FIXED POINT THEORY OF NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

BY

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This thesis work is dedicated to my wife Wisal and my son Amro, who have been a constant source of support and encouragement during the challenges of graduate school and life. This work is also dedicated to my parents Atif & Fayzah, brothers Abdullah, Basil, Wasil and sisters Eman & Doa'a, who have always loved me unconditionally and have taught me to work hard for the things that I aspire to achieve.

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THESIS ABSTRACT

NAME: Sami Atif Shukri

TITLE OF STUDY: Fixed Point Theory of Nonexpansive Mappings in Hyperbolic Spaces

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In this thesis, we establish analogues of classical results for nonexpansive mappings in hyperbolic spaces. Some fundamental fixed point results in partially ordered Banach spaces are extended to hyperbolic spaces. A new characterization of reflexive and strictly convex Banach spaces is established. We also extend this characterization to hyperbolic spaces. An extension of the Banach Contraction Principle for best proximity points in $CAT(0)$ *spaces is obtained. Moreover, the case of nonexpansive mappings is discussed in this setting. An extension of the Gromov geometric definition of* $CAT(0)$ *spaces is introduced. Finally, iterative approximation of common fixed points of nonexpansive and quasi-nonexpansive mappings defined on a convex metric space is studied.*

ملخص الرسالة

الاسم الكامل: سامي عاطف شكري

عنوان الرسالة: نظرية النقطة الثابتة للاقترانات غير التوسعية في فضاءات مترية زائدية

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في أطروحة رسالة الدكتوراة، تم تعميم أهم نظريات النقطة الثابتة الى مجالات غير خطية. فقد قمنا بدراسة نتائج النقطة الثابتة للاقرانات المنكمشة و غير التوسعية في فضاءات زائدية. بالاضافة الى تعميم تلك النتائج الى فضاءات غير خطية مرتبة. تم تعميم تعريف جروموف الهندسي لفضاء كات(0) و دراسة نظرية النقطة الثابتة في هذا الفضاء. تم وضع تعريف جديد متري لفضاء بانخ المحدب المتماثل، باستجدام هذا التعريف الجديد قمنا بتعميم اهم نظريات النقطة الثابتة للاقترانات غير التوسعية الى نظريات اقرب نقطة. و أخيرا، قمنا ببناء طرق تقريب و حساب النقاط الثابتة للاقترانات المعرفة على مجالات غير خطية محدبة و زائدية.

CHAPTER 1

INTRODUCTION

"The theory of fixed points is one of the most powerful tools of modern mathematics" quoted by Felix Browder, which gave a new impetus to the modern fixed point theory via the development of nonlinear functional analysis as an active and vital branch of mathematics. The flourishing field of fixed point theory started in the early days of topology (the work of Poincare, Lefschetz-Hopf, and Leray-Schauder). For example, the existence problems are usually translated into a fixed point problem like the existence of solutions to elliptic partial differential equations, or the existence of closed periodic orbits in dynamical systems, and more recently the existence of answer sets in logic programming.

The fixed point problem (as the basis of fixed point theory) may be stated as:

Problem 1.0.1 *Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $T : A \to B$ be a mapping. When does a point $x \in A$ exist such that $Tx = x$?*

A point x is called a fixed point of T whenever $Tx = x$. The set of fixed points of

T will be denoted by $F(T)$.

Banach Contraction Principle [9] is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in analysis and other related subjects. This principle asserts that a contraction on a complete metric space has a unique fixed point. A contraction mapping is continuous and hence this principle has a drawback that it is not applicable to discontinuous functions. Recently, fixed point theorems for discontinuous mappings in Banach spaces setting have been established by Berinde and Păcurar [15]. In particular, they studied fixed points of almost contraction mappings in Banach spaces.

Most of the problems in various disciplines of science are nonlinear in nature. In Section 3.2.1, we set a metric analogue of Berinde and Păcurar Theorem for almost contraction mappings in hyperbolic spaces.

Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. They are a natural extension of contractive mappings. However, the fixed point problem for nonexpansive mappings differ sharply from that of the contractive mappings. Indeed, the existence of the fixed points of nonexpansive mappings requires restrictive conditions on the domain. This explains why it took more than four decades to prove the earliest fixed point results for nonexpansive mappings in Banach spaces by Browder [20], Göhde [46] and Kirk [64]. Kirk's fixed point theorem is strongly connected to the linear convex structure of linear spaces. As early as 1965, many have tried to weaken this tiding. Takahashi [107] was the first one to give a metric analogue of Kirk Theorem.

Goebel and Kirk [41] extended Browder and Göhde Theorem for nonexpansive mappings to the case of asymptotically nonexpansive mappings in uniformly convex Banach spaces. A metric analogue of Goebel and Kirk Theorem is obtained by Kohlenbach and Leustean [73] in uniformly convex hyperbolic spaces.

Later on, Kirk [65] substantially weakened the assumption of asymptotic nonexpansiveness to generalize Goebel and Kirk Theorem for non-Lipschitzian mappings of asymptotically nonexpansive type. In Section 3.2.2, We give a metric analogue of Kirk Theorem in uniformly convex hyperbolic spaces.

Recently, a new direction has been discovered in dealing with extension of the Banach Contraction Principle to metric spaces endowed with a partial order. Ran and Reurings [98] successfully carried out the first attempt. In particular, they showed how this extension is useful when dealing with some special matrix equations. Another similar approach was given by Nieto and Rodríguez-López [91] and they used it in solving some differential equations. Jachymski [53] gave a more general unified version of these extensions by considering the graph instead of a partial order.

Recently, Bachar and Khamsi [8] studied the existence of fixed points of nonexpansive mappings defined on partially ordered Banach spaces. This work is a continuation of the previous work of Ran and Reurings [98], Nieto and Rodríguez-López [91] and Jachimsky [53] for contraction mappings.

Banach Contraction Principle has been extended nicely to set-valued mappings by Nadler [90]. Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multivalued mappings. The earliest results in this direction were established by Markin [87] in a Hilbert space setting and by Browder [21] for spaces having a weakly continuous duality mapping. Lami Dozo [79] generalized these results to a Banach space satisfying Opial condition. In Section 3.3.1, we give a natural generalization of Bachar and Khamsi Theorm [8] for monotone multivalued nonexpansive mappings, which also provides an extension of Lami Dozo Theorem [79] in partially ordered Banach spaces.

Very recently, Buthinah and Khamsi [25] gave an analogue of the fixed point theorem of Browder and Göhde for nonexpansive mappings in a partially ordered uniformly convex hyperbolic space. In Section 3.3.2, we give a more general version of Buthinah and Khamsi Theorem by replacing partial order with the graph [4, 5].

In Problem 1.0.1, $A \cap B$ is nonempty, is necessary (but not sufficient) condition for the existence of a fixed point of T. If the necessary condition fails, then the mapping T does not have any fixed point. This standpoint forces us to think of a point x in A such that x is closest to Tx in some sense. Best proximity point analysis has been developed in this direction [1, 33, 34, 35, 36, 70, 74, 97].

In [2], Alber and Guerre-Delabriere defined the concept of a weakly contractive mapping which provides a generalization of contraction mappings and they proved a fixed point result for such mappings in Hilbert spaces. After four years, Rhoades [101] extended the theorem of Alber and Guerre-Delabriere to Banach spaces. Recently, Raj [97], introduced the so-called P-property to extend Rhoades Theorem for best proximity points. As an example of a metric space where the P-property holds, one may consider any pair of closed, convex and bounded subsets of a real Hilbert space. In Section 3.4, we give a characterization of reflexivity in terms of the P-property. Moreover, we discuss this characterization in hyperbolic spaces. Following the work of Raj, we give an extension of the Banach Contraction Principle and Takahashi Theorem of nonexpansive mappings for best proximity points.

An example of linear hyperbolic space is a normed space. Hadamard manifolds [24], the Hilbert open unit ball equipped with the hyperbolic mteric [45], and $CAT(0)$ spaces [68] are examples of nonlinear hyperbolic spaces which play a major role in metric fixed point theory.

In Section 4.2, we give an example of a nonlinear hyperbolic space in which the P-property holds; one may consider any pair of closed, convex and bounded subsets of a $CAT(0)$ space. Moreover, we extend the Banach Contraction Principle for best proximity point in a $CAT(0)$ space. In [67], Kirk extended his fundamental result for nonexpansive mappings in $CAT(0)$ spaces. As an application of the P-property, we seek an extension of Kirk Theorm for best proximity points.

Recently, Sintunavarat and Kumam [105] introduced the concept of coupled best proximity points and they studied the existence of these points for a pair of cyclic contraction mappings in a uniformly convex Banach space. They posed an open problem: extend their main result for another class of spaces. In Section 4.3, we give an answer to this open problem in the setting of Hilbert ball which constitutes a nice subclass of $CAT(0)$ spaces.

In order to study best proximity points in partially ordered $CAT(0)$ spaces, in Section 4.4, we introduce the concept of proximally monotone Lipschitzian mappings and give an extension of Ran and Reurings Theorem and Buthianah and Khamsi Theorem for monotone nonexpansive mappings for best proximity points in a partially ordered $CAT(0)$ space.

To broaden the scope of study of $CAT(0)$ spaces, in Section 4.5, we extend the Gromov geometric definition of $CAT(0)$ spaces [47] to the case when the comparison triangles lie in a general Banach space rather than the Euclidean plane; in particular, we study the case of the Banach space l_p, $p > 2$. As an application of our results, we study the fixed point property for nonexpansive mappings in these spaces.

A plethora of metrical fixed point theorems have been obtained in this thesis, more or less important from a theoretical point of view, which usually deal with the existence, or the existence and uniqueness of fixed points for certain mappings.

Finally, to provide fixed point results which are important from a practical point of view (with constructive method for finding fixed points), in Sections 5.3 and 5.4, we study iterative construction of common fixed points of two (respectively, a finite family of) nonexpansive mappings on a hyperbolic space.

CHAPTER 2

METRIC FIXED POINT THEORY

2.1 Metric Contraction Principles

2.1.1 The metric topology

A *topology* on a set X is any family $\mathcal{F}$ of subsets of X which satisfies the following simple axioms:

1. $\emptyset$ and X are in $\mathcal{F}$.

2. The union of any subcollection of $\mathcal{F}$ is a member of $\mathcal{F}$.

3. The intersection of any finite subcollection of $\mathcal{F}$ is a member of $\mathcal{F}$.

The pair $(X, \mathcal{F})$ is called a *topological space*.

A subset U of X is said to be an *open* set if $U \in \mathcal{F}$. A *closed* set in X is a set whose complement is open. Thus $B \subseteq X$ is closed if $X \backslash B \in \mathcal{F}$, where

$$X \backslash B = \{x \in X : x \notin B\}.$$

If $(X, \mathcal{F})$ is a topological space, then it is clear from the definition (and very elementary properties of sets) that:

1. $\emptyset$ and X are closed sets.

2. The intersection of any subcollection of closed sets is a closed set.

3. The union of any finite subcollection of closed sets is a closed set.

Many topological spaces, and especially those which arise naturally in the study of analysis, satisfy an additional assumption. A topological space X is said to be *Hausdorff* if given any two points $x, y \in X$, there are open sets U and V in X such that $x \in U$, $y \in V$, and $U \cap V \neq \emptyset$. A sequence $\{x_n\}$ of elements of a topological space X is said to *converge* to $x \in X$ (written $\lim\limits_{n\to\infty} x_n = x$) if given any open set U containing x, there is an integer N such that for $n \geq N$, $x_n \in U$. The assumption that the space is Hausdorff, assures that limit of a sequence is always unique.

There are two natural ways of introducing the metric topology in a metric space. Given a metric space (M, d), define for $x \in M$ and $r > 0$, *open ball* centered at x with radius r as:

$$U(x; r) = \{y \in M : d(x, y) < r\}.$$

The *metric topology* on a metric space M is the topology obtained by taking open sets as the collection of all sets $\mathcal{F}$ in M and which have the property that $S \in \mathcal{F}$ provided each point $x \in S$ is the center of some open ball $U(x;r)$ which lies completely in S. It easy to check that $\mathcal{F}$ is indeed a topology (As required, $\emptyset$ is an open set in this topology). And with this topology, the topological notion of limit is consistent with the one defined as: $\lim_{n\to\infty} x_n = x$ if given any $\varepsilon > 0$ there exists an integer N such that if $n \geq N$, $d(x_n, x) < \varepsilon$.

This gives rise to an important characterization of closed sets in a metric space.

Theorem 2.1.1 *A subset B of a metric space M is closed if and only if*

$$(*) \qquad \{x_n\} \subseteq B \text{ and } \lim_{n\to\infty} x_n = x \Rightarrow x \in B.$$

Another efficient way of introducing the metric topology in a metric space is to first define 'closed sets'. Call a point $x \in M$ a *limit point* of $B \subseteq M$, if there exists a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} x_n = x$. Now define *closed sets* in M to be precisely those sets which contain all of their limit points, and take as *open sets* those sets whose complements are closed. In view of the preceding theorem the topology obtained in this way is known as the metric topology.

If B is a subset of a topological space X, then the closure $\overline{B}$ of B is defined as the intersection of all closed subsets of X which contain B. It is easy to see that a set B in a topological space is closed if and only if $\overline{B} = B$. Another easy consequence of the previous theorem is the following.

Theorem 2.1.2 *If B is a subset of a metric space (M, d), then $x \in \overline{B}$ if and only if there exists a sequence $\{x_n\} \subseteq B$ such that $\lim_{n\to\infty} x_n = x$.*

A subset S of a topological space $(X, \mathcal{F})$ is said to be *compact* if whenever B is contained in the union of a collection $\mathcal{U}$ of sets of $\mathcal{F}$ (that is, when B has an *open cover* $\mathcal{U}$), it is the case that some finite subcollection of $\mathcal{U}$ contains B. This definition applies to metric spaces as well. However, in the case of metric spaces, there is a characterization of compactness that is very useful.

Theorem 2.1.3 *A subset B of a metric space M is compact if and only if any sequence $\{x_n\}$ of points of B has a subsequence $\{x_{n_k}\}$ which converges to a point of B.*

The previous two theorems yield an important fact: *A subset of a compact metric space is itself compact if and only if it is closed.*

2.1.2 Completeness

Probably the first person to recognize fundamental role, completeness plays in spaces which usually arise in analysis, was the Polish mathematician Stefan Banach [9]. Indeed, in recognition of its importance Banach took it as an axiom in considering what are now known as Banach spaces. It is, however, an entirely metric condition. First let us recall the definition of Cauchy sequences.

Definition 2.1.1 *A sequence $\{x_n\}$ in a metric space (M, d) is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $d(x_m, x_n) < \varepsilon$.*

Definition 2.1.2 *A metric space (M, d) is said to be complete if each Cauchy sequence $\{x_n\}$ in M has a limit.*

The following general fact about completeness is also quite useful.

Proposition 2.1.1 *Every closed subspace of a complete metric space is itself complete.*

2.1.3 Banach Contraction Principle

Banach Contraction Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in many areas of mathematics. This is because the contractive condition on the mapping is simple and easy to test, in a complete metric space. It finds almost canonical applications in the theory of differential and integral equations. Although the basic idea was known to others earlier, the principle in its present form first appeared in 1922 in Banach's thesis [9].

Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be *Lipschitzian* if there is a constant $k \geq 0$ such that for all $x, y \in M$

$$d(T(x), T(y)) \leq k\, d(x, y). \tag{2.1.1}$$

The smallest number k for which (2.1.1) holds is called the *Lipschitz* constant of T.

Definition 2.1.3 *A Lipschitzian mapping $T : M \to M$ with Lipschitz constant $k < 1$ is said to be a contraction mapping.*

Theorem 2.1.4 *(Banach Contraction Principle) Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping with Lipschitz constant k. Then T has a unique fixed point x_0. Moreover, for each $x \in M$, we have*

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x),$$

for $n = 1, 2, \ldots$ In particular, we have $\lim_{n\to\infty} T^n(x) = x_0$.

2.1.4 Set-valued contractions

Banach Contraction Principle has been extended nicely to set-valued mappings by Nadler [90].

First, we introduce the Hausdorff metric space. Let (M, d) be a metric space and let $\mathcal{M}$ denote the family of all nonempty, bounded and closed subsets of M. For $A \in \mathcal{M}$ and $\varepsilon > 0$, define the ε-neighborhood of A to be the set

$$N_\varepsilon(A) = \{x \in M : dist(x, A) < \varepsilon\}.$$

where $dist(x, A) = \inf_{y\in A} d(x, y)$. Now for $A, B \in \mathcal{M}$, set

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}.$$

Then $(\mathcal{M}, H)$ is a metric space, and H is called the Hausdorff metric on $\mathcal{M}$.

The key idea in Nadler's extention is the following. If A and B are nonempty, closed and bounded subsets of a metric space and if $x \in A$, then given $\varepsilon > 0$, there must exist a point $y \in B$ such that

$$d(x, y) \leq H(A, B) + \varepsilon.$$

This is so because the definition of Hausdorff distance assures that for any $\mu > 0$

$$A \subseteq N_{\rho+\mu}(B)$$

where $\rho = H(A, B)$.

Theorem 2.1.5 *Let (M, d) be a complete metric space, and let $\mathcal{M}$ be the collection of all nonempty, bounded and closed subsets of M endowed with the Hausdorff metric H. Suppose $T : M \to \mathcal{M}$ is a contraction mapping in the sense that for some $k < 1$:*

$$H(T(x), T(y)) \leq k\, d(x, y), \qquad x, y \in M.$$

Then there exists a point $x \in M$ such that $x \in T(x)$.

Two things are worth noting about the preceding theorem. First, in contrast to Banach's theorem, it is not asserted that the fixed point x is unique. Indeed, it need not be so. Also, since M is complete, $(\mathcal{M}, H)$ is complete as well, but this fact is not needed in its proof.

2.2 Fixed Point Theory in Banach Spaces

2.2.1 Fixed point theorems for continuous mappings

The concept of 'continuity' lies at the heart of Calculus.

Definition 2.2.1 *If S is a subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^n$, then f is said to be continuous if for each $\mathbf{a} \in S$,*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

Define for a point $\mathbf{m} \in \mathbb{R}^n$ and $r \geq 0$, closed ball centered at $\mathbf{m}$ of radius r as:

$$B(\mathbf{m}; r) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{m}, \mathbf{x}) = ||\mathbf{m} - \mathbf{x}|| \leq r\}.$$

We are now in a position to state one of the most fundamental of all 'fixed point theorems' ever proved.

Theorem 2.2.1 *(Brouwer's Fixed Point Theorem). Let B be closed ball in $\mathbb{R}^n$. Then any continuous mapping $f : B \to B$ has at least one fixed point.*

Brouwer's theorem has a long history. Ideas leading to the proof of Brouwer's theorem were discovered by Henri Poincaré as early as 1886. Brouwer himself proved the theorem for $n = 3$ in 1909. In 1910, Hadamard gave the first proof for arbitrary n, and Brouwer gave another proof in 1912.

Brouwer's Theorem fails in some infinite dimensional vector spaces ([56], Example 7.1). On the other hand, bounded and closed subsets of $\mathbb{R}^n$ are compact. It turns out that this is precisely the assumption needed to assure the validity of an infinite dimensional version of Brouwer's Fixed Point Theorem.

Definition 2.2.2 *A subset K of a normed linear space is said to be convex if $\lambda x + (1-\lambda)y \in K$ for each $x, y \in K$ and each scalar $\lambda \in [0,1]$.*

The next fact follows from a routine induction argument.

Proposition 2.2.1 *[56] A subset K of a normed linear space is convex if and only if $\sum_{i=1}^{n} \lambda_i x_i \in K$ for any finite set $\{x_1, x_2, \cdots, x_n\} \subseteq K$ and any scalars $\lambda_i \geq 0$ for which $\sum_{i=1}^{n} \lambda_i = 1$.*

Theorem 2.2.2 *(Schauder's Fixed Point Theorem) Let K be a nonempty, compact and convex subset of a Banach space E, and $f : K \to K$ be continuous. Then f has at least one fixed point.*

2.2.2 Basic theorems for nonexpansive mappings

Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. They are a natural extension of contraction mappings. However, the fixed point problem for nonexpansive mappings differ sharply from that of the contraction mappings. A nonexpansive mapping need not have a fixed point, even if it transforms a bounded, convex and closed subset in a Banach space into itself (see, Example 2.2.1). In contrast to the case of contraction mappings, the role played by the geometry of the ambient Banach space in the theory of fixed points of nonexpansive mappings is essential, the same being true for the metric and the topological structures of the set, where the nonexpansive mapping is defined. This explains why it took more than four decades to have fixed point results for nonexpansive mappings in Banach spaces by Browder [20], Göhde [46], and Kirk [64].

Example 2.2.1 *[64] Consider in the Banach space $C[0,1]$ of continuous functions; the bounded, convex and closed subset $K = \{x(t) \in C[0,1] : 0 \leq x(t) \leq 1,\ x(0) = 0$ and $x(1) = 1\}$. Define the mapping T as follows:*

$$T(x(t)) = tx(t).$$

It is easily seen that T maps K into itself, it is nonexpansive and has no fixed point.

The fixed point problem in Banach spaces becomes:

Problem 2.2.1 *Let E be a Banach space, and K a nonempty, bounded, closed and convex subset of E. When does any nonexpansive mapping $T : K \to K$ have a fixed point?*

Interest in modulus of convexity arose from a careful study of geometric properties of the Hilbert space ℓ_2 space. Indeed, let x and y be any vectors in ℓ_2. The parallelogram law implies

$$\|x+y\|^2 = 2\Big[\,\|x\|^2 + \|y\|^2\,\Big] - \|x-y\|^2 .$$

So if we assume that x and y are in the unit ball and bounded away from each other (which means that $\|x-y\| \geq \varepsilon > 0$), then we obtain,

$$\|x+y\|^2 \leq 4-\varepsilon^2,$$

which implies

$$\left\|\frac{x+y}{2}\right\| \leq \sqrt{1-\frac{\varepsilon^2}{2}}\ .$$

So the middle point is uniformly inside the unit ball. This property is known as *uniform convexity.*

Definition 2.2.3 *[56] The modulus of convexity of a Banach space E is the function $\delta : [0,2] \to [0,1]$ defined by*

$$\delta(\varepsilon) = \inf\left\{1-\left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}. \qquad (2.2.1)$$

Definition 2.2.4 *A Banach space E with modulus of convexity δ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$.*

Theorem 2.2.3 ***(Browder-Göhde's Theorem)***

If K is a bounded, closed and convex subset of a uniformly convex Banach space E and $T : K \to K$ is nonexpansive, then T has a fixed point. Moreover, the fixed point set of T is a closed and convex subset of K.

If E is a real Banach space, then a mapping $f : E \to \mathbb{R}$ is called a *linear* *functional* if for each $x, y \in E$ and $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

The space of all *continuous* linear functionals on E is denoted by E^*.

We shall say that a sequenc $\{x_n\}$ *converges to* $x \in E$ *in the weak topology* (or *converges weakly to* x) if $\lim_{n\to\infty} f(x_n) = f(x)$ for each $f \in E^*$. When this is true, we write

$$w\text{-}\lim_{n\to\infty} x_n = x.$$

Another common notation for the weak topology is $\sigma(E, E^*)$. A subset K of E is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of each of its weakly convergent sequences. The resulting topology on E is called the *weak topology* on E. Sets which are compact in this topology are said to be *weakly compact*. It can be shown that the weak topology is the weakest topology for which all the functionals $f \in E^*$ are continuous.

