EFFECTS OF OSCILLATORY MOTION ON THE WAVE PROPAGATION IN BEAMS

BY

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This thesis is dedicated to my mother for her endless love, support and encouragement

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LIST OF ABBREVIATIONS

A	area of cross-section
b	width of the beam
h	thickness of the beam
E	Young's Modulus of material
G	Shear Modulus of material
I	Moment of inertia of cross-section
L	beam length
l	elemental length
U	strain energy
T	kinetic energy
V	work done
δU	variation of strain energy
δT	variation of kinetic energy
δV	variation of work done
u	x-axis displacement
w	z-axis displacement
u_0	midplane displacement in x-axis
w_0	midplane displacement in z-axis
φ	Angle of rotation of the normal to the mid surface
FT	Fourier Transformation
C_o	Oscillatory velocity of the beam
$\dot{C}_o$	Oscillatory acceleration of the beam

$\hat{C}_o$ Fourier transformation of velocity

$\hat{\dot{C}}_o$ Fourier transformation of acceleration

IFFT Inverse Fast Fourier Transformation

Greek Symbols

ζ wave number

ρ density of material

η damping coefficient

κ shear correction factor

ω circular frequency

β angle due to shear distortion

Ω Frequency of oscillatory motion in Hertz

ABSTRACT

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In this present work, the effects of axial oscillatory motion on the propagation of ultrasonic waves in isotropic infinite beams were studied numerically. The beam oscillates axially with an oscillating frequency **Ω.** The beam is modeled by using Timoshenko beam theory where the rotary inertia as well as shearing effect is considered and the equations of motion are obtained by employing Hamilton's principle**.** The normal surface displacement produced by a transient point force is calculated using a multiple transform technique. The equations are then transformed from the time-space domain to frequency-wave number domain by employing Fourier Transformation (FT). To recover the displacement in time-space domain the inversion is applied by using a combination of Romberg's integration and Inverse Fast Fourier Transformation (IFFT). During the transformation process, the dispersion relations are formulated. The effects of the oscillation frequency, **Ω**, on the calculated normal displacement signals due to point-force excitation were investigated for different values of **Ω**. Also, the effect of **Ω** on the dispersion curves of the two modes of propagation is studied.

ملخص الرسالة

الاسم الكامل: **خرم قيوم**

عنوان الرسالة: **تأثيرات الحركة الترددية على إنتشار الموجات في العوارض**

التخصص: **الهندسة الميكانيكية**

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تم في هذا العمل دراسة عددية لتأثير الحركة التذبذبية المحورية على إنتشار الموجات فوق الصوتية في العوارض متحدة الخواص لا نهائية الطول ـ تتذبذب العارضة محورياً بتردد تأرجحي Ω ـ تم صياغة نموزج العارضة باستخدام نظرية تيموشينكو للعوارض حيث أخذ في الأعتبار القصور الذاتي الدوار وكذلك تأثير القص وبعد ذلك تم الحصول على معادلات الحركة من خلال توظيف مبدأ هاملتون ـ تم حساب الإزاحة السطحية العامودية الناتجة عن القوة النقطية العابرة بإستخدام تقنية تحويل متعددة ـ ثم تم بعد ذلك تحويل المعادلات من نطاق الزمن- الفضاء إلى نطاق التردد- رقم الموجة من خلال توظيف تحويل فورييه (FT) ـ لإسترداد الإزاحة في نطاق الزمن-الفضاء، تم إستعمال تكامل رومبيرغ ومعكوس تحويل فورييه السريع (IFFT) ـ في أثناء عملية التحول هذه تمت صياغة علاقات التشتت الموجي ـ تم دراسة تأثير وتيرة التذبذب التأرجحي على إشارات الإزاحة العامودية الناتجة بسبب إثارة القوةالنقطية ، و ذلك لقيم متعددة من Ω ـ كماتم أيضا دراسة تأثير Ω على منحنيات التشتت لنمطي إنتشار الموجي ـ

CHAPTER 1

INTRODUCTION

The use of elastic wave propagation through the monitored part of structure is of great interest. The most important example is detection of material difference and damage in the investigated structure. By investigating the characteristics of propagating waves, the type and position of defects can be determined.

Use of ultrasonic waves for Nondestructive testing is about 40 years old technique. For detection of flaws in beams and structures use of ultrasonic oscillations has become a classical test method by considering all important influencing factors. Nowadays it is expected that ultrasonic testing along with great advances in instrument technology give reproducible test results within narrow tolerance. Ultrasonic waves are emitted from a transducer into an object and the returning waves are captured and then analyzed. If an impurity or a crack is present in structure, the wave will bounce back of them and be seen in the returned signal.

In the past, radiography was only technique for the detection of internal flaws in structures. After World War II use of ultrasonic methods for testing as explained by Sokolovin [1] and then applied by Firestonein [2] was developed and instruments were available for testing. The ultrasonic testing principle is based on the fact that waves in structures are reflected at interfaces as well as by internal flaws. Ultrasonic testing can be used to determine the location of discontinuity and to determine the thickness of materials within a structure by accurately measuring the time required for an ultrasonic pulse to travel and to reflect from the discontinuity through the material. There are mainly two types of elastic waves that can propagate in a homogenous and isotropic beam of infinite length and they are longitudinal and shear waves respectively

Although beam is a word in common usage for engineering design, it has a very particular definition. A beam is a structural element that carries loads which are perpendicular to its longitudinal axis and that develops internal stresses due to its internal bending moment and shear force. These internal moments and shear forces result from externally applied loads. The first approximation model of many structural elements is that they behaves as a beam in bending. Some examples are beams in a building, railroad rails, power transmission shafting and airplane wings. An understanding of beam theory and behavior is essential to the understanding of many kind of structures. Study of the behavior of structures that are oscillating in nature or attached to a moving base has been pursed for with a number of diverse disciplines, such as machine design, aircraft dynamics and robotics.

In the present work, transient wave propagation analysis in an axially oscillating infinite Timoshenko beam will be studied using a point force. Governing equations of motion will be derived using the dynamic version of the Hamilton's principle. Two dimensional transformation method will be used here in conjunction with the Timoshenko beam theory. Equations of motion will be transformed from time domain (t) to the frequency domain (ω) by using Fourier transforms. Romberg's method of integration will be used here to compute the inverse transformation from wave-number domain (ζ) to the spatial domain (x). Then the resultant displacement vector will be obtained by using IFFT. The transient response as well as the dispersion curves will be evaluated for several values of the oscillatory frequencies.

The thesis is organized as follows. Chapter 2 defines Literature review, motivation and objectives of proposed work. In Chapter 3, theoretical formulation, governing equations of motion and other related terminology is defined. Two dimensional Spectral formulation, temporal and spatial Fourier transformation are presented in Chapter 4. Method of solution is discussed in chapter 5. In Chapter 6 results and discussion for the present study is presented. Finally, the thesis ends in Section 7 with conclusions and perspectives.

CHAPTER 2

LITERATURE REVIEW

The problem of wave propagation in elastic beams is long standing. In studying wave propagation both temporal and spectral parameters are taken into account. Dynamic analysis applied to beams, can be broadly classified into two categories:

- Euler-Bernoulli beams analysis
- Timoshenko beams analysis

The classical beam theory (Euler-Bernoulli beam theory) based on assumption that if lateral dimensions of the beam are less than $\frac{1}{10}^{th}$ of its length, then the effect of shear deformation and rotary inertia can be ignored. Due to its simpler mathematical form, some analytical solutions are available for the homogeneous infinite beam to dynamic loads (Kenney [3]; McGhie [4]; Sun [5]).

However, only the flexural wave could propagate in the Euler-Bernoulli beam, whereas the shear wave would be ignored. This drawback limits the application of the Euler-Bernoulli beam theory only to motion where wavelength is larger compared to the thickness of the beam. As the wavelengths generated by most dynamic loads, used in ultrasonic testing, can be expected as the same order as lateral dimensions of the beam, it is almost necessary to use the Timoshenko beam theory in which the shear deformation and rotary inertia are included to model the beam (Achenbach and Sun [6]; Wang and Gagnon [7]; Billger and Folkow [8]; Folkow et al., [9]; Chen et al., [10]).

Transverse vibration problems of axially moving beams received a lot attention due to its association with various applications, such as band saw blades, textile machines, power transmission belts and high-speed magnetic tapes. Many analytical methods and numerical approaches have been used to study the vibration behavior and dynamic characteristics of axially moving beams. Pakdemirli et al.[11] investigated the non-linear transverse vibrations of an axially moving Euler-Bernoulli beam, where the multiple-scale method was employed to the motion equation for finding approximate solutions. Yang and Chen [12] analyzed the stability of an axially moving viscoelastic beam with simply supported ends and with pulsating speed.

Lee and Oh [13] proposed a spectral element model for the transverse response of an axially moving viscoelastic beam, and conducted numerical studies to analyze the viscoelasticity effect on the dynamic characteristics and stability of the moving beams. Liu and Deng [14] introduced a subspace-based identification algorithm to obtain the pseudo-modal parameters of an axially moving cantilever beam, and checked the capabilities of the algorithm by comparing the results with the ones observed from the experimental study. Chen and Yang [15] defined the governing equation of nonlinear vibration of axially moving beams, obtained the integro partial-differential formulation through replacing the axial tension by its average over the full beam length, and evaluated the natural frequencies using the multiple-scale method. Jakˇsi´c [16] presented a numerical algorithm for the solution of natural frequencies of axially moving beams, the validity of which was proved by comparing the calculated results with the analytical solutions of its limiting cases. Ponomareva and van Horssen [17] studied the transverse vibration of an axially moving beam by adopting a combined model that is a string model at lower frequencies and a stretched beam model at higher frequencies. Based on the Hamilton principle,Chang et al.[18] derived the governing equations of an axially moving Rayleigh beam, the natural frequencies and the transverse response of which were obtained from the eigenvalue problem and by using the time integration method, respectively.

Huang et al. [19] examined the nonlinear vibration behavior of an axially moving beam with periodic excitation. Wang [20] investigated the weakly forced vibration of an axially accelerating beam defined by the standard linear solid model, and used the multiple-scale method to analyze the steady-state response. Marynowski [21] investigated the dynamics of the axially moving sandwich beams with elastic faces and a viscoelastic core, where the two-parameter Kelvin-Voigt rheological model was employed to define the material characteristics of the core.

In order to take into account the shear deformation and rotational inertia there are comprehensive studies based on the Timoshenko theory. Lee et al. [22] presented a spectral model for the axially moving Timoshenko beam, and analyzed the effects of the axial speed and the axial tension on the vibration behavior and stability of the moving beam. To investigate the vibration response of the railway track due to a moving train, Cojocaru et al.[23] modeled the train as a Timoshenko beam with distributed stiffness moving at a constant velocity, where the derived eighth-order system of ordinary differential equations was solved by the mathematical software Maple. Using the Galerkin method, Yang et al. [24] discretized the governing equation for transverse vibration of a simply-supported axially accelerating Timoshenko beam, and analyzed the instability conditions of the system by the averaging method. Tang et al. [25] investigated vibration characteristics and critical speed of axially moving Timoshenko beams, the results of which were compared to the ones for Rayleigh beams, shear beams and Euler-Bernoulli beams. Tang et al. [26] extended their previous work by performing the parametric resonance analysis of axially moving Timoshenko beams with variable velocity, where the multiple-scale method was applied to study the steady-state behavior. Subsequently, Tang et al. [27] studied the nonlinear vibration of simply-supported axially moving Timoshenko beams under external excitations, and adopted the multiple-scale method to consider super harmonic resonance and sub harmonic resonance. Moreover, the results of nonlinear transverse vibration of an axially moving Timoshenko beam with two free edges were reported by Li et al. [28].To avoid the approximate boundary conditions in previous studies, Tang et al. [29] derived the governing equations for axially accelerating viscoelastic Timoshenko beams with the accurate boundary conditions by means of the generalized Hamilton principle. Ghayesh and Balar [30] obtained two dynamic models of axially moving Timoshenko beams including one regarding only the transverse displacement and the other regarding both longitudinal and transverse displacements, and employed the multiple-scale method to obtain the natural frequencies and steady-state behavior of the system. By considering the longitudinal, transverse, and rotational displacements, Ghayesh and Amabili [31] investigated the three-dimensional nonlinear dynamic phenomenon of axially moving Timoshenko beams, and solved the reduced set of nonlinear ODEs using the direct time integration method.

In the above studies, mainly vibration analysis were conducted for oscillating beams. To the best of our knowledge, it seems, that no work has been performed to investigate the effects of wave propagation in axially oscillating beams.

