for small data.
- Provision of a toy model and well-posedness estimates for the non-linear elasticity equations.

To accomplish these goals, the outline of the article is a follows. We introduce notation and the main theorem in Section 2. Section 3 contains a linearisation argument and a construction of a new and accurate parametrix in terms of ϵ and solutions to a linear system of equations. Section 4 shows that the trace of the source-to-solution map behaves appropriately for the reconstruction of the linear problem from the nonlinear problem, and some explicit examples for non-trapping sound speeds are given which satisfy Theorem 1 and Corollary 1. The appendix contains the well posedness results needed for the problem to make sense. These results are at the end as they are essentially self-contained.

Notation:

We let Ω' be an extended domain with smooth boundary containing Ω. In practice Ω' can be arbitrarily large—practically all of $\mathbb{R}^3$. We assume both Ω' and Ω are open. In this paper we use the Einstein summation convention. For two matrices A and B, the inner product is denoted by

$$A : B = a_{ij} b_{ji},$$

and we write $|A|^2 = A : A$. For vector-valued functions

$$f(x) = (f_1(x), f_2(x), f_3(x)) : \Omega' \to \mathbb{R}^3 \ ,$$

the Hilbert space $H_0^m(\Omega')^3$, $m \in \mathbb{N}$ is defined as the completion of the space $\mathcal{C}_c^\infty(\Omega')^3$ with respect to the norm

$$\|f\|_m^2 = \|f\|_{m,\Omega'}^2 = \sum_{|i|=1}^{m} \int_{\Omega'} \left(|\nabla^i f(x)|^2 + |f(x)|^2 \right) dx,$$

where we write $\nabla^i = \partial^{i_1} \partial^{i_2} \partial^{i_3}$ for $i = (i_1, i_2, i_3)$ for the higher-order derivative.

In general, we assume the sound speed coefficients are $C^s(\overline{\Omega})$ with s an integer such that $s - 1 > 3/2$ in order to use Sobolev embedding on the actual solutions. We consider the 3d case here, but many of the results generalise to other dimensions and different types of power semi-linearities provided the underlying equations are well-posed. Let m_1 and m_0 be nonzero constants with $m_1 \geq m_0$. We define the admissible class of conformal factors depending on s as

$$\mathcal{A}_0^s = \{c^2(x); \quad m_1 \geq c^2(x) \geq m_0; \ \forall x \in \overline{\Omega} \quad \text{and} \quad c^2 \in C^s(\overline{\Omega})\} \tag{1.1}$$

We consider a coupled system with three sound speeds c_i^2. We assume $c_i^2 \in \mathcal{A}_0^s$ for all $i = 1, 2, 3$. Moreover we also assume there exists a ball $\overline{\Omega} \subset B_R(0)$ such that $c_i \equiv 1$ on $(B_R(0))^c$, and that c_i is extended in a smooth way outside $\overline{\Omega}$ so this is possible. The extended sound speeds we denote as $\tilde{c}_i^2$.

2. Statement of the main theorem

We now examine a coupled system of semi-linear wave equations, which is a toy model for the linearisation of the nonlinear elasticity problem. We could extend these results with appropriate modifications to arbitrary quadratic nonlinearities. Recall we have the following inclusions $\overline{\Omega} \subset \Omega' \subset \mathbb{R}^3$. Let $u = (u_1, u_2, u_3)$ and we consider the system:

$$\begin{aligned} &\partial_t^2 u_i - \tilde{c}_i^2(x) \Delta u_i = |u|^2 + f(t,x) \quad \text{in } (0,T) \times \Omega', \quad i = 1,2,3 \\ &u(0,x) = b_0(x) \quad \partial_t u(0,x) = b_1(x) \quad \text{in} \quad \Omega' \\ &u(t,x)|_{\partial\Omega' \times (0,T)} = 0 \end{aligned} \tag{2.1}$$

Assume $c_i^2 \in \mathcal{A}_0^s$, and $\tilde{c}_i^2$ its corresponding extension to $\mathbb{R}^3$ as defined in the end of last section. This equation is well posed with $u(t,x) \in C([0,T]; H_0^s(\Omega')^3) \cap C^1([0,T]; H_0^{s-1}(\Omega')^3))$ for $s - 1 > 3/2$, when $\|f(t,x)\|_{L^2([0,T]; H_0^{s-1}(\Omega')^3)}$ $\|u_0(x)\|_{H_0^s((\Omega')^3)}$, $\|u_1(x)\|_{H_0^{s-1}((\Omega')^3)}$ are all bounded. The constant T is finite depending on a uniform bound of the following norms: $\|f(t,x)\|_{L^2([0,T];H_0^{s-1}(\Omega')^3)}$, $\|b_0(x)\|_{H_0^s((\Omega')^3)}$, and $\|b_1(x)\|_{H_0^{s-1}((\Omega')^3)}$, $\|c_i(x)\|_{H_0^s(\Omega)^3}$, $i=1,2,3$, and m_0, m_1. This local well posedness result does not appear to have been stated in the literature in this form and proved in the Appendix, where the dependence of the various parameters is detailed. A more classical, similar result for well posedness of hyperbolic coupled systems with variable coefficients can be found in [14], but this is only for first order systems. One could perhaps prove this theorem using an abstract semi-group argument which would use the results in [14], however the dependence of the various parameters is important for the proof of Theorem 1 which is why all the details are spelled out in the Appendix. However the Appendix is stand alone, meaning that it could be read independently of the body of text.