Recall that a closed and convex subset K of a Banach space E is said to have the normal structure property [18] if any bounded and convex subset H of K which contains more than one point, contains a nondiametral point, i.e. there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - x\| : x \in H\} < diam(H),$$

where $diam(H) = \sup\{||x - y|| : x, y \in H\}$.

Theorem 2.2.4 ***(Kirk's Theorem)***

Let K be a weakly-compact convex subset of a Banach space E. Assume that K has the normal structure property. Then any nonexpansive mapping $T : K \to K$ has a fixed point.

Note that in Example 2.2.1, K dose not have the normal structure property [48].

CHAPTER 3

RESULTS IN HYPERBOLIC SPACES

3.1 Introduction

Most of the problems in various disciplines of science are nonlinear in nature. Therefore, translating linear version of a known problem into its equivalent nonlinear version is of paramount interest. Furthermore, investigation of numerous problems in spaces without linear structure has its own importance in pure and applied sciences.

Convexity is often the way to weaken a linearity requirement while leaving a problem tractable. Several attempts have been made to introduce a convex structure on a metric space. Historically, there are two types of convexity considered in metric spaces. One definition, involves metric segment, usually known as Menger convexity [88], while the other one involves the concept of convexity structure [93].

In this thesis, we will mainly use the Menger convexity.

3.1.1 Basic definitions and properties

Let (M, d) be a metric space. Assume that for any x and y in M, there exists a unique metric segment $[x, y]$, which is an isometric copy of the real line interval $[0, d(x, y)]$. Denote this family of metric segments in M by $\mathcal{F}$. If for any $\beta \in [0, 1]$, there exists a unique point $z \in [x, y]$ in $\mathcal{F}$ such that

$$d(x, z) = (1 - \beta)d(x, y), \quad \text{and} \quad d(z, y) = \beta d(x, y),$$

then we denote this point z by $\beta x \oplus (1 - \beta)y$. Metric spaces having this property are usually called *convex metric spaces* [88]. Moreover, if we have

$$d\Big(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y\Big) \leq \alpha d(p, q) + (1 - \alpha)d(x, y),$$

for all p, q, x, y in M, and $\alpha \in [0, 1]$, then M is said to be a *hyperbolic space* [99].

Recall that a subset C of a hyperbolic space M is said to be convex whenever $[x, y] \subset C$ for any $x, y \in C$.

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [24], $CAT(0)$ spaces [66, 67, 80] (see, Section 4.1) and the Hilbert open unit ball equipped with the hyperbolic [45] (see, Section 4.3).

In 1970, Takahashi [107] introduced a concept of convexity in a metric space (M, d) as follows.

A mapping $W : M^2 \times [0,1] \to M$ is a convex structure in M if

$$d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$$

for all $x, y \in M$ and $\alpha \in [0,1]$. The metric space M together with a convex structure W is known as a convex metric space.

Kohlenbach [71] enriched the concept of convex metric space as "hyperbolic space" by including the following additional conditions in the definition of a convex metric space.

$$\begin{aligned} &(1)\ \ d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\, d(x, y) \\ &(2)\ \ W(x, y, \alpha) = W(y, x, 1 - \alpha) \\ &(3)\ \ d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w) \end{aligned} \tag{3.1.1}$$

for all $x, y, z, w \in M$ and $\alpha, \beta \in [0,1]$.

3.1.2 Metric analogues of classical results for nonexpansive mappings

The Banach Contraction Principle is a metric result and does not depend on any linear structure. Kirk Theorem heavily depends on convexity in linear spaces. As early as 1965, many authors have tried to weaken this tiding. Takahashi [107] is the first one to give a metric analogue of Kirk Theorem.

Recall that a hyperbolic space (X, d) is said to have the property (R) if any non-increasing sequence of nonempty, convex, bounded and closed sets, has a nonempty intersection [38].

Note that any hyperbolic space X which satisfies the property (R) is complete [55].

Theorem 3.1.1 *[55] Suppose that M is a convex metric space which satisfies the property (R). Let K be a nonempty, bounded, closed and convex subset of M with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.*

In 1972, Goebel and Kirk [41] proved the following:

Theorem 3.1.2 *If E is uniformly convex Banach space, K is a bounded, closed and convex subset of E and $T : K \to K$ is asymptotically nonexpansive on K, that is, if there exists a sequence $\{k_n \geq 1\}$ of numbers such that $k_n \to 1$ as $n \to \infty$ and*

$$||T^n x - T^n y|| \leq k_n ||x - y||, \;\; x, y \in K, \;\; n > N_0.$$

Then T has a fixed point.

This generalizes fixed point theorem of Browder [20], Göhde [46] and Kirk [64] for nonexpansive mapping.

Uniformly convex Banach spaces form an important subclass of Banach spaces. We can define uniform convexity for hyperbolic spaces too.

Definition 3.1.1 *[45] Let (X,d) be a hyperbolic space. For any $r > 0$, $a \in X$ and $\varepsilon > 0$, set*

$$\delta(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\Big\}.$$

If $\delta(r,\varepsilon) > 0$, then X is said to be uniformly convex.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ).

Remark 3.1.1 *[55] If (X,d) is uniformly convex hyperbolic space, then (X,d) is strictly convex hyperbolic space, i.e., whenever*

$$d\Big(\alpha x \oplus (1-\alpha)y, a\Big) = d(x,a) = d(y,a)$$

for $\alpha \in (0,1)$ and $x, y, a \in X$, then we must have $x = y$.

The following result is an analogue of the well known fact that a uniformly convex Banach space is reflexive. For a reference the reader may consult Theorem 2.1 in [45].

Theorem 3.1.3 *[55] If (X,d) is a complete uniformly convex hyperbolic space, then (X,d) has the property (R).*

We close this section with a metric analogue of Goebel and Kirk theorem due to Kohlenbach and Leustean:

Theorem 3.1.4 *[73] Let C be a nonempty, closed, convex and bounded subset of a complete uniformly convex hyperbolic space X. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Then T has a fixed point in C.*

3.2 Fixed Points of non-Lipschitzian Mappings

In this section, some fixed point theorems for discontinuous mappings in Banach spaces due to Berinde and Păcurar [15] and Kirk [65] are extended to hyperbolic spaces.

3.2.1 Non-self almost contraction mappings

A contraction mapping is continuous and hence Banach Contraction Principle has a drawback that it is not applicable to discontinuous functions. Fixed point theorems for discontinuous mappings in Banach space setting have been established in [15] for single-valued non-self almost contractions. Since almost contractions form a large class of contractive type mappings and it includes, amongst others, the Banach contraction mappings, therefore the results in [15] are significant generalization of some important metric fixed point theorems for single-valued self and non-self mappings; see for example [6, 30, 100].

We state the main result in [15]:

Theorem 3.2.1 *Let E be a Banach space, K a nonempty closed subset of E and $T : K \to E$ a non-self almost contraction, that is, a mapping for which there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$||Tx - Ty|| \leq \delta \cdot ||x - y|| + L||y - Tx|| , \quad \textit{for all } x, y \in K. \tag{3.2.1}$$

If T has property (M) (see Definition 3.2.3) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \textit{ where } \partial K \textit{ stands for the boundary of } K,$$

then T has a fixed point in K.

Note that here T may be discontinuous but T is continuous at the fixed point.

Banach space setting is far away from being the most general setting in which Theorem 3.2.1 can be established. Moreover, property (M), a fundamental concept used in the proof, could also naturally be adapted in uniformly convex hyperbolic spaces (see Example 3.2.1).

Let X be a uniformly convex hyperbolic space, C a nonempty closed subset of X and $T : C \to X$ a non-self mapping. If $x \in C$ is such that $Tx \notin C$, then we suppose throughout this section that there exists $y \in \partial C$ with $y = (1 - \lambda)x \oplus \lambda Tx\, (0 < \lambda < 1)$, such that

$$d(x, Tx) = d(x, y) + d(y, Tx),\ y \in \partial C. \tag{3.2.2}$$

Definition 3.2.1 *Let X be a uniformly convex hyperbolic space, C a nonempty closed subset of X and $T : C \to X$ a non-self mapping. Let $x \in C$ with $Tx \notin C$ and let $y \in \partial C$ be the element given by (3.2.2). If, for any such element x, we have*

$$d(y, Ty) \leq d(x, Tx), \tag{3.2.3}$$

then we say that T has property (M).

Let (M, d) be a metric space. Let C be a nonempty subset of M. Define the nearest point projection $P_C : M \to 2^C$ by

$$P_C(x) = \Big\{c \in C;\ d(x, c) = d(x, C)\Big\}.$$

If $P_C(x) \neq \emptyset$, for every x in M, then C is said to be proximinal. In case $P_C(x)$ reduces to one point, for every x in M, then C is said to be a Chebyshev set. In this case, the nearest point projection P_C is seen as a singlevalued mapping, i.e., $P_C : M \to C$ is defined by

$$d(x, P_C(x)) = d(x, C),$$

for any $x \in M$.

Lemma 3.2.1 *[55] Let (X, d) be a complete uniformly convex hyperbolic space. Let C be nonempty convex and closed subset of X. Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C, i.e., there* *exists a unique* $c_0 \in C$ *such that*

$$d(x, c_0) = d(x, C),$$

i.e., C *is Chebyshev.*

We introduce a non-self mapping which has property (M).

Example 3.2.1 *Let* C *be a nonempty convex and closed subset of a complete uniformly convex hyperbolic space* X*. For a fixed* $x \in X \setminus C$*, set* $c_o = P_C(x)$ *where* P_C *is the nearest point projection from* X *onto* C*. Let* $B = B(c_o, \frac{d(c_0,x)}{2})$ *be the closed ball centered at* c_o *with radius* $\frac{d(c_0,x)}{2}$*.*

Define $T : B \to X$ *by* $Tb = \frac{1}{2}b \oplus \frac{1}{2}c_0$*, the midpoint of* $[b, c_0]$*, if* $b \neq c_0$ *and* $Tb = x$*, if* $b = c_0$*. Then* T *has property (M).*

Indeed, the only $b \in B$ *with* $Tb \notin B$ *is* $b = c_0$*; let* $y \in \partial B$ *be the element as in* (3.2.2)*. The equation*

$$d(y, Ty) = d(y, \frac{1}{2}y \oplus \frac{1}{2}c_0) = \frac{1}{2}d(y, c_0) = \frac{1}{4}d(c_0, Tc_0)$$

shows that (3.2.3) holds.

Now we set a metric analogue of Theorem 3.2.1.

Theorem 3.2.2 *Let* X *be a complete uniformly convex hyperbolic space,* C *a nonempty closed subset of* X *and* $T : C \to X$ *a non-self almost contraction. If* T

has property (M) and satisfies Rothe's boundary condition

$$T(\partial C) \subset C, \tag{3.2.4}$$

then T has a fixed point in C.

Proof.

Let $x_0 \in \partial C$. By (3.2.4), we know that $Tx_0 \in C$. Let $x_1 = Tx_0$. Now, if $Tx_1 \in C$, set $x_2 = Tx_1$. If $Tx_1 \notin C$, then there exists unique x_2 on the segment $[x_1, Tx_1]$ which also belongs to ∂C, that is,

$$x_2 = (1-\lambda)x_1 \oplus \lambda Tx_1 \, (0 < \lambda < 1).$$

Continuing in this way, we obtain a sequence $\{x_n\}$ whose terms satisfy one of the following properties:

i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in C$;

ii) $x_n = (1-\lambda)x_{n-1} \oplus \lambda Tx_{n-1} \in \partial C \, (0 < \lambda < 1)$, if $Tx_{n-1} \notin C$.

To simplify the argument in the proof, we put

$$P = \{x_k \in \{x_n\} : \, x_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : \, x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset C$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, in view of (3.2.4), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence. To prove this, we must discuss following three different cases:

Case I. $x_n,\, x_{n+1} \in P$.

In this case, we have $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$ and so by (3.2.1), we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1})$$

.

As $x_n = Tx_{n-1}$, so we have

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}). \tag{3.2.5}$$

Case II. $x_n \in P,\; x_{n+1} \in Q$.

In this case, we have $x_n = Tx_{n-1}$, $x_{n+1} \neq Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Hence

$$d(x_n, x_{n+1}) \leq d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n)$$

and so by (3.2.1), we get

$$d(x_n, x_{n+1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1}) = \delta d(x_n, x_{n-1}),$$

which again yields inequality (3.2.5).

Case III. $x_n \in Q,\ x_{n+1} \in P$.

In this situation, we have $x_{n-1} \in P$. By property (M), we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_{n-1}, Tx_{n-1})).$$

From $x_{n-1} \in P$, we have $x_{n-1} = Tx_{n-2}$ and so by (3.2.1), we get

$$d(Tx_{n-2}, Tx_{n-1}) \leq \delta d(x_{n-2}, x_{n-1}) + Ld(x_{n-1}, Tx_{n-2}) = \delta d(x_{n-2}, x_{n-1}).$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-2}, x_{n-1}). \tag{3.2.6}$$

Therefore, summarizing all the three cases and using (3.2.5) and (3.2.6), it follows that the sequence $\{x_n\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq \delta \max\{d(x_{n-2}, x_{n-1}),\ d(x_{n-1}, x_n)\}, \tag{3.2.7}$$

for all $n \geq 2$. Now, by induction for $n \geq 2$, from (3.2.7) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1),\ d(x_1, x_2)\},$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \le \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \le 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset C$ and C is closed, $\{x_n\}$ converges to some point in C.

Denote

$$x^* = \lim_{n\to\infty} x_n\,, \tag{3.2.8}$$

and let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$ (such a subsequence always exists) that we denote for simplicity by $\{x_n\}$ too.

Then

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*)\,.$$

By (3.2.1), we have

$$d(Tx_n, Tx^*) \le \delta\, d(x_n, x^*) + L\, d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \le (1+L)d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*), \text{ for all } n \ge 0. \tag{3.2.9}$$

Letting $n \to \infty$ in (3.2.9), we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T.

∎

Berinde [13] has shown that it is possible to obtain uniqueness of the fixed point of an almost contraction, by imposing an additional contractive condition, quite similar to (3.2.1).

The uniqueness of fixed point of an almost contraction on a nonlinear domain is given below; its proof is simple and so omitted.

Theorem 3.2.3 *Let X be a complete uniformly convex hyperbolic space, C a nonempty closed subset of X and $T : C \to X$ a non-self almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx)\,, \quad \textit{for all } \; x, y \in C\,.$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial C) \subset C,$$

then T has a unique fixed point in C.

3.2.2 Non-Lipschitzian mappings of asymptotically nonexpansive type

In [65], Kirk substantially weakened the assumption of asymptotic nonexpansiveness of T in Theorem 3.1.2 as:

$$\limsup_{n\to\infty}\{\sup_{y\in k}[||T^n x - T^n y|| - ||x-y||]\} \leq 0, \text{ for each } x \in K,$$

which may hold even if none of the iterates of T is Lipschitzian. Although, it is assumed that at least one of its iterates is continuous, the mapping itself need not be so.

Before we obtain a metric analogue of Kirk Theorem, we need the following

Remark 3.2.1 *[55] For a complete uniformly convex hyperbolic space (X, d), with modulus of convexity $\delta(r, \varepsilon)$, we observe the following:*

1. *$\delta(r, \varepsilon)$ is an increasing function of ε for every fixed r and $\delta(r, 0) = 0$.*
2. *For $r_1 \leq r_2$ there holds*

$$1 - \frac{r_2}{r_1}\Big(1 - \delta\Big(r_2, \varepsilon\ \frac{r_1}{r_2}\Big)\Big) \leq \delta(r_1, \varepsilon).$$

Theorem 3.2.4 *Let (X, d) be a complete uniformly convex hyperbolic space and $C \subset X$ be nonempty, bounded, closed and convex. Suppose that $T : C \to C$ has the property "T^N is continuous for some positive integer N", and T satisfies:*

$$\limsup_{n\to\infty}\{\sup_{y\in k}[d(T^n x, T^n y) - d(x,y)]\} \leq 0, \text{ for each } x \in C. \tag{3.2.10}$$

Then T has a fixed point in C.

Proof. For each $y \in C$ and $r > 0$, let $S(y, r)$ denote the ball centered at y with radius r. Let $x \in C$ be fixed, and let the set R_x consists of those numbers ρ for which there exists an integer k such that

$$C \cap \big(\bigcap_{n=k}^{\infty} S(T^n x, \rho)\big) \neq \phi.$$

If D is the diameter of C, then $D \in R_x$, so $R_x \neq \phi$. Let ρ_0 =g.l.b. R_x, and for each $\varepsilon > 0$, define $K_\varepsilon = \bigcup_{k=1}^{\infty}\big(\bigcap_{n=k}^{\infty} S(T^n x, \rho_0 + \varepsilon)\big)$. Thus for each $\varepsilon > 0$, the sets $K_\varepsilon \cap C$ are nonempty and convex and so the property (R) of (X, d) implies that

$$K = \bigcap_{\varepsilon>0}\big(\overline{K}_\varepsilon \cap C\big) \neq \phi.$$

Now let $z \in K$, and let

$$\tau(z) = \limsup_{n\to\infty} d(z, T^n z).$$

Suppose $\rho_0 = 0$. Then clearly $T^i x \to z$ as $i \to \infty$. Let $\eta > 0$. By (3.2.10), $n > M$

$$\sup_{y\in C}[d(T^n z, T^n y) - d(z,y)] \le \frac{1}{3}\eta, \text{ where } \ n > M.$$

As $T^i x \to z$ so there exists $m > n$ such that $d(T^m x, z) \le \frac{1}{3}\eta$ and $d(T^{m-n}x, z) \le \frac{1}{3}\eta$. Thus if $n \ge M$, then we get

$$\begin{aligned} d(z, T^n z) &\le d(z, T^m x) + d(T^m x, T^n z) \\ &\le d(z, T^m x) + d(T^n z, T^n(T^{m-n}x)) - d(z, T^{m-n}x) + d(z, T^{m-n}x) \\ &\le \frac{1}{3}\eta + \sup_{y\in C}[d(T^n z, T^n y) - d(z,y)] + \frac{1}{3}\eta \\ &= \eta. \end{aligned}$$

This proves that $T^i z \to z$ as $i \to \infty$, that is, $\tau(z) = 0$. But $\tau(z) = 0$ implies $T^{Nn} z \to z$ as $n \to \infty$ and the continuity of T^N yields $T^N z = z$. Thus

$$Tz = T(T^{Nn})z = T^{Nn+1}z \to z \text{ as } n \to \infty, \tag{3.2.11}$$

and $Tz = z$. Therefore, we may assume $\rho_0 > 0$ and $\tau(z) > 0$ (In fact, we may assume this for any $x, z \in C$.)

Now let $\varepsilon > 0,\ \varepsilon \le \tau(z)$. By the definition of ρ_0, there exists an integer N^* such that for $n \ge N^*$ we have

$$d(z, T^n x) \le \rho_0 + \varepsilon.$$

By (3.2.10), there exists N^{**} such that for $n \geq N^{**}$, we have

$$\sup_{y \in C}[d(T^n z, T^n y) - d(z, y)] \leq \varepsilon.$$

Select j so that $j \geq N^{**}$ and hence

$$d(z, T^j z) \geq \tau(z) - \varepsilon.$$

Thus if $n - j \geq N^*$, then we have

$$\begin{aligned} d(T^j z, T^n x) &= \{d(T^j z, T^j(T^{n-j}x)) - d(z, T^{n-j}x)\} + d(z, T^{n-j}x) \\ &= \varepsilon + (\rho_0 + \varepsilon) \\ &= \rho_0 + 2\varepsilon. \end{aligned}$$

For $m = \frac{1}{2}z \oplus \frac{1}{2}T^j z$, we have by uniform convexity of (X, d),

$$d(m, T^n x) \leq \big(1 - \delta\big(\rho_0 + 2\varepsilon, \frac{\tau(z) - \varepsilon}{\rho_0 + 2\varepsilon}\big)\big)(\rho_0 + 2\varepsilon),\ n \geq N^* + j.$$

By the minimality of ρ_0, this implies

$$\rho_0 \leq \big(1 - \delta\big(\rho_0 + 2\varepsilon, \frac{\tau(z) - \varepsilon}{\rho_0 + 2\varepsilon}\big)\big)(\rho_0 + 2\varepsilon);$$

letting $\varepsilon \to 0$,

$$\rho_0 \leq \left(1 - \delta\left(\rho_0, \frac{\tau(z)}{\rho_0}\right)\right)\rho_0.$$

So $\left(1 - \delta\left(\rho_0, \frac{\tau(z)}{\rho_0}\right)\right) \geq 1$ and hence $\delta\left(\rho_0, \frac{\tau(z)}{\rho_0}\right) = 0$; this implies that $\tau(z) = 0$. Hence as shown before in (3.2.11), $Tz = z$.

∎

Corollary 3.2.1 *Let (X, d) be a complete uniformly convex hyperbolic space and $C \subset X$ be nonempty, bounded, closed and convex. Suppose $T : C \to C$ is an asymptotically nonexpansive. Then T has a fixed point in C.*

Theorem 3.2.4 shows that $F(T)$, set of fixed points of T is not empty. The next theorem illustrates the structure of set $F(T)$.

Theorem 3.2.5 *Under the assumptions of Theorem 3.2.4, $F(T)$ is closed and convex.*

Proof. For the closeness of $F(T)$, let $\{x_n\} \subset F(T)$ be such that $x_n \to x$. Then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^N x_n = T^N \lim_{n\to\infty} x_n = T^N x$. Hence as shown before in (3.2.11), $Tx = x$.

To show convexity, it is sufficient to prove that $z = \frac{1}{2}x \oplus \frac{1}{2}y \in F(T)$ for all $x, y \in F(T)$. We have

$$\limsup_{i\to\infty} d(T^i z, x) = \limsup_{i\to\infty} d(T^i z, T^i x) \leq d(z, x) = \frac{1}{2}d(x, y),$$

$$\limsup_{i\to\infty} d(T^i z, y) = \limsup_{i\to\infty} d(T^i z, T^i y) \leq d(z, y) = \frac{1}{2}d(x, y).$$

Thus

$$\limsup_{i\to\infty} d(T^i z, z) \le \frac{1}{2}\big(1-\delta\big(\frac{1}{2}d(x,y),2\big)\big)d(x,y)$$

and hence

$$z = \lim_{i\to\infty} T^i z = \lim_{i\to\infty} T^{i+N} z = T^N \lim_{i\to\infty} T^i z = T^N z.$$

Once again, as in (3.2.11), $Tz = z$.

∎

3.3 Fixed Points of Monotone Mappings

In this section, iterative construction of fixed points of monotone multivalued nonexpansive mappings defined on a Banach space is introduced. This result generalizes a fixed point theorem for multivalued nonexpansive mappings proved by Lami Dozo [79]. Moreover, we examine the existence of fixed points of G-monotone nonexpansive mappings. Our main result sets analogue of Browder and Göhde's fixed point theorem for monotone nonexpansive mappings. This serves as a bridge between graph theory and metric fixed point theory.

3.3.1 Monotone multivalued nonexpansive mappings

Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued mappings. The earliest fundamental results in this direction were established by Markin [87] in a Hilbert space setting and by Browder [21] for spaces having a weakly continuous duality mapping. Lami Dozo [79] generalized these results to a Banach space satisfying Opial condition. By using Edelsteins method of asymptotic centers, Lim [82] obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space. Kirk and Massa [69] gave an extension of Lim's theorem to prove the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed, bounded and convex subset is nonempty and compact.

Let $\preceq$ be a partial order on a Banach space $(E, ||.||)$ Recall that $\preceq$ satisfies:

(i) $x \preceq x$ for all $x \in E$;

(ii) $x \preceq y$ and $y \preceq x \Rightarrow x = y$;

(iii) $x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$.