2.1 Motivation

From the literature review it is clear that there is a lot of research work done for non-oscillatory beams, especially for infinite and semi-infinite structures. Studies on elastic wave propagation in axially moving beams or rotating beams can also be found, but with different interests. None of these studies dealt with the effect of oscillatory motion on the wave propagation in beams.

It is hoped that the results of this proposed work will assist in ultrasonic testing of oscillatory beams and to understand how oscillatory motion can affect the waves used in ultrasonic nondestructive testing. Further, it is hoped that the outcome of this study will help us to separate the effect of oscillation from the captured signal and to recover pure propagation of the wave.

2.2 Objectives of the Present Work

The overall objective of the proposed thesis work is to investigate the effects of the oscillatory frequency on the propagating waves generated in the axially oscillating beam.

The specific objectives are as follows;

- To develop a numerical model for predicting the wave propagation inside the axially oscillating isotropic infinite beam based on Timoshenko beam theory.
- To obtain the solution for the resulting equations of motion using the multiple transform techniques.
- To study the effects of the frequency of oscillation on the transient wave response and on the characteristics of the dispersion curves.

CHAPTER 3

THEORETICAL FORMULATION

If the cross-sectional dimensions are not small as compared to the length of the beam, we have to consider the effect of shear deformation and rotary inertia as well. Timoshenko in 1921 [32] included these two effects to get results in accordance with the exact beam theory. This procedure that is presented by Timoshenko is later known as thick beam theory or Timoshenko beam theory.

The study of wave propagation in thin beam theory at high frequencies showed that improbable infinite phase velocities are predicted. By ignoring the rotary inertia effect for thin beam theory will obviously play its role. At very high frequencies the contribution of this rotary effect in form of rotational acceleration could be significant. An understated assumption that is used in elementary theory is that the shear deformation is zero. In present work we include both of these effects for Timoshenko beam model.

3.1 General Assumptions

For present study following basic assumptions are considered:

1. Beam is unstressed and initially straight.
2. The material used for Timoshenko beam is perfectly elastic, isotropic and homogeneous.
3. There is no resultant force perpendicular to the neutral axis of beam.
4. The effect of rotary inertia and shear is included.

5. The deflection is small as compared to the beam thickness.

6. Material properties are constant throughout the analysis.

7. Plain sections remain plain during bending.

8. Every cross-section will remain symmetric about the plane of bending.

3.2 Governing Equations of Motion

Consider an element of beam subjected to shear force, distributed load and bending moment as shown in Figure (1). The transverse displacement is measured by $w = w(x,t)$ and the slope of the centroidal axis is given by $\frac{\partial w}{\partial x}$. This slope is considered to be made of two components. The first component is $\varphi = \varphi(x,t)$ which is angle of rotation of the normal to the mid-surface and an additional contribution is $\beta = \beta(x,t)$, which defines the effect of shear.

Thus we have:

$$\frac{\partial w}{\partial x} = \varphi - \beta \tag{3.1}$$

The bending moment $M(x,t)$ at the cross section under consideration is given by,

$$M(x,t) = EI\frac{\partial \varphi(x,t)}{\partial x} \tag{3.2}$$

Where E is the modulus of elasticity of the material and I is the moment of inertia of cross section. The relation between the shear force $V(x,t)$ and the shear deformation is,

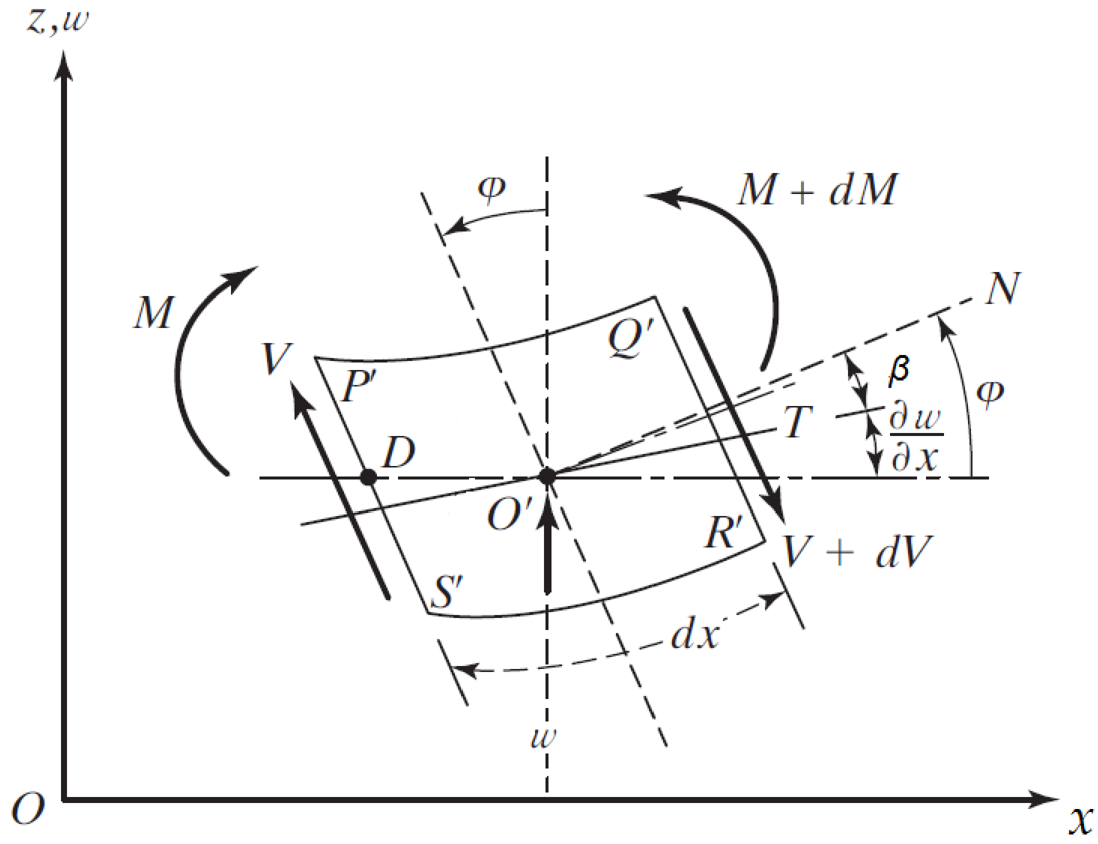

Figure 1: Differential element of Timoshenko beam

$$V(x,t) = kGA\beta(x,t) \tag{3.3}$$

where κ is the shear correction factor , G defines the material modulus of rigidity, which is assumed to be constant and $A = A(x)$ is the area of cross-section.The value of κ will depends on the shape of the cross-section of beam and is determined, usually by stress analysis means. In present study, κ is taken to be equal to $\frac{2}{3}$. By substituting the value of $\beta(x,t)$ in equation (3.1) into equation (3.3) we obtain,

$$V(x,t) = kGA\left(\varphi - \frac{\partial w}{\partial x}\right) \tag{3.4}$$

In the present study we are considering an infinite Timoshenko beam having an axially oscillatory motion, beam will oscillate sinusoidally with a frequency of Ω having an amplitude A_0. Let O be the arbitrary origin of infinite Timoshenko beam at a point where we apply a forcing function $f(t)$, beam have thickness $'h'$ and width 'b' as shown in Figure (2). Let the displacements in the *x* and *z* directions to be $u(t)$ and $w(t)$ respectively.

Now the governing equations of motions will be derived by using approximate Timoshenko beam theory and Hamilton's principle. First we introduce the axial oscillatory effect in beam which will result in modification of velocity terms. The axial displacement in the x-axis direction is expressed mathematically as,

$$X(t) = A_o sin\Omega t \tag{3.5}$$

Oscillatory axial velocity can be define as time derivative of displacement and expressed mathematically,

$$C_o(t) = C_o = \Omega A_o cos\Omega t \tag{3.6}$$

The velocity components in the x and z directions of an arbitrary point of coordinates $O(x\,,z)$ at any time t on the middle surface of the beam are denoted by $\dot{u}(\mathrm{t})$ and $\dot{w}(t)$ respectively.

The governing equations will be derived using the dynamic version of Hamilton's Principle which can be written as:

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta T)dt = 0 \tag{3.7}$$

Where,

δU = the virtual strain energy

δV = virtual work done by applied force

δT = the virtual kinetic energy

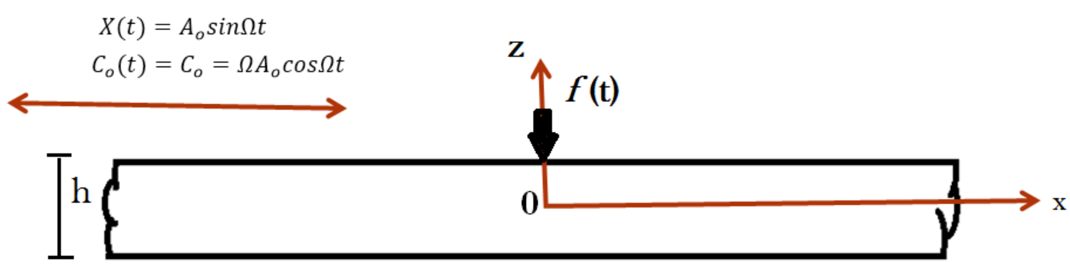

Figure 2: Timoshenko beam with axial oscillation

Kinetic energy (T) in terms of $\dot{u}(\mathrm{t})$ and $\dot{w}(t)$ can be expressed as follows:

$$T = \frac{1}{2} \int_{Volume} \rho[\dot{u}^2 + \dot{w}^2] \, d_{volume} \tag{3.8}$$

$$T = \frac{1}{2} \int_0^L \int_A \rho[\dot{u}^2 + \dot{w}^2] \, d_A dx \tag{3.9}$$

$$T = \frac{1}{2} \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho[\dot{u}^2 + \dot{w}^2] \; bdzdx \tag{3.10}$$

Where ρ is the mass density of the beam and L is the assumed length of the beam in x direction.

Note that

$$\int_A z^2 \, d_A = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 bdz = I \tag{3.11}$$

Where I is the cross-section area moment of inertia. According to the Timoshenko beam theory the displacements are expressed as:

$$u(x,z,t) = u_o(x,t) - z\varphi(x,t) \tag{3.12}$$

$$w(x,z,t) = w_o(x,t) \tag{3.13}$$

Where u_o and w_o are the mid-plane displacements components in the x-axis and z-axis directions respectively. Now the kinetic energy (T) in terms of $\dot{u}_o{}^2$, $\dot{w}_o{}^2$ and $\dot{\varphi}^2$ can be expressed as follows:

$$T = \frac{1}{2}\int_0^L \rho A \dot{u}_o{}^2 \, dx + \frac{1}{2}\int_0^L \rho A \dot{w}_o{}^2 dx + \frac{1}{2}\int_0^L \rho I \dot{\varphi}^2 dx \tag{3.14}$$

Where the velocities $\dot{u_0}$, $\dot{w}_0$ and $\dot{\varphi}$ can be expressed due to the effect of oscillatory motion as:

$$\dot{u_0} = C_o + \frac{\partial u_0}{\partial t} + C_o \frac{\partial u_0}{\partial x} \tag{3.15}$$

$$\dot{w_0} = \frac{\partial w_o}{\partial t} + C_o \frac{\partial w_o}{\partial x} \tag{3.16}$$

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t} + C_o \frac{\partial \varphi}{\partial x} \tag{3.17}$$

Therefore,

$$\begin{aligned} T = \frac{1}{2}\int_0^L \rho A (C_o + \frac{\partial u_0}{\partial t} + C_o \frac{\partial u_0}{\partial x})^2 dx + \frac{1}{2}\int_0^L \rho A (\frac{\partial w_0}{\partial t} + C_o \frac{\partial w_0}{\partial x})^2 dx \\ + \frac{1}{2}\int_0^L \rho I \left(\frac{\partial \varphi}{\partial t} + C_o \frac{\partial \varphi}{\partial x}\right)^2 dx \end{aligned} \tag{3.18}$$

Where,

$$(C_o + \frac{\partial u_o}{\partial t} + C_o \frac{\partial u_o}{\partial x})^2$$
$$= {C_o}^2 + 2C_o \frac{\partial u_o}{\partial t} + 2{C_o}^2 \frac{\partial u_o}{\partial x} + 2C_o \frac{\partial u_o}{\partial t}\frac{\partial u_o}{\partial x}$$
$$+ (\frac{\partial u_o}{\partial t})^2 + (C_o)^2(\frac{\partial u_o}{\partial x})^2 \tag{3.19}$$

$$(\frac{\partial w_o}{\partial t} + C_o \frac{\partial w_o}{\partial x})^2 = \left(\frac{\partial w_o}{\partial t}\right)^2 + 2C_o \frac{\partial w_o}{\partial t}\frac{\partial w_o}{\partial x} + {C_o}^2 \left(\frac{\partial w_o}{\partial x}\right)^2 \tag{3.20}$$

$$\left(\frac{\partial \varphi}{\partial t} + C_o \frac{\partial \varphi}{\partial x}\right)^2 = \left(\frac{\partial \varphi}{\partial t}\right)^2 + 2C_o \frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x} + {C_o}^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 \tag{3.21}$$