We recall that as a consequence of Sobolev embedding for all $\alpha > 3/2$, we have $H^\alpha(\Omega') \subseteq L^\infty(\Omega')$. This embedding is the only time we use the fact Ω' is bounded because it does not hold for unbounded domains. The reason we do not assume everything is bounded in the first place is that the proof techniques are based on energy estimates. We notice that because $s > 5/2$, by Sobolev embedding, we automatically obtain $u(t,x) \in C([0,T];C^1(\Omega')^3) \cap C^1([0,T];C(\Omega')^3)$. For simplicity we assume $s=3$, for the rest of this article except the Appendix and while the regularity in the proof techniques for recovery of the coefficients could be reduced, it is unclear if the system data propagates regularly in any sense for $s \le 5/2$.

We let the vector valued source-to-solution map Λ associated to u solving (2.1) be a map which is defined by

$$(\Lambda(b_0, b_1, f)) = (u_1, u_2, u_3)|_{[0,T]\times\partial\Omega}.$$

The map Λ is defined as an operator provided the input is in the regularity class in the main theorem because the trace theorem (see the Appendix, Lemma 4) gives immediately that the map is well defined with range in $L^2([0,T];L^2(\partial\Omega)^3)$. This point is important because the map Λ is NOT linear from the source terms to the solution. Furthermore, the statement of the main theorem is still true for the restriction of the operator to one with an input domain with any one, or combination of the inputs b_0, b_1, or f set equal to 0.

Analogously we let the linear source-to-solution map Λ^{lin} associated to u_{lin} solving (2.1) with 0 right hand side be the map of the source to trace of the solution. It is a key point that we restrict the domain of Λ to a subclass of data F of the form $F = (b_0, b_1, f) = \epsilon F_1 = \epsilon(b_0', b_1', f_1)$, with F_1 *independent* of ϵ and such that

$$\|b_0'\|_{H_0^3(\Omega')^3} + \|b_1'\|_{H_0^2(\Omega')^3} + \|f_1\|_{L^2([0,T];H_0^2(\Omega')^3)} = \|F_1\|_* \le 1 \tag{2.2}$$

and not all possible data. (The number 1 is arbitrary, it could be a different finite constant.) As a consequence of the proof techniques, the domain of the operator Λ^{lin} we determine takes a subclass of data F of the form $F = F_1$ with $\|F\|_* \le 1$, for a particular finite maximum T as detailed below. The T in consideration is then *independent* of ϵ.

We assume the parameter ϵ is such that $\epsilon \in (0, \epsilon_1)$, for some finite $\epsilon_1 < 1$. Let $T_0(\epsilon)$ be the maximal time for which the system (2.1) is well posed, which is inversely proportional to ϵ. We assume T fixed is such that $T < T_0(\epsilon_1)$. (Again, the timescale T_0 and its dependence on ϵ is detailed in the Appendix).

Our main result is the following

Theorem 1. *Let $\mathcal{U}_1(t,x) = (u_{11}, u_{12}, u_{13})$ and $\mathcal{U}_2(t,x) = (u_{21}, u_{22}, u_{23})$, satisfy (2.1) with distinct sound speed coefficients, $c_{i,1}$ and $c_{i,2} \in \mathcal{A}_0^3$, for $i = 1,2,3$. If $\Lambda_1 = \Lambda_2$ on $[0,T]\times\partial\Omega$, then $\Lambda_1^{lin} = \Lambda_2^{lin}$ on $[0,T]\times\partial\Omega$.*

As a result we have the following Corollaries:

Corollary 1. *Assume that* $\Lambda_1 = \Lambda_2$ *on* $[0,T]\times\partial\Omega$, *then* $c_{i,1}^2 = c_{i,2}^2$, *for all* $i = 1,2,3$, *whenever it is known that the source to solution map for the linear problem uniquely determines the conformal factors (up to a diffeomorphism).*

Remark 1. The proof of Theorem 1 does not require any assumptions on Ω, only that $\overline{\Omega}$ be compact for the well-posedness estimates in Theorem 3 to hold, and that $c_i \in \mathcal{A}_0^3$ and an appropriate assumption on the timescale T in terms of the input data. The proof of Theorem 1 involving the trace operators does not involve any other assumptions.

In spite of the main theorem being devoid of non-trapping assumptions, in practice some non trapping assumptions on the domain Ω are required for the hypothesis of Corollary 1 to hold c.f. [19,34,35]. These non trapping assumptions are not required if using the boundary control method and the full source to solution map [2,3]. Typically this Corollary enforces a condition of the form $\mathrm{diam}(\Omega) \le T$ where the diameter of Ω is taken with respect to the maximum of the sound speeds. In the Appendix we show that such a condition is possible e.g., a nonzero ϵ_1 is proven to exist in the Appendix in Lemma 3.

3. Linearisation of the inverse problem

We consider the linear system of wave equations

$$\begin{aligned}
&\partial_t^2 u_i - \tilde{c}_i^2(x)\Delta u_i = f_i(t,x), \quad i = 1,2,3 \quad \text{in} \quad (0,T)\times\Omega' \\
&u(0,x) = b_0(x) \quad \partial_t u(0,x) = b_1(x) \quad \text{in} \quad \Omega' \\
&u(t,x)|_{\partial\Omega'\times(0,T)} = 0
\end{aligned} \tag{3.1}$$

and the linear operator $\Box_S$ which is associated to the system if we let $u = (u_1,u_2,u_3)^t$. Through abuse of notation, we let $\Box_S^{-1}F(t,x)$ denote the solution to the Cauchy problem (3.1) above. As such, $\Box_S^{-1}$ is associated to the diagonal matrix