Assume that we have a partial order $\preceq$ defined on E such that order intervals are convex and τ-closed, where τ is a Hausdorff topology on E. Recall that an order interval is any of the subsets $[a, b] = \{x \in E; a \preceq x \preceq b\}$, $[a, \rightarrow) = \{x \in E; a \preceq x\}$, $(\leftarrow, a] = \{x \in E; x \preceq a\}$, for any $a, b \in E$.

Definition 3.3.1 *[8] Let C be a nonempty subset of E. Let $T : C \rightarrow C$ be a mapping.*

(1) T is said to be monotone if $T(x) \preceq T(y)$ whenever $x \preceq y$ for any $x, y \in C$.

(2) T is said to be monotone nonexpansive if and only if T is monotone and

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad whenever \ \ x \preceq y.$$

Husain and Tarafdar [49] gave a definition of nonexpansive multivalued mappings which, in singlevalued case, coincides with the usual definition of nonexpansive mappings.

Definition 3.3.2 *[49] Let C be a subset of a metric space (M, d). A multivalued mapping $T : C \to 2^C$ (nonempty subsets of C) is said to be nonexpansive if for any $x, y \in C$ and any $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq d(x, y).$$

Next, we define the concept of monotone nonexpansive multivalued mappings in a partially ordered metric space which, in singlevalued case, coincides with the definition of monotone nonexpansive mappings. The definition of monotone multivalued mappings has roots in [57].

Definition 3.3.3 *Let $(M, d, \preceq)$ be a metric space endowed with a partial order and C a nonempty subset of M. A multivalued mapping $T : C \to 2^C$ is said to be monotone increasing (resp. decreasing) nonexpansive if for any $x, y \in C$ with $x \preceq y$ and any $u \in T(x)$, there exists $v \in T(y)$ such that*

$$u \preceq v \ (resp.\ v \preceq u) \quad and \quad d(u, v) \leq \ d(x, y).$$

For a multivalued mapping T, x is a fixed point if and only if $x \in T(x)$. The set of all fixed points of a mapping T is denoted by $F(T)$.

Throughout this section, we assume that C is a convex and bounded subset (not reducible to one point) of a Banach space E. Let $\mathcal{K}(C)$ denote the set of all nonempty compact subsets of C. Let $T : C \to \mathcal{K}(C)$ be a monotone increasing multivalued nonexpansive mapping such that $C_T := \{x \in C;\ x \preceq y\ for\ some\ y \in T(x)\}$ is not empty.

Fix $\lambda \in (0,1)$ and $x_0 \in C_T$. Under the above assumptions, there exists $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. Set $x_1 = \lambda x_0 + (1-\lambda)y_0$. Since order intervals are convex, we have $x_0 \preceq x_1 \preceq y_0$. Since T is monotone increasing multivalued nonexpansive mapping, therefore there is $y_1 \in T(x_1)$ such that $y_0 \preceq y_1$ and $||y_1 - y_0||) \leq ||x_1 - x_0||$. Continuing in this manner we get the Krasnoselskii-Ishikawa [52, 76] iteration sequence $\{x_n\}$ in C defined by

$$x_{n+1} = \lambda x_n + (1-\lambda)y_n, \quad n \geq 0. \tag{KIS}$$

By induction, we obtain that

$$x_n \preceq x_{n+1} \preceq y_n \preceq y_{n+1}$$

and

$$||y_{n+1} - y_n|| \leq ||x_{n+1} - x_n||,$$

for any $n \geq 0$.

In order to proceed further, we will need the following fundamental result. Its origin may be found in [43, 44].

Proposition 3.3.1 *Under the above assumptions, we have*

$$(GK) \quad (1+n\lambda)\,||y_i - x_i|| \;\leq\; ||y_{i+n} - x_i|| + (1-\lambda)^{-n}\Big(||y_i - x_i|| - ||y_{i+n} - x_{i+n}||\Big),$$

for any $i, n \in \mathbb{N}$. This inequality implies

$$\lim_{n \to +\infty} \|x_n - y_n\| = 0.$$

Proof. The first part of this proposition is easy to prove via an induction argument on the index i. As for the second part, note that $\{\|x_n - y_n\|\}$ is decreasing. Indeed, we have $x_{n+1} - x_n = (1-\lambda)(y_n - x_n)$, for any $n \geq 1$. Therefore, $\{\|x_n - y_n\|\}$ is decreasing if and only if $\{\|x_{n+1} - x_n\|\}$ is decreasing which holds in view of

$$\|x_{n+2} - x_{n+1}\| \leq \lambda\|x_{n+1} - x_n\| + (1-\lambda)\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|,$$

for any $n \geq 0$. Set $\lim\limits_{n \to +\infty} \|x_n - y_n\| = R$. Then we let $i \to +\infty$ in the inequality (GK) to obtain

$$(1+n\lambda)R \leq \delta(C),$$

for any $n \in \mathbb{N}$, where $\delta(C) = \sup\{\|x-y\|,\ x, y \in C\} < +\infty$. Hence

$$R \leq \frac{\delta(C)}{(1+n\lambda)},\ n = 1, 2, \cdots$$

which implies $R = 0$, i.e., $\lim\limits_{n \to +\infty} \|x_n - y_n\| = 0$. In particular, we have

$\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. i.e., T has an approximate fixed point sequence $\{x_n\} \in C$.

∎

Before we state the main result of this section, let us recall the definition of Opial condition [92].

Definition 3.3.4 *A Banach space E is said to satisfy the τ-Opial condition if whenever any sequence $\{y_n\}$ in E which τ-converges to y, then we have*

$$\limsup_{n\to+\infty} \|y_n - y\| < \limsup_{n\to+\infty} \|y_n - z\|,$$

for any $z \in E$ with $z \neq y$.

Now we are ready to state the main result of this section.

Theorem 3.3.1 *Let E be a Banach space. Let τ be a topology on E such that E satisfies the τ-Opial condition. Let $\preceq$ be a partial order on E such that order intervals are convex and τ-closed. Let C be a bounded convex τ-compact nonempty subset of E. Set $\mathcal{K}(C)$ to be the set of all nonempty compact subsets of C. Let $T : C \to \mathcal{K}(C)$ be a monotone increasing multivalued nonexpansive mapping. If $C_T := \{x \in C;\ x \preceq y \ for\ some\ y \in T(x)\}$ is not empty, then T has a fixed point.*

Proof. Fix $\lambda \in (0, 1)$ and $x_0 \in C_T$. Consider the sequence $\{x_n\}$ defined in (KIS) which starts at x_0. Since C is τ-compact, therefore $\{x_n\}$ will have a subsequence $\{x_{\phi(n)}\}$ which τ-converges to some point $w \in C$. Since order intervals are τ-closed and convex, we conclude that $x_n \preceq w$, for any $n \geq 0$. Indeed, fix $k \geq 0$. The order interval $[x_k, \rightarrow)$ contains all the elements from the sequence $\{x_{\phi(n)}\}$ except finitely many. As the order intervals are τ-closed, so we conclude that $w \in [x_k, \rightarrow)$, for any $k \geq 0$. Let z be the τ-limit of another subsequence of $\{x_n\}$. Therefore, we must have $z \preceq w$. By reversing the roles of w and z, we get $w \preceq z$. The properties of the partial order will force $w = z$ which implies that $\{x_n\}$ τ-converges to w and $x_n \preceq w$, for any $n \in \mathbb{N}$. Consider the type function

$$r(x) = \limsup_{n \to +\infty} \|x_n - x\|, \quad x \in C.$$

Now Proposition 3.3.1 implies $r(x) = \limsup_{n\to+\infty} \|y_n - x\|$, for any $x \in C$. As T is monotone increasing multivalued nonexpansive mapping, so there exists $w_n \in T(w)$ such that $y_n \preceq w_n$ and $||y_n - w_n|| \leq ||x_n - w||$, for any n. Since $T(w)$ is compact, there exists a subsequence $\{w_{\phi(n)}\}$ of $\{w_n\}$ such that $w_{\phi(n)} \to v \in T(w)$. It follows that

$$\begin{aligned}
\limsup_{n\to+\infty} \|x_{\phi(n)} - v\| &= \limsup_{n\to+\infty} \|y_{\phi(n)} - v\| \\
&\leq \limsup_{n\to+\infty} \|y_{\phi(n)} - w_{\phi(n)}\| + \limsup_{n\to+\infty} \|w_{\phi(n)} - v\| \\
&= \limsup_{n\to+\infty} \|y_{\phi(n)} - w_{\phi(n)}\| \\
&\leq \limsup_{n\to+\infty} \|x_{\phi(n)} - w\|.
\end{aligned}$$

Finally, E satisfies the τ-Opial condition, so we must have $w = v \in T(w)$, i.e., w is a fixed point of T. ∎

The following results are direct consequences of Theorem 3.3.1.

Corollary 3.3.1 *Let C be a nonempty, bounded, closed and convex subset of l_p, $1 < p < +\infty$. Let τ be the weak topology. Consider the pointwise partial ordering in l_p, i.e. $\{\alpha_n\} \preceq \{\beta_n\}$ iff $\alpha_n \leq \beta_n$, for any $n \geq 1$. Then any monotone increasing multivalued nonexpansive mapping $T : C \to \mathcal{K}(C)$ has a fixed point provided $C_T := \{x \in C;\ x \preceq y\ for\ some\ y \in T(x)\}$ is not empty.*

Remark 3.3.1 *The case of $p = 1$ is not interesting for the weak-topology since l_1 is a Schur Banach space. Recall that a bounded sequence $\{u^k\}$ in ℓ_1 converges weakly to $u \in \ell_1$ if and only if $\lim\limits_{k\to\infty} \left\|u^k - u\right\| = 0$. Spaces which have this property are said to have the Schur property. Thus the weakly compact subsets of such spaces coincide with the norm compact subsets.*

But if we consider the weak-topology $\sigma(l_1, c_0)$ on l_1, then l_1 satisfies the Opial condition [8]. In this case, we have a conclusion similar to Corollary 3.3.1 for l_1.*

3.3.2 G-monotone nonexpansive mappings

We start with the basic definitions and properties of graph theory.

A graph G is a nonempty set $V(G)$ of elements called vertices together with a possibly empty subset $E(G)$ of $V(G) \times V(G)$ called edges. We assume that all graphs are reflexive, i.e., $(x, x) \in E(G)$ for each $x \in V(G)$. Moreover, we assume that there exists a distance function d defined on the set of vertices $V(G)$. We could treat G as a weighted graph by giving each edge the metric distance between its vertices. The conversion graph G^{-1} is obtained by reversing the direction of

$E(G)$. In this case, we have

$$E(G^{-1}) = \{(y, x) \mid (x, y) \in E(G)\}.$$

An oriented graph G is a digraph provide $(x, y) \in E(G)$, implies that $(y, x) \notin E(G)$. The graph $\widetilde{G}$ is obtained from G by removing the direction of edges, i.e., $(x, y) \in \widetilde{G}$ if $(x, y) \in G$ or $(y, x) \in G$.

Let x and y be in $V(G)$. A (directed) path from x to y is a finite sequence $(x_i)_{i=1}^{i=N}$ of vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. In this case, the length of the path $(x_i)_{i=1}^{i=N}$ is $N + 1$. The graph G is said to be connected if there exists a path between any two vertices. The graph G is said to be weakly connected if $\widetilde{G}$ is connected.

Definition 3.3.5 *The graph G is said to be transitive whenever $(x, z) \in E(G)$ provided $(x, y) \in E(G)$ and $(y, z) \in E(G)$, for any $x, y, z \in V(G)$. In other words, G is transitive if for any two vertices x and y that are connected by a directed finite path, we have $(x, y) \in E(G)$.*

The following lemma is needed.

Lemma 3.3.1 *[25] Let C be a nonempty, closed and convex subset of a uniformly convex hyperbolic space (X, d). Let $\tau : C \to [0, +\infty)$ be a type function, i.e., there* *exists a bounded sequence* $\{x_n\} \in X$ *such that*

$$\tau(x) = \limsup_{n \to +\infty} d(x_n, x),$$

for any $x \in C$*. Then* τ *is continuous. Since* X *is hyperbolic, therefore* τ *is convex, i.e., the subset* $\{x \in C;\ \tau(x) \leq r\}$ *is convex for any* $r \geq 0$*. Moreover, there exists a unique minimum point* $z \in C$ *such that*

$$\tau(z) = \inf\{\tau(x);\ x \in C\}.$$

Throughout this section, we assume that (X, d) is a hyperbolic space endowed with a graph G. Let C be a nonempty, closed, convex and bounded subset of X not reducible to one point. Assume that G is transitive and G-intervals are convex and closed, G-intervals are any of the subsets $[a, \rightarrow) = \{x \in C; (a, x) \in E(G)\}$ and $(\leftarrow, b] = \{x \in C; (x, b) \in E(G)\}$, for any $a, b \in C$.

Definition 3.3.6 *Let* C *be a nonempty subset of* X*. A mapping* $T : C \to C$ *is called*

(i) *G-monotone if for any* $x, y \in C$ *such that* $(x, y) \in E(G)$*, we have* $(T(x), T(y)) \in E(G)$*.*

(ii) *G-monotone nonexpansive if* T *is G-monotone and*

$$d(T(x), T(y)) \leq d(x, y),$$

for any $x, y \in C$ such that $(x, y) \in E(G)$.

Let $T : C \to C$ be G-monotone nonexpansive mapping. Fix $\lambda \in (0, 1)$ and $x_0 \in C$. Define the Krasnoselskii-Ishikawa iteration sequence [52, 76] sequence $\{x_n\}$ in C by

$$x_{n+1} = (1 - \lambda)x_n \oplus \lambda T(x_n), \quad n \geq 0. \tag{KIS}$$

Assume that $(x_0, T(x_0)) \in E(G)$. Since G-intervals are convex and T is G-monotone, we have $(x_0, x_1), (x_1, T(x_0)), (T(x_0), T(x_1)) \in E(G)$. By induction, we have

$$(x_n, x_{n+1}), (x_{n+1}, T(x_n), (T(x_n), T(x_{n+1})) \in E(G),$$

for any $n \geq 1$, which implies, (by G-monotone nonexpansiveness of T),

$$d(T(x_{n+1}), T(x_n)) \leq d(x_{n+1}, x_n).$$

In order to proceed further, we need the following fundamental result [43, 44].

Proposition 3.3.2 *under the above assumptions, we have*

$$(GK) \quad (1 + n\lambda)\, d(T(x_i), x_i) \;\leq\; d(T(x_{i+n}), x_i) + (1 - \lambda)^{-n}\Big(d(T(x_i), x_i) - d(T(x_{i+n}), x_{i+n})\Big),$$

for any $i, n \in \mathbb{N}$. This inequality implies

$$\lim_{n \to +\infty} d(x_n, T(x_n)) = 0,$$

i.e., $\{x_n\}$ *is an approximate fixed point sequence of* T*.*

Here is the main theorem of this section.

Theorem 3.3.2 *Let the triplet* (X, d, G) *be as described above. Assume that* (X, d) *is uniformly convex hyperbolic space. Let* C *be a nonempty, closed, convex and bounded subset of* X *not reducible to one point. Let* $T : C \to C$ *be a G-monotone nonexpansive mapping. Then* T *has a fixed point provided there exists* $x_0 \in C$ *such that* $(x_0, T(x_0)) \in E(G)$*.*

Proof. Consider the Krasnoselskii-Ishikawa sequence $\{x_n\}$ generated by (KIS) starting at x_0 with $\lambda \in (0, 1)$. Using the properties of $\{x_n\}$ and the transitivity of G, the subsets $[x_n, \to)$, $n \geq 0$, are nonempty, non-increasing, convex and closed. Since X is uniformly convex hyperbolic space, Property (R) implies that

$$C_\infty = \bigcap_{n\geq 0}[x_n, \to) \cap C = \bigcap_{n\geq 0}\{x \in C;\ (x_n, x) \in E(G)\} \neq \emptyset.$$

Let $x \in C_\infty$. Then $(x_n, x) \in E(G)$ for any $n \geq 0$. Since T is G-monotone, we have $(T(x_n), T(x)) \in E(G)$. As $(x_{n+1}, T(x_n)) \in E(G)$, so by transitivity of G, we get $(x_{n+1}, T(x)) \in E(G)$ for any $n \geq 0$, i.e., $T(C_\infty) \subset C_\infty$. Consider the type function $\tau : C_\infty \to [0, +\infty)$ generated by $\{x_n\}$, i.e., $\tau(x) = \limsup_{n\to+\infty} d(x_n, x)$. Since $\lim_{n\to+\infty} d(x_n, T(x_n)) = 0$, we get $\tau(x) = \limsup_{n\to+\infty} d(T(x_n), x)$, for any $x \in C_\infty$. Lemma 3.3.1 implies the existence of a unique $z \in C_\infty$ such that $\tau(z) = \inf\{\tau(x);\ x \in C_\infty\}$. Since $z \in C_\infty$, we have $(x_n, z) \in E(G)$, for any $n \geq 1$, which implies

$$\tau(T(z)) = \limsup_{n\to+\infty} d(T(x_n), T(z)) \leq \limsup_{n\to+\infty} d(x_n, z) = \tau(z).$$

The uniqueness of the minimum point implies that $z = T(z)$, i.e., z is a fixed point of T. ∎

Next, we show how to weaken uniform convex property when we assume that X is a linear space.

Let $(E, \|.\|)$ be a Banach space. We say that E is uniformly convex in the direction $z \in E$, with $\|z\| = 1$, if $\delta(\varepsilon, z) > 0$, where

$$\delta(\varepsilon, z) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|; \|x\| \leq 1,\ \|y\| \leq 1,\ \ x - y = \alpha\ z,\ and\ \|x - y\| \geq \varepsilon\right\},$$

for any $\varepsilon \in (0, 2]$. Uniform convexity in every direction was introduced by Garkavi [40] in connection with his study of Chebyshev centers. Zizler [116] proved that any separable Banach space has an equivalent norm which is uniformly convex in every direction. It is also known that uniformly convex Banach spaces are super-reflexive [12] which shows that the class of uniformly convex spaces is much smaller than the class of uniformly convex in every direction.

The following lemma is an analogue of Lemma 3.3.1 in Banach spaces that are uniformly convex in every direction.

Lemma 3.3.2 *[25] Let $(E, \|.\|)$ be a Banach space which is uniformly convex in* *every direction. Let C be a nonempty, weakly compact and convex subset of X. Let $\tau : C \to [0, +\infty)$ be a type function. Then there exists a unique minimum point $z \in C$ such that*

$$\tau(z) = \inf\{\tau(x);\ x \in C\}.$$

The following proposition is an analogue of Proposition 3.3.2 of Banach spaces as they are hyperbolic spaces.

Proposition 3.3.3 *Let the triple $(E, \|.\|, G)$ be a Banach space endowed with a directed graph G. Let C be a nonempty, convex and bounded subset of E not reducible to one point such that $V(G) = C$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Fix $\lambda \in (0,1)$ and $x_0 \in C$ such that $(x_0, T(x_0)) \in E(G)$. Consider the sequence $\{x_n\}$ in C defined by (KIS). Hence*

$$(GK) \quad (1 + n\lambda)\,\|T(x_i) - x_i\| \ \leq\ \|T(x_{i+n}) - x_i\| + (1-\lambda)^{-n}\Big(\|T(x_i) - x_i\| - \|T(x_{i+n}) - x_{i+n}\|\Big),$$

for any $i, n \in \mathbb{N}$. Then we have $\lim\limits_{n\to+\infty} \|x_n - T(x_n)\| = 0$, i.e., $\{x_n\}$ is an approximate fixed point sequence of T.

Definition 3.3.7 *[3] The triple $(E, \|.\|, G)$ has property (P) if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in C such that $(x_n, x_{n+1}) \in E(G)$, for any $n \geq 0$, and if a subsequence $\{x_{k_n}\}$ converges weakly to x, then $(x_{k_n}, x) \in E(G)$, for all n.*

Lemma 3.3.2 in conjunction with the ideas of the proof of Theorem 3.3.2, we get the following fixed point result.

Theorem 3.3.3 *Let $(E, \|.\|, G)$ be a Banach space endowed with a directed reflexive and transitive graph G such that Property (P) is satisfied. Assume that E is uniformly convex in every direction. Let C be a nonempty weakly compact and convex subset of E. Assume that G-intervals are convex and closed. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Then T has a fixed point provided $X_T = \{x \in C; (x, T(x)) \in E(G)\} \neq \emptyset$.*

Let us finish this section with the following two examples.

Example 3.3.1 *Consider the Hilbert space ℓ_2 defined by*

$$\ell_2 = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} |x_n|^2 < +\infty\Big\}.$$

Define the digraph G on ℓ_2 by:

$$(x, y) \in E(G) \quad \text{if and only if} \quad x_n \leq y_n, \ \ n \geq 2,$$

where $x = (x_n)$ and $y = (y_n)$ are in ℓ_2. Then G is transitive. Moreover, it is easy to check that G-intervals are convex and closed. Consider

$$x = (1, 0, 0, \cdots) \ \text{ and } \ y = (2, 0, 0, \cdots) \in \ell_2.$$

Then, we have $(x, y) \in E(G)$ and $(y, x) \in E(G)$, i.e., G contains a cycle. Therefore, the graph G will not be generated by a partial order. This example suggests *that the idea of replacing partial order by a graph, is worth consideration.*

Example 3.3.2 *[110] Let C be the closed unit ball of the space l_1 with the norm $\|\{x_k\}\| = \sum_k |x_k|$. Let $G = (C, E(G))$ be the graph on C defined by*

$$E(G) = \{(x_k, y_k) \ : \ |x_k| + |y_k| \leq 1 \text{ and } \|\{x_k\} - \{y_k\}\| \leq \frac{3}{8}\}.$$

It is easy to show that $E(G)$ is convex. Define $T : C \to C$ by

$$T(\{x_k\}) = \{x_k^2\}, \ \ \{x_k\} \in C$$

We can easily check that T is G-monotone nonexpansive. However, it is not nonexpansive because $\|Tx - Ty\| > \|x - y\|$ where $\{x\} = \{\frac{1}{2}, 0, 0, \dots\}$ and $\{y\} = \{1, 0, 0, \dots\}$.

3.4 Metric Characterization of Reflexivity

In this section, we prove that a Banach space is reflexive and strictly convex if and only if it satisfies the metric property known as P-property. We also discuss this characterization in hyperbolic spaces. As an application, we obtain an extension of the Banach Contraction Principle for best proximity points. The case of nonexpansive mappings is also discussed.

3.4.1 Introduction

If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for this equation. In other words, we search for an element in the domain of the mapping, whose image is as close to it as possible. This situation motivates to develop "best proximity point theory" (see, [1, 33, 34, 35, 36, 70, 74, 97]). The best proximity point theorems can be viewed as a generalization of fixed point theorems, since most of the fixed point theorems can be derived as corollaries of best proximity point theorems.

Definition 3.4.1 *Let (M, d) be a metric space. Let A and B be nonempty subsets of M. Let $T : A \to B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if*

$$d(x, Tx) = d(A, B) = \inf\{d(a, b);\ a \in A,\ b \in B\}.$$

Note that if $A \cap B \neq \emptyset$, then x is a best proximity point of T if $T(x) = x$, i.e., x is a fixed point of T.

The proximity pair associated with the pair (A, B), denoted by (A_0, B_0), is defined by

$$A_0 = \{x \in A : d(x, y) = d(A, B);\ \text{for some } y \in B\},$$

and

$$B_0 = \{y \in B : d(x, y) = d(A, B);\ \text{for some } x \in A\}.$$

It is clear that A_0 is nonempty if and only if B_0 is so.