Now taking the variation of ${\dot{u}_o}^2$,

$$2\dot{u}_o\delta\dot{u}_o = 2[C_o\delta\dot{u}_o + {C_o}^2\delta {u_o}' + C_o {u_o}'\delta\dot{u}_o + C_o\dot{u}_o\delta {u_o}' + \dot{u}_o\delta\dot{u}_o + {C_o}^2 {u_o}'\delta {u_o}'] \tag{3.22}$$

Where

$$\dot{u}_o = \frac{\partial u_o}{\partial t} \; and \; {u_0}' = \frac{\partial u_o}{\partial x}$$

Now the variation for ${\dot{w}_o}^2$,

$$2\dot{w}_o\delta\dot{w}_o = 2[\dot{w}_o\delta\dot{w}_o + {C_o}^2 {w_o}'\delta {w_o}' + C_o {w_o}'\delta\dot{w}_o + C_o\dot{w}_o\delta {w_o}'] \tag{3.23}$$

Where,

$$\dot{w}_o = \frac{\partial w_o}{\partial t} \; and \; w_o{}' = \frac{\partial w_o}{\partial x}$$

and variation for the $\dot{\varphi}^2$,

$$2\dot{\varphi}\delta\dot{\varphi} = 2[\dot{\varphi}\delta\dot{\varphi} + C_o{}^2\varphi'\delta\varphi' + C_o\varphi'\delta\dot{\varphi} + C_o\dot{\varphi}\delta\varphi'] \tag{3.24}$$

Where,

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t} \; and \; \varphi' = \frac{\partial \varphi}{\partial x}$$

The total virtual kinetic energy δT:

$$\delta T = \int_0^L \rho A\dot{u}_o\delta\dot{u}_o dx + \int_0^L \rho A\dot{w}_o\delta\dot{w}_o dx + \int_0^L \rho I\dot{\varphi}\delta\dot{\varphi} dx \tag{3.25}$$

Now determine $\int_{t_1}^{t_2} \delta T dt$ term by term,

$$\int_{t_1}^{t_2}\int_0^L \rho A(C_o\delta\dot{u}_o)dx\,dt = \int_0^L \rho A(C_o)\delta u_0 \Big|_{t_1}^{t_2} dx - \int_0^L\int_{t_1}^{t_2} \rho A\left(\frac{\partial C_o}{\partial t}\right)\delta\, u_o dt\, dx \tag{3.26}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A\left(C_o{}^2\delta u_o{}'\right)dx\,dt = \int_{t_1}^{t_2} \rho A C_o{}^2\delta u_o\Big|_0^L dt \tag{3.27}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A C_o u_o{}'\delta\dot{u}_o dx\,dt$$

$$= \int_0^L \rho A C_o u_o{}'\delta u_o \Big|_{t_1}^{t_2} dx - \int_0^L\int_{t_1}^{t_2} \rho A \frac{\partial}{\partial t}\left(C_o \frac{\partial u_o}{\partial x}\right)\delta u_o dt\,dx \tag{3.28}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A C_o \dot{u}_o \delta u_o{}' dx\,dt$$

$$= \int_{t_1}^{t_2} \rho A C_o \dot{u}_o \delta u_o \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho A C_o \frac{\partial^2 u_o}{\partial t \partial x}\delta u_o dx\,dt \tag{3.29}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A \dot{u}_o \delta\dot{u}_o dx\,dt = \int_0^L \rho A \dot{u}_o \delta u_o \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \rho A \ddot{u}_o \delta u_o dt \tag{3.30}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A C_o{}^2 u_o{}'\delta u_o{}' dx\,dt$$

$$= \int_{t_1}^{t_2} \rho A C_o{}^2 u_o{}'\delta u_o \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho A C_o{}^2 u_o{}''\delta u_o dx\,dt \tag{3.31}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A \dot{w}_o \delta\dot{w}_o dx\,dt = \int_0^L \rho A \dot{w}_o \delta w_o \Big|_{t_1}^{t_2} dx - \int_0^L\int_{t_1}^{t_2} \rho A \ddot{w}_o \delta w_o dt\,dx \tag{3.32}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A C_o {w_o}'\delta \dot{w}_o dx\,dt$$
$$= \int_0^L \rho A C_o {w_o}'\delta\, w_o \Big|_{t_1}^{t_2} dx - \int_0^L\int_{t_1}^{t_2} \rho A \frac{\partial}{\partial t}\left(C_o \frac{\partial w_o}{\partial x}\right)\delta w_o dt\,dx \tag{3.33}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A C_o \dot{w}_o \delta {w_o}' dx\,dt$$
$$= \int_{t_1}^{t_2} \rho A C_o \dot{w}_o \delta w_o \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho A C_o \left(\frac{\partial^2 w_o}{\partial t \partial x}\right)\delta w_o dx\,dt \tag{3.34}$$

$$\int_{t_1}^{t_2}\int_0^L \rho A {C_o}^2 {w_o}'\delta {w_o}' dx\,dt$$
$$= \int_{t_1}^{t_2} \rho A {C_o}^2 {w_o}'\delta w_o \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho A {C_o}^2 {w_o}''\delta w_o dx\,dt \tag{3.35}$$

$$\int_{t_1}^{t_2}\int_0^L \rho I\, \dot{\varphi}\delta\, \dot{\varphi} dx\,dt = \int_0^L \rho I\, \dot{\varphi}\delta\varphi \Big|_{t_1}^{t_2} dx - \int_0^L\int_{t_1}^{t_2} \rho I \ddot{\varphi}\delta\, \varphi dt\,dx \tag{3.36}$$

$$\int_{t_1}^{t_2}\int_0^L \rho I C_o \varphi' \delta\,\dot{\varphi} dx\,dt$$

$$= \int_0^L \rho I C_o \varphi' \delta\varphi \Big|_{t_1}^{t_2} dx$$

$$- \int_0^L \int_{t_1}^{t_2} \rho I \frac{\partial}{\partial t}\left(C_o \frac{\partial \varphi}{\partial x}\right) \delta\,\varphi dt\,dx \tag{3.37}$$

$$\int_{t_1}^{t_2}\int_0^L \rho I C_o\, \dot{\varphi} \delta\varphi' dx\,dt$$

$$= \int_{t_1}^{t_2} \rho I C_o\, \dot{\varphi} \delta\varphi \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho I C_o \frac{\partial^2 \varphi}{\partial t \partial x} \delta\,\varphi dx\,dt \tag{3.38}$$

$$\int_{t_1}^{t_2}\int_0^L \rho I {C_o}^2 \varphi' \delta\varphi' dx\,dt$$

$$= \int_{t_1}^{t_2} \rho I {C_o}^2 \varphi' \delta\varphi \Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \rho A {C_o}^2 \varphi'' \delta\varphi dx\,dt \tag{3.39}$$

The strain energy:

$$U = \frac{1}{2}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} [Ez^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 + kG\left(\varphi - \frac{\partial w_o}{\partial x}\right)^2]\, bdzdx \tag{3.40}$$

$$U = \frac{1}{2}\int_0^L [EI\left(\frac{\partial\varphi}{\partial x}\right)^2 + kGA\left(\varphi - \frac{\partial w_o}{\partial x}\right)^2]\,dx \tag{3.41}$$

We took into account that δu_o , δw_o and $\delta\varphi$ vanish $at\ t = t_1\ , t_2$

The virtual strain energy:

$$\begin{aligned}\delta U = &\int_0^L EI\varphi'\delta\varphi' dx \\ &+ \int_0^L kGA\ \varphi\delta\ \varphi dx - \int_0^L kGA\left(\frac{\partial w_o}{\partial x}\delta\ \varphi + \varphi\delta w_o{}'\right)dx \\ &+ \int_0^L kGA(w_o{}'\delta\ w_o{}')dx\end{aligned} \tag{3.42}$$

Now determine $\int_{t_1}^{t_2}\delta U dt$ term by term,

$$\int_{t_1}^{t_2}\int_0^L EI\varphi'\delta\varphi'\ dxdt = \int_{t_1}^{t_2} EI\varphi'\delta\varphi\,\Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L EI\varphi''\delta\varphi dx\ dt \tag{3.43}$$

$$\int_{t_1}^{t_2}\int_0^L kGA\ \varphi\delta\ \varphi dx\ dt = \int_{t_1}^{t_2}\int_0^L kGA\varphi\delta\varphi\ dxdt \tag{3.44}$$

$$\int_{t_1}^{t_2}\int_0^L kGA\ \frac{\partial w_o}{\partial x}\delta\ \varphi dx\ dt = -\int_{t_1}^{t_2}\int_0^L kGAw_o{}'\delta\varphi\ dxdt \tag{3.45}$$

$$-\int_{t_1}^{t_2}\int_0^L kGA\varphi\delta\, w_o{}'\, dxdt$$

$$= -\int_{t_1}^{t_2} kGA\varphi\delta w_o\Big|_0^L dt + \int_{t_1}^{t_2}\int_0^L kGA\varphi'\delta w_o dx\, dt \tag{3.46}$$

$$\int_{t_1}^{t_2}\int_0^L kGAw_o{}'\delta w_o{}'\, dxdt$$

$$= \int_{t_1}^{t_2} kGAw_o{}'\delta w_o\Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L kGAw_o{}''\delta w_o dx\, dt \tag{3.47}$$

The virtual work done by the applied force,

$$V = -\int_A f(x,t)w_o\, d_A = -\int_0^L f(x,t)w_o dx \tag{3.48}$$

Variation of the work done by the applied force,

$$\delta V = -\int_0^L f(x,t)\delta w_o dx \tag{3.49}$$

$$\int_{t_1}^{t_2}\int_0^L \delta V\, dxdt = -\int_{t_1}^{t_2}\int_0^L f(x,t)\delta w_o dx\, dt \tag{3.50}$$

We can now substitute into the extended Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta T)dt = 0 \tag{3.51}$$

To proceed, we note that $\delta w_o, \delta u_o$ and $\delta\varphi$ are arbitrary and independent. Therefore, let

$$\delta w_o = 0 \; at \; x = 0, x = L$$

$$\delta u_o = 0 \; at \; x = 0, x = L$$

$$\delta \varphi = 0 \; at \; x = 0, x = L$$

δu_o , δw_o and $\delta \varphi$ are arbitrary for $x < 0 < L$. Now collecting the coefficients of δu_o , δw_o and $\delta \varphi$ then setting them equal to zero, we obtain equations of motion.