$$\Box_S^{-1} = \begin{pmatrix} \Box_{\tilde{c}_1}^{-1} & 0 & 0 \\ 0 & \Box_{\tilde{c}_2}^{-1} & 0 \\ 0 & 0 & \Box_{\tilde{c}_3}^{-1} \end{pmatrix} \tag{3.2}$$

with $\Box_{\tilde{c}_i}^{-1}$, $i = 1,2,3$ is the inverse operator associated to each $\Box_{\tilde{c}_i} = \partial_t^2 - \tilde{c}_i^2\Delta$. For *any* fixed and finite T and $\beta \in \mathbb{N}$, we know from [18] that there exists a unique $u_i = \Box_{\tilde{c}_i}^{-1}(b_{0i}, b_{1i}, f_i)$ with $u_i \in C([0,T]; H_0^\beta(\Omega')) \cap C^1([0,T]; H_0^{\beta-1}(\Omega'))$, if F is bounded in the $*$ norm and ϵ is sufficiently small. As a result the operator $\Box_S^{-1}$ is diagonal in each component and is a bounded operator

$$(H_0^\beta(\Omega')^3, H_0^{\beta-1}(\Omega')^3, L^2([0,T]; H_0^{\beta-1}(\Omega')^3)) \mapsto C([0,T]; H_0^\beta(\Omega')^3) \cap C^1([0,T]; H_0^{\beta-1}(\Omega')^3). \tag{3.3}$$

We consider the 'open source problem' for the nonlinear waves now

$$\begin{aligned}
&\partial_t^2 u_i - \tilde{c}_i^2(x)\Delta u_i = |u|^2 + f_i(t,x), \quad i = 1,2,3 \quad \text{in} \quad \mathbb{R}_t^+ \times\Omega' \\
&u(0,x) = b_0(x) \quad \partial_t u(0,x) = b_1(x) \quad \text{in} \quad \Omega' \\
&u(t,x)|_{\partial\Omega'\times(0,T)} = 0.
\end{aligned} \tag{3.4}$$

Let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be three component vectors and we set N as the quadratic nonlinearity $N(v, w) = (v \cdot w, v \cdot w, v \cdot w)$, although this construction is applicable for any quadratic nonlinearity. While this is only a lemma, the parametrix itself tells us that the solutions to the non-linear problem can be tractable if the Cauchy data is sufficiently small, without having to use a tedious Duhamel principle argument. A related parametrix idea is in [17], but they do not show the terms are in a bounded hierarchy as they are using low regularity distributional solutions.

Lemma 1. *Let $\epsilon > 0$, $f_1(t, x)$ in $L^2([0, T]; H_0^2(\Omega')^3)$, b_0', b_1' in $H_0^3(\Omega')^3$, $H_0^2(\Omega')^3$ respectively, with*

$$\|f_1(t, x)\|_{L^2([0,T];H_0^2(\Omega')^3)} + \|b_0'\|_{H_0^3(\Omega')^3} + \|b_1'\|_{H_0^2(\Omega')^3} = \|F_1\|_* \leq 1 \tag{3.5}$$

a parametrix solution to (3.4) when $F = \epsilon F_1 = \epsilon(b_0', b_1', f_1)$ with ϵ small, is represented by the following

$$w = \epsilon w_1 + \epsilon^2 w_2 + E_\epsilon \tag{3.6}$$

with individual terms given by

$$w_1 = \Box_S^{-1} F \tag{3.7}$$

$$w_2 = -\Box_S^{-1}(0, 0, N(w_1 \cdot w_1))$$

$$\|E_\epsilon\|_{C([0,T];H_0^1(\Omega')^3) \cap C^1([0,T];L_0^2(\Omega')^3)} \leq 2D_1(T)^3 \epsilon^3$$

and $w \in C([0, T]; H_0^3(\Omega')^3) \cap C^1([0, T]; H_0^2(\Omega')^3)$. Moreover for $F = \epsilon F_1$ we have that

$$\|w_i\|_{C([0,T];H_0^1(\Omega')^3) \cap C^1([0,T];L_0^2(\Omega')^3)} \leq (D_1(T))^i \quad i = 1, 2 \tag{3.8}$$

where $D_1(T) = C_1(1 + T + (1 + \tilde{A}_1 T) \exp(\tilde{A}_1 T)) \exp(\tilde{A}_1 T)$) is the constant in Theorem 2 determined by (A.12) from Theorem 3.

Proof. By plugging in (3.6) into (3.4), and matching up the terms in powers of ϵ one gets a set of recursive formulae. Solving the equations recursively gives the expansion for the coefficients. To prove inequality (3.8) one remarks that

$$\|w_1\|_{C([0,T];H_0^1(\Omega')^3) \cap C^1([0,T];L_0^2(\Omega')^3)} \leq D_1(\|F_1\|_*) \tag{3.9}$$

which is essentially inequality (A.12) from Theorem 3 in the Appendix. We use this fact and Gargliano–Nirenberg–Sobolev to see

$$\|\Box_S^{-1}(N(w_1, w_1))\|_{C([0,T];H_0^1(\Omega')^3)} \leq D_1 \|w_1^2\|_{C([0,T];L_0^2(\Omega')^3)} \leq D_1 \|w_1\|^2_{C([0,T];L_0^4(\Omega')^3)} \leq \tag{3.10}$$

$$D_1 \left(\|w_1\|_{C([0,T];\dot{H}_0^1(\Omega')^3)} \right)^{3/2} \left(\|w_1\|_{C([0,T];L_0^2(\Omega')^3)} \right)^{1/2} \leq (D_1)^2$$

where in the last inequality we used the fact x^α is monotone increasing in α for $\alpha \geq 0$ and the requirement $\|F_1\|_* \leq 1$, by our choice of domain for the operator Λ.