Recently, Raj [97] introduced the so-called P-property. Using this property, some best proximity point results were proved for various classes of non-self mappings in Banach and metric spaces [1, 97].

Definition 3.4.2 *[97] A pair (A, B) of nonempty subsets of a metric space (M, d), with $A_0 \neq \emptyset$, is said to have the P-property if and only if*

$$\left.\begin{array}{l} d(a,b) = d(A,B) \\ d(x,y) = d(A,B) \end{array}\right\} \Longrightarrow d(a,x) = d(b,y),$$

whenever $a, x \in A_0$ and $b, y \in B_0$.

We prove the following technical result.

Lemma 3.4.1 *Let (X, d) be a hyperbolic space. If X satisfies the property (R), then any nonempty, closed and convex subset A of X is proximinal. Moreover, if we assume that X is strictly convex, then A is a Chebyshev subset.*

Proof. Assume that (X, d) is a hyperbolic space which satisfies the property (R). Let A be a nonempty, closed and convex subset of X. In order to prove that A is proximinal, let $x \in X$. Set

$$A_n = \left\{a \in A;\ d(x,a) \leq d(x,A) + \frac{1}{n}\right\},$$

for any $n \geq 1$. It is obvious that A_n is a nonempty, closed, convex and bounded subset of A, for any $n \geq 1$. Moreover, the sequence $\{A_n\}$ is decreasing. Using the property (R) in X, we conclude that $P_A(x) = \bigcap\limits_{n\geq 1} A_n \neq \emptyset$. Hence A is proximinal. Next, we assume that X is strictly convex. Let A be a nonempty, closed and convex subset of X. Let us prove that $P_A(x)$ is a singleton for any $x \in X$. We have already proved that $P_A(x)$ is not empty. Let $a_1, a_2 \in P_A(x)$. Let us prove that $a_1 = a_2$. Since A is convex, we have $\alpha a_1 \oplus (1-\alpha)a_2 \in A$, for any $\alpha \in [0,1]$. As (X,d) is hyperbolic, so we have

$$d(x,A) \leq d(x, \alpha a_1 \oplus (1-\alpha)a_2) \leq \alpha\ d(x,a_1) + (1-\alpha)d(x,a_2) = d(x,A),$$

for any $\alpha \in [0,1]$. Hence

$$d(x, \alpha a_1 \oplus (1-\alpha)a_2) = d(x,a_1) = d(x,a_2),$$

for any $\alpha \in [0,1]$. Since X is strictly convex, therefore we conclude that $a_1 = a_2$.

∎

3.4.2 Characterization of reflexive strictly convex Banach spaces

Let us start with the following well known result.

Lemma 3.4.2 *Let $(E, \|.\|)$ be a Banach space.*

(1) E is reflexive if and only if any nonempty, closed and convex subset C of E is proximinal [16].

(2) If E is strictly convex, then $P_C(x)$ is either empty or a singleton, for each $x \in E$ and C a nonempty, closed and convex subset of X.

In particular, if E is strictly convex and reflexive, then any nonempty, closed and convex subset C of E is a Chebyshev subset.

Proof. (1). Assume that E is reflexive. Then E satisfies the property (R). Lemma 3.4.1 implies that any nonempty, closed and convex subset of E is proximinal. Conversely, assume that any nonempty, closed and convex subset C of E is proximinal. Let us prove that E is reflexive. Let $x^* \in E^*$, the dual space of E, such that $\|x^*\| = 1$. Set $C = \{c \in E;\ x^*(c) = 0\}$. Then C is a closed subspace of E. Let $x \in E$ be such that $x^*(x) \neq 0$. Since C is proximinal, $P_C(x) \neq \emptyset$. Pick $c_0 \in P_C(x)$. Then we have $\|x - c_0\| \leq \|x - c\|$, for any $c \in C$. Since C is a subspace, we may assume that $c_0 = 0$, i.e., $\|x\| \leq \|x - c\|$, for any $c \in C$. Let $z \in E$. Then

$$x^*\left(z - \frac{x^*(z)}{x^*(x)}x\right) = 0, i.e.,\ \ c = z - \frac{x^*(z)}{x^*(x)}x \in C.$$

Assume that $x^*(z) \neq 0$. Then

$$\frac{x^*(x)}{x^*(z)}z = x + \frac{x^*(x)}{x^*(z)}c,$$

which implies

$$\|x\| \leq \left\|x + \frac{x^*(x)}{x^*(z)}c\right\| = \left\|\frac{x^*(x)}{x^*(z)}z\right\|.$$

Hence

$$\frac{|x^*(z)|}{\|z\|} \leq \frac{|x^*(x)|}{\|x\|},$$

for any $z \in E$ such that $x^*(z) \neq 0$. Clearly, this will imply

$$\sup\Big\{|x^*(z)|;\ z \in E \ \textit{such that}\ \|z\| = 1\Big\} = \frac{|x^*(x)|}{\|x\|}.$$

In other words, x^* achieves its maximum at a point in the unit sphere of E. James's characterization of reflexivity [54] implies that E is reflexive.

As for (2), the proof of Lemma 3.4.1 implies that if E is strictly convex, then for any nonempty, closed and convex subset C of E, $P_C(x)$ is either empty or a singleton, for any $x \in E$. ▮

Now let us state a new characterization of reflexive strictly convex Banach spaces.

Theorem 3.4.1 *A Banach space* $(E, \|.\|)$ *is reflexive and strictly convex if and only if every pair* (A, B) *of nonempty, bounded, closed and convex subsets of* E *has the P-property.*

Proof. Let E be a reflexive strictly convex Banach space and let A and B be nonempty, closed and convex subsets of E. By Lemma 3.4.2, B is a Chebyshev subset. Let P_B be the nearest point projection onto B. Consider the set

$$A_n = \Big\{x \in A;\ d(x, B) = ||x - P_B(x)|| \leq d(A, B) + \frac{1}{n}\Big\},$$

for any $n \geq 1$. From the definition of $d(A, B)$ and continuity and convexity of the function $x \to d(x, B)$, we know that A_n is a nonempty, bounded, closed and convex subset of A, for any $n \geq 1$. Obviously, $\{A_n\}$ is decreasing. Using the reflexivity of E, we conclude that $A_\infty = \bigcap_{n\geq 1} A_n \neq \emptyset$. Let $u \in A_\infty$. Hence

$$d(u, B) = \|u - P_B(u)\| \leq d(A, B) + \frac{1}{n},$$

for any $n \geq 1$, which implies that $d(u, B) = \|u - P_B(u)\| \leq d(A, B)$. By definition of $d(A, B)$, we have $d(A, B) \leq \|u - P_B(u)\|$, so we get $\|u - P_B(u)\| = d(A, B)$, i.e., $u \in A_0$ and $P_B(u) \in B_0$. Therefore, A_0 and B_0 are nonempty. In order to show that (A, B) has the P-property, let $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ such that

$$\|x_1 - y_1\| = \|x_2 - y_2\| = d(A, B).$$

Assume that $x_1 - y_1 \neq x_2 - y_2$. Using the strict convexity of E and the convexity of A and B, we get

$$\left\|\frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2}\right\| = \left\|\frac{x_1 - y_1}{2} + \frac{x_2 - y_2}{2}\right\| < d(A, B),$$

which is a contradiction. Hence $x_1 - y_1 = x_2 - y_2$ which implies $\|x_1 - x_2\| = \|x_1 - x_2\|$. Therefore, the pair (A, B) has the P-property. Conversely, assume that any pair (A, B) of nonempty, bounded, closed and convex subsets of E has the P-property. Let us prove that E is reflexive and strictly convex. Let A be a nonempty, bounded, closed and convex subset of E. For any $b \notin A$, set $B = \{b\}$. As the pair (A, B) has the P-property, so $A_0 \neq \emptyset$, i.e., A is proximinal. Therefore, Lemma 3.4.2 implies that E is reflexive. Finally, assume that E is not strictly convex. Then, there exist $x, y \in E$ such that $x \neq y$ and $\|\alpha\, x + (1-\alpha)y\| = 1$, for any $\alpha \in [0,1]$. Set $A = [x, y]$ and $B = \{0\}$. It is clear that (A, B) is a pair of nonempty, bounded, closed and convex subsets of E, with $d(A, B) = 1$. Hence $x, y \in A_0$ and $B_0 = \{0\}$. By $x \neq y$, we have $\|x - y\| \neq \|0 - 0\| = 0$. Thus the pair (A, B) fails to have the P-property. This contradiction implies that E is strictly convex. ∎

Next, we seek extension of Theorem 3.4.1 to the case of hyperbolic spaces.

Theorem 3.4.2 *Let (X, d) be a hyperbolic space. Assume that X satisfies the property (R) and is strictly convex. Let (A, B) be a pair of two nonempty, bounded, closed and convex subsets of X. If P_A and P_B are nonexpansive mappings, then (A, B) has the P-property.*

Proof. First, we prove that $A_0 \neq \emptyset$. Note that the function $x \to d(x, B)$ is continuous and convex. As before, in the proof of Theorem 3.4.1, consider the set

$$A_n = \left\{x \in A;\ d(x, B) \leq d(A, B) + \frac{1}{n}\right\},$$

for any $n \geq 1$. From the definition of $d(A, B)$, we know that A_n is a nonempty, bounded, closed and convex subset of A, for any $n \geq 1$. Obviously, $\{A_n\}$ is decreasing. Using the property (R) satisfied by X, we conclude that $A_\infty = \bigcap_{n\geq 1} A_n \neq \emptyset$. Let $u \in A_\infty$. Hence

$$d(u, B) \leq d(A, B) + \frac{1}{n},$$

for any $n \geq 1$, which implies that $d(u, B) \leq d(A, B)$. By definition of $d(A, B)$, we have $d(A, B) \leq d(u, B)$, and so we get $d(u, B) = d(A, B)$, i.e., $u \in A_0$. Therefore, A_0 is nonempty. Similarly, we will prove that B_0 is also nonempty. Next, we show that (A, B) has the P-property. Let $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ be such that

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B).$$

Let us prove that $d(x_1, x_2) = d(y_1, y_2)$. We claim that $P_B(x_i) = y_i$ and $P_A(y_i) = x_i$, for $i = 1, 2$. Indeed, we have

$$d(A, B) \leq d(x_1, P_B(x_1)) \leq d(x_1, y_1) = d(A, B).$$

Hence $d(x_1, P_B(x_1)) = d(x_1, y_1)$ which implies on the basis of B is a Chebyshev subset, $P_B(x_1) = y_1$. Similarly, we can prove that $P_B(x_2) = y_2$ and $P_A(y_i) = x_i$, for $i = 1, 2$. From our assumptions on A and B, we know that P_A and P_B are nonexpansive mappings. Hence

$$d(x_1, x_2) = d(P_A(y_1), P_A(y_2)) \leq d(y_1, y_2) = d(P_B(x_1), P_B(x_2)) \leq d(x_1, x_2),$$

which implies $d(x_1, x_2) = d(y_1, y_2)$. ∎

3.4.3 Applications to best proximity points

Let A and B be nonempty and closed subsets of a complete metric space (M, d). A mapping T is said to be a cyclic mapping on $A \cup B$ if $T(A) \subset B$ and $T(B) \subset A$. If T is a contraction, then $A \cap B \neq \emptyset$. Indeed, fix $a \in A$, therefore $T^{2n}(a) \in A$ and $T^{2n+1}(a) \in B$, for any $n \in \mathbb{N}$. Since T is a contraction, therefore $\{T^n(a)\}$ is Cauchy. Using the completeness of M, we conclude that $\{T^n(a)\}$ converges to $x \in A \cap B$ which is the unique fixed point of T.
Motivated by this, Eldred and Veeramani in [33] introduced the concept of cyclic contraction mappings and gave sufficient conditions for the existence of a unique point $x \in A$ such that $d(x, T(x)) = d(A, B)$.

Note that if $A \cap B \neq \emptyset$, then x is a best proximity point of T if $T(x) = x$, i.e., x is a fixed point of T. Assume that A_0 is a Chebyshev subset of M and $T(A_0) \subset B_0$. Then $x \in A$ is a best proximity point of T if and only if $P_{A_0}(T(x)) = x$, i.e., x is a fixed point of $P_{A_0} \circ T$. Indeed, let $x \in A$ be a best proximity point of T, i.e., $d(x, T(x)) = d(A, B)$. In particular, we have $x \in A_0$. As $T(A_0) \subset B_0$, so $T(x) \in B_0$. Since $d(T(x), x) = d(T(x), A_0)$, therefore we conclude that $x = P_{A_0}(T(x))$. Conversely, assume that x is a fixed point of $P_{A_0} \circ T$, i.e., $x = P_{A_0}(T(x))$. Then, we have $x \in A_0$ and $T(x) \in B_0$. Hence

$$d(x, T(x)) = d(P_{A_0}(T(x)), T(x)) = d(T(x), A_0) = d(A, B),$$

in view of the fact that $T(x) \in B_0$. Therefore, x is a best proximity point of T in

A.

Theorem 3.4.3 *Let (M,d) be a complete metric space. Let (A,B) be a pair of nonempty, bounded, and closed subsets of M. Assume that $A_0 \neq \emptyset$ and (A,B) has the P-property. Let $T : A \to B$ be a contraction mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point x in A.*

Proof. First, we prove the existence of a best proximity point of T in A. As $A_0 \neq \emptyset$, so we pick $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, therefore $Tx_0 \in B_0$. So, there exists an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$. Again, since $Tx_1 \in B_0$, there exists an element $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. By induction, we construct a sequence $\{x_n\}$ such that

(i) $x_n \in A_0$, for any $n \in \mathbb{N}$;

(ii) $d(x_{n+1}, T(x_n)) = d(A,B)$.

Since the pair (A,B) has the P-property, we have

$$\left.\begin{array}{l} d(x_{n+1}, T(x_n)) = d(A,B) \\ d(x_n, T(x_{n-1})) = d(A,B) \end{array}\right\} \Longrightarrow d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})),$$

for any $n \geq 1$. Since T is a contraction, there exists $k < 1$ such that

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k \ d(x_n, x_{n-1}),$$

for any $n \geq 1$, which implies that

$$d(x_{n+1}, x_n) \leq k^n \, d(x_1, x_0),$$

for any $n \geq 1$. Hence $\sum_{n \in \mathbb{N}} d(x_{n+1}, x_n)$ is convergent which implies that $\{x_n\}$ is Cauchy. Since M is complete, there exists $x \in M$ such that $\{x_n\}$ converges to x. As $\{x_n\} \subset A$ and A is closed, so we conclude that $x \in A$. Since T is a continuous mapping and B is closed, therefore we have $\{T(x_n)\}$ converges to $T(x) \in B$. From $d(x_{n+1}, Tx_n) = d(A, B)$, we get $d(x, T(x)) = d(A, B)$, i.e., $x \in A_0$ and $T(x) \in B_0$. Clearly, x is a best proximity point of T in A. Next, we prove that T has a unique best proximity point in A. Suppose that there exist $x, y \in A$ such that

$$d(x, T(x)) = d(y, T(y)) = d(A, B).$$

The pair (A, B) satisfies the P-property and T is a contraction mapping, so we get

$$d(x, y) = d(Tx, Ty) < d(x, y),$$

which implies $d(x, y) = 0$, i.e., $x = y$. Hence T has a unique best proximity point in A. ▌

In fact, Theorem 3.4.3 may be extended to hyperbolic spaces with the λ-property. We say that a hyperbolic space X has the λ-property if, for any nonempty, closed, convex and Chebyshev subset C of X, we have

$$d(P_C(x), P_C(y)) \leq \lambda\, d(x, y),$$

for any $x, y \in X$, $\lambda \geq 0$ i.e., P_C is Lipschitzian with λ as a Lipschitz constant.

Theorem 3.4.4 *Let (X, d) be a complete hyperbolic space. Assume that X has the λ-property. Let (A, B) be a pair of nonempty, bounded, closed, and convex subsets of X. Assume that $A_0 \neq \emptyset$. Let $T : A \to B$ be a Lipschitzian mapping with a Lipschitz constant k which satisfies $k\lambda < 1$. Assume that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point x in A.*

Proof. Consider the mapping $P_{A_0} \circ T : A_0 \to A_0$. Our assumption on the Lipschitz constant of T implies that $P_{A_0} \circ T$ is a contraction. Since A_0 is a nonempty and closed subset of a complete metric space, it is complete. Hence $P_{A_0} \circ T$ has a unique fixed point. Therefore, T has a unique best proximity point in A. ∎

CHAPTER 4

RESULTS IN $CAT(0)$ SPACES

An example of a linear hyperbolic space is normed space. Hadamard manifolds [24], the Hilbert open unit ball equipped with the hyperbolic [45], and $CAT(0)$ spaces [68] are examples of nonlinear hyperbolic spaces which play a major role in metric fixed point theory.

In this chapter, we continue study of fixed point theory of nonexpansive mappings in a special class of nonlinear hyperbolic spaces, namely, $CAT(0)$ spaces.

4.1 Introduction

A metric space M is said to be a $CAT(0)$ space (the term is due to Gromov, see, e.g., [17], page 159) if it is geodesically connected, and if every geodesic triangle in M is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space [17]. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [19]), the complex Hilbert ball with a hyperbolic metric (see [45]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [17]. Burago et al. [23] present somewhat more elementary treatment, while Gromov [47] deals with a deeper study on this subject.

Let (M, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to M is called a *path*. A path $\gamma : [0, 1] \to M$ is called a *geodesic* if $d(\gamma(s), \gamma(t)) = |s - t| d(\gamma(0), \gamma(1))$, for every $s, t \in [0, 1]$. We will say that (M, d) is a *geodesic space* if every two points $x, y \in M$ are connected by a geodesic, i.e., there exists a geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In this case, we denote such geodesic by $[x, y]$. Note that in general such geodesic is not uniquely determined by its endpoints. For a point $z \in [x, y]$, we will use the notation $z = (1 - t)x \oplus ty$, where $t = d(x, z)/d(x, y)$ assuming $x \neq y$. The metric space (M, d) is called *uniquely geodesic* if every two points of M are connected by a unique geodesic. In this case $[x, y]$ will denote the unique geodesic connecting x and y in M.

The most fundamental examples of geodesic spaces are normed vector spaces, complete Riemannian manifolds, and polyhedral complexes of piecewise constant curvature. In the last two cases the existence of geodesic paths is not so obvious; determining when such spaces are uniquely geodesic is also a non-trivial matter. The case of normed vector spaces is much easier [17].

Let (X, d) be a uniquely geodesic space. A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $||\bar{x}_i - \bar{x}_j|| = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A point $\overline{x} \in [\overline{x_1}, \overline{x_2}]$ is called a comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = ||\overline{x_1} - \overline{x}||$.

Definition 4.1.1 *[17] Let X be a geodesic space and Δ be a geodesic triangle in X and $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,*

$$d(x, y) \leq ||\overline{x} - \overline{y}||.$$

X is called a $CAT(0)$ space if X is a geodesic space all of whose geodesic triangles satisfy the $CAT(0)$ inequality.

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [67]). Let (X, d) be a $CAT(0)$ space. Let $x, y_1, y_2 \in X$, and $\frac{1}{2}y_1 \oplus \frac{1}{2}y_2$ be the midpoint of the segment $[y_1, y_2]$. The $CAT(0)$ inequality implies:

$$d^2\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This is (CN) inequality of Bruhat and Tits [22]. The (CN) inequality implies that $CAT(0)$ spaces are uniformly convex hyperbolic space (see [55]) with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Theorem 4.1.1 *[55] If (X, d) is a complete $CAT(0)$ space, then (X, d) is strictly convex and has the property (R).*

4.2 Best Proximity Points in $CAT(0)$ Spaces

In this section, we give an example of a nonlinear hyperbolic space where the P-property holds; one may consider any pair of bounded, closed and convex subsets of a $CAT(0)$ space. As application, we give an extension of the Banach Contraction Principle for best proximity points in $CAT(0)$ spaces. In [67], Kirk extended his fundamental result for nonexpansive mappings [64] to $CAT(0)$ spaces. As an application of the P-property, we give an extension of Kirk Theorem for best proximity points.

In the previous chapter, it is not clear whether the conclusion of Theorem 3.4.2 is still valid if we drop the nonexpansiveness of the nearest point projections since this condition is not necessary in the linear case. In the case of $CAT(0)$ spaces, we have:

Lemma 4.2.1 *[17] Let C be a closed and convex subset of a complete $CAT(0)$ space (X, d). Then the following hold:*

(i) C is a Chebyshev set.

(ii) The nearest point projection P_C is a nonexpansive mapping.

Using Lemma 4.2.1, Theorem 4.1.1 and Theorem 3.4.2, we get the following result:

Theorem 4.2.1 *Let (X, d) be a complete $CAT(0)$ space. Any pair (A, B) of nonempty, bounded, closed, and convex subsets of X has the P-property.*

As a corollary of Theorem 3.4.3 or Theorem 3.4.4 with the aid of Theorem 4.2.1, we get the following result:

Corollary 4.2.1 *Let (A, B) be a pair of nonempty, bounded, closed and convex subsets of a complete $CAT(0)$ space (X, d). Let $T : A \to B$ be a contraction mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point in A.*

One may wonder what happens to the conclusion of Corollary 4.2.1 if T is assumed to be T is nonexpansive. As for $CAT(0)$ spaces, Kirk [68] proved the following generalization of the famous Browder- Goehde-Kirk fixed point theorem for nonexpansive mappings.

Theorem 4.2.2 *Let C be a nonempty closed convex and bounded subset of a complete $CAT(0)$ space X. Let $T : C \to C$ be a nonexpansive mapping defined on C. Then T has a fixed point in C.*

Armed with Theorem 4.2.2, we are ready to extend the conclusion of Corollary 4.2.1 to nonexpansive mappings.

Theorem 4.2.3 *Let (X, d) be a complete $CAT(0)$ space. Let (A, B) be a pair of nonempty, bounded, closed and convex subsets of X. Let $T : A \to B$ be nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then T has a best proximity point in A.*

Proof. Consider the mapping $P_{A_0} \circ T : A_0 \to A_0$. Since the nearest point projection in $CAT(0)$ spaces is nonexpansive, our assumption on the mapping T implies that $P_{A_0} \circ T$ is nonexpansive. Since A_0 is a nonempty, closed and convex subset of A, therefore A_0 is bounded. Theorem 4.2.2 implies that $P_{A_0} \circ T$ has a fixed point. Therefore, T has a best proximity point in A. ∎

Since nonexpansive mappings may fail to have a fixed point, there is no reason here to search for the uniqueness of the best proximity point.

4.3 Coupled Best Proximity Points in the Hilbert Ball

Let $(\mathbb{H}, \langle ., . \rangle)$ be a complex Hilbert space with the inner product $\langle ., . \rangle$ and induced norm $|.|$, and let $\mathbb{B} = \{x \in \mathbb{H} : |x| < 1\}$ be its open unit ball. The hyperbolic metric $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}^+$ is defined by

$$\rho(x, y) = \arg\tanh(1 - \sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - \langle x, y \rangle|^2}, \; x, y \in \mathbb{B}.$$

This metric is the infinite-dimensional analogue of the Poincaré metric on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. By $B(a, r) = \{x \in \mathbb{B} : \rho(x, a) < r\}$

[75] , we mean the ρ-ball of center a and radius r. A subset of $\mathbb{B}$ is called ρ-bounded if it is contained in a ρ-ball. We say that a mapping $e : \mathbb{R} \to \mathbb{B}$ is a metric embedding if $\rho(e(s), e(t)) = |s - t|$ for all reals s and t. The image of $\mathbb{R}$ under a metric embedding is called a metric line. The image of a real interval $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$ under such a mapping is called a metric segment. It is known ([45] , page 102) that for any two distinct points x and y in $\mathbb{B}$, there is a unique line (also called a geodesic) which passes through x and y. This metric line determines a unique metric segment joining x and y denoted by $[x, y]$. For each $0 \leq t \leq 1$, there is a unique point z on this metric segment such that $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1 - t)\rho(x, y)$. This point z will be denoted by $(1 - t)x \oplus ty$.