δu_o:

$$\rho A \frac{\partial C_o}{\partial t} + \rho A \frac{\partial \left(C_o \frac{\partial u_o}{\partial x} \right)}{\partial t} + \rho A C_o \frac{\partial^2 u_o}{\partial t \partial x} + \rho A {C_o}^2 {u_o}'' + \rho A \ddot{u}_o = 0 \tag{3.52}$$

δw_o:

$$kGA\varphi' - kGA\, {w_o}'' + \rho A \frac{\partial \left(C_o \frac{\partial w_o}{\partial x} \right)}{\partial t} + \rho A {C_o}^2 {w_o}'' + \rho A C_o \frac{\partial^2 w_o}{\partial t \partial x} + \rho A \ddot{w}_o = f(x,t) \tag{3.53}$$

$\delta \varphi$:

$$kGA\, \varphi - EI\varphi'' - kGA{w_o}' + \rho I \frac{\partial \left(C_o \frac{\partial \varphi}{\partial x} \right)}{\partial t} + \rho I C_o \frac{\partial^2 \varphi}{\partial t \partial x} + \rho I {C_o}^2 \varphi'' + \rho I \ddot{\varphi} = 0 \tag{3.54}$$

Where,

$$\frac{\partial \left(C_o \frac{\partial u_o}{\partial x} \right)}{\partial t} = \dot{C}_o \frac{\partial u_o}{\partial x} + C_o \frac{\partial^2 u_o}{\partial t \partial x} \tag{3.55}$$

$$\frac{\partial\left(C_o \frac{\partial w_o}{\partial x}\right)}{\partial t} = \dot{C}_o \frac{\partial w_o}{\partial x} + C_o \frac{\partial^2 w_o}{\partial t \partial x} \tag{3.56}$$

$$\frac{\partial\left(C_o \frac{\partial \varphi}{\partial x}\right)}{\partial t} = \dot{C}_o \frac{\partial \varphi}{\partial x} + C_o \frac{\partial^2 \varphi}{\partial t \partial x} \tag{3.57}$$

and

$$\dot{C}_o = -\Omega^2 A_o \sin \Omega t$$

Then the two equations of motion perpendicular to the direction of propagation of wave, equation (3.53) and equation (3.54) become,

$$\begin{aligned} kGA\left(\frac{\partial \varphi}{\partial x} - \frac{\partial^2 w_o}{\partial x^2}\right) + \rho A {C_o}^2 \frac{\partial^2 w_o}{\partial x^2} + 2\,\rho A C_o \frac{\partial^2 w_o}{\partial t \partial x} + \rho A \dot{C}_o \frac{\partial w_o}{\partial x} \\ + \rho A \frac{\partial^2 w_o}{\partial t^2} = f(x,t) \end{aligned} \tag{3.58}$$

$$\begin{aligned} kGA\left(\varphi - \frac{\partial w_o}{\partial x}\right) - EI \frac{\partial^2 \varphi}{\partial x^2} + \rho I {C_o}^2 \frac{\partial^2 \varphi}{\partial x^2} + 2\,\rho I C_o \frac{\partial^2 \varphi}{\partial t \partial x} + \rho I \dot{C}_o \frac{\partial \varphi}{\partial x} \\ + \rho I \ddot{\varphi} = 0 \end{aligned} \tag{3.59}$$

The equation of motion parallel to the direction of propagation of wave, equation (3.52) becomes,

$$\rho A \dot{C}_o + \rho A \dot{C}_o \frac{\partial u_o}{\partial x} + 2\,\rho A C_o \frac{\partial^2 u_o}{\partial t \partial x} + \rho A {C_o}^2 \frac{\partial^2 u_o}{\partial x^2} + \rho A \ddot{u}_o = 0 \tag{3.60}$$

CHAPTER 4

SPECTRAL FORMULATION

To obtain the solution of the derived equations of motion, the equations are transformed first from the time domain (t) to the frequency domain (ω) by applying the Fourier time transforms of all time-dependent variables. Then we apply spatial Fourier transforms of the variables to go from space domain (x) to the wavenumber (ζ) domain. Then we solve for the unknown displacements. Finally to recover the physical displacements, in the time domain at any axial location, we evaluate the integral involved in the spatial (wavenumber) inverse transform using numerical integration. This is done at each frequency (ω) within the power spectrum of the applied forcing function. Then, the resulting frequency spectra for the displacement are inverted using inverse fast Fourier transformation IFFT.

4.1 Temporal Fourier Transformation

Although the temporal Fourier transformation methods are well developed and well documented, we will review the basic concepts for the completeness. Let's assume that $g(x,t)$ represents any time-dependent variable. Then the time Fourier transform of $g(x,t)$ is given by

$$G(x,\omega) = \int_0^{\infty} g(x,t)\, e^{i\omega t} dt \tag{4.1}$$

$$g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(x,\omega)\, e^{-i\omega t} d\omega \tag{4.2}$$

Where ω is the frequency ($\omega = 2\pi f$). Now transforming equation (3.58) and equation (3.59) from time domain to frequency domain by using the above integral expression.

$$kGA\left(\frac{\partial \hat{\varphi}}{\partial x} - \frac{\partial^2 \widehat{w}_o}{\partial x^2}\right) + \rho A \hat{C}_o{}^2 \frac{\partial^2 \widehat{w}_o}{\partial x^2} + 2\, i\omega\rho A \hat{C}_o \frac{\partial \widehat{w}_o}{\partial x} + \rho A \hat{\dot{C}}_o \frac{\partial \widehat{w}_o}{\partial x} - \omega^2 \rho A \widehat{w}_o = \hat{f}(\omega) \tag{4.3}$$

$$kGA\left(\hat{\varphi} - \frac{\partial \widehat{w}_o}{\partial x}\right) - EI \frac{\partial^2 \hat{\varphi}}{\partial x^2} + \rho I \hat{C}_o{}^2 \frac{\partial^2 \hat{\varphi}}{\partial x^2} + 2\, i\omega\rho I \hat{C}_o \frac{\partial^2 \varphi}{\partial t \partial x} + \rho I \hat{\dot{C}}_o \frac{\partial \hat{\varphi}}{\partial x} - \omega^2 \rho I \hat{\varphi} = 0 \tag{4.4}$$

Similarly by applying time Fourier transformation to the equation (3.60) we have,

$$\rho A \hat{\dot{C}}_o + \rho A \hat{\dot{C}}_o \frac{\partial \hat{u}_o}{\partial x} + 2i\omega\, \rho A \frac{\partial \hat{u}_o}{\partial x} + \rho A \hat{C}_o{}^2 \frac{\partial^2 \hat{u}_o}{\partial x^2} - \omega^2 \rho A \hat{u}_o = 0 \tag{4.5}$$

Where '^' indicates the time Fourier transform.

4.2 Spatial Fourier Transformation

For spatial transformation (*i.e.* to transform from space variable (*x*) domain to the wave-number (ζ) domain) we will perform following integral for the equations of motion.

$$G(\zeta,\omega) = \int_{-\infty}^{\infty} \hat{g}(x,\omega) e^{i\zeta x}\, dx \tag{4.6}$$

and the inverse is given by,

$$\hat{g}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\zeta,\omega)e^{-i\zeta x}\,dx \tag{4.7}$$

Applying spatial Fourier transformation to equations (4.3) , (4.4) and (4.5) respectively,

$$\begin{aligned} &kGA(i\zeta\boldsymbol{\psi} + \zeta^2\boldsymbol{W}) - \zeta^2\,\rho A\hat{C}_o{}^2\boldsymbol{W} - 2\zeta\omega\,\rho A\hat{C}_o\boldsymbol{W} + i\zeta\rho A\hat{\dot{C}}_o\boldsymbol{W} \\ &- \omega^2\,\rho A\boldsymbol{W} = \hat{f} \end{aligned} \tag{4.8}$$

$$\begin{aligned} &kGA(\boldsymbol{\psi} - i\zeta\boldsymbol{W}) + EI\zeta^2\boldsymbol{\psi} - \zeta^2\,\rho I\hat{C}_o{}^2\boldsymbol{\psi} - 2\,\zeta\omega\rho I\hat{C}_o\boldsymbol{\psi} + i\zeta\rho I\hat{\dot{C}}_o\boldsymbol{\psi} \\ &- \omega^2\,\rho I\boldsymbol{\psi} = 0 \end{aligned} \tag{4.9}$$

$$\rho A\hat{\dot{C}}_o + i\zeta\rho A\hat{\dot{C}}_o\boldsymbol{U} - 2\zeta\omega\,\rho A\boldsymbol{U} - \zeta^2\,\rho A\hat{C}_o{}^2\boldsymbol{U} - \omega^2\,\rho A\boldsymbol{U} = 0 \tag{4.10}$$

Where,

$$\boldsymbol{U} = u(\omega,\zeta)$$

$$\boldsymbol{W} = w(\omega,\zeta)$$

$$\boldsymbol{\psi} = \varphi(\omega,\zeta)$$

Now writing transformed equations of motion (4.8) and (4.9) in matrix form we have,

$$\underbrace{\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}}_{M} \begin{Bmatrix} \boldsymbol{W} \\ \boldsymbol{\psi} \end{Bmatrix} = \begin{Bmatrix} \hat{f} \\ 0 \end{Bmatrix} \tag{4.11}$$

Where,

$$M_{11} = \kappa GA\zeta^2 - \zeta^2\,\rho A\hat{C}_o{}^2 - 2\,\zeta\omega\rho A\hat{C}_o + i\zeta\rho A\dot{\hat{C}}_o - \omega^2\,\rho A$$

$$M_{12} = i\zeta kGA$$

$$M_{21} = -i\zeta kGA$$

$$M_{22} = kGA + EI\zeta^2 - \zeta^2\,\rho I\hat{C}_o{}^2 - 2\,\zeta\omega\rho I\hat{C}_o + i\zeta\rho I\dot{\hat{C}}_o - \omega^2\,\rho I$$

4.3 Dispersion Relations

Dispersion relation relates the wavenumber of a wave to its frequency. For an axially oscillatory Timoshenko beam we get dispersion equation by letting determinant of 'M' matrix equal to zero and solving for ζ at each frequency, that can be expressed mathematically as ,

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = 0 \tag{4.12}$$

Since the solution of the above expression is not possible analytically, we may solve this using numerical root-finding techniques. Solution of equation (4.12) gives two modes of wave propagation, which are flexural and shear modes respectively.

4.4 Phase Velocity

The phase velocity (C_p) can be defined as the velocity at which a wave composed of one frequency component propagates. Mathematically it can be expressed as,

$$C_p = \frac{\omega}{\zeta}$$

Where,

ω = Angular frequency of wave

ζ = Wave number

4.5 Group Velocity

The group velocity (C_g) describes how fast the entire waveform moves, where the waveform is composed of several individual waves superimposed. Mathematically group velocity is expressed as,

$$C_g = \frac{\partial \omega}{\partial \zeta}$$

Phase velocity and group velocity depends on the sectional properties of the beam in term of area $A = bh$ and $I = \frac{bh^3}{12}$ as well as elastic properties. Both of these velocities are dispersive in nature and approach infinity for very high frequencies [33]. The phase and group velocities will reach constant values at very high frequency and there values for flexural mode are;

$$C_p = \sqrt{\frac{GA\kappa}{\rho A}}$$

$$C_g = \sqrt{\frac{EI}{\rho I \kappa}}$$

CHAPTER 5

METHOD OF SOLUTION

For the transient response of Timoshenko beam by using approximate method, the input excitation force at discrete time sample is converted to frequency domain by using Fast Fourier Transformation (FFT) algorithm. To obtain the FFT for excitation forcing care must be taken about number of data points, N which should be in powers of 2. After transformation from time to frequency domain power spectrum of excitation force is obtained. The sampling rate Δt must be selected according to the Nyquist critical frequency. For any sampling rate Δt, there is a special frequency f_c called Nyquist critical frequency which is related to Δt by,

$$\Delta t = \frac{1}{2f_c}$$

Practically, f_c is the maximum frequency contained in the signal

The Fourier transform is estimated using frequency sampling interval Δf which is calculated by,

$$\Delta f = \frac{1}{N\Delta t}$$

After getting the power spectrum for the amplitude modulated forcing (excitation) function, the resultant signal is band-limited between minimum and maximum frequencies, $f_{\min}$ and $f_{\max}$ respectively. Now to get the normal displacement vector, we have to perform numerical integration along the wave-number at each frequency. In the present study, integration is performed using Romberg's method for integration. To improve the stability of the numerical integration, small amount of damping is introduced in the elastic constants E and G in the form,

$$E = E_o(1 - i\eta)$$

$$G = G_o(1 - i\eta)$$

Where E_o $and G_o$ is the undamped valued for elastic constants of the beam. In present work the value of η is taken as 0.025. After getting response in frequency dependent vector it is then reconstructed in the time domain by using Inverse Fast Fourier Transform (IFFT) technique. This method of solution is shown schematically in Figure (3).