To find a bound on the error, we see that if u is the true solution to (2.1), and w is the Ansatz solution, the error $u - w = E_\epsilon(t, x)$ satisfies the equation

$$\Box_S E_\epsilon = |u|^2 - |w|^2 + \tilde{E}_\epsilon \tag{3.11}$$

where for all $i = 1, 2, 3$

$$\tilde{E}_{\epsilon i} = 2\epsilon^3 w_2 \cdot w_1 + \epsilon^4 w_2^2 \tag{3.12}$$

which implies

$$\Box_S E_\epsilon = E(u+w) + \tilde{E}_\epsilon. \tag{3.13}$$

Using (3.8), and Theorem 3, the main part of the parametrix and error are bounded appropriately. Indeed, we have that

$$\begin{aligned} &\|E_\epsilon\|_{C([0,T];H_0^1(\Omega')^3)\cap C^1([0,T];L_0^2(\Omega')^3)} \leq \\ &D_1(T)\|E_\epsilon(u+w)\|_{L^2([0,T];L_0^2(\Omega')^3)} + D_1(T)\|\tilde{E}_\epsilon\|_{L^2([0,T];L_0^2(\Omega')^3)} \leq \\ &D_1(T)T\|E_\epsilon\|_{C([0,T];L_0^2(\Omega')^3)}\|(u+w)\|_{C([0,T];L_0^2(\Omega')^3)} + D_1(T)\|\tilde{E}_\epsilon\|_{L^2([0,T];L_0^2(\Omega')^3)} \leq \\ &2T\epsilon D_1(T)\|E_\epsilon\|_{C([0,T];L_0^2(\Omega')^3)} + D_1(T)\|\tilde{E}_\epsilon\|_{L^2([0,T];L_0^2(\Omega')^3)}. \end{aligned} \tag{3.14}$$

The result follows provided

$$2T\epsilon D_1(T) < 1 \tag{3.15}$$

which is already satisfied by (A.35). $\Box$

4. Testing of the waves: A new construction

The difficulty in constructing accurate approximations to solutions of nonlinear PDE is existence of singularities which can propagate forward in time when the waves interact. When $\phi(x)$ is smooth and compactly supported, then convolution with

$$f_k(x) = k^{d/2}\phi\left(\frac{x}{k}\right) \tag{4.1}$$

as $k \to \infty$ approximates a Dirac mass δ_0 with d the dimension of the space in consideration. We see the function $f_k(x)$ is in $L^2(\mathbb{R}^d)$ but $f_k^2(x)$ is not when $k \to \infty$. This causes problems when considering a parametrix for a semi-linear wave equation of the form $\Box_g u = |u|^2$ and indeed, there are examples where the wave front sets of the nonlinear hyperbolic PDE do not coincide with those of the linear hyperbolic PDE, c.f. [1] Theorem 2.1 for example.

In [28], they proved that the initial and subsequent crossings wave solutions to the linear PDE are the only source of nonlinear singularities. Thus, for $H^\alpha(\mathbb{R}^d)$ $\alpha > d/2$ compactly supported initial data we no longer have this problem, and the data propagates regularly (provided there are no derivatives in the nonlinearity). Using theorems in [27,28], and [1] we could lower the assumptions on the initial data regularity for the problem, using the same techniques here, but this is not the main focus of the article. Lowering the Cauchy data regularity often comes at the cost of shortening the validity of the timescale of the solutions.

We show that one can recover the coefficients of the toy model for the elasticity coefficients and show that the wave interaction is nonzero given sufficient regularity.

Proof of Theorem 1. The components in the parametrix as in (3.6) for each of them we denote as u_{jik} where j denotes the vector component $j = 1, 2, 3$, i denotes the index of the system $i = 1, 2$ and k denotes the power in the expansion of ϵ, $k = 1, 2$. Therefore

$$\begin{aligned} (\mathcal{U}_1 - \mathcal{U}_2) = &\epsilon(u_{111} - u_{211}, u_{121} - u_{221}, u_{131} - u_{231}) + \\ &\epsilon^2(u_{112} - u_{212}, u_{122} - u_{222}, u_{132} - u_{232}) + \epsilon^3(E_\epsilon^1 - E_\epsilon^2) \end{aligned} \tag{4.2}$$

where $E_\epsilon^0(t,x) = (E_\epsilon^1 - E_\epsilon^2)$ is a three term component of the error. From Lemma 1, this error is bounded by $D^3(T)\epsilon^3$ in $C([0,T]; H^1(\Omega)^3)$ norm. Here is where we use the fact u, w and E_ϵ are bounded in $C([0,T]; C^1(\Omega)^3)$ norm so we know the data propagates regularly, and we do not have to check any singularity crossings.

If $\Lambda_1 = \Lambda_2$ then it follows that $\Lambda_1^{lin} = \Lambda_2^{lin}$, by matching up the $\mathcal{O}(\epsilon)$ terms in the expansion and varying over all data F_1. Indeed, otherwise one has that E_ϵ^0, $(w_{1,1} - w_{2,1})$, and $(w_{1,2} - w_{2,2})$ are all nonzero and

$$\frac{\|(w_{1,1} - w_{2,1}) + \epsilon(w_{1,2} - w_{2,2})\|_{L^2([0,T];L^2(\partial\Omega)^3)}}{\epsilon^2} = \|E_\epsilon^0\|_{L^2([0,T];L^2(\partial\Omega)^3)} \tag{4.3}$$

for all possible choices of data F_1 and for all ϵ. The left hand side blows up as ϵ goes to 0. However, the right hand side involving E_ϵ^0 is uniformly bounded by $4TD_1(T)^3 < 2\epsilon_1^{-1}[D_1(\epsilon_1)]^2 \approx \epsilon_1^{-3}$ from (A.35) and Lemma 4 in the Appendix. Thus this statement is impossible. The key point is that for each ϵ, the maximal lifespan of the solution is $T(\epsilon)$ with $T(\epsilon) > T(\epsilon_1)$. This is a bit tricky to understand as we restrict to T such that $T < T(\epsilon_1)$, so even though a larger lifespan may exist, this is not what timescale we use for the *family* of source data. □

We now recall some definitions in the literature to provide an example of metrics which satisfy the necessary conditions for Theorem 1.