From now onwards we assume that $\mathbb{B}$ is the open unit Hilbert ball equipped with the hyperbolic metric ρ.

Lemma 4.3.1 *([103], Lemma 2.3) For any three points a, b and x in $\mathbb{B}$ and any number $0 \leq t \leq 1$, we have*

$$\rho^2((1 - t)a \oplus tx, b) \leq (1 - t)\rho^2(a, b) + t\rho^2(x, b) - t(1 - t)\rho^2(a, x).$$

It shows that the (CN) inequality holds in the Hilbert ball. So by ([17], page 163) $\mathbb{B}$ is a $CAT(0)$ space . Since $\mathbb{B}$ is complete ([7], page 18), therefore, it is a Hadamard space.

Recently, Sintunavarat and Kumam [105] introduced the notion of coupled best proximity point in the following manner.

Definition 4.3.1 *Let A and B be nonempty subsets of a metric space M and $T : A \times A \to B$. The point $(x, x') \in A \times A$ is called a coupled best proximity point of T if $d(x, T(x, x')) = d(x', T(x', x)) = d(A, B)$.*

In this section, we study existence and convergence of coupled best proximity points of a pair of cyclic contraction mappings in the Hilbert ball.

Definition 4.3.2 *[105] Let A and B be nonempty subsets of a metric space M, $F : A \times A \to B$ and $G : B \times B \to A$. The pair (F, G) is said to be cyclic contraction if there exists non-negative number $\alpha < 1$ such that*

$$d(F(x, x'), G(y, y')) \leq \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B)$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Note that if (F, G) is a cyclic contraction, then (G, F) is also a cyclic contraction.

The following lemma is needed.

Lemma 4.3.2 *Let A be a nonempty ρ-closed and ρ-convex subset and B a nonempty ρ-closed subset of $\mathbb{B}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

1. *$\rho(z_n, y_n) \to d(A, B)$.*

2. *For every $\epsilon > 0$, there exists N_0 such that for all $m > n \geq N_0$, $\rho(x_m, y_n) \leq d(A, B) + \epsilon$.*

Then, for every $\epsilon > 0$, there exists N_1 such that for all $m > n \geq N_1$, $\rho(x_m, z_n) \leq \epsilon$.

Proof. The proof is similar to that of ([33], Lemma 3.7). Indeed, assume on the contrary; then there exists $\epsilon_0 > 0$ such that for every $k \in N$, we have $\rho(x_{m_k} - z_{n_k}) \geq \epsilon_0$ where $m_k > n_k \geq k$.

Choose $0 < \gamma < 1$ such that $\frac{\epsilon_0}{\gamma} > d(A, B)$ and ϵ which satisfies $0 < \epsilon < \min(\frac{\epsilon_0}{\gamma} - d(A, B), \frac{d(A,B)(1-\sqrt{1-\frac{1}{4}\gamma^2})}{\sqrt{1-\frac{1}{4}\gamma^2}})$.

For this $\epsilon > 0$, there exists N_0 such that for all $m_k > n_k \geq N_0$, $\rho(x_{m_k}, y_{n_k}) \leq d(A, B) + \epsilon$. Also, there exists N_2 such that $\rho(z_{n_k}, y_{n_k}) \leq d(A, B) + \epsilon$ for all $n_k \geq N_2$. Choose $N_1 = \max(N_0, N_2)$. By Lemma 4.3.1, for all $m_k > n_k \geq N_1$, we get

$$\begin{aligned}
\rho^2(\frac{1}{2}x_{m_k} \oplus \frac{1}{2}z_{n_k}, y_{n_k}) &\leq \frac{1}{2}\rho^2(x_{m_k}, y_{n_k}) + \frac{1}{2}\rho^2(z_{n_k}, y_{n_k}) - \frac{1}{4}\rho^2(x_{m_k}, z_{n_k}) \\
&\leq \frac{1}{2}(d(A, B) + \epsilon)^2 + \frac{1}{2}(d(A, B) + \epsilon)^2 - \frac{1}{4}\epsilon_0^2 \\
&= (1 - \frac{1}{4}(\frac{\epsilon_0}{d(A, B) + \epsilon})^2)(d(A, B) + \epsilon)^2.
\end{aligned}$$

Hence,

$$\rho(\frac{1}{2}x_{m_k} \oplus \frac{1}{2}z_{n_k}, y_{n_k}) \leq \sqrt{1 - \frac{1}{4}(\frac{\epsilon_0}{d(A, B) + \epsilon})^2}(d(A, B) + \epsilon).$$

By the choice of ϵ, we have $\rho(\frac{1}{2}x_{m_k} \oplus \frac{1}{2}z_{n_k}, y_{n_k}) < d(A, B)$, for all $m_k > n_k \geq N_1$, which is a contradiction.

∎

Here is the main result of this section.

Theorem 4.3.1 *Let A and B be nonempty ρ-closed and ρ-convex subsets of $\mathbb{B}$, $F : A \times A \to B$, $G : B \times B \to A$ and (F, G) be a pair of cyclic contractions. Then F and G have a coupled best proximity point.*

Proof.

Let $(x_0, x'_0) \in A \times A$. Define

$$x_{2n+1} = F(x_{2n}, x'_{2n}), \quad x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), \quad x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. We now obtain:

$$
\begin{aligned}
\rho(x_{2n}, x_{2n+1}) &= \rho(x_{2n}, F(x_{2n}, x'_{2n})) \\
&= \rho(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\leq \frac{\alpha}{2}[\rho(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + \rho(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] \\
&\quad +(1-\alpha)d(A, B) \\
&\leq \frac{\alpha^2}{2}[\rho(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + \rho(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2}))] \\
&\quad +(1-\alpha^2)d(A, B).
\end{aligned}
$$

By induction, we get

$$
\rho(x_{2n}, x_{2n+1}) \leq \frac{\alpha^{2n}}{2}[\rho(x_0, F(x_0, x'_0)) + \rho(x'_0, F(x'_0, x_0))] + (1-\alpha^{2n})d(A, B).
$$

When $n \to \infty$, we obtain

$$
\rho(x_{2n}, x_{2n+1}) \to d(A, B). \tag{4.3.1}
$$

By a similar argument, we have

$$
\rho(x_{2n+1}, x_{2n+2}) \to d(A, B).
$$

So by Lemma 4.3.2, we get $\rho(x_{2n}, x_{2n+2}) \to 0$, and $\rho(x_{2n+1}, x_{2n+3}) \to 0$. A similar argument shows that $\rho(x'_{2n}, x'_{2n+2}) \to 0$, and $\rho(x'_{2n+1}, x'_{2n+3}) \to 0$.

Moreover, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ with $m > n \geq N$ such that

$$\frac{1}{2}[\rho(x_{2m}, x_{2n+1}) + \rho(x'_{2m}, x'_{2n+1})] < d(A, B) + \epsilon. \tag{4.3.2}$$

To show this, suppose that (4.3.2) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ satisfying

$$\frac{1}{2}[\rho(x_{2m_k}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2n_k+1})] \geq d(A, B) + \epsilon'.$$

and

$$\frac{1}{2}[\rho(x_{2m_k-2}, x_{2n_k+1}) + \rho(x'_{2m_k-2}, x'_{2n_k+1})] < d(A, B) + \epsilon'.$$

Therefore, we get

$$\begin{aligned} d(A, B) + \epsilon' &< \frac{1}{2}[\rho(x_{2m_k}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2n_k+1})] \\ &\leq \frac{1}{2}[\rho(x_{2m_k}, x_{2m_k-2}) + \rho(x_{2m_k-2}, x_{2n_k+1}) \\ &\quad +\rho(x'_{2m_k}, x'_{2m_k-2}) + \rho(x'_{2m_k-2}, x'_{2n_k+1})] \\ &< \frac{1}{2}[\rho(x_{2m_k}, x_{2m_k-2}) + \rho(x'_{2m_k}, x'_{2m_k-2})] + d(A, B) + \epsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\frac{1}{2}[\rho(x_{2m_k}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2n_k+1})] \to d(A, B) + \epsilon'.$$

By using the triangle inequality, we get

$\frac{1}{2}[\rho(x_{2m_k}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2n_k+1})]$

$$
\begin{aligned}
&\leq \frac{1}{2}[\rho(x_{2m_k}, x_{2m_k+2}) + \rho(x_{2m_k+2}, x_{2n_k+3}) + \rho(x_{2n_k+3}, x_{2n_k+1}) \\
&\quad +\rho(x'_{2m_k}, x'_{2m_k+2}) + \rho(x'_{2m_k+2}, x'_{2n_k+3}) + \rho(x'_{2n_k+3}, x'_{2n_k+1})] \\
&\leq \frac{1}{2}[\rho(x_{2m_k}, x_{2m_k+2}) + \rho(x_{2n_k+3}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2m_k+2}) + \rho(x'_{2n_k+3}, x'_{2n_k+1})] \\
&\quad \frac{\alpha^2}{2}[\rho(x_{2m_k}, x_{2n_k+1}) + \rho(x'_{2m_k}, x'_{2n_k+1})] + (1-\alpha^2)d(A,B).
\end{aligned}
$$

Letting $k \to \infty$, we get

$$
d(A,B) + \epsilon' \leq \alpha^2[d(A,B) + \epsilon'] + (1-\alpha^2)d(A,B) = d(A,B) + \alpha^2\epsilon'.
$$

which is a contradiction. Therefore, (4.3.2) holds.

We now show that for every $\epsilon > 0$, there exists N such that

$$
\rho(x_{2m}, x_{2n+1}) \leq d(A,B) + \epsilon \tag{4.3.3}
$$

for all $m > n \geq N$.

Suppose not, then there exists $\bar{\epsilon} > 0$ such that for all $k \in \mathbb{N}$ there are $m_k > n_k \geq k$,

$$
\rho(x_{2m_k}, x_{2n_k+1}) > d(A,B) + \bar{\epsilon},
$$

Moreover, let m_k be the least integer greater than n_k that satisfies the above inequality. Now we have

$$
\begin{aligned}
d(A,B)+\overline{\epsilon} &< \rho(x_{2m_k}, x_{2n_k+1}) \\
&\leq \rho(x_{2m_k}, x_{2n_k-1}) + \rho(x_{2n_k-1}, x_{2n_k+1}) \\
&\leq d(A,B)+\overline{\epsilon} + \rho(x_{2n_k-1}, x_{2n_k+1}).
\end{aligned}
$$

Letting $k \to \infty$, we have $\rho(x_{2m_k}, x_{2n_k-1}) \to d(A,B)+\overline{\epsilon}$.

Furthermore, for all $m > n \geq N$, we get

$$
\begin{aligned}
\rho(x_{2m_k}, x_{2n_k+1}) &\leq \rho(x_{2m_k}, x_{2m_k+2}) + \rho(x_{2m_k+2}, x_{2n_k+3}) + \rho(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq d(A,B)+\alpha^2\overline{\epsilon} + \rho(x_{2m_k}, x_{2m_k+2}) + +\rho(x_{2n_k+3}, x_{2n_k+1}).
\end{aligned}
$$

Letting $k \to \infty$, we have

$$d(A,B)+\overline{\epsilon} \leq d(A,B)+\alpha^2\overline{\epsilon}.$$

which is a contradiction. Therefore, condition (4.3.3) holds. Now on the basis of (4.3.1) and (4.3.3), we get by Lemma 4.3.2 that $\{x_{2n}\}$ is a Cauchy sequence. In a similar way, we can prove that $\{x'_{2n}\}$ is a Cauchy sequence. As $\mathbb{B}$ is complete, so there exist $p, q \in A$ such that $x_{2n} \to p$ and $x'_{2n} \to q$.

$$d(A,B) \leq \rho(p, x_{2n-1}) \leq \rho(p, x_{2n}) + \rho(x_{2n}, x_{2n-1}). \tag{4.3.4}$$

Letting $n \to \infty$ in (4.3.4), we have, $\rho(p, x_{2n-1}) \to d(A, B)$. By a similar argument, we get $\rho(q, x'_{2n-1}) \to d(A, B)$. Now it follows that

$$\begin{aligned} \rho(x_{2n}, F(p, q)) &= \rho(G(x_{2n-1}, x'_{2n-1}), F(p, q)) \\ &\leq \frac{\alpha}{2}[\rho(p, x_{2n-1}) + \rho(q, x'_{2n-1})] + (1-\alpha)d(A, B). \end{aligned}$$

If $n \to \infty$ in the above inequality, then we get $\rho(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $\rho(q, F(q, p)) = d(A, B)$. Therefore, (p, q) is a coupled best proximity point of F. In a similar way, we can prove that there exist $p', q' \in B$ such that $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$. Moreover, we have $\rho(p', G(p', q')) = d(A, B)$ and $\rho(q', G(q', p')) = d(A, B)$ and so (p', q') is a coupled best proximity point of G. ∎

Remark 4.3.1 *Theorem 4.3.1 extends ([105], Theorem 3.10) to $\mathbb{B}$ (an example of a nonlinear hyperbolic space). Hence, it provides an answer to the questions posed by Sintunavarat and Kumam [105].*

4.4 Best Proximity Points in Partially Ordered $CAT(0)$ Spaces

In this section, we define the concept of proximally monotone Lipschitzian mappings on partially ordered metric spaces. Then we obtain sufficient conditions for the existence and uniqueness of best proximity points for such mappings in

$CAT(0)$ spaces. This work is a continuation of the work of Ran and Reurings, Nieto and Rodríguez-López, and Khamsi et al. for monotone mappings.

4.4.1 The Pythagorean property

Recently, based on geometrical properties of a Hilbert space, the so-called Pythagorean property is introduced as follows:

Definition 4.4.1 *[37] A pair of subsets (A, B) of a metric space M is said to be proximinal iff $A = A_0$ and $B = B_0$. It is said to be sharp proximinal if anf only if for any $(x, y) \in A \times B$, there exist a unique $x' \in B$ and $y' \in A$ such that*

$$d(x, x') = d(y, y') = d(A, B).$$

A sharp proximinal pair (A, B) in a metric space M is said to have the Pythagorean property if and only if, for each $(x, y) \in A \times B$, we have

$$d(x, y)^2 = d(x, y')^2 + d(y', y)^2 \text{ and } d(x, y)^2 = d(y, x')^2 + d(x', x)^2,$$

where x' and y' are the (unique) points in B and A, respectively, with $d(x, x') = d(A, B)$ and $d(y', y) = (A, B)$.

It is shown in [37] that the following facts hold in a $CAT(0)$ space.

Proposition 4.4.1 *Let A and B be nonempty closed and convex subsets of a complete $CAT(0)$ space X. Then the pair (A_0, B_0) is nonempty, closed and convex* *in X.*

Lemma 4.4.1 *Let (A, B) be a nonempty, closed and convex pair in a complete $CAT(0)$ space X. Then the pair (A_0, B_0) is a sharp proximinal pair.*

Theorem 4.4.1 *Let X be a complete $CAT(0)$ space. Then nonempty, closed, and convex proximinal pairs (A, B) of subsets of X have the Pythagorean property.*

We close this section, with one additional fact which will be needed in the next section.

Theorem 4.4.2 *Let A and B be nonempty closed and convex subsets of a complete $CAT(0)$ space X. Then the pair (A_0, B_0) has the Pythagorean property.*

Proof.

By Lemma 4.4.1, (A_0, B_0) is a sharp proximinal pair. Proposition 4.4.1 implies that the pair (A_0, B_0) is nonempty, closed and convex in X. If (A_0, B_0) is proximinal pair, then Theorem 4.4.1 implies that the pair (A_0, B_0) has the Pythagorean property. To see this, let (A_{00}, B_{00}) be the proximity pair associated with the pair (A_0, B_0). Indeed, $A_{00} \subseteq A_0$. Conversely, let $x \in A_0$. Then there exists $y \in B$ such that $d(x, y) = d(A, B)$. Hence, $y \in B_0$ and $d(x, y) = d(A, B) = d(A_0, B_0)$. i.e., $x \in A_{00}$. Therefore, $A_0 \subseteq A_{00}$. Hence, $A_0 = A_{00}$. In similar way, we show that $B_0 = B_{00}$. Therefor, the pair (A_0, B_0) is a proximinal pair.

∎

4.4.2 Proximally monotone Lipschitzian mappings

In this section, we obtain sufficient conditions for the existence and uniqueness of best proximity points for proximally monotone mappings in $CAT(0)$ spaces.

We define the concept of proximally monotone Lipschitzian mappings on a partially ordered metric space. The definition of proximally monotone mappings has its roots in [10].

Definition 4.4.2 *Let A, B be nonempty subsets of metric space M and $T : A \to B$ be a mapping.*

(1) T is said to be proximally monotone if it satisfies the condition:

$$x \preceq y,\ d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B) \text{ imply } u \preceq v$$

for all $x, y, u, v \in A$.

(2) T is said to be proximally monotone Lipschitzian mapping if T is proximally monotone and there exists $k \geq 0$ such that

$$d(Tx, Ty) \leq k\ d(x, y),$$

for any $x, y \in A$ where x and y are comparable. If $k < 1$ (k=1) , then we say that T is a monotone contraction (nonexpansive) mapping.

If $A = B$, the above definition coincides with the definition of monotone Lipschitzian mappings [8]. Moreover, the best proximity point x reduces to a fixed of T.

The following best proximity point theorem provides an extended version of the Banach Contraction Principle for proximally monotone contraction mappings on partially ordered $CAT(0)$ spaces.

Theorem 4.4.3 *Let (A, B) be a pair of nonempty, bounded, closed, and convex subsets of a partially ordered $CAT(0)$ space $(X, d, \preceq)$ in which order intervals are closed. Let $T : A \to B$ be a proximally monotone contraction mapping such that $T(A_0) \subseteq B_0$. If there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$, then T has a best proximity point x in A. Moreover, if y in A is a best proximity point of T comparable to x, then $y = x$.*

Proof. Since $Tx_1 \in T(A_0) \subseteq B_0$, therefore there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. By the definition of proximally monotone mappings for $x = x_0, y = u = x_1, v = x_2$, we obtain $x_1 \preceq x_2$.

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that, for all $n \in \mathbb{N}$, $x_{n-1} \preceq x_n$ and $d(x_n, Tx_{n-1}) = d(A, B)$.

By Theorem 4.4.2, the pair (A_0, B_0) has the Pythagorean property, hence

$$\left.\begin{aligned} d(x_{n+1}, Tx_n)^2 + d(A, B)^2 = d(x_n, Tx_n)^2 \\ d(x_n, Tx_{n-1})^2 + d(A, B)^2 = d(x_n, Tx_n)^2 \end{aligned}\right\} \Longrightarrow d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}),$$

for any $n \geq 1$.

As T is a proximally monotone contraction, so there exists $k < 1$ such that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}),$$

for any $n \geq 1$, which implies that

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0),$$

for any $n \geq 1$. Hence $\sum_{n \in \mathbb{N}} d(x_{n+1}, x_n)$ is convergent which implies that $\{x_n\}$ is Cauchy. Since X is complete, therefore there exists $x \in X$ such that $\{x_n\}$ converges to x. As $\{x_n\} \subset A_0$ and by Proposition 4.4.1, A_0 is closed, so we conclude that $x \in A_0$. Since the order intervals are closed, we conclude that $x_n \preceq x$, for any $n \in \mathbb{N}$. Furthermore, $d(Tx_n, Tx) \leq kd(x_n, x) \to 0$ as $n \to \infty$. So $\{Tx_n\}$ converges to $Tx \in B_0$, By $d(x_{n+1}, Tx_n) = d(A, B)$, we get $d(x, Tx) = d(A, B)$, i.e., $x \in A_0$ and $Tx \in B_0$. Clearly, x is a best proximity point of T in A.

Next, we prove that if y in A is a best proximity point of T comparable to x, then $y = x$. Without loss of any generality, assume that $x \preceq y$. Since T is a proximally monotone contraction mapping, therefore by the Pythagorean property, we have as before

$$\left.\begin{aligned} d(x, Tx) = d(A, B) \\ d(y, Ty) = d(A, B) \end{aligned}\right\} \Longrightarrow d(x, y) = d(Tx, Ty).$$

As T is a contraction mapping, so we get

$$d(x, y) = d(Tx, Ty) < d(x, y),$$

which implies $d(x, y) = 0$, i.e., $y = x$.

∎

Remark 4.4.1 *In Theorem 4.4.3, if $A = B$, then $d(A, B) = 0$ i.e., $x_1 = Tx_0$. Therefore, $x_0 \preceq Tx_0$. Hence, our result extends the work of Ran and Reurings [98] and Nieto and Rodríguez-López [91] for monotone contraction mappings.*

One may wonder what happens to the conclusion of Theorem 4.4.3 if T is not assumed to be a proximally monotone contraction. In particular, what happens when we assume that T is proximally monotone nonexpansive. To answer this question, we recall the following theorem valid in hyperbolic spaces

Theorem 4.4.4 *[25] Let $(X, d, \preceq)$ be a partially ordered hyperbolic space in which order intervals are closed and convex. Assume that (X, d) is uniformly convex hyperbolic space. Let C be a nonempty, bounded, closed and convex subset of X not reducible to one point. Let $T : C \to C$ be a monotone nonexpansive mapping. If there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable, then T has a fixed point.*

Armed with Theorem 4.4.4, we are ready to extend the conclusion of Theorem 4.4.3 for proximally monotone nonexpansive mappings.

Theorem 4.4.5 *Let (A, B) be a pair of nonempty, bounded, closed, and convex subsets of a partially ordered $CAT(0)$ space $(X, d, \preceq)$ in which order intervals are closed and convex and A_0 not reducible to one point. Let $T : A \to B$ be a proximally monotone nonexpansive mapping such that $T(A_0) \subseteq B_0$. If there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$, then T has a best proximity point x in A.*

Proof.

Note that, for any $y_0 \in B_0$, there exists unique $x_0 \in A_0$ such that $d(x_0, y_0) = d(A, B)$. Hence

$$d(A, B) \leq d(A, y_0) \leq d(A_0, y_0) \leq d(x_0, y_0) = d(A, B).$$

That is, $d(A, y) = d(A_0, y) = d(A, B)$, for all $y \in B_0$.

Consider the mapping $P_{A_0} \circ T : A_0 \to A_0$. Since the nearest point projection in $CAT(0)$ spaces is nonexpansive, our assumption on the mapping T implies that $P_{A_0} \circ T$ is monotone nonexpansive.

Moreover, let $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$. As A_0 is a Chebyshev subset, so $P_{A_0}(T(x_0)) = x_1$. Hence, $x_0 \preceq P_{A_0}(T(x_0))$.

Since A_0 is a nonempty, closed and convex subset of A, therefore A_0 is bounded. As $CAT(0)$ spaces are uniformly convex hyperbolic spaces, so Theorem 4.4.4 implies that $P_{A_0} \circ T$ has a fixed point.