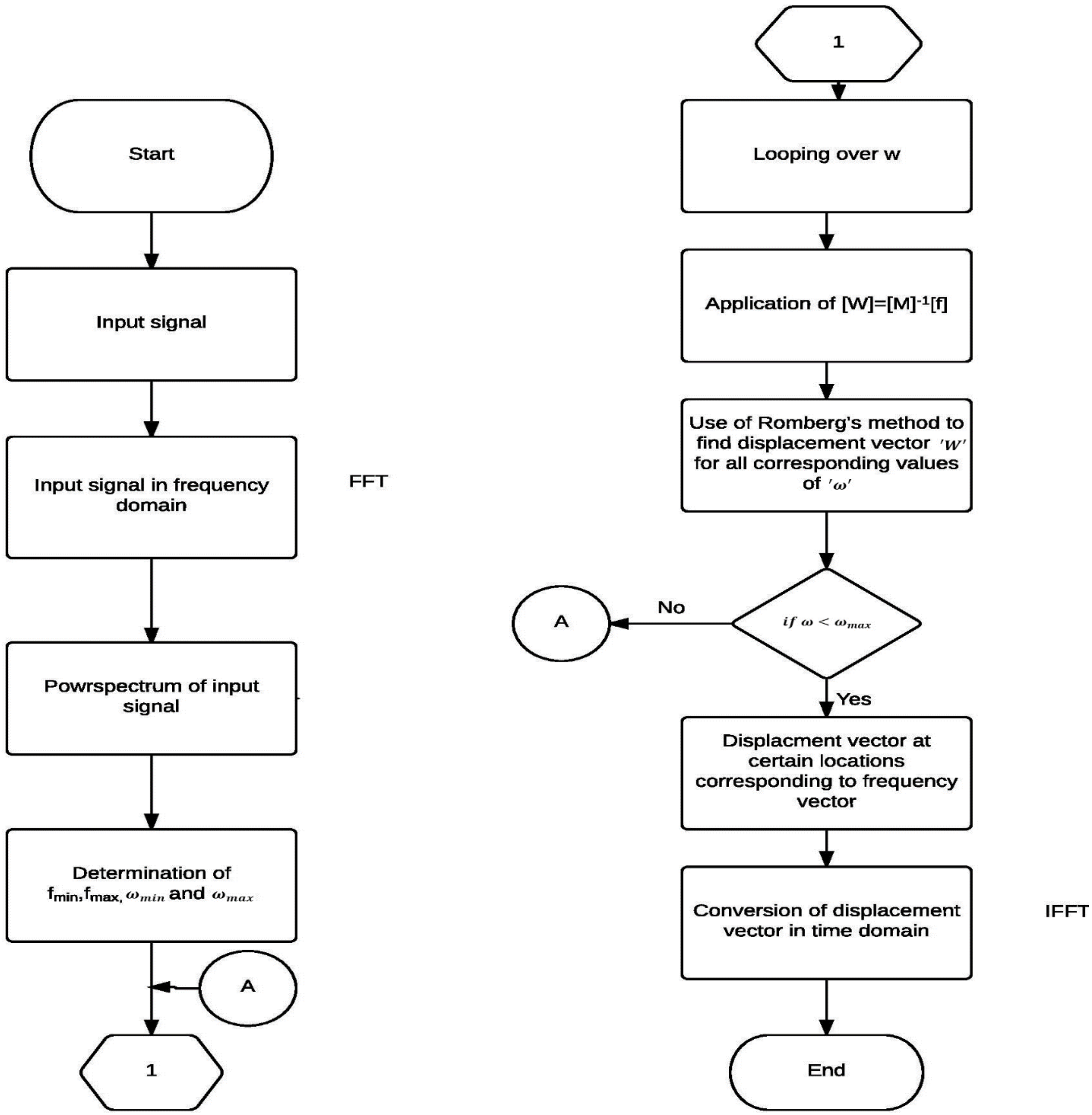

Figure 3: The method of solution flow chart

CHAPTER 6

RESULTS AND DISCUSSION

For the present study a steel beam of infinite length is assumed to study the effects of the oscillatory motion on wave propagation in the beams. We have considered a stationary beam (Ω=0) and an axially oscillating beam with amplitude of oscillation A_0=1μm, and oscillatory frequency Ω=82 KHz, 180 KHz and 1500 KHz respectively. The beam has the following properties as shown in Table (1).

Table 1: Assumed geometrical and material properties for the beam

Parameter	Description	Value used
E	Young's Modulus	2.011×10^{11} N/m^2
G	Shear Modulus	7.522×10^{10} N/m^2
h	Thickness of the beam	1.0×10^{-2}m
ρ	Mass density of the beam	7830 kg/m^3

6.1 The Excitation Force

A modulated sinusoidal (tone burst) force having a central frequency of 0.180 MHz is used to excite the beam. The purpose of modulation here is to limit the bandwidth of input signal. The excitation force and its power spectrum density is shown in Figure (4) The lower end of the frequency spectrum as calculated from the lower bound of power spectrum is 50 KHz and the higher end is 300 KHz

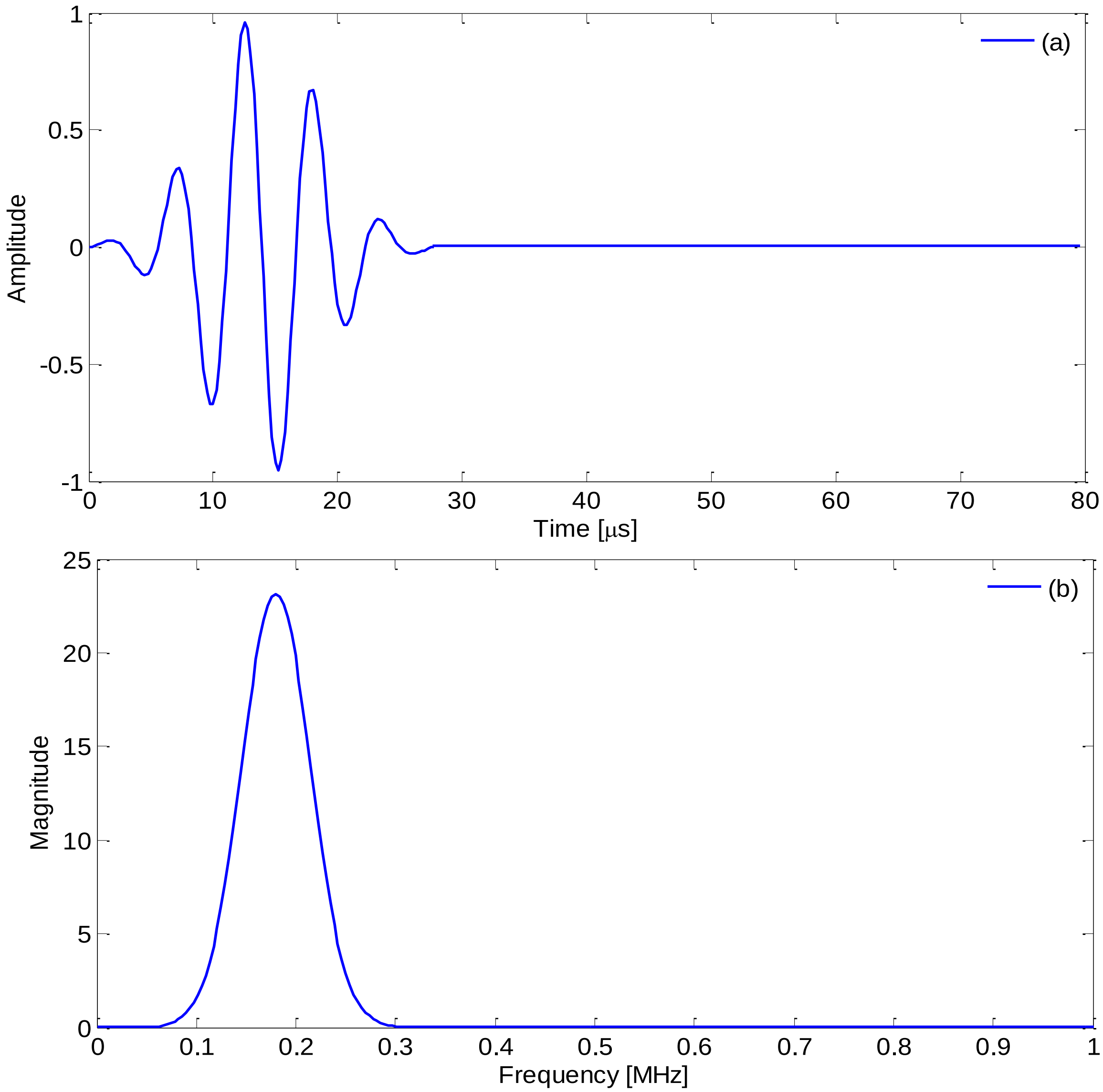

Figure 4: (a) The excitation force used in the study (b) The Power spectrum of the excitation force

6.2 The Dispersion Curves

The relationship between wave speed and frequency is called the dispersion curve. From the governing equations of motion we have developed the dispersion relations for the two different formats:

1. Phase velocity (C_p) versus frequency*thickness (f*h)
2. Group velocity (C_g) versus frequency*thickness (f*h)

The dispersion curves are studied for the flexural and the shear modes. Curves are obtained considering the stationary beam (Ω=0) and the axially oscillatory beam having frequency of oscillation Ω=82 KHz, Ω=180 KHz and Ω=1500 KHz respectively.

Figure (5) shows the phase velocity dispersion curves for four oscillatory cases, that are (1) Ω=0 (stationary) (2) Ω=82 KHz (3) Ω=180 KHz and (4) Ω=1500 KHz. By observing these dispersion curves, we can see that they start from zero with normal curved path as expected until one of the oscillatory frequency is reached. It can be seen that there is a sudden vertical jump when the frequency coincides with the frequency of oscillation. Then the dispersion curve will tend to converge to a constant phase velocity at a faster rate compared to the stationary case. At the jump points (*i.e.* when $\Omega = \omega$), the sudden change of phase velocity becomes more significant at lower frequency range. As an example, at $\Omega=\omega$=82 KHz, C_p jumps from 1008 m/s to 2421 m/s which means a 140% increase. When $\Omega=\omega$=180 KHz, C_p jumps from 1444 m/s to 2551 m/s with 76.7% increase. While it is only 7.6% increase for the case of $\omega=\Omega$=1500 KHz. Also we can observe that the higher the oscillatory frequency, the faster the convergence rate. Moreover, the convergence value of the phase velocity for the oscillatory case in general is larger than that for the non-oscillatory (stationary) beam. The convergence phase velocities as measured from the results of Figure 5) for the stationary beam is about 2701 m/s while for the oscillatory beam (Ω=1500 KHz) it is about 2810 m/s, which is approximately a 4 % increase.

Figure (6) shows the phase velocity dispersion curve for the shear mode for several values of Ω. Unlike the flexural wave where the dispersion curve starts from the zero frequency, the shear wave will not propagate below their cut-off frequencies. The effect of oscillation on this cut-off frequency can be seen to be more pronounced when oscillatory frequency Ω is lower that the cut-off frequency of the shear wave for the stationary beam. This can be clearly noticed at Ω=82 KHz and Ω=180 KHz where the cut-off frequency is shifted to the left and coincide with the frequency of oscillation (Ω). This means that, the oscillation will excite the shear mode at its oscillatory frequency. For the stationary beam, the cut-off frequency is about 1 MHz. When the oscillatory frequency is larger than the cut-off frequency value of the stationary beam, it will have negligible effect on the cut-off frequency. However in this case there is a sudden decrease in the phase velocity when the frequency coincides with Ω. As it can be noticed that the convergence value of the phase velocity is larger in the case of oscillatory beam than the value for stationary beam. As an example, the convergence value for of phase velocity (C_p) for the stationary beam ≈ 3042 m/s while it is equal to 5120 m/s for the oscillatory beam corresponding to Ω=82 KHz. This increase in convergence is about 68%.