Definition 1 (*Definition in [37]*)**.** Let (M_0, g) be a compact Riemannian manifold with boundary. We say that M_0 satisfies the foliation condition by strictly convex hyper surfaces if M_0 is equipped with a smooth function $\rho : M_0 \to [0,\infty)$ which level sets $\sigma_t = \rho^{-1}(t)$, $t < T$ with some $T > 0$ finite, are strictly convex as viewed from $\rho^{-1}((0,t))$ for g, $d\rho$ is non-zero on these level sets, and $\Sigma_0 = \partial M_0$ and $M_0 \setminus \bigcup_{t\in[0,T)} \Sigma_t$ has empty interior.

The global geometric condition of [37] is a natural analog of the condition

$$\frac{\partial}{\partial r}\frac{r}{c(r)} > 0 \quad \text{s.t.} \quad \frac{\partial}{\partial r} = \frac{x}{|x|}\cdot\partial_x \tag{4.4}$$

the radial derivative as proposed by Herglotz [9] and Wiechert & Zoeppritz [38] for an isotropic radial sound speed $c(r)$. In this case the geodesic spheres are strictly convex.

In fact [34], c.f. Section 6. extends the Herglotz and Wiechert & Zoeppritz results to not necessarily radial speeds $c(x)$ which satisfy the radial decay condition (4.4). Let $B(0,R)$ $R > 0$ be the ball in $\mathbb{R}^d$ with $d \geq 3$ which is entered at the origin with radius $R > 0$. Let $0 < c(x)$ be a smooth function in $B(0,R)$.

Proposition 1. *The Herglotz and Wieckert & Zoeppritz condition is equivalent to the condition that the Euclidean spheres $S_r = \{|x| = r\}$ are strictly convex in the metric $c^{-2}dx^2$ for $0 < r \leq R$.*

Example 1 (*Herglotz Wiechert and Zoeppritz Systems*)**.** Let Ω be the unit ball, so $M_0 = \overline{\Omega}$ then for any $c_i \in C^3(\Omega)$, $i = 1,2,3$ such that

$$\frac{1}{1+r^2} \leq c_i(r) \leq 1 \tag{4.5}$$

satisfy the convexity condition (4.4), and the conditions of Theorem 1 for equations of the form (2.1). Using known results on injectivity in [34], systems with coefficients of this type provide an example of a case where Corollary 1 holds. Here we remark that $\partial_t^2 - c^2\Delta$ and $\partial_t^2 - \Delta_g$ have the same principal symbols if $g = c^{-2}dx$ and $c^2 \in \mathcal{A}_0^3$ (they coincide in dimension 2). In particular, in [34] they show for the scalar valued wave equation with $f_1(t,x) = 0$,

$$\begin{aligned}
&\partial_t^2 u - \tilde{c}^2(x)\Delta u = 0 \quad \text{in } [0,T] \times \Omega', \\
&u(0,x) = u_0(x) \quad \partial_t u(0,x) = u_1(x) \quad \text{in} \quad \Omega' \\
&u(t,x)|_{\partial\Omega' \times [0,T]} = 0
\end{aligned} \tag{4.6}$$

in $\mathbb{R}^3$, that the linear source to solution map Λ is enough to determine the lens relation on the subset Ω. For sound speeds of the above form, they can reconstruct the sound speed from the lens relation. Note in this case the Cauchy data outside Ω becomes the boundary data they are measuring as there is no well-defined definition of boundary data for the nonlinear problem.

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Appendix. Well-posedness estimates for the semi-linear wave equations

We set $\overline{\Omega} \subset \Omega'$, where Ω' is a larger domain in $\mathbb{R}^3$, with Dirichlet boundary conditions and smooth boundary. In the appendix, we prove the following theorem:

Theorem 2. *Let $s > 5/2$ be an arbitrary integer. Assume that $c_i(x) \in \mathcal{A}_0^s, \forall i = 1,2,3$. Let $F(t,x) = (u_0, u_1, f) = \epsilon F_1(t,x) = \epsilon(b_0, b_1, f_1)$ with*

$$\|b_0\|_{H_0^s(\Omega')^3} + \|b_1\|_{H_0^{s-1}(\Omega')^3} + \|f_1\|_{L^2([0,T];H_0^{s-1}(\Omega')^3)} = \|F_1(t,x)\|_* \leq 1, \tag{A.1}$$

then there exists a unique solution $u(t,x)$ with $u(t,x) \in C([0,T]; H_0^s(\Omega')^3) \cap C^1([0,T]; H_0^{s-1}(\Omega')^3)$ to the coupled system:

$$\begin{aligned}
&\partial_t^2 u_i - \tilde{c}_i^2(x)\Delta u_i = |u|^2 + f_i(t,x) \quad \textit{in } [0,T] \times \Omega', \quad i = 1,2,3 \\
&u(0,x) = u_0(x) \quad \partial_t u(0,x) = u_1(x) \quad \textit{in} \quad (\Omega')^3 \\
&u(t,x)|_{\partial\Omega' \times [0,T]} = 0
\end{aligned} \tag{A.2}$$

provided $C(s)T < \log((12\epsilon)^{-1}) - C'(s)$ where $C(s), C'(s)$ depend on s and the $C^s(\Omega')$ norm of the $c_i's$.