Finally, assume that x is a fixed point of $P_{A_0} \circ T$, i.e., $x = P_{A_0}(Tx)$. Then, we have $x \in A_0$ and $Tx \in B_0$. Hence

$$d(x, Tx) = d(P_{A_0}(Tx), Tx) = d(Tx, A_0) = d(A, B).$$

Therefore, x is a best proximity point of T in A.

∎

We now give an example to illustrate the main results of this work.

Example 4.4.1 *Let E be the Euclidean vector space $\mathbb{R}^2$. Obviously, the $CAT(0)$ inequality holds in E and so it is a $CAT(0)$ space.*

Consider the product order $\preceq$ on $\mathbb{R}^2$, i.e. $(a, b) \preceq (c, d)$ iff $a \leq c$ and $b \leq d$. Let $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(x, 1) : 0 \leq x \leq 1\}$.

Then, $(E, d, \preceq)$ is a partially ordered complete $CAT(0)$ space in which order intervals are closed and convex, (A, B) is a pair of nonempty, bounded, closed, and convex subsets of X, $A_0 = A, B_0 = B$ and $d(A, B) = 1$.

Define a mapping $T : A \to B$ by

$$T((x, 0)) = (kx, 1),$$

for $k \in [0, 1]$.

Clearly, $T(A_0) \subseteq B_0$. Let $x_0 = x_1 = (0, 0)$. Then $x_0 \preceq x_1$ and $d(x_1 T x_0) = d((0, 0), T((0, 0))) = d((0, 0), (0, 1)) = 1$.

We show now that T is a proximally monotone Lipschitzian mapping. For $(x,0),(y,0),(u,0),(v,0) \in A$ with $(x,0) \preceq (y,0)$, $d((u,0),T((x,0))) = 1$ and $d((v,0),T((y,0))) = 1$, we have $(u,0) = (kx,0)$ and $(v,0) = (ky,0)$. Hence, $(u,0) \preceq (v,0)$.

Moreover, $d(T((x,0)),T((y,0)))= d((kx,1),(ky,1)) = kd((x,0),(y,0))$.

If $k < 1$, then T is proximally monotone contraction mapping. Indeed, in this case, point $(0,0) \in A$ is the best proximity point of T.

Finally, If $k = 1$, then T is proximally monotone nonexpansive mapping and any $x \in A$ is a best proximity point of T.

4.5 Generalized $CAT(0)$ Spaces

In this section, we extend the Gromov geometric definition of $CAT(0)$ spaces to the case when the comparison triangles are not from the Euclidean plane but they belong to a general Banach space; in particular, we study the case of the Banach space l_p, $p > 2$. This generalization is being offered for the first time.

4.5.1 $CAT_p(0)$ spaces

In the definition of a $CAT(0)$ space, the comparison triangle is a subset of the Euclidean vector space $\mathbb{R}^2$. What structure and properties one may get if we allow the comparison triangles to lie in some normed space E. For this, first we introduce the following definition:

Definition 4.5.1 *Let (X, d) be a geodesic space and $(E, \|.\|)$ be a normed vector space. We say that X is a generalized $CAT(0)$ space if for any geodesic triangle Δ in X, there exists a comparison triangle $\overline{\Delta}$ in E such that the comparison axiom is satisfied, i.e., for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, we have*

$$d(x, y) \leq \|\overline{x} - \overline{y}\|.$$

Obviously, a normed vector space $(E, \|.\|)$ is itself a generalized $CAT(0)$ space in the sense of Definition 4.5.1.

As for a normed vector space to be a $CAT(0)$ space, we have the following result:

Theorem 4.5.1 *([17], Proposition 1.14.) If a real normed vector space $(E, \|.\|)$ is $CAT(0)$ space, then it is a pre-Hilbert space.*

According to Theorem 4.5.1, our definition gives a new class of $CAT(0)$ spaces provided $(E, \|.\|)$ is not a pre-Hilbert space. As an example, we take $E = l_p$, for $p \geq 2$, the Banach space closest to a Hilbert space (without being so).

Definition 4.5.2 *Let (X, d) be a geodesic space. The space X is said to be a $CAT_p(0)$ space, for $p > 2$, if for any geodesic triangle Δ in X, there exists a comparison triangle $\overline{\Delta}$ in l_p such that the comparison axiom is satisfied, i.e., for* *all* $x, y \in \Delta$ *and all comparison points* $\overline{x}, \overline{y} \in \overline{\Delta}$, *we have*

$$d(x, y) \leq \|\overline{x} - \overline{y}\|.$$

It is obvious that l_p, for $p > 2$, is a $CAT_p(0)$ space. We suspect that a nonlinear example will be given by the open unit ball of l_p endowed with the Kobayashi distance [78].

We discuss some of the properties of $CAT_p(0)$ spaces.

Theorem 4.5.2 *Let* (X, d) *be a* $CAT_p(0)$ *space, with* $p \geq 2$. *Then for any* x, y_1, y_2 *in* X, *we have*

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^p(x, y_1) + \frac{1}{2} d^p(x, y_2) - \frac{1}{2^p} d^p(y_1, y_2), \tag{4.5.1}$$

which we call the (CN_p) *inequality.*

Proof. Let x, y_1, y_2 be in X and Δ be the associated geodesic triangle in X. As X is a $CAT_p(0)$ space, so there exists a comparison geodesic triangle $\overline{\Delta}$ in l_p, with $p \geq 2$. The associated comparison points in l_p will be denoted by $\overline{x}, \overline{y}_1$ and $\overline{y}_2$. The comparison axiom implies:

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \left\|\overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2}\right\|,$$

which implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^p \leq \left\|\overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2}\right\|^p.$$

Recall the Clarkson's inequality [31] :

$$\|a+b\|^p + \|a-b\|^p \leq 2^{p-1}\ \Big(\|a\|^p + \|b\|^p\Big), \tag{4.5.2}$$

for any a, b in l_p, for $p \geq 2$. As application of this inequality to $a = \dfrac{\overline{x} - \overline{y}_1}{2}$ and $b = \dfrac{\overline{x} - \overline{y}_2}{2}$, yields:

$$\left\|\frac{\overline{x} - \overline{y}_1}{2} + \frac{\overline{x} - \overline{y}_2}{2}\right\|^p + \left\|\frac{\overline{x} - \overline{y}_1}{2} - \frac{\overline{x} - \overline{y}_2}{2}\right\|^p \leq 2^{p-1}\ \left(\left\|\frac{\overline{x} - \overline{y}_1}{2}\right\|^p + \left\|\frac{\overline{x} - \overline{y}_2}{2}\right\|^p\right).$$

Or,

$$\left\|\overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2}\right\|^p \leq \frac{1}{2}\|\overline{x} - \overline{y}_1\|^p + \frac{1}{2}\|\overline{x} - \overline{y}_2\|^p - \frac{1}{2^p}\|\overline{y}_1 - \overline{y}_2\|^p.$$

Hence,

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}\|\overline{x} - \overline{y}_1\|^p + \frac{1}{2}\|\overline{x} - \overline{y}_2\|^p - \frac{1}{2^p}\|\overline{y}_1 - \overline{y}_2\|^p.$$

Now from $\|\bar{x} - \bar{y}_j\| = d\,(x, y_j)$, for $j \in \{1, 2\}$, we get

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^p(x, y_1) + \frac{1}{2}d^p(x, y_2) - \frac{1}{2^p}d^p(y_1, y_2).$$

∎

Note that if $p = 2$, then the (CN_p) inequality coincides with the classical (CN)

inequality.

One of the implications of the (CN) inequality is the uniform convexity of the distance in a $CAT(0)$ space.

We discuss the case of uniform convexity of the $CAT_p(0)$ spaces.

A direct consequence of the (CN_p) inequality is the following result:

Theorem 4.5.3 *Any $CAT_p(0)$ space, with $p \geq 2$, is uniformly convex hyperbolic space. Moreover, we have*

$$\delta(r, \varepsilon) \geq 1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p},$$

for every $r > 0$ and for each $\varepsilon > 0$.

The Banach spaces l_p, $p > 1$, are not only uniformly convex Banach space but they admit a geometric property known as p-uniform convexity (see [12] p. 310). Theorem 4.5.3 implies that $CAT_p(0)$ spaces enjoy the p-uniform convexity as well.

Next we discuss behavior of the type functions in $CAT_p(0)$ spaces.

Theorem 4.5.4 *Let (X, d) be a complete $CAT_p(0)$ space, with $p \geq 2$. Let C be any nonempty, closed, convex and bounded subset of X. Let τ be a type defined on C. Then any minimizing sequence of τ is convergent. Its limit z is the unique minimum of τ and satisfies*

$$\tau^p(z) + \frac{1}{2^{p-1}}\, d^p(z, x) \leq \tau^p(x), \tag{4.5.3}$$

for any $x \in C$.

Proof. Let $\{x_n\}$ be a sequence in C such that $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$. Denote $\tau_0 = \inf\{\tau(x); x \in C\}$. Let $\{y_k\}$ be a minimizing sequence of τ. Since C is bounded, there exists $R > 0$ such that $d(x, y) \leq R$ for any $x, y \in C$. Now Theorem 4.5.2 implies

$$d^p\left(x_n, \frac{y_m \oplus y_k}{2}\right) \leq \frac{1}{2}d^p(x_n, y_m) + \frac{1}{2}d^p(x_n, y_k) - \frac{1}{2^p}d^p(y_m, y_k),$$

for any $n, m, k \in \mathbb{N}$. If we let n go to infinity, then we get

$$\tau^p\left(\frac{1}{2}y_m \oplus \frac{1}{2}y_k\right) \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \frac{1}{2^p}d^p(y_m, y_k),$$

which implies

$$\tau_0^p \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \frac{1}{2^p}d^p(y_m, y_k),$$

or

$$\frac{1}{2^p}d^p(y_m, y_k) \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \tau_0^p,$$

for any $k, m \geq 1$. Since $\{y_n\}$ is a minimizing sequence of τ, therefore we conclude that

$$\lim_{k,m\to\infty} d(y_m, y_k) = 0,$$

i.e., the sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to some point $z \in C$. As τ is continuous, so we get $\tau_0 = \tau(z)$. Next, we establish inequality (4.5.3). Let $x \in C$. The (CN_p) inequality implies

$$d^p\Big(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\Big) \leq \frac{1}{2}d^p(x, x_n) + \frac{1}{2}d^p(z, x_n) - \frac{1}{2^p}d^p(x, z),$$

for any n. Hence

$$\limsup_{n\to\infty} d^p\Big(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\Big) \leq \frac{1}{2}\limsup_{n\to\infty} d^p(x, x_n) + \frac{1}{2}\limsup_{n\to\infty} d^p(z, x_n) - \frac{1}{2^p}d^p(x, z).$$

The definition of z implies that

$$\limsup_{n\to\infty} d^p(z, x_n) \leq \limsup_{n\to\infty} d^p\Big(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\Big).$$

Hence

$$\frac{1}{2}\limsup_{n\to\infty} d^p(z, x_n) \leq \frac{1}{2}\limsup_{n\to\infty} d^p(x, x_n) - \frac{1}{2^p}d^p(x, z),$$

implies the desired inequality. ∎

Note that the inequality (4.5.3) is similar to the Opial condition defined in Banach spaces [92].

4.5.2 Application: fixed point results

In this section, we discuss the existence of fixed points of uniformly Lipschitzian mappings defined on a $CAT_p(0)$ space.

Definition 4.5.3 *Let C be a nonempty subset of a metric (M, d). Let $T : C \to C$ be a* Lipschitzian *mapping. The mapping T is called* uniformly Lipschitzian *if*

$\sup_{n\geq 1} Lip(T^n) < \infty$, *where* $Lip(T)$ *denotes the* Lipschitz constant *of* T.

It is well-known that if a mapping is uniformly Lipschitzian, then one may find an equivalent distance for which the mapping is nonexpansive; see ([45] pages 34-38). Indeed, let $T : C \to C$ be uniformly Lipschitzian. Set

$$\rho(x, y) = \sup\{d(T^n(x), T^n(y)),\ n = 0, 1, , \cdots\}$$

for all $x, y \in C$, one can obtain a metric ρ on C which is equivalent to the metric d and relative to which T is nonexpansive. In this context, it is natural to ask the question: if a set C has the fixed point property (fpp) for nonexpansive mappings with respect to the metric d, then does C also have (fpp) for mappings which are nonexpansive relative to an equivalent metric? This is known as the stability of (fpp). The first result in this direction is due to Goebel and Kirk [42]. Motivated by this question, we investigate the fixed point property of uniformly Lipschitzian mappings in $CAT_P(0)$, for $p \geq 2$.

Let (X, d) be $CAT_p(0)$, $p \geq 2$. Define the normal structure coefficient $N(X)$ (see [26]) by :

$$N(X) = \inf \ \frac{\text{diam}(C)}{R(C)},$$

where the infimum is taken over all C: nonempty, bounded and convex subset of X not reducible to one point, diam(C) is diameter of C, and $R(C) = \inf\left\{\sup_{y\in C} d(x, y);\ x \in C\right\}$ is the Chebyshev radius of C. Note that X satisfies the property (R) (see [55]), so for any nonempty, bounded, convex and closed subset

C of X, there exists $x \in C$ such that $R(C) = \sup_{y \in C} d(x, y)$. It is easy to check that $N(X) \leq 2$. Using the (CN_p) inequality, we can show that

$$N(X) \geq \left(1 - \frac{1}{2^p}\right)^{-1/p} > 1.$$

The main result of this section is an analogue of Theorem 3 in [111].

Theorem 4.5.5 *Let (X, d) be complete $CAT_p(0)$, $p \geq 2$, space. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be uniformly Lipschitzian with*

$$\lambda(T) = \sup_{n \geq 1} \; Lip(T^n) < \left(\frac{1 + \sqrt{1 + 8(N(X)/2)^p}}{2}\right)^{1/p}.$$

Then T has a fixed point.

Proof. Fix $x_0 \in C$. Using an induction argument, we will construct a sequence $\{x_m\}$ in C such that x_{m+1} is the point z found in Theorem 4.5.4 associated with the sequence $\{T^n(x_m)\}$, for any $m \geq 0$. For any $m \geq 0$, denote

$$r_m = \limsup_{n \to \infty} \; d(x_{m+1}, T^n(x_m)) \text{ and } \; R_m = \sup_{n \geq 1} \; d(x_m, T^n(x_m)).$$

Set $C^* = \overline{conv}(\{T^n(x_m); n \geq 1\}$. By the properties of $CAT_p(0)$ spaces, there exists $z \in C^*$ such that $R(C^*) = \sup_{x \in C^*} d(x, z)$. In particular, we have

$$\sup_{n \geq n_0} d(z, T^n(x_m)) \leq \frac{1}{N(X)} \; \text{diam}(C^*) = \frac{1}{N(X)} \; \text{diam}(\{T^n(x_m); n \geq 1\}),$$

for any $n_0 \geq 1$. Since $r_m \leq \limsup_{n\to\infty} \; d(z, T^n(x_m))$ and

$$\operatorname{diam}(\{T^n(x_m); n \geq 1\}) \leq \lambda(T) \; \sup_{n\geq 1} d(x_m, T^n(x_m)),$$

we get

$$r_m \leq \frac{\lambda(T)}{N(X)} \; R_m, \;\; m = 1, \cdots .$$

This result is similar to Theorem 1 in [84]. Using Theorem 4.5.4, we get

$$r_m^p + \frac{1}{2^p} \; d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{1}{2} \limsup_{n\to\infty} \; d^p(T^s(x_{m+1}), T^n(x_m)),$$

which implies that

$$r_m^p + \frac{1}{2^p} \; d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{\lambda(T)^p}{2} \limsup_{n\to\infty} \; d^p(x_{m+1}, T^{n-s}(x_m)),$$

or

$$r_m^p + \frac{1}{2^p} \; d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{\lambda(T)^p}{2} r_m^p.$$

Hence

$$\frac{1}{2^p} R_{m+1}^p = \frac{1}{2^p} \; \sup_{s\geq 1} d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{\lambda(T)^p - 1}{2} r_m^p \leq \frac{(\lambda(T)^p - 1)}{2} \frac{\lambda(T)^p}{N(X)^p} R_m^p,$$

which implies that $R_{m+1} \leq A \; R_m$, for any $m \geq 1$, where

$$A = \left(\frac{(\lambda(T)^p - 1)\lambda(T)^p}{2(N(X)/2)^p} \right)^{1/p}.$$

Our assumption on $\lambda(T)$ leads to $A < 1$. Since $R_m \leq A^{m-1}\, R_1$, for any $m \geq 1$, we conclude that $\sum_{m\geq 1} R_m$ is convergent. Since $d(x_m, x_{m+1}) \leq r_m + R_m \leq 2R_m$, for any $m \geq 1$, the series $\sum d(x_m, x_{m+1})$ is also convergent, and therefore $\{x_m\}$ is Cauchy. By the completeness of X, $\{x_m\}$ converges to some point $z \in C$. Since

$$d(z, T(z)) \leq d(z, x_m) + d(x_m, T(x_m)) + d(T(x_m), T(z)),$$

therefore, we get $d(z, T(z)) \leq (1 + Lip(T))d(z, x_m) + R_m$, for any $m \geq 1$. If we let $m \to \infty$, we get $d(z, T(z)) = 0$, i.e., $T(z) = z$. ∎

As a corollary of Theorem 4.5.5, we get the following result:

Theorem 4.5.6 *Let (X, d) be complete $CAT_p(0)$, $p \geq 2$, space. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be a nonexpansive mapping. Then T has a fixed point.*

CHAPTER 5

APPROXIMATION OF COMMON FIXED POINTS

A plenty of metric fixed point theorems have been obtained, in earlier chapters, which usually establish the existence, or the existence and uniqueness of fixed points for certain mappings. Among these fixed point theorems, only a small number is important from a practical point of view, that is, which offer a constructive method for finding the fixed points. In this chapter, we study iterative construction of fixed points on certain general nonlinear domains.

5.1 Constructive Methods

In this section, we aim to survey some of the most used fixed point iteration procedures: the Picard iteration, the Krasnoselskij iteration, the Mann iteration and the Ishikawa iteration.

Let X be any set and $T : X \to X$ a mapping. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the n^{th} iterate of x under T. In order to simplify the notations we will often use Tx instead of $T(x)$. The mapping $T^n (n \geq 1)$ is called the n^{th} iterate of T.

Let (M, d) be a metric space. For any $x_0 \in M$, the sequence $\{x_n\}_{n\geq 0} \subset M$ given by

$$x_n = Tx_{n-1} = T^n x_0,\ n = 1, 2, ... \tag{5.1.1}$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

The method of successive approximations appears to have been introduced by Liouville [85] and used by Cauchy. It was developed systematically for the first time by Picard [94] in his classical and well-known proof of the existence and uniqueness of the solution of initial value problems for ordinary differential equations, dating back to 1890.

All the other fixed point iteration schemes are introduced in a real normed space E. Let $T : E \to E$ be a mapping, $x_0 \in E$ and $\lambda \in [0, 1]$. The sequence $\{x_n\}_{n\geq 0}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n = 0, 1, 2, ... \tag{5.1.2}$$

is called the Krasnoselskii iteration procedure or, simply, Krasnoselskii iteration.

The Krasnoselskii iteration, for the particular case $\lambda = \frac{1}{2}$, was first introduced by Krasnoselskii [76] in 1955, and in the general form by Schaefer [102] in 1957.

It is easy to see that the Krasnoselskii iteration $\{x_n\}_{n\geq 0}$ given by (5.1.2) is exactly the Picard iteration corresponding to the averaged operator

$$T_\lambda = (1-\lambda)I + \lambda T,\ I = \text{ the identity operator} \tag{5.1.3}$$

and that for $\lambda = 1$ the Krasnoselskii iteration reduces to Picard iteration. Moreover, we have $F(T) = F(T_\lambda)$, for all $\lambda \in (0, 1]$.

The normal Mann iteration or Mann iteration, starting from $x_0 \in E$, is the sequence $\{x_n\}_{n\geq 0}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n = 0, 1, 2, ..., \tag{5.1.4}$$

where $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ satisfies certain appropriate conditions.

The original Mann iteration was defined in a matrix formulation by Mann [86] in 1953.

If we consider

$$T_n = (1-\alpha_n)I + \alpha_n T,$$

then we have $F(T) = F(T_n)$, for all $\alpha_n \in (0, 1]$.

If the sequence $\alpha_n = \lambda$ (constant), then the Mann iteration process obviously reduces to the Krasnoselskii iteration.

In 1974, Ishikawa [51] introduced an iteration process. The Ishikawa iteration scheme or, simply, Ishikawa iteration defined by $x_0 \in E$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n],\ n = 0, 1, 2, ..., \tag{5.1.5}$$

where $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \subset [0, 1]$ satisfy certain appropriate conditions.

If we rewrite (5.1.5) in a systematic form

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n,\ n = 0, 1, 2, ..., \end{aligned}$$

then we can regard the Ishikawa iteration as a sort of two-step Mann iteration, with two different parameter sequences.

Despite this apparent similarity and the fact that, for $\beta_n = 0$, Ishikawa iteration reduces to the Mann iteration, generally, there is no dependence between convergence results for Mann iteration and Ishikawa iteration.

The Krasnoselskii, Mann and Ishikawa iteration procedures are mainly used to generate successive approximations for fixed points of various classes of mappings in normed linear spaces, for which the Picard iteration does not converge.

5.2 Approximate Fixed Point Sequences

Let (M, d) be a metric space and $T : M \to M$ be a mapping. A sequence $\{x_n\} \subset M$ satisfying

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

is called an approximate fixed point sequence for T.

Convergence theorems for various mappings, through different iterative methods have been obtained by a number of authors (e.g., [95, 104, 115] and the references therein). For more on the study of fixed point iteration process, the interested reader is referred to Berinde [14] and Ciric [28, 29].

To discuss the convergence of approximate fixed point sequences in hyperbolic spaces, we introduce a different notion of convergence; namely, $\triangle$-convergence.

The concept of $\triangle$-convergence was introduced several years ago independently by Kuczumow [77] and Lim [83], which behaves in $CAT(0)$ spaces as weak convergence in Banach spaces [32].

Let $\{x_n\}$ be a bounded sequence in a metric space M and C be a nonempty subset of M. Define $r\left(., \{x_n\}\right) : C \to [0, \infty)$, by:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius ρ_C of $\{x_n\}$ with respect to C is given by

$$\rho_C = \inf\Big\{r(x, \{x_n\}) : x \in C\Big\}.$$

The asymptotic radius of $\{x_n\}$ with respect to M will be denoted by ρ. A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to C if $r(\xi, \{x_n\}) = r(C, \{x_n\}) = \min\{r(x, \{x_n\}) : x \in C\}$. We denote by $A(C, \{x_n\})$, the set of asymptotic centers of $\{x_n\}$ with respect to C. When $C = M$, we call ξ an asymptotic center of $\{x_n\}$ and simply use the notation $A(\{x_n\})$. In general, the set $A(C, \{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or may even contain infinitely many points.

Definition 5.2.1 *A bounded sequence $\{x_n\}$ in a metric space M is said to $\Delta-$converge to $x \in M$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. Symbolically, $x_n \overset{\Delta}{\to} x$.*

It is known that uniformly convex Banach spaces and even $CAT(0)$ spaces enjoy the property that "bounded sequences have unique asymptotic center with respect to closed and convex subsets" [32]. The following lemma is due to Leustean [81] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 5.2.1 *[81] Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed and convex subset C of X.*

Lemma 5.2.2 *([61]) Let C be a nonempty, closed and convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in C such that*

$A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.