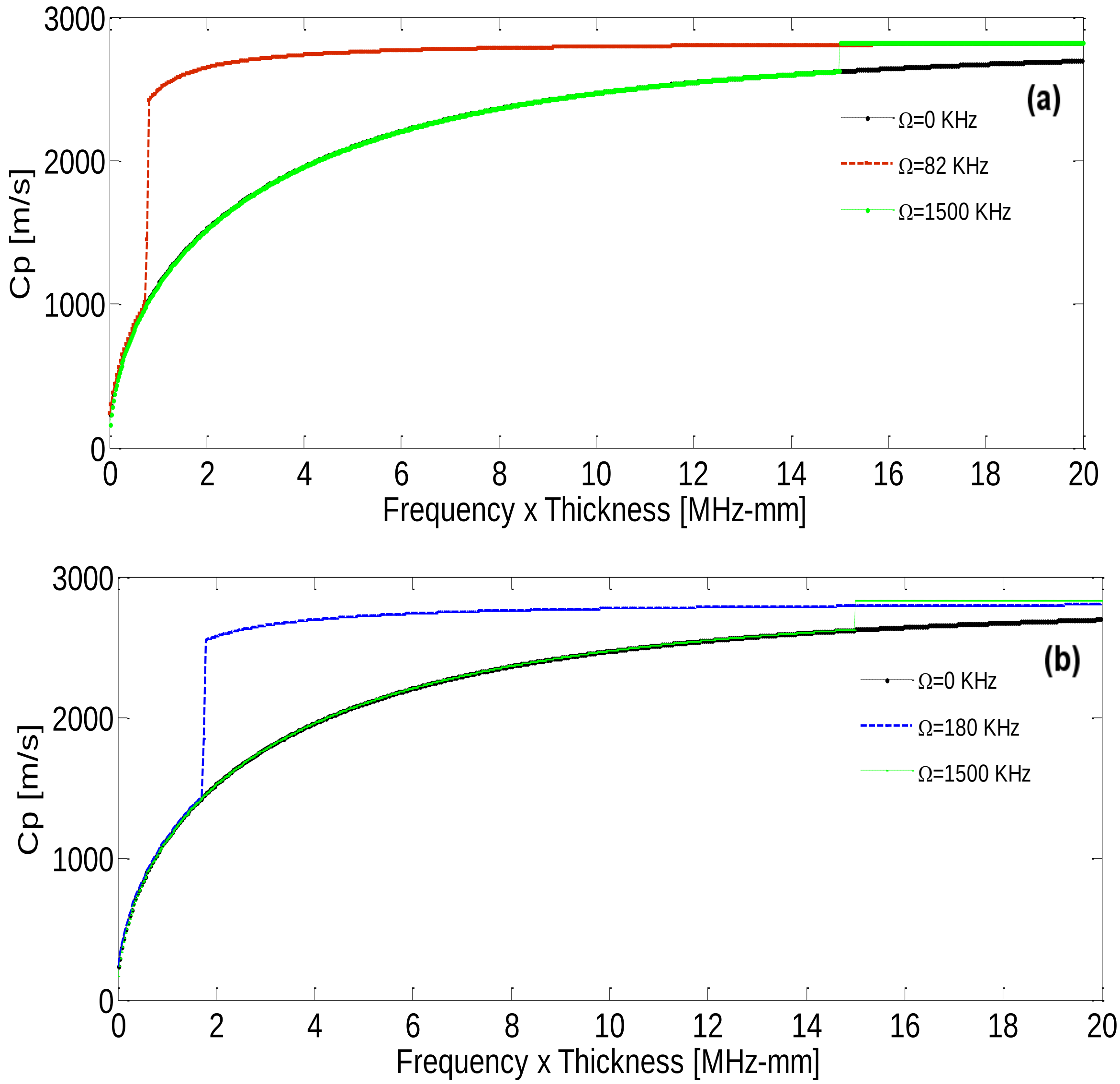

Figure 5: The phase velocity curves comparison for the flexural mode (a) Ω=0, 82 KHz and 1500 KHz (b) Ω =0, 180 KHz and 1500 KHz

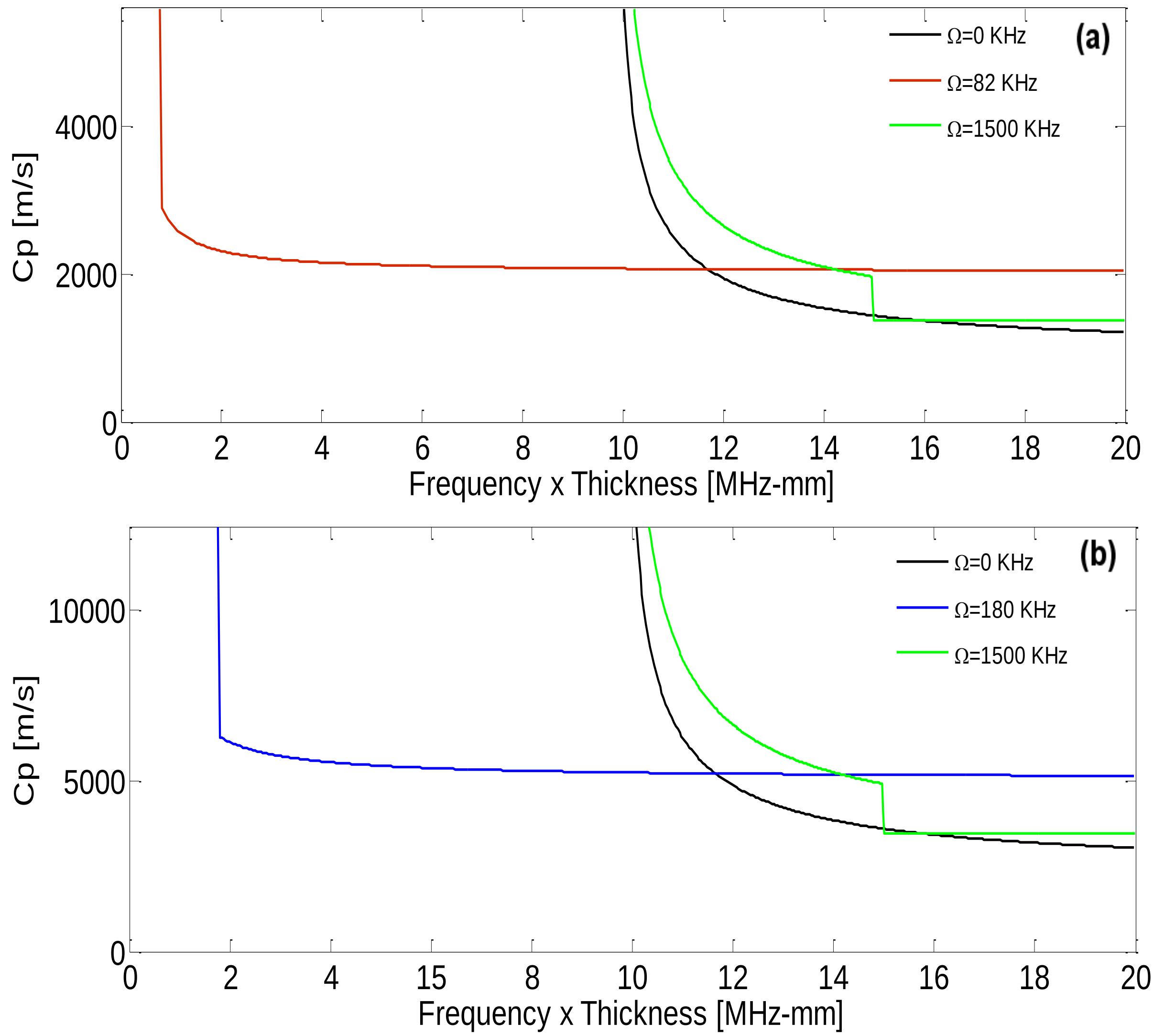

Figure 6: The phase velocity curves comparison for the shear mode (a) Ω=0, 82 KHz and 1500 KHz (b) Ω =0, 180 KHz and 1500 KHz

Figure (7) and Figure (8) shows the group velocity dispersion curves for the flexural and the shear mode respectively. The dispersion curves are obtained for four oscillatory frequencies, that are (1) Ω=0 (stationary) (2) Ω=82 KHz (3) Ω=180 KHz and (4) Ω=1500 KHz. By observing these curves, we have noticed that these start from zero with normal curved path as expected until one of the oscillatory frequency (Ω) is reached. It can be seen that there is a sudden sharp spike when the Ω coincides with ω. Then the dispersion curves for the flexural mode tend to converge to a constant value at once. This convergence rate for the oscillatory group velocity (C_g) is slower compared to the stationary case. At the points where $\Omega \approx \omega$, the sudden change of the group velocity becomes more significant at lower frequency range. As an example, at Ω=ω=82 KHz, C_g experiences a vertical sharp spike from 1848 m/s to 430 m/s and then becomes constant for the higher values of the frequency. Similarly for Ω=ω=180 KHz, C_g jumps from 2443 m/s to 648 m/s and then becomes constant. Further increase in frequency value will not effect the group velocity and it will tends to converge at a 4.7 % slower rate compared to the stationary beam case. Also, we can observe that for all the oscillatory frequency (Ω) values, the convergence value of the group velocity for stationary beam is 2829 m/s and for the oscillatory it is about 2971 m/s.

Figure (8) shows the group velocity dispersion curves for the shear mode for several values of the oscillatory frequency. Unlike the flexural mode, where the dispersion curve starts from zero frequency values, the shear wave will not start propagating below their cut-off frequency value. This can be clearly noticed that the cut-off frequency at Ω=82 KHz and Ω=180 KHz is now shifted to the left and coincides with the frequency of oscillation (Ω). Which means that the oscillation will excite shear mode at its oscillation frequency.

For the stationary beam (Ω=0), the cut-off frequency is ≈ 1MHz. When the oscillatory frequency is larger than the cut-off value of the stationary beam, it will not have considerable effect on the cut-off value. However when Ω=1500 KHz there is a vertical spike at $\Omega \approx \omega$ and then the group velocity will tends to a constant value. The convergence value of the group velocity for the shear mode is larger in the oscillatory case than the value for the stationary beam. This convergence value of velocity is almost same for all three oscillating frequencies. The convergence value of C_g for the stationary beam is ≈4146 m/s, while it is ≈5055 m/s for the oscillatory beam. There is about 22% increase in the convergence value for the oscillatory beam.

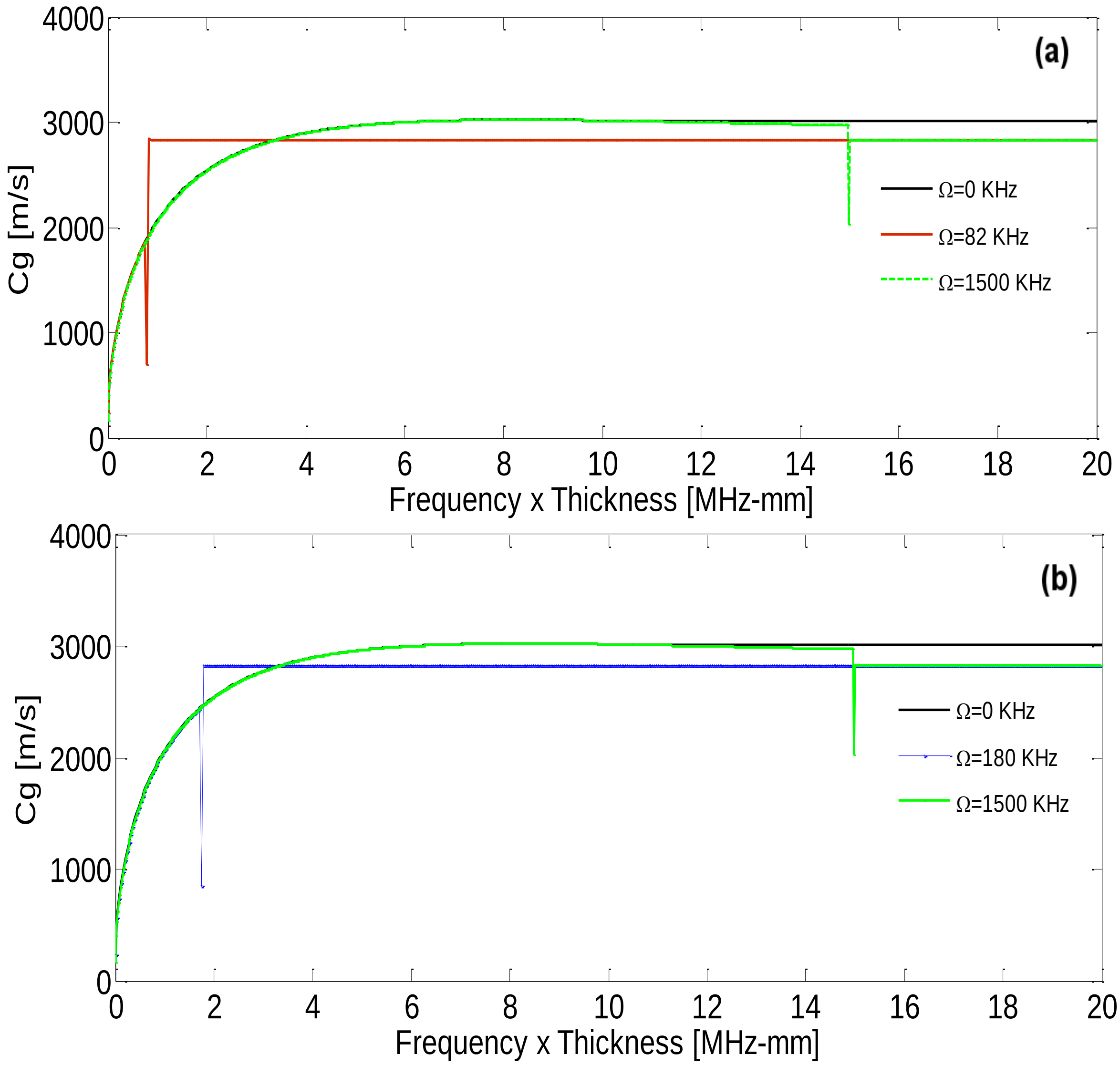

Figure 7: The group velocity curves comparison for the flexural mode (a) Ω=0, 82 KHz and 1500 KHz (b) Ω =0, 180 KHz and 1500 KHz