We prove the local well posedness theorem via an abstract Duhamel iteration argument. We recall Duhamel's principle.

Definition 2 (*Duhamel's Principle*)**.** Let $\mathcal{D}$ be a finite dimensional vector space, and let I be a time interval. The point t_0 is a time t in I. The operator L and the functions v, f are such that:

$$L \in \mathrm{End}(\mathcal{D}) \quad v \in C^1(I \to \mathcal{D}), \quad f \in C^0(I \to \mathcal{D}) \tag{A.3}$$

then we have that

$$\partial_t v(t) - Lv(t) = f(t) \quad \forall t \in I \tag{A.4}$$

if and only if

$$v(t) = \exp((t-t_0)L)v(t_0) + \int_{t_0}^{t} \exp((t-s)L)f(s)\,ds \quad \forall t \in I. \tag{A.5}$$

We view the general equation as

$$v = v_{lin} + JN(f) \tag{A.6}$$

with J a linear operator. We also have the following abstract iteration result:

Lemma 2 (*[36] Prop 1.38*)**.** *Let $\mathcal{N}, \mathcal{S}$ be two Banach spaces and suppose we are given a linear operator $J : \mathcal{N} \to \mathcal{S}$ with the bound*

$$\|JF\|_{\mathcal{S}} \leq C_0 \|F\|_{\mathcal{N}} \tag{A.7}$$

for all $F \in \mathcal{N}$ and some $C_0 > 0$. Suppose that we are given a nonlinear operator $N : \mathcal{S} \to \mathcal{N}$ which is a sum of a u dependent part and a u independent part. Assume the u dependent part N_u is such that $N_u(0) = 0$ and obeys the following Lipschitz bounds

$$\|N(u) - N(v)\|_{\mathcal{N}} \leq \frac{1}{2C_0}\|u - v\|_{\mathcal{S}} \tag{A.8}$$

for all $u, v \in B_\epsilon = \{u \in \mathcal{S} : \|u\|_S \leq \epsilon\}$ for some $\epsilon > 0$. In other words we have that $\|N\|_{\dot{C}^{0,1}(B_\epsilon \to \mathcal{N})} \leq \frac{1}{2C_0}$. Then, for all $u_{lin} \in B_{\epsilon/2}$ there exists a unique solution $u \in B_\epsilon$ with the map $u_{lin} \mapsto u$ Lipschitz with constant at most 2. In particular we have that

$$\|u\|_{\mathcal{S}} \leq 2\|u_{lin}\|_{\mathcal{S}}. \tag{A.9}$$

We start by proving general energy estimates for the linear problem. We have the following classical result, for all $\beta \in \mathbb{N}$.

Theorem 3. *Let $c \in \mathcal{A}_0^\beta$, and $f(t,x) \in L^2([0,T]; H_0^{\beta-1}(\Omega'))$, $u_0(x) \in H_0^\beta(\Omega')$, $u_1(x) \in H_0^{\beta-1}(\Omega')$. If u is a solution to*

$$\begin{aligned} &\partial_t^2 u - \tilde{c}^2(x)\Delta u = f(t,x) \quad \text{in} \quad [0,T] \times \Omega' \\ &\partial_t u(0,x) = u_1(x) \quad u(0,x) = u_0(x) \text{ in } \Omega' \\ &u(t,x) = 0 \quad \text{on} \quad [0,T] \times \partial\Omega' \end{aligned} \tag{A.10}$$

we have the following set of estimates:

- *There exists C depending on m_0 and $\|c^2\|_{C^1(\Omega')}$ and $\tilde{A}_1$ depending on $\|c^2\|_{C^1(\Omega')}$ such that*

$$\begin{aligned} &\|u\|_{C([0,T];\dot{H}_0^1(\Omega')) \cap C^1([0,T];L_0^2(\Omega'))} \leq \\ &C\left(\|u_0\|_{H_0^1(\Omega')} + \|u_1\|_{L_0^2(\Omega')} + \|f(t,x)\|_{L_0^2(\Omega' \times [0,T])}\right)\exp(\tilde{A}_1 T). \end{aligned} \tag{A.11}$$

  and

- *There exists C_1 which depends on m_0 and $\|c_i^2(x)\|_{H^\beta(\Omega')}$ and $\tilde{A}_\beta$ which depends on $\|c_i^2(x)\|_{H^\beta(\Omega')}$ such that*

$$\|u\|_{C([0,T];H_0^\beta(\Omega'))} + \|\partial_t u\|_{C([0,T];H_0^{\beta-1}(\Omega'))} \leq \tag{A.12}$$

$$C_1(1+T)\exp(\tilde{A}_\beta T)\times(\|u_0\|_{H_0^\beta(\Omega')}+\|u_1\|_{H_0^{\beta-1}(\Omega')}+$$
$$\tilde{A}_\beta T(\|u\|_{C([0,T];H_0^{\beta-1}(\Omega'))}+\|\partial_t u\|_{C([0,T];H_0^{\beta-2}(\Omega'))})+\|f\|_{L^2([0,T];H_0^{\beta-1}(\Omega'))}).$$

Proof. The proofs below are loosely based on Theorem 4.6 and Corollary 4.9 in [20] which have been adapted for our setting. By definition we have

$$\int_0^t\int_{\Omega'}(\partial_s^2 u-\tilde{c}^2\Delta u)\partial_s u\,dx\,ds=\int_0^t\int_{\Omega'}f(s,x)\partial_s u\,dx\,ds \tag{A.13}$$

Notice that even though u is not necessarily in $C^2([0,T]\times\Omega)$ the integral on the left hand side makes sense as $f(t,x)\in L^2([0,T];H_0^{\beta-1}(\Omega'))$ for $\beta\geq 1$, and $\partial_t u\in L^1([0,T];L_0^2(\Omega'))$ by [18], or a finite speed of propagation argument. While we could refer the well-posedness estimates in [18], which have similar structure as above, it is important to understand what the constants in the norm bounds are in terms of T actually are for later use.