5.3 Nonexpansive Iteration in Hyperbolic Spaces

In this section, we establish strong convergence and Δ-convergence theorems of an implicit iteration algorithm associated with a pair of nonexpansive mappings on a nonlinear domain. In particular, we prove that such an algorithm converges to a common fixed point of the mappings. Our results generalize well-known similar results in the linear setting.

5.3.1 Introduction

Recent developments in fixed point theory reflect that the iterative construction of fixed points is vigorously proposed and analyzed for various classes of mappings in different spaces [58]. Implicit algorithms provide better approximation of fixed points than explicit algorithms. The number of steps of an algorithm also plays an important role in iterative approximation methods. The case of two mappings has a direct link with the minimization problem [108].

Recently, Hou and Du [50] introduced the following new general implicit iteration scheme for approximating common fixed points of a pair of nonexpansive mappings in a uniformly convex Banach space. Given $x_0 \in C$ (a subset of a

Banach space),

$$\begin{aligned} x_n &= a_n x_{n-1} + b_n T y_n + c_n S x_n, \\ y_n &= a'_n x_{n-1} + b'_n x_n + c'_n S x_{n-1} + d'_n T x_n, \quad n \geq 0, \end{aligned} \tag{5.3.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{d'_n\}$ are sequences of real numbers in $[0,1]$ satisfying $a_n + b_n + c_n = 1,\ a'_n + b'_n + c'_n + d'_n = 1$, and $T,\ S : C \to C$ are nonexpansive mappings.

We investigate $\triangle$-convergence as well as strong convergence through the two-step implicit algorithm (5.3.1) of two nonexpansive mappings in the more general setup of hyperbolic spaces.

The two-step algorithm (5.3.1) can be defined in a hyperbolic space in the sense of Kohlenbach [71] as follows:

$$\begin{aligned} x_n &= W\left(Ty_n, W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), b_n\right), \\ y_n &= W\left(Sx_{n-1}, W\left(x_n, W\left(Tx_n, x_{n-1}, \frac{d'_n}{1-b'_n-c'_n}\right), \frac{b'_n}{1-c'_n}\right), c'_n\right), \quad n \geq 1 \end{aligned} \tag{5.3.2}$$

provided $b_n < 1$ and $b'_n + c'_n < 1$.

In order to establish that algorithm (5.3.2) exists, we define a mapping $G_1 : C \to C$ by: $G_1(x) = W\left(Ty, W\left(Sx, x_0, \frac{c_1}{1-b_1}\right), b_1\right)$, where $y = W\left(Sx_0, W\left(x, W\left(Tx, x_0, \frac{d'_1}{1-b'_1-c'_1}\right), \frac{b'_1}{1-c'_1}\right), c'_1\right)$. For a given $x_0 \in C$, the existence of x_1 is guaranteed if G_1 has a fixed point. Now for any $u, v \in C$, we have

$$\begin{aligned}
d(G_1u, G_1v) &= d(W(TW(Sx_0, W(u, W(Tu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1'), \\
&\quad W(Su, x_0, \frac{c_1}{1-b_1}), b_1), W(TW(Sx_0, W(\nu, \\
&\quad W(T\nu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1'), W(S\nu, x_0, \frac{c_1}{1-b_1}), b_1)) \\
&\leq b_1 d(TW(Sx_0, W(u, W(Tu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1'), \\
&\quad TW(Sx_0, W(\nu, W(T\nu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1')) \\
&\quad +(1-b_1) d(W(Su, x_0, \frac{c_1}{1-b_1}), W(S\nu, x_0, \frac{c_1}{1-b_1})) \\
&\leq b_1 d(\ W(Sx_0, W(u, W(Tu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1'), \\
&\quad W(Sx_0, W(\nu, W(T\nu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), c_1')) + c_1 d(\ u,\ \nu) \\
&\leq b_1(1-c_1') d(\ W(u, W(Tu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'}), \\
&\quad W(\nu, W(T\nu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \frac{b_1'}{1-c_1'})) +\ c_1 d(\ u,\ \nu) \\
&\leq b_1 b_1' d(\ u,\ \nu) + b_1(1-c_1')(1-\frac{b_1'}{1-c_1'}) d(\ W(Tu, x_0, \frac{d_1'}{1-b_1'-c_1'}), \\
&\quad W(T\nu, x_0, \frac{d_1'}{1-b_1'-c_1'})) +\ c_1 d(\ u,\ \nu) \\
&\leq b_1 b_1' d(\ u,\ \nu) + b_1 (1-c_1') \left(1-\frac{b_1'}{1-c_1'}\right) (\frac{d_1'}{1-b_1'-c_1'}) d(\ u,\ \nu) + \\
&\quad c_1 d(\ u,\ \nu) \\
&< d(\ u,\ \nu).
\end{aligned}$$

Therefore G_1 is a contraction. By Banach Contraction Principle, G_1 has a unique fixed point. Thus the existence of x_1 is established. Continuing in this way, we can establish the existence of x_2, x_3 and so on. Thus the implicit algorithm (5.3.2) is well-defined.

From now on, for two mappings T and S, we set $F = F(T) \cap F(S)$.

We need the following.

Lemma 5.3.1 *The convex structure W in the sense of Kohlenbach [71] is a continuous mapping.*

Proof. Let (x_n, y_n, α_n) converges to (x, y, α). We claim, $W(x_n, y_n, \alpha_n)$ converges to $W(x, y, \alpha)$.

To show this, $d(W(x_n, y_n, \alpha_n), W(x, y, \alpha)) \leq d(W(x_n, y_n, \alpha_n), W(x_n, y_n, \alpha)) + d(W(x_n, y_n, \alpha), W(x, y, \alpha)) \leq |\alpha_n - \alpha| d(x_n, y_n) + \alpha d(x_n, x) + (1 - \alpha) d(y_n, y)$.

Taking $\lim_{n\to\infty}$ yields, $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), W(x, y, \alpha)) = 0$.

∎

Lemma 5.3.2 *[61] Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(x_n, x) \leq r$, $\limsup_{n\longrightarrow\infty} d(y_n, x) \leq r$ and $\lim_{n\longrightarrow\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 5.3.3 *Let C be a nonempty, closed and convex subset of a hyperbolic space X and let T and S be nonexpansive selfmappings on C such that $F \neq \phi$. If the sequence $\{x_n\}$ in (5.3.2) satisfies the following conditions:*

1. $a_n \to 0, a'_n \to 0, b'_n \to 0, as\ n \to \infty$;

2. $b_n, c_n, c'_n, d'_n \in [\delta, 1-\delta],\ \delta \in (0, \frac{1}{2})$;

3. $c'_n + d'_n \le \gamma,\ \gamma \in (0,1)$,

then we have,

1. $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in F$. *and* $x_n,\ Tx_n$ *and* Sx_n *are all bounded;*

2. $\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0$.

Proof. For any $p \in F$,

$$\begin{aligned}
d(x_n,p) =& d(W(Ty_n, W(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}), b_n), p) \\
\le& b_n d(Ty_n, p) + (1-b_n) d(W(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}), p) \\
\le& b_n d(\ y_n, p) + c_n d(\ x_n, p) + (1-b_n)(1 - \frac{c_n}{1-b_n}) d(x_{n-1}, p) \\
\le& b_n c'_n d(\ x_{n-1}, p) + b_n (1-c'_n) d(\ W(x_n, W(Tx_n, x_{n-1}, \frac{d'_n}{1-b'_n-c'_n}), \frac{b'_n}{1-c'_n}),\ p) \\
&+ c_n d(\ x_n, p) + (1-b_n)(1 - \frac{c_n}{1-b_n}) d(x_{n-1}, p) \\
&= [1 - b_n b'_n - b_n d'_n - c_n]\, d\,(x_{n-1}, p) + [b_n b'_n + b_n d'_n + c_n]\, d\,(\ x_n, p)\,,
\end{aligned}$$

Therefore,

$$d(x_n, p) \le d(x_{n-1}, p).$$

It follows, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Consequently, $\lim_{n\to\infty} d(x_n, F)$ exists and $\{x_n\},\ \{Tx_n\}$ and $\{Sx_n\}$ are all bounded.

Next, assume that $\lim_{n\to\infty} d(x_n, p) = c$.

The case $c = 0$ is trivial. Next, we deal with the case $c > 0$.

As S is nonexpansive, so $\limsup_{n\to\infty} d(Sx_n, p) \leq c$.

Note that

$$\begin{aligned}
d(y_n, p) =& d(W(Sx_{n-1}, W(x_n, W(Tx_n, x_{n-1}, \frac{d'_n}{1-b'_n-c'_n}), \frac{b'_n}{1-c'_n}), c'_n), p) \\
\leq& c'_n d(x_{n-1}, p) + (1-c'_n) d(\ W(x_n, W(Tx_n, x_{n-1}, \frac{d'_n}{1-b'_n-c'_n}, \frac{b'_n}{1-c'_n})\ p) \\
\leq& c'_n d(x_{n-1}, p) + b'_n d(\ x_n, p) + (1-c'_n)(1-\frac{b'_n}{1-c'_n})(\frac{d'_n}{1-b'_n-c'_n}) d(\ x_n, p) \\
&+ (1-c'_n)(1-\frac{b'_n}{1-c'_n})(1-\frac{d'_n}{1-b'_n-c'_n}) d(x_{n-1}, p) \\
\leq& d(x_{n-1}, p) + (b'_n + d'_n)(d(x_n, p) - d(x_{n-1}, p))
\end{aligned}$$

Taking lim sup on both sides in the above estimate, we have

$$\limsup_{n\to\infty} d(y_n, p) \leq c.$$

Since T is nonexpansive, therefore $\limsup_{n\to\infty} d(Ty_n, p) \leq c$.

Moreover,

$$\begin{aligned} d\left(x_n, p\right) &= d\left(W\left(Ty_n, W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), b_n\right), p\right) \\ &= d\left(W\left(Ty_n, W\left(Sx_n, x_{n-1}, 1-\frac{a_n}{1-b_n}\right), b_n\right), p\right), \end{aligned}$$

since $\lim_{n\to\infty} W\left(Sx_n, x_{n-1}, 1-\frac{a_n}{1-b_n}\right) = \lim_{n\to\infty} Sx_n$,

we have $\lim_{n\to\infty} d\left(x_n, p\right) = \lim_{n\to\infty} d\left(W\left(Ty_n, Sx_n, b_n\right), p\right) = c$.

Therefore, by Lemma 5.3.2, $\lim_{n\to\infty} d\left(Ty_n, Sx_n\right) = 0$.

Next,

$$d\left(W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), p\right) \leq d\left(x_{n-1}, p\right) + \frac{c_n}{1-b_n}\left(d\left(x_n, p\right) - d\left(x_{n-1}, p\right)\right),$$

hence

$$\limsup_{n\to\infty} d\left(W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), p\right) \leq c.$$

Since $\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d\left(W\left(Ty_n, W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), b_n\right), p\right) = c$.

So, by Lemma 5.3.2, we have

$$\lim_{n\to\infty} d\left(Ty_n, W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right)\right) = 0.$$

Moreover,

$$
\begin{aligned}
d\left(x_n, p\right) &= d\left(W\left(Ty_n, W\left(Sx_n, x_{n-1}, 1-\frac{a_n}{1-b_n}\right), b_n\right), p\right) \\
&\leq b_n d\left(Ty_n, p\right)+\left(1-b_n\right) d\left(W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), p\right) \\
&\leq d\left(Ty_n, p\right)+\left(1-b_n\right) d\left(W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), Ty_n\right),
\end{aligned}
$$

Taking lim inf on both sides, we have

$$
c \leq \liminf_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq c.
$$

That is,

$$
\lim_{n\to\infty} d(y_n, p) = c.
$$

Moreover,

$$
d\left(y_n, p\right) = d\left(W\left(Sx_{n-1}, W\left(x_n, W\left(Tx_n, x_{n-1}, \frac{d_n'}{1-b_n'-c_n'}\right), \frac{b_n'}{1-c_n'}\right), c_n'\right), p\right),
$$

so,

$$
\begin{aligned}
\lim_{n\to\infty} d(y_n, p) &= lim_{n\to\infty} d\left(W\left(Sx_{n-1}, W\left(x_n, W\left(Tx_n, x_{n-1}, 1\right), 0\right), c_n'\right), p\right) \\
&= \lim_{n\to\infty} d\left(W\left(Sx_{n-1}, W\left(x_n, Tx_n, 0\right), c_n'\right), p\right) \\
&= \lim_{n\to\infty} d\left(W\left(Sx_{n-1}, Tx_n, c_n'\right), p\right).
\end{aligned}
$$

So $\lim_{n\to\infty} d\left(W\left(Sx_{n-1}, Tx_n, c'_n\right), p\right) = c$.

Since, $\limsup_{n\to\infty} d(Sx_{n-1}, p) \leq c$ and $\limsup_{n\to\infty} d(Tx_n, p) \leq c$ Thus, by Lemma 5.3.2,

$\lim_{n\to\infty} d\left(Sx_{n-1}, Tx_n\right) = 0$.

Moreover,

$$\begin{aligned} d\left(x_n, \mathrm{Ty}_n\right) &= d\left(W\left(Ty_n, W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), b_n\right), \mathrm{Ty}_n\right) \\ &\leq b_n d\left(Ty_n, \mathrm{Ty}_n\right) + (1-b_n)\, d\left(W\left(Sx_n, x_{n-1}, \frac{c_n}{1-b_n}\right), Ty_n\right) \\ &\leq c_n d\left(\mathrm{Sx}_n, \mathrm{Ty}_n\right) + a_n d\left(x_{n-1}, Ty_n\right). \end{aligned}$$

So,

$$\lim_{n\to\infty} d\left(x_n, \mathrm{Ty}_n\right) = 0$$

Now, $d(x_n, Sx_n) \leq d(x_n, Ty_n) + d(Ty_n, Sx_n)$, gives by Lemma 5.3.2, $\lim_{n\to\infty} d(x_n, Sx_n) = 0$.

Next,

$$\begin{aligned} d\left(y_n, x_n\right) &= d\left(W\left(Sx_{n-1}, W\left(x_n, W\left(Tx_n, x_{n-1}, \frac{d'_n}{1-b'_n-c'_n}\right), \frac{b'_n}{1-c'_n}\right), c'_n\right), x_n\right) \\ &\leq c'_n d\left(Sx_{n-1}, Tx_n\right) + c'_n d\left(\ x_n, Tx_n\right) + d'_n d\left(\ x_n, Tx_n\right) + a'_n d\left(\ x_n, x_{n-1}\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
d\left(x_n, Tx_n\right) \leq & d(x_n, Sx_n) + d(Sx_n, Ty_n) + d(Ty_n, Tx_n) \\
\leq & d(x_n, Sx_n) + d(Sx_n, Ty_n) + d(y_n, x_n) \\
\leq & d(x_n, Sx_n) + d(Sx_n, Ty_n) + c'_n d(Sx_{n-1}, Tx_n) + c'_n d(\ x_n, Tx_n) \\
& + d'_n d(\ x_n, Tx_n) + a'_n d(\ x_n, x_{n-1})
\end{aligned}
$$

Or,

$$
d\left(x_n, Tx_n\right) = \frac{1}{1 - c'_n - d'_n}\left(d\left(x_n, Sx_n\right) + d\left(Sx_n, Ty_n\right) + c'_n d\left(Sx_{n-1}, Tx_n\right) + a'_n d\left(\ x_n, x_{n-1}\right)\right),
$$

hence

$$
d\left(x_n, Tx_n\right) \leq \frac{1}{1-\gamma}\left(d\left(x_n, Sx_n\right) + d\left(Sx_n, Ty_n\right) + c'_n d\left(Sx_{n-1}, Tx_n\right) + a'_n d\left(\ x_n, x_{n-1}\right)\right)
$$

Therefore, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

∎

5.3.2 Convergence in hyperbolic spaces

In this section, we establish △- convergence and strong convergence of the implicit algorithm (5.3.2).

Theorem 5.3.1 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity*

η and let T and S be nonexpansive selfmappings on C such that $F \neq \phi$. Then the sequence $\{x_n\}$ in (5.3.2), $\triangle-$converges to a common fixed point of T and S.

Proof.

It follows from Lemma 5.3.3 that $\{x_n\}$ is bounded. Therefore, by Lemma 5.2.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 5.3.3, we have $\lim_{n\to\infty} d(u_n, Tu_n) = 0 = \lim_{n\to\infty} d(u_n, Su_n)$.

We claim that u is the common fixed point of T and S.

Clearly,

$$d\left(Tu, u_n\right) \leq d\left(Tu, Tu_n\right) + d\left(Tu_n, u_n\right),$$

taking lim sup,

$$\limsup_{n\to\infty} d\left(Tu, u_n\right) \leq \limsup_{n\to\infty} d\left(u, u_n\right),$$

we set, $r(Tu, u_n) \leq r(u, u_n)$. i. e., $Tu \in A(u_n)$. Hence, $Tu = u$. Similarly, we can show that $Su = u$.

Therefore, u is the common fixed point of T and S.

Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 5.3.3.

Suppose $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, u) \\
&= \limsup_{n\to\infty} d(u_n, u),
\end{aligned}$$

a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle-$converges to a common fixed point of T and S.

∎

Recall that a sequence $\{x_n\}$ in a metric space M is said to be *Fejér monotone* with respect to C (a subset of M) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in C$ and for all $n \geq 1$. A mapping $T : C \to C$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Let f be a nondecreasing selfmapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$. Then a selfmapping T on C with $F(T) \neq \phi$, satisfies condition (A) if

$$d\,(x, Tx) \geq f(d(x, F(T))) \quad \text{for all } x \in C.$$

Different modifications of the condition (A) for two selfmappings have been made recently in the literature [62, 96] as follows:

Let T and S be two nonexpansive selfmappings on C with $F \neq \phi$. Then the mappings T and S are said to satisfy:

(i) condition (B) on C if

$$d\,(x, Tx) \geq f(d(x, F)) \text{ or } d\,(x, Sx) \geq f(d(x, F)) \quad \text{for all } x \in C;$$

(ii) condition (C) on C if

$$\frac{1}{2}\left\{d\,(x, Tx) + d\,(x, Sx)\right\} \geq f(d(x, F)) \quad \text{for all } x \in C.$$

Note that the condition (B) and the condition (C) are equivalent to the condition (A) if $T = S$.

We shall use conditions (B) and (C) to study strong convergence of the algorithm (5.3.2).

For further development, we need the following technical result.

Lemma 5.3.4 *[11] Let C be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to C. Then $\{x_n\}$ converges to some $p \in C$ if and only if* $\lim_{n\to\infty} d(x_n, C) = 0$.

Lemma 5.3.5 *Let C be a nonempty, closed and convex subset of a complete* *uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let T and S be nonexpansive selfmappings on C such that $F \neq \phi$. Then the sequence $\{x_n\}$ in (5.3.2) converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

We now establish strong convergence of the algorithm (5.3.2) based on Lemma 5.3.5.

Theorem 5.3.2 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T and S be nonexpansive selfmappings on C with $F \neq \phi$ and satisfying condition (B). Then the sequence $\{x_n\}$ in (5.3.2) converges strongly to $p \in F$.*

Proof. It follows from Lemma 5.3.3 that $\lim_{n\to\infty} d(x_n, F)$ exists. Moreover, Lemma 5.3.3 implies that $\lim_{n\to\infty} d(x_n, Tx_n) = d(x_n, Sx_n) = 0$. So condition (B) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 5.3.5 implies that $\{x_n\}$ converges strongly to a point p in F. ∎

Theorem 5.3.3 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T and S be nonexpansive selfmappings on C with $F \neq \phi$and satisfying* *condition (C). Then the sequence $\{x_n\}$ in (5.3.2) converges strongly to $p \in F$.*

Proof. It follows from Lemma 5.3.3 that $\lim_{n\to\infty} d(x_n, F)$ exists. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$, By Lemma 5.3.3 $\lim_{n\to\infty} d(x_n, Tx_n) = d(x_n, Sx_n) = 0$. Moreover, $d(x_n, p) \leq d(x_{n-1}, p)$ for all p in F, which gives

$$\inf_{p\in F} d(x_n, p) \leq \inf_{p\in F} d(x_{n-1}, p).$$

That is, $d(x_n, F) \leq d(x_{n-1}, F)$ which shows that $\lim_{n\to\infty} d(x_n, F)$ exists. So condition (C) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 5.3.5 implies that $\{x_n\}$ converges strongly to a point p in F. ∎

Theorem 5.3.4 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T and S be nonexpansive selfmappings on C with $F \neq \phi$. Suppose that the mappings T and S are semi-compact. Then the sequence $\{x_n\}$in (5.3.2) converges strongly to $p \in F$.*

Proof. Since one of T and S is semicompact, therefore by Lemma 5.3.3, there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that it converges strongly to u. Since C is closed, $u \in C$. i. e., $\lim_{n\to\infty} d(x_{n_j}, C) = 0$ which implies

$\lim_{n\to\infty} d(x_{nj}, F) = 0$. It follows from Lemma 5.3.3 that $\lim_{n\to\infty} d(x_n, F)$ exists, so $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 5.3.5 implies that $\{x_n\}$ converges strongly to a point p in F. ∎

5.4 Quasi-Nonexpansive Iteration in Convex Metric Spaces

In this section, a general viscosity iterative method for a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space is introduced. The special cases of new iterative method are viscosity iterative method of Chang et al. [27], analogue of viscosity iterative method of Fukhar-ud-din et al. [39] and an extension of the multistep iterative method of Yildirim and Özdemir [114]. Our results generalize and extend the corresponding known results in uniformly convex Banach spaces and $CAT(0)$ spaces simultaneously [60].

5.4.1 Introduction

Let C be a nonempty subset of a metric space M and $T : C \to C$ be a mapping. We assume that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$ for $I = \{1, 2, 3, ..., r\}$. The mapping T is (i) quasi-nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x \in C, y \in F(T)$ (ii) asymptotically quasi-nonexpansive if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n) d(x, p)$ for all $x \in C, p \in F(T)$ and $n \geq 1$ (iii) generalized asymptotically quasi-nonexpansive [104] if there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} c_n$ such that $d(T^n x, p) \leq d(x, p) + u_n d(x, p) + c_n$ for all $x \in C, p \in F(T)$ and $n \geq 1$ (iv) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)$, for all $x, y \in C$ and $n \geq 1$ and (v) uniformly Hölder continuous if there are constants $L > 0, \gamma > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)^\gamma$ for all $x, y \in C$ and $n \geq 1$.

Clearly, the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

The following example improves and extends ([104], Example 3.2) to a finite family of generalized asymptotically quasi-nonexpansive mappings

Example 5.4.1 *Let $E = \mathbb{R}$ and $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$. Define $T_i x = \frac{x}{i+1} sin(\frac{1}{x})$ if $x \neq 0$ and $T_i x = 0$ if $x = 0$ for all $x \in C$ and $i \in I$. Then $T_i^n x \to 0$ uniformly (see [63]). For each fixed n, define $f_{in}(x) = ||T_i^n x|| - ||x||$ for all x in C and $i \in I$. Set $c_{in} = \sup_{x\in C}\{f_{in}(x), 0\}$. Then $\lim_{n\to\infty} c_{in} = 0$ and*

$$||T_i^n x|| \leq ||x|| + c_{in}.$$

This shows that $\{T_i : i \in I\}$ is a finite family of generalized asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$.