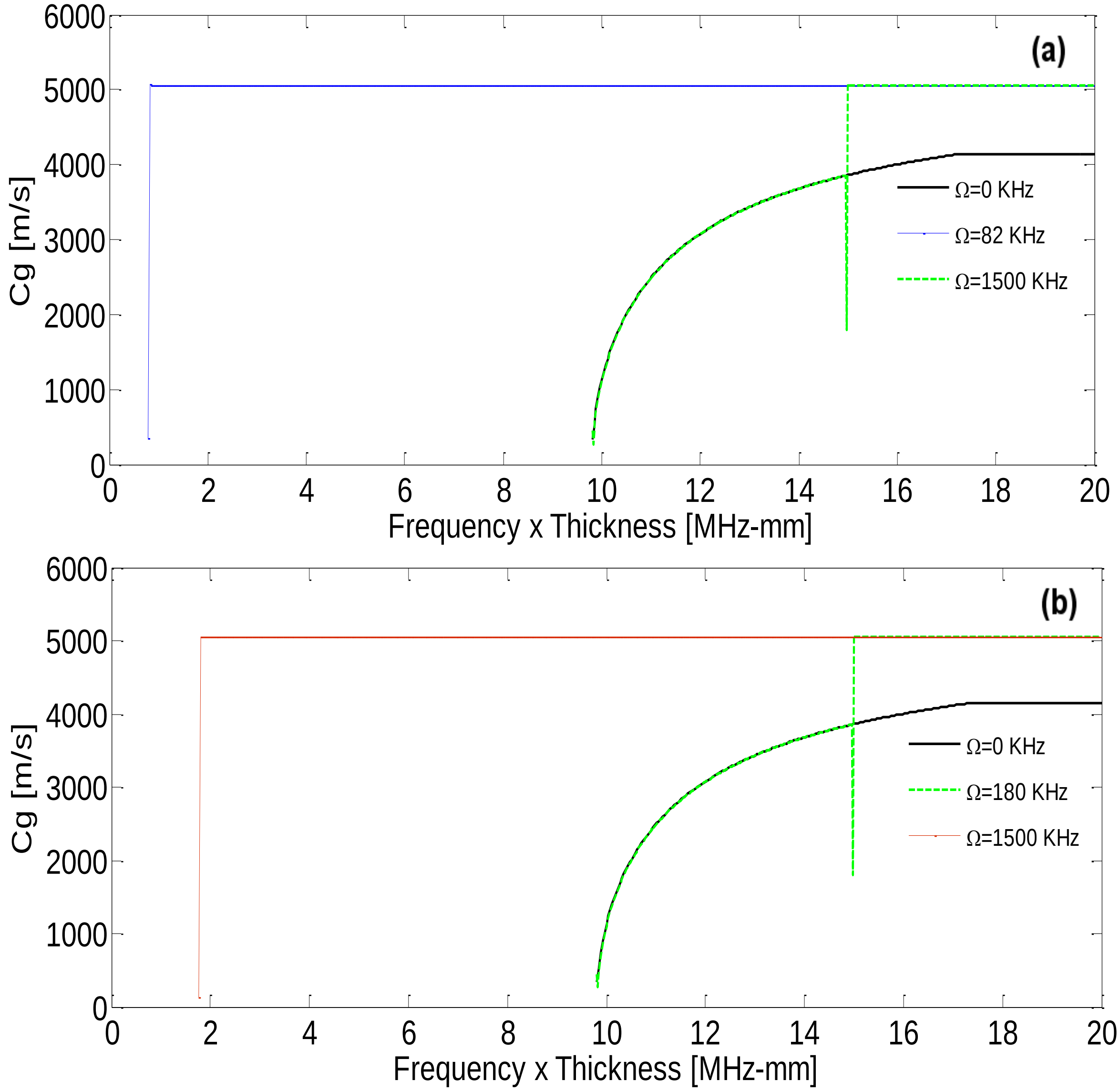

Figure 8: The group velocity curves comparison for the shear mode (a) Ω=0, 82 KHz and 1500 KHz (b) Ω =0, 180 KHz and 1500 KHz

6.3 The Normal Displacement Response

The normal displacement response of the stationary and axially oscillatory beam is studied for the two oscillatory frequencies Ω=82 KHz and Ω=180 KHz respectively. The normal response is captured for the stationary beam at three different locations (x=1h, x=2h and x=5h). By analyzing the response curves for the stationary beam as shown in Figure (9), it is observed that the flexural mode is the only propagation mode, as it is expected because the cut-off frequency for the shear mode is 1 MHz. The excitation force is band limited between 50 KHz to 300 KHz. The group velocity of this waveform was calculated as 2476 m/s using the time of flight between the initial and final locations. The shape of these waveforms indicates that this propagating wave is almost non-dispersive. Also it is noted from the resultant waveform that there is notable decrease in the amplitude of signal. This may be due to the introduction of small amount of damping in form of elastic constants and secondary due to the distance variation from the source of excitation.

The response for the oscillatory beam having oscillation frequency of Ω=82 KHz is captured at three locations as shown in Figure (10). It is clear that as we go away form the origin (arbitrary point 'O' where force of excitation is applied), there is more distortion in the response pulse due to the dispersion. Only the flexural propagating mode is predominant here while the cut-off frequency for the shear mode is now 82 KHz. The amplitude of response has increased here as compared to the normal response of the stationary beam and significantly distorted. Also the response stays longer than that of the non-oscillatory beam as compared to the stationary case.

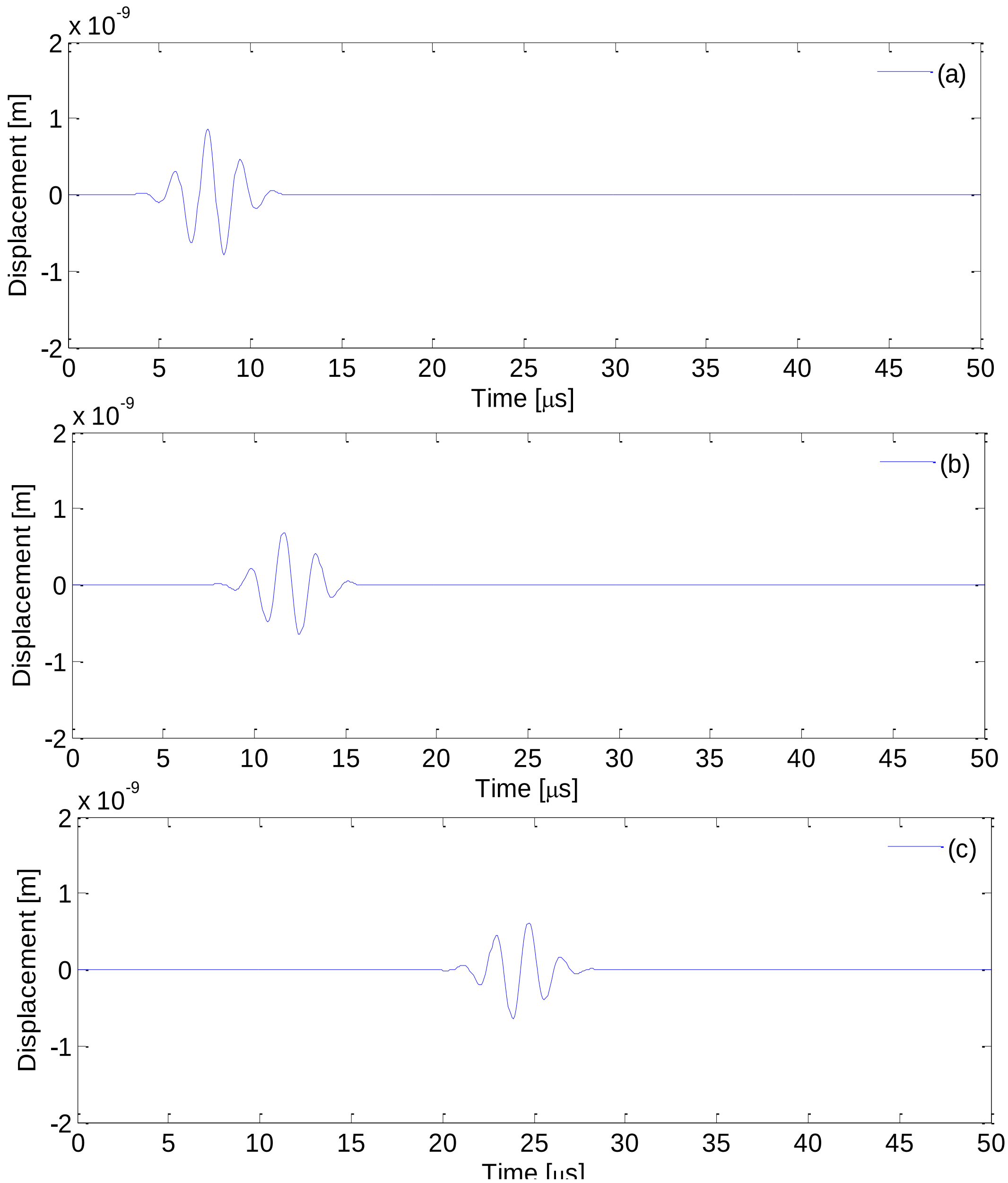

Figure 9: The response for the stationary (Ω=0) beam at (a) x=1h (b) x=2h and (c) x=5h

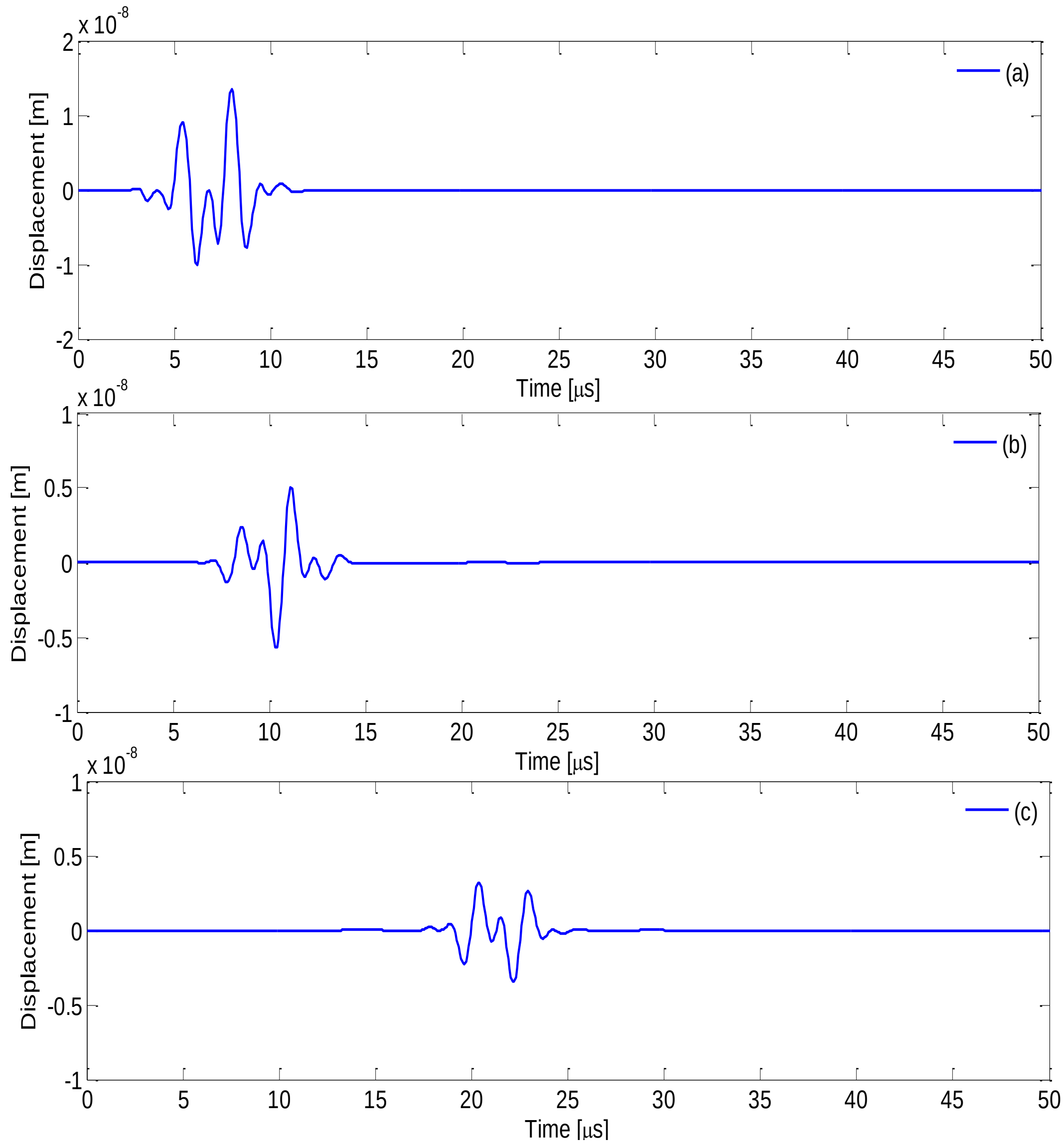

Figure 10: The response for the axially oscillatory (Ω=82 KHz) beam at (a) x=1h (b) x=2h and (c) x=5h

Finally the normal response for an axially oscillatory beam is captured having an oscillatory frequency of 180 KHz as shown in Figure (11). The shape of the waveform indicates that this propagating wave is much distorted. The group speed of the waveform was calculated as 2830 m/s using the time of flight between the initial and the final locations. It is noted that the maximum amplitude of the waveform increases as Ω increases. Also the time of stay lengthens a little more. The waveform shows decay in the amplitude during propagating. Here predominantly only the flexural mode is present.