We also have

$$\nabla\cdot(\tilde{c}^2\nabla u)=\tilde{c}^2\Delta u+\nabla\tilde{c}^2\cdot\nabla u. \tag{A.14}$$

We also have by the divergence theorem

$$\int_0^t\int_{\Omega'}\partial_s u(\nabla\cdot(\tilde{c}^2\nabla u))\,dx\,ds= \tag{A.15}$$
$$-\int_0^t\int_{\Omega'}\partial_s(\nabla u)\cdot(\tilde{c}^2\nabla u)\,dx\,ds+\int_0^t\int_{\partial\Omega'}\partial_s u\frac{\partial(\tilde{c}^2 u)}{\partial\nu}\,dS\,ds.$$

We set

$$\|u\|_E^2(t)=\frac{1}{2}\left(\int_0^t\int_{\Omega'}|\nabla u(s,x)|^2+|\partial_s u(s,x)|^2\,dx\,ds\right) \tag{A.16}$$

and

$$\|u\|_{Ec}^2(t)=\frac{1}{2}\left(\int_0^t\int_{\Omega'}\tilde{c}^2|\nabla u(s,x)|^2+|\partial_s u(s,x)|^2\,dx\,ds\right). \tag{A.17}$$

The end result of plugging the equalities into (A.13) is that

$$\frac{d}{ds}\|u\|_{Ec}^2(T)= \tag{A.18}$$
$$\int_0^T\int_{\Omega'}f\partial_s u\,dx\,ds+\int_0^T\int_{\partial\Omega'}\partial_s u\frac{\partial(\tilde{c}^2 u)}{\partial\nu}\,dS\,ds-\int_0^T\int_{\Omega'}\nabla\tilde{c}^2\cdot\nabla u\partial_s u\,dx\,ds$$

We let $C=\min\{m_0,1\}$. Taking the absolute values of both sides and remarking that $2ab\leq a^2+b^2$ for all real valued functions a,b we obtain

$$C\frac{d}{dt}\|u\|_E^2(T)\leq\tilde{A}\|f\|_{L^2(\Omega'\times[0,T])}^2+\tilde{A}\|u\|_E^2(T) \tag{A.19}$$

Applying Grownwall's inequality gives the desired result. For the second estimate, differentiating Eq. (A.13) (e.g. applying the operator ∇^k successively) gives control over

$$\|u\|_{C([0,T];\dot{H}_0^k(\Omega'))\cap C^1([0,T];\dot{H}_0^{k-1}(\Omega'))} \tag{A.20}$$

it remains to control $\|u\|_{C([0,T];L^2_0(\Omega'))}$ but it is easy to see

$$\|u\|_{C([0,T];L^2(\Omega'))} \le \|u_0\|_{L^2(M)} + \int_0^T \|\partial_t u\|^2_{L^2(\Omega')}(t)\,dt \tag{A.21}$$

which gives the desired result. □

Proof of Theorem 2. Recall that $H^\alpha(M) \subseteq L^\infty(M)$ if $\alpha > d/2$, which is an assumption we will use here. If we reformulate the wave equation (A.10) as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ c^2\Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{A.22}$$

with

$$\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ c^2\Delta & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \Phi = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{A.23}$$

One can write the inhomogeneous scalar valued wave equation as

$$\begin{aligned} \mathcal{U}_t &= A\mathcal{U} + F \\ \mathcal{U}(0) &= \Phi \end{aligned} \tag{A.24}$$

Using this as our model, we can re-write the more complicated system (A.2)

$$\begin{aligned} \mathcal{W}_t &= \tilde{A}\mathcal{W} + \tilde{F} \\ \mathcal{W}(0) &= (u_{01}, u_{10}, u_{02}, u_{12}, u_{03}, u_{13})^t \end{aligned} \tag{A.25}$$

(where the second subscript denotes the components of u_0, u_1, respectively) with

$$\begin{aligned} \mathcal{W} &= (u_1, v_1, u_2, v_2, u_3, v_3)^t \\ \tilde{F} &= (0, |u|^2, 0, |u|^2, 0, |u|^2)^t + (0, \epsilon f_{1i}, 0, \epsilon f_{1i}, 0, \epsilon f_{1i}) \end{aligned} \tag{A.26}$$

and

$$A_i = \begin{pmatrix} 0 & 1 \\ \tilde{c}_i^2\Delta & 0 \end{pmatrix} \tag{A.27}$$

elements of the block diagonal matrix

$$\tilde{A} = \begin{pmatrix} A_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_3 \end{pmatrix} \tag{A.28}$$