Let C be a convex subset of a normed space. Yildirim and Özdemir [114] introduced the following multistep iterative method:

$$
\begin{aligned}
x_1 &\in C, \\
x_{n+1} &= (1-a_{1n})y_{n+r-2} + a_{1n}T_1^n\ y_{n+r-2}, \\
y_{n+r-2} &= (1-a_{2n})y_{n+r-3} + a_{2n}T_2^n y_{n+r-3}, \\
&. \\
&. \\
&. \\
y_{n+1} &= (1-a_{(r-1)n})y_n + a_{(r-1)n}T_{(r-1)}^n\ y_n, \\
y_n &= (1-a_{rn})x_n + a_{rn}T_r^n\ x_n,\ r \geq 2,\ n \geq 1,
\end{aligned} \tag{5.4.1}
$$

where $\{T_i : i \in I\}$ is a family of selfmappings of C, $a_{in} \in [\epsilon, 1-\epsilon]$, for some $\epsilon \in (0, \frac{1}{2})$, for all $n \geq 1$.

If $T_1 = T_2 = ... = T_r$ and $\alpha_{jn} = 0$ for $j = 1, ..., r$ and $r \geq 1$, then the iterative method (5.4.1) reduces to Mann iterative method [86]. Let us note that the scheme (5.4.1) and multistep scheme (1.3) in [59] are independent of each other.

Moudafi [89] proposed viscosity iterative method by selecting a particular fixed point of a given nonexpansive mapping. The so-called viscosity iterative method has been studied by many authors (see, for example, see [95, 112]). These methods are very important because of their applicability to convex optimization, linear programming, monotone inclusions and elliptic differential equations [89].

Recently, Chang et al. [27] introduced and studied the following viscosity iterative method:

$$x_{n+1} = (1-\alpha_n)f(x_n) + \alpha_n T^n y_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1 \tag{5.4.2}$$

where T is an asymptotically nonexpansive mapping [41] and f is a fixed contraction.

We now devise a general iterative method which extends the methods in (5.4.1) and (5.4.2), simultaneously in a convex metric space in the sense of Takahashi [107].

We define S_n- mapping generated by a family $\{T_i : i \in I\}$ of generalized asymptotically quasi-nonexpansive mappings on C as:

$$S_n x = U_{rn} x \tag{5.4.3}$$

where $U_{0n} = I$ (the identity mapping), $U_{1n}x = W(T_r^n U_{0n}x, U_{0n}x, a_{rn})$,
$U_{2n}x = W(T_{r-1}^n U_{1n}x, U_{1n}x, a_{(r-1)n}), ..., U_{rn}x = W(T_1^n U_{(r-1)n}x, U_{(r-1)n}x, a_{1n})$.

For $\{\alpha_n\} \subset J$,a fixed contractive mapping f on C and S_n given in (5.4.3), we define $\{x_n\}$ as follows:

$$x_1 \in C, x_{n+1} = W(f(x_n), S_n x_n, \alpha_n), \tag{5.4.4}$$

and call it a general viscosity iterative method in a convex metric space.

In general, a convex structure W in the sense of Takahashi [107] is not continuous [109]. Throughout this section, we assume that W is continuous.

We need the following known results for our convergence analysis.

Lemma 5.4.1 *(cf. [106]) Let the sequences* $\{a_n\}$ *and* $\{u_n\}$ *of real numbers satisfy:*

$$a_{n+1} \leq (1+u_n)a_n,\ a_n \geq 0,\ u_n \geq 0, \sum_{n=1}^{\infty} u_n < +\infty.$$

Then (i) $\lim_{n\to\infty} a_n$ *exists; (ii) if* $\liminf_{n\to\infty} a_n = 0$*, then* $\lim_{n\to\infty} a_n = 0$.

5.4.2 Convergence in convex metric spaces

In this section, we prove some results for the viscosity iterative method (5.4.4) to converge to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space.

Lemma 5.4.2 *Let* C *be a nonempty, closed and convex subset of a convex metric space* X *and* $\{T_i : i \in I\}$ *be a family of generalized asymptotically quasi-nonexpansive selfmappings of* C*, i.e.,* $d(T_i^n x, p_i) \leq (1+u_{in})d(x, p_i) + c_{in}$ *for all* $x \in C$ *and* $p_i \in F(T_i)$*,* $i \in I$ *where* $\{u_{in}\}$ *and* $\{c_{in}\}$ *are sequences in* $[0, \infty)$ *with* $\sum_{n=1}^{\infty} u_{in} < \infty, \sum_{n=1}^{\infty} c_{in} < \infty$ *for each* i*. Then, for the sequence* $\{x_n\}$ *in (5.4.4) with* $\sum_{n=1}^{\infty} \alpha_n < \infty$*, there are sequences* $\{\nu_n\}$ *and* $\{\xi_n\}$ *in* $[0, \infty)$ *satisfying* $\sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$ *such that*

(a) $d(x_{n+1}, p) \leq (1+\nu_n)^r d(x_n, p) + \xi_n$*, for all* $p \in F$ *and all* $n \geq 1$*;*

(b) $d(x_{n+m}, p) \leq M_1 (d(x_n, p) + \sum_{n=1}^{\infty} \xi_n)$*, for all* $p \in F$ *and* $n \geq 1, m \geq 1, M_1 > 0$.

Proof. (a) Let $p \in F$ and $\nu_n = \max_{i\in I} u_{in}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} u_{in} < \infty$ for each i, therefore $\sum_{n=1}^{\infty} \nu_n < \infty$.

Now we have

$$\begin{aligned} d(U_{1n}x_n, p) &= d(W(T_r^n U_{0n}x_n, U_{0n}x_n, a_{rn}), p) \\ &\leq (1-a_{rn})d(x_n,p) + a_{rn}d(T_r^n x_n, p) \\ &\leq (1-a_{rn})d(x_n,p) + a_{rn}\left[(1+u_{rn})d(x_n,p) + c_{rn}\right] \\ &\leq (1+u_{rn})d(x_n,p) + c_{rn} \\ &\leq (1+\nu_n)^1 d(x_n,p) + c_{rn}. \end{aligned}$$

Assume that $d(U_{kn}x_n, p) \leq (1+\nu_n)^k d(x_n,p) + (1+\nu_n)^{k-1}\sum_{i=1}^{k} c_{(r-i+1)n}$ holds for some $1 < k$.

Consider

$$\begin{aligned} d\left(U_{(k+1)n}x_n, p\right) &= d\left(W(T_{r-k}^n U_{kn}x_n, U_{kn}x_n, a_{(r-k)n}), p\right) \\ &\leq (1-a_{(r-k)n})d(U_{kn}x_n,p) + a_{(r-k)n}d\left(T_{r-k}^n U_{kn}x_n, p\right) \\ &\leq (1-a_{(r-k)n})d(U_{kn}x_n,p) + a_{(r-k)n}[(1+u_{(r-k)n})d(U_{kn}x_n,p) \\ &\quad + c_{(r-k)n}] \\ &\leq (1+\nu_n)d(U_{kn}x_n,p) + c_{(r-k)n} \\ &\leq (1+\nu_n)[(1+\nu_n)^k d(x_n,p) + (1+\nu_n)^{k-1}\sum_{i=1}^{k} c_{(r-i+1)n}] \\ &\quad + c_{(r-k)n} \\ &\leq (1+\nu_n)^{k+1} d(x_n,p) + (1+\nu_n)^k \sum_{i=1}^{k+1} c_{(r-i+1)n} \end{aligned}$$

By mathematical induction, we have

$$d\left(U_{jn}x_n,p\right)\leq(1+\nu_n)^j d\left(x_n,p\right)+(1+\nu_n)^{j-1}\sum_{i=1}^{j}c_{(r-i+1)n},\quad 1\leq j\leq r. \quad (5.4.5)$$

Hence

$$d\left(S_nx_n,p\right)=d\left(U_{rn}x_n,p\right)\leq(1+\nu_n)^r d\left(x_n,p\right)+(1+\nu_n)^{r-1}\sum_{i=1}^{r}c_{(r-i+1)n}. \quad (5.4.6)$$

Now, by (5.4.4) and (5.4.6), we obtain

$$\begin{aligned}
d\left(x_{n+1},p\right) &= d\left(W(f\left(x_n\right),S_nx_n,\alpha_n),p\right)\\
&\leq \alpha_n d\left(f\left(x_n\right),p\right)+\left(1-\alpha_n\right)d\left(S_nx_n,p\right)\\
&\leq \alpha_n d\left(x_n,p\right)+\alpha_n d\left(f\left(p\right),p\right)\\
&\quad +\left(1-\alpha_n\right)\left((1+\nu_n)^r d\left(x_n,p\right)+(1+\nu_n)^{r-1}\sum_{i=1}^{r}c_{(r-i+1)n}\right)\\
&\leq (1+\nu_n)^r d\left(x_n,p\right)+\left(1-\alpha_n\right)\left(1+\nu_n\right)^{r-1}\sum_{i=1}^{r}c_{(r-i+1)n}\\
&\quad +\alpha_n d\left(f\left(p\right),p\right)\\
&\leq (1+\nu_n)^r d\left(x_n,p\right)+\alpha_n d\left(f\left(p\right),p\right)+(1+\nu_n)^{r-1}\sum_{i=1}^{r}c_{(r-i+1)n}.
\end{aligned}$$

Setting $\max\left\{d\left(f\left(p\right),p\right),\sup(1+\nu_n)^{r-1}\right\}=M$, we get that

$$d\left(x_{n+1},p\right)\leq(1+\nu_n)^r d\left(x_n,p\right)+M\left(\alpha_n+\sum_{i=1}^{r}c_{(r-i+1)n}\right).$$

That is,

$$d(x_{n+1}, p) \leq (1+\nu_n)^r d(x_n, p) + \xi_n,$$

where $\xi_n = M\left(\alpha_n + \sum_{i=1}^{r} c_{(r-i+1)n}\right)$ and $\sum_{n=1}^{\infty} \xi_n < \infty$.

(b) We know that $1 + t \leq e^t$ for $t \geq 0$. Thus, by part (a), we have

$$\begin{aligned}
d(x_{n+m}, p) &\leq (1+\nu_{n+m-1})^r d(x_{n+m-1}, p) + \xi_{n+m-1} \\
&\leq e^{r\nu_{n+m-1}} d(x_{n+m-1}, p) + \xi_{n+m-1} \\
&\leq e^{r(\nu_{n+m-1}+\nu_{n+m-2})} d(x_{n+m-2}, p) + \xi_{n+m-1} + \xi_{n+m-2} \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq e^{r\sum_{i=n}^{n+m-1} v_i} d(x_n, p) + \sum_{i=n+1}^{n+m-1} v_i \sum_{i=n}^{n+m-1} \xi_i \\
&\leq e^{r\sum_{i=1}^{\infty} v_i} \left(d(x_n, p) + \sum_{i=1}^{\infty} \xi_i \right) \\
&= M_1 \left(d(x_n, p) + \sum_{i=1}^{\infty} \xi_i \right), \text{where} M_1 = e^{r\sum_{i=1}^{\infty} v_i}.
\end{aligned}$$

∎

The next result deals with a necessary and sufficient condition for the convergence of $\{x_n\}$ in (5.4.4) to a point of F.

Theorem 5.4.1 *Let C, $\{T_i : i \in I\}$, F, $\{u_{in}\}$ and $\{c_{in}\}$ be as in Lemma 5.4.2. Let X be complete. The sequence $\{x_n\}$ in (5.4.4) with $\sum_{n=1}^{\infty} \alpha_n < \infty$, converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) =$*

$\inf_{p \in F}(x, p)$.

Proof. The necessity is obvious; we only prove the sufficiency. By Lemma 5.4.2 (a), we have

$$d(x_{n+1}, p) \leq (1+\nu_n)^r d(x_n, p) + \xi_n \text{ for all } p \in F \text{ and } n \geq 1.$$

Therefore,

$$\begin{aligned} d(x_{n+1}, F) &\leq (1+\nu_n)^r d(x_n, F) + \xi_n, \\ &= \left(1 + \sum_{k=1}^{r} \frac{r(r-1)\cdots(r-k+1)}{k!}\nu_n^k\right) d(x_n, F) + \xi_n. \end{aligned}$$

As $\sum_{n=1}^{\infty} \nu_n < +\infty$, so $\sum_{n=1}^{\infty}\sum_{k=1}^{r} \frac{r(r-1)\cdots(r-k+1)}{k!}\nu_n^k < \infty$. Now $\sum_{n=1}^{\infty} \xi_n < \infty$ in Lemma 5.4.2 (a), so by by Lemma 5.4.1 and $\liminf_{n\to\infty} d(x_n, F) = 0$, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$. From the proof of Lemma 5.4.2 (b), we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, F) + d(x_n, F) \leq (1 + M_1)\, d(x_n, F) + M_1 \sum_{i=n}^{\infty} \xi_i, \quad (5.4.7)$$

As $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty} \xi_i < \infty$, so there exists a natural number n_0 such that

$$d(x_n, F) \leq \frac{\varepsilon}{2(1+M_1)} \text{ and } \sum_{i=n}^{\infty} \xi_i < \frac{\varepsilon}{2M_1} \text{ for all } n \geq n_0.$$

So for all integers $n \geq n_0, m \geq 1$, we obtain from (5.4.7) that

$$d(x_{n+m}, x_n) < (M_1 + 1)\frac{\varepsilon}{2(1 + M_1)} + M_1\frac{\varepsilon}{2M_1} = \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X and so converges to $q \in X$. Finally, we show that $q \in F$. For any $\overline{\varepsilon} > 0$, there exists natural number n_1 such that

$$d(x_n, F) = \inf_{p \in F} d(x_n, p) < \frac{\overline{\varepsilon}}{3} \text{ and } d(x_n, q) < \frac{\overline{\varepsilon}}{2}, \text{for all } n \geq n_1.$$

There must exist $p^* \in F$ such that $d(x_n, p^*) < \frac{\overline{\varepsilon}}{2}$ for all $n \geq n_1$; in particular, $d(x_{n_1}, p^*) < \frac{\overline{\varepsilon}}{2}$ and $d(x_{n_1}, q) < \frac{\overline{\varepsilon}}{2}$.

Hence

$$d(p^*, q) \leq d(x_{n_1}, p^*) + d(x_{n_1}, q) < \overline{\varepsilon}.$$

Since $\overline{\varepsilon}$ is arbitrary, therefore $d(p^*, q) = 0$. That is, $q = p^* \in F$. ∎

Remark 5.4.1 *A generalized asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mappings. So Theorem 5.4.1 holds good for the class of generalized asymptotically nonexpansive mappings.*

5.4.3 Convergence in uniformly convex hyperbolic spaces

The aim of this section is to establish some convergence results for the iterative method (5.4.4) of generalized asymptotically quasi-nonexpansive mappings on a uniformly convex hyperbolic space.

Lemma 5.4.3 *Let C be a nonempty, closed and convex subset of a uniformly convex metric space X and $\{T_i : i \in I\}$ be a family of uniformly Hölder continuous and generalized asymptotically quasi-nonexpansive selfmappings of C, i.e., $d(T_i^n x, p_i) \leq (1+u_{in})d(x, p_i) + c_{in}$ for all $x \in C$ and $p_i \in F(T_i)$, where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$, respectively, for each $i \in I$. Then, for the sequence $\{x_n\}$ in (5.4.4) with $a_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, we have $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$, for each $j \in I$.*

Proof. Let $p \in F$ and $\nu_n = \max_{i \in I} u_{in}$, for all $n \geq 1$.By Lemma 5.4.1 (i) and Lemma 5.4.5 (a), it follows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. Assume that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{5.4.8}$$

The inequality (5.4.5) together with (5.4.8) gives that

$$\limsup_{n\to\infty} d(U_{jn} x_n, p) \leq c, 1 \leq j \leq r. \tag{5.4.9}$$

By (5.4.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(f(x_n), S_n x_n, \alpha_n), p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(S_n x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + \alpha_n d(f(p), p) + (1-\alpha_n) d(U_{rn} x_n, p), \end{aligned}$$

and hence

$$c \leq \liminf_{n\to\infty} d\left(U_{rn}x_n, p\right) \tag{5.4.10}$$

Combining (5.4.9) and (5.4.10), we get

$$\lim_{n\to\infty} d\left(U_{rn}x_n, p\right) = c.$$

Note that

$$\begin{aligned}
d\left(U_{rn}x_n, p\right) &= d(W(T_1^n U_{(r-1)n}x_n, U_{(r-1)n}x_n, a_{1n}), p) \\
&\leq a_{1n}d(T_1^n U_{(r-1)n}x_n, p) + (1-a_{1n})d(U_{(r-1)n}x_n, p) \\
&\leq a_{1n}[(1+u_{1n})d\left(U_{(r-1)n}x_n, p\right) + c_{1n}] + (1-a_{1n})d(U_{(r-1)n}x_n, p) \\
&\leq a_{1n}(1+\nu_n)d\left(U_{(r-1)n}x_n, p\right) + a_{1n}c_{1n} \\
&\leq a_{1n}(1+\nu_n)[a_{2n}(1+\nu_n)d\left(U_{(r-2)n}x_n, p\right) + a_{2n}c_{2n}] + a_{1n}(1+\nu_n)c_{1n} \\
&\leq a_{1n}a_{2n}(1+\nu_n)^2 d\left(U_{(r-2)n}x_n, p\right) + a_{1n}a_{2n}(1+\nu_n)c_{2n} + a_{1n}c_{1n} \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq a_{1n}a_{2n}...a_{(j-1)n}(1+\nu_n)^{j-1}d\left(U_{(r-(j-1))n}x_n, p\right) + \\
&\quad a_{1n}a_{2n}...a_{(j-1)n}(1+\nu_n)^{(j-1)-1}c_{(j-1)n} + \\
&\quad a_{1n}a_{2n}...a_{((j-1)-1)n}(1+\nu_n)^{(j-1)-2}c_{((j-1)-1)n} + ... + \\
&\quad a_{1n}a_{2n}(1+\nu_n)c_{2n} + a_{1n}c_{1n}.
\end{aligned}$$

Hence

$$c \leq \liminf_{n\to\infty} d\left(U_{(r-(j-1))n}x_n, p\right), 1 \leq j \leq r. \tag{5.4.11}$$

Using (5.4.9) and (5.4.11), we have

$$\lim_{n\to\infty} d\left(U_{(r-(j-1))n}x_n, p\right) = c.$$

That is,

$$\lim_{n\to\infty} d\left(W(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n, a_{jn}), p\right) = c \text{ for } 1 \leq j \leq r.$$

This together with (5.4.8), (5.4.9), and Lemma 5.3.2 gives that

$$\lim_{n\to\infty} d\left(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n,\right) = 0 \text{ for } 1 \leq j \leq r. \tag{5.4.12}$$

If $j = r$,we have by (5.4.12),

$$\lim_{n\to\infty} d\left(T_r^n x_n, x_n\right) = 0.$$

In case $j \in \{1, 2, 3, ..., r-1\}$, we observe that

$$
\begin{aligned}
d\left(x_n, U_{(r-j)n}x_n\right) &= d\left(x_n, W\left(T_{j+1}^n U_{(r-(j+1))n}x_n, U_{(r-(j+1))n}x_n, a_{(j+1)n}\right)\right) \\
&\leq a_{(j+1)n} d\left(T_{j+1}^n U_{(r-(j+1))n}x_n, x_n\right) + (1-a_{(j+1)n}) d\left(U_{(r-(j+1))n}x_n, x_n\right) \\
&\leq (1+\nu_n) d\left(U_{(r-(j+1))n}x_n, x_n\right) + c_{(j+1)n} \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq (1+\nu_n)^{r-j} d\left(U_{0n}x_n, x_n\right) + (1+\nu_n)^{r-j-1} c_{rn} + \\
&\quad (1+\nu_n)^{r-j-2} c_{(r-1)n} + \ldots + (1+\nu_n) c_{(j+2)n} + c_{(j+1)n}.
\end{aligned}
$$

Hence,

$$
\lim_{n\to\infty} d\left(x_n, U_{(r-j)n}x_n\right) = 0. \tag{5.4.13}
$$

Since T_j is uniformly Hölder continuous, therefore the inequality

$$
\begin{aligned}
d\left(T_j^n x_n, x_n\right) &\leq d\left(T_j^n x_n, T_j^n U_{(r-j)n}x_n\right) + d\left(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n\right) + \\
&\quad d(U_{(r-j)n}x_n, x_n) \\
&\leq L d\left(x_n, U_{(r-j)n}x_n\right)^{\gamma} + d\left(x_n, U_{(r-j)n}x_n\right) + d\left(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n\right),
\end{aligned}
$$

together with (5.4.12) and (5.4.13) gives that

$$
\lim_{n\to\infty} d\left(T_j^n x_n, x_n\right) = 0.
$$

Hence,

$$d\left(T_j^n x_n, x_n\right) \to 0 \text{ as } n \to \infty \text{ for } 1 \leq j \leq r. \tag{5.4.14}$$

As before, we can show that

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(x_n, W(f\left(x_n\right), S_n x_n, \alpha_n)\right) \\ &\leq \alpha_n\left(1+\alpha\right) d\left(x_n, p\right) + \alpha_n d\left(p, f\left(p\right)\right) \\ &\quad +(1-\alpha_n)[a_{1n} d\left(U_{(r-1)n} x_n, T_1^n U_{(r-1)n} x_n\right) + d\left(x_n . U_{(r-1)n} x_n\right)], \end{aligned}$$

Therefore, by (5.4.12) and (5.4.13), we get

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{5.4.15}$$

Let us observe that:

$$\begin{aligned} d\left(x_n, T_j x_n\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) \\ &\quad + d\left(T_j^{n+1} x_{n+1}, T_j^{n+1} x_n\right) + d\left(T_j^{n+1} x_n, T_j x_n\right) \\ &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) \\ &\quad + L d\left(x_{n+1}, x_n\right)^\gamma + L d\left(T_j^n x_n, x_n\right)^\gamma . \end{aligned}$$

By uniform Hölder continuity of T_j, (5.4.14) and (5.4.15) , we get

$$\lim_{n\to\infty} d\left(x_n, T_j x_n\right) = 0, 1 \leq j \leq r. \tag{5.4.16}$$

■

Theorem 5.4.2 *Under the hypotheses of Lemma 5.4.3, assume, for some $1 \leq j \leq r$, T_j^m is semi-compact for some positive integer m. If X is complete, then $\{x_n\}$ in (5.4.4), converges strongly to a point in F.*

Proof. Fix $j \in I$ and suppose T_j^m to be semi-compact for some $m \geq 1$. By (5.4.16), we obtain

$$\begin{aligned} d\left(T_j^m x_n, x_n\right) &\leq d\left(T_j^m x_n, T_j^{m-1} x_n\right) + d\left(T_j^{m-1} x_n, T_j^{m-2} x_n\right) \\ &\quad + \cdots + d\left(T_j^2 x_n, T_j x_n\right) + d\left(T_j x_n, x_n\right) \\ &\leq d\left(T_j x_n, x_n\right) + (m-1) L d\left(T_j x_n, x_n\right)^{\gamma} \to 0. \end{aligned}$$

Since $\{x_n\}$ is bounded and T_j^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q \in C$. Hence, by (5.4.16), we have

$$d(q, T_i q) = \lim_{n\to\infty} d\left(x_{n_j}, T_i x_{n_j}\right) = 0, i \in I.$$

Thus $q \in F$ and so by Theorem 5.4.1, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i \in I\}$. ■

An immediate consequence of Lemma 5.4.3 and Theorem 5.4.2 is the following strong convergence result in uniformly convex hyperbolic spaces.

Theorem 5.4.3 *Let C, $\{T_i : i \in I\}$, $F, \{u_{in}\}$ and $\{c_{in}\}$ be as in Lemma 5.4.3. If* *there exists a constant M such that $d(x_n, T_j x_n) \geq Md(x_n, F)$, for all $n \geq 1$ and X is complete, then the sequence $\{x_n\}$ in (