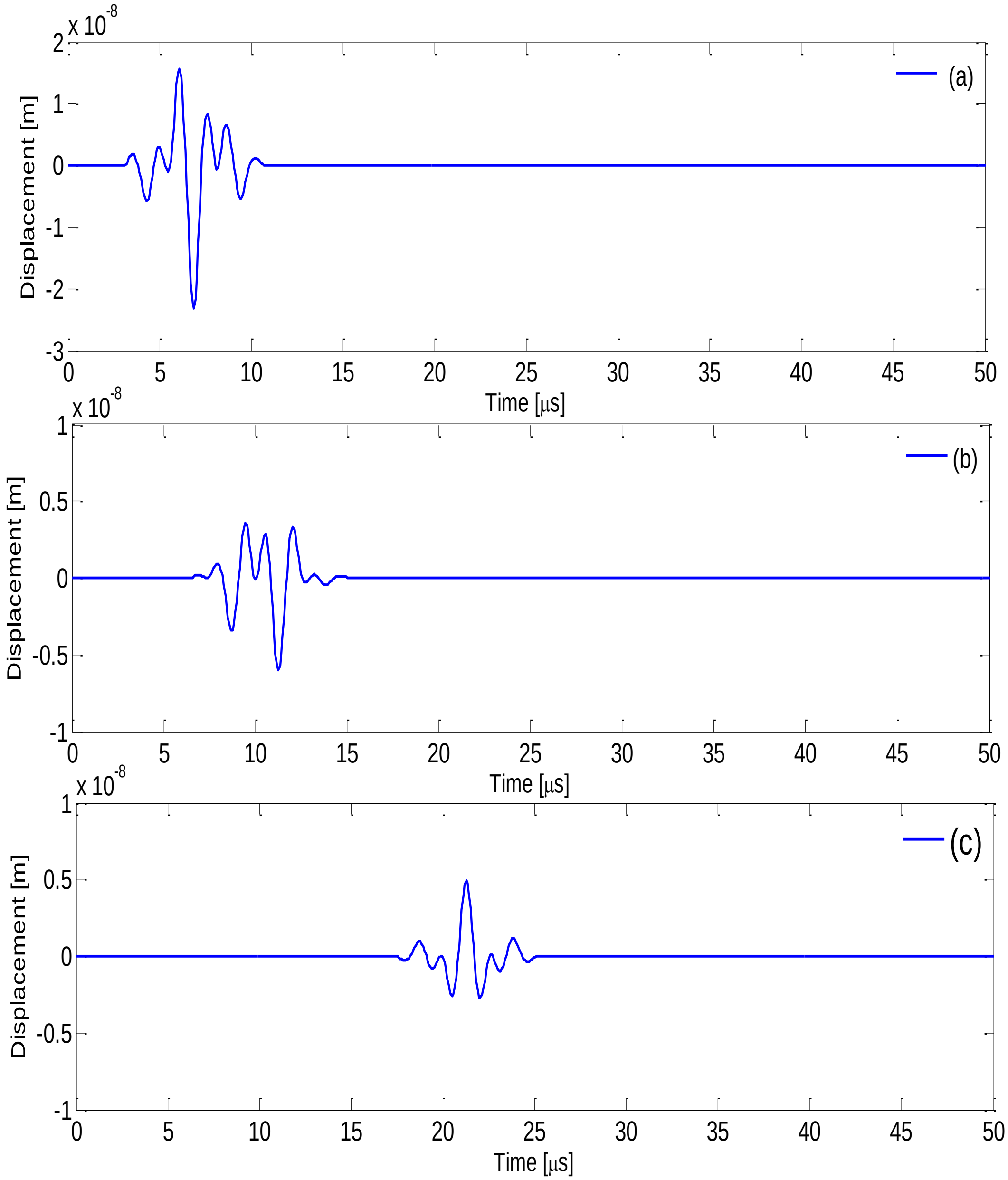

Figure 11: The response for the axially oscillatory (Ω=180 KHZ) beam at (a) x=1h (b) x=2h and (c) x=5h

For the normal displacement analysis, a comparison of the power spectrum density is also studied. Power Spectral Density (PSD) is a measure of a signal's power intensity in the frequency domain. In practice, the PSD is computed from the FFT spectrum of a signal. The PSD provides a useful way to characterize the magnitude versus frequency content of a response signal. As shown in Figure (12), a comparison of the spectral density is made for the stationary and the oscillatory beam at location x=5h. The upper and lower bounds of the PSD are 50 KHz and 300 KHz respectively. For the stationary beam there is smooth distribution over the frequencies. For an axially oscillatory beam with Ω=82 KHz there is a small peak at respective Ω value, which shows a distortion in the frequency distribution.

By observing the PSD for Ω=180 KHz it is noted that there is irregular behavior of frequency distribution, this is due to the distortion caused when the Ω coincides with the central frequency of the excitation force. It is worth saying that the original non-distorted waveform corresponding to the stationary beam can be recovered by the means of using digital filtering around the oscillatory frequency.

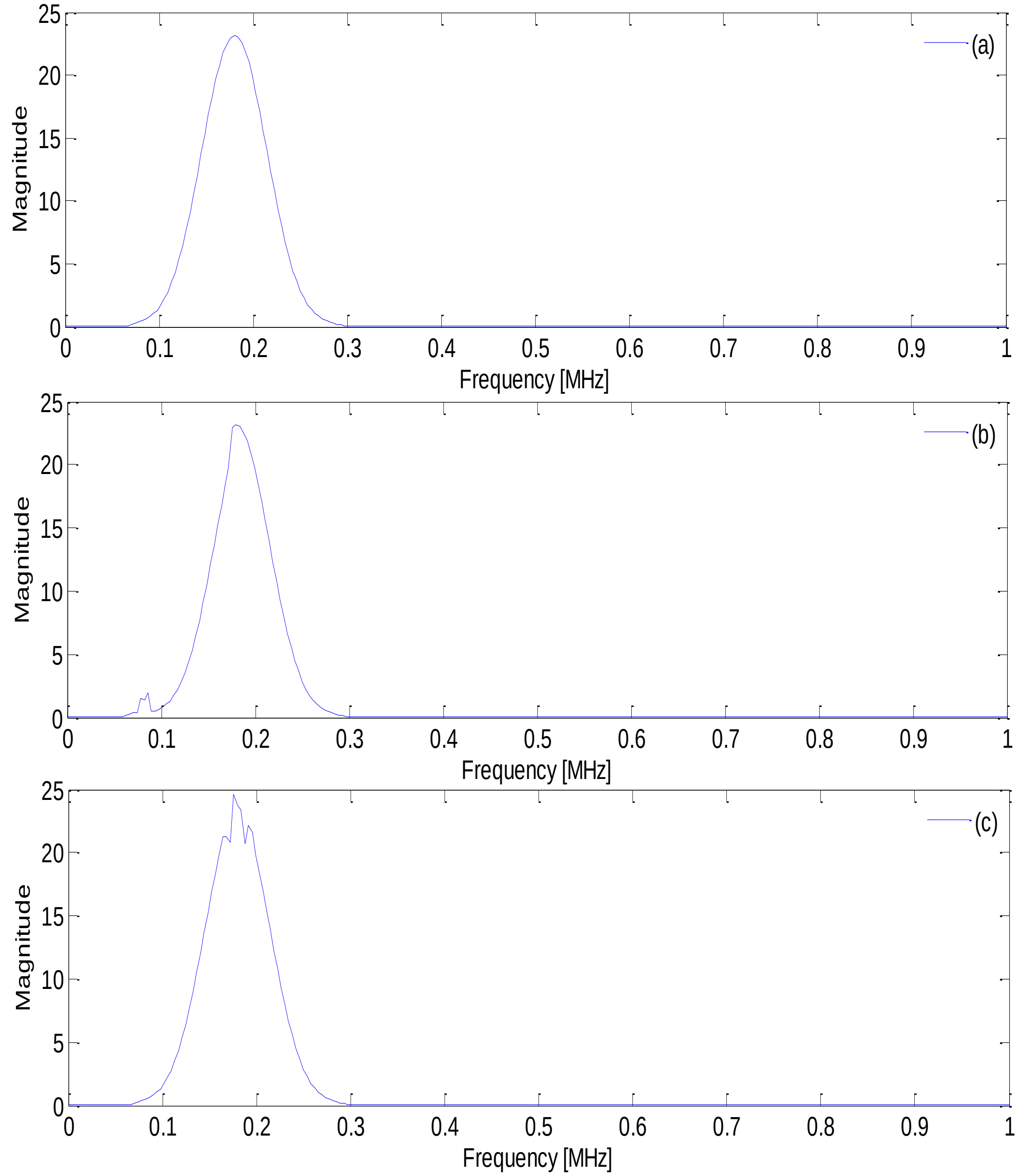

Figure 12: The power spectrum compariosn for the normal response at location x=5h for (a) Ω=0 (b) Ω=82 KHz and (c) Ω=180 KHz

CHAPTER 7

CONCLUSIONS

The effects of the axially oscillatory frequency on the dispersion curves and the normal displacement responses of Timoshenko beam has been investigated. Two dimensional transformation method is used here in conjunction with the modified Timoshenko beam theory. The equations of motion are obtained by using the dynamic version of the Hamilton's principle. Equations of motion are then transformed from time *(t)* domain to frequency domain *(ω)* by using a multiple transformation technique. Romberg's method of integration is being used to compute the inverse transformation from wave-number (ζ) domain to spatial domain *(x)*. The resultant normal displacement vector is reconstructed in time domain *(t)* by using IFFT. Key conclusions drawn from the present study are;

- The phase velocity curve experiences a sudden vertical jump when the oscillatory frequency (Ω) coincides with the frequency (ω).
- The phase velocity dispersion curves tend to converge to a constant value at a faster rate as compared to the stationary beam.
- At frequency values when $\Omega \approx \omega$ the sudden change in amplitude of the phase velocity becomes more significant at lower frequency range.
- It is noticed that, the introduction of oscillation will set a value for the shear mode to start propagating.

- When the Ω value is larger than the cut-off frequency value of the stationary beam, it will have negligible effect on the cut-off frequency.
- For the group velocity dispersion curves there is a sudden sharp spike when the oscillatory frequency (Ω) coincides with the frequency of propagation (ω).
- The group velocity dispersion curves of the flexural mode tend to converge to a constant value at once when $\Omega \approx \omega$.
- The convergence rate for the oscillatory group velocity is slower as compared to the stationary case.
- For the shear mode group velocity curves, the cut-off frequency at Ω=82 KHz and Ω=180 KHz is decreased and coincide with the frequency of oscillation (Ω).
- When the oscillatory frequency (Ω) is larger than the cut-off value of the stationary beam, it will not have notable effect on the cut-off value of the frequency.
- The convergence value of the group velocity for shear mode is larger in the oscillatory case, it is about 22% more as compared to the stationary beam.
- Waveforms for the normal displacement response show that the flexural mode is the only propagating mode, as expected due to the use of an approximate beam theory.
- Response curves for the oscillatory beam indicate that the propagating wave is significantly distorted as compared to the stationary beam.
- The displacement waveforms show decay in the amplitude during propagating.

The method used here shows that it is suitable to predict the wave propagation in axially oscillating beams in the lower Frequency*Thickness ranges. The use of modified higher beam theory will be of great help in studying the wave propagation effects in the higher Frequency*Thickness ranges.

There are number of additional areas for further research, some of these may be;

- Study the effects of rotary oscillations in the beams.
- Use of modified higher beam theories.
- Study the combined effects of rotary and axially oscillations in beam.
- Extend the work to investigate composite oscillatory beams.

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