where the bold face $\mathbf{0}$ is a 2×2 matrix of 0's. We then apply the abstract Duhamel iteration argument with $\mathcal{S} = (C([0,T]; H_0^s(\Omega')), C([0,T]; H_0^{s-1}(\Omega')))^3$ (equivalent to $C([0,T]; H_0^s(\Omega')^3) \cap C^1([0,T]; H_0^{s-1}(\Omega')^3))$ if we note $v = \partial_t u$) and $\mathcal{N}$ is the $L^2([0,T]; H_0^{s-1}(\Omega')^6)$ norm as implied by (2.2). We leave the s as an arbitrary integer, so if we set J the Duhamel propagator associated to $\tilde{A}$ with $\tilde{F} = (0, F_1, 0, F_2, 0, F_3) \in L^2([0,T]; H_0^{s-1}(\Omega')^6)$, then the inequality $\|J\tilde{F}\|_{\mathcal{S}} \le C_0\|\tilde{F}\|_{\mathcal{N}}$ is satisfied with $C_0 = D_s(T)$ given to us by Theorem 3, as $u = J\tilde{F}$ with corresponding source data applied with $F = (0, 0, F')$, $F' = (F_1, F_2, F_3)$ (the constant $D_s(T)$ is the maximum over the conformal factors). In practice for the rest of the article we only need $s = 3$.

The key observation is that

$$\|\tilde{F}(W_1) - \tilde{F}(W_2)\|_{\mathcal{N}} \leq B\|W_1 - W_2\|_{\mathcal{S}}. \tag{A.29}$$

for some positive constant B, depending on ϵ and T with

$$W_1 = (w_{1,1}, v_{1,1}, w_{1,2}, v_{1,2}, w_{1,3}, v_{1,3})^t \tag{A.30}$$
$$\text{and } W_2 = (w_{2,1}, v_{3,1}, w_{2,2}, v_{2,2}, w_{2,3}, v_{2,3})^t.$$

By definition, we have

$$\|\tilde{F}(W_1) - \tilde{F}(W_2)\|_{\mathcal{N}} = 3\|(0, w_{1,1}^2 - w_{2,1}^2, 0, w_{1,2}^2 - w_{2,2}^2, 0, w_{1,3}^2 - w_{2,3}^2)\|_{\mathcal{N}}.$$

We see

$$\sup_{i=1,2} \|w_i\|_{\mathcal{S}} \leq \epsilon \tag{A.31}$$

where we used the upper bound implied by the hypothesis $\mathcal{W}_1, \mathcal{W}_2 \in B_\epsilon$. We then obtain

$$\|\tilde{F}(W_1) - \tilde{F}(W_2)\|_{\mathcal{N}} \leq 6\epsilon T\|W_1 - W_2\|_{\mathcal{S}} \tag{A.32}$$

with and the result (A.29) follows with $B = 6\epsilon T$.

The corresponding Duhamel iterates are

$$\mathcal{W}^0 = \mathcal{W}_{lin} \quad \mathcal{W}^n = \mathcal{W}_{lin}^{n-1} + JN(\mathcal{W}^{n-1}) \tag{A.33}$$

and from Lemma 2 we can conclude

$$\lim_{n\to\infty} \mathcal{W}^n = \mathcal{W}^* \tag{A.34}$$

is the unique solution $W^* \in B_\epsilon$ whenever T is sufficiently small, by Lemma 2. In particular, for the Theorem to hold we must have

$$6T\epsilon < \frac{1}{2D_s(T)} \quad \Rightarrow \quad TD_s(T) < (12\epsilon)^{-1}. \tag{A.35}$$

As $D_s(T)$ is a polynomial in T and $\exp(\tilde{A}T)$ and since $\log(R) \leq R$ for all $R \in \mathbb{R}^+$,

$$C(s)T < \log((12\epsilon)^{-1}) - C(s') \tag{A.36}$$

for some $C(s)$, $C(s')$ depending on s and $\tilde{A}_s$. For a similar argument without using the abstract iteration result, for a scalar wave equation with quadratic nonlinearity one can see [29]. □

We have the following Lemma which is only necessary in the case of non-trapping sound speed example, not the main result.

Lemma 3. *Let $T(\epsilon)$ denote the maximal timespan for well-posedness of the system (2.1). There exists $\epsilon_1 \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_1)$, the inequality*

$$\mathrm{diam}(\Omega) < T(\epsilon_1) < T(\epsilon) \tag{A.37}$$

holds.

Proof. For each ϵ, we know the timescale $T(\epsilon)$ must be such that (A.36) holds with $s = 3$. Then the condition (A.37) is satisfied if (A.36) holds with T replaced by $\mathrm{diam}(\Omega)$. This is clearly possible as $\mathrm{diam}(\Omega)$ is finite, whence the conclusion is possible. □

Lemma 4. *The operator Λ as a nonlinear operator is bounded when acting on $u \in L^2([0,T]; H^1(\Omega)) \cap C([0,T]; C(\overline{\Omega}))$,*

$$\|\Lambda u\|_{L^2([0,T];L^2(\partial\Omega))} \leq C\|u\|_{L^2([0,T];H^1(\Omega))} \tag{A.38}$$

where C is a constant depending only on the geometry of Ω.

Remember that the boundary of Ω' necessarily is smooth, but that of Ω does not have to be for this definition bound to hold.

We recall the trace theorem

Theorem 4. *Assume that Ω is a bounded domain with C^1 boundary, then $\exists$ a bounded linear operator*

$$Tv = v|_{\partial\Omega} \quad \text{for} \quad v \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \tag{A.39}$$

and a constant $c(p, \Omega)$ depending only on p and the geometry of Ω such that

$$\|Tv\|_{L^p(\partial\Omega)} \leq c(p, \Omega)\|v\|_{W^{1,p}(\Omega)} \tag{A.40}$$

The proof of Lemma 4 now follows immediately.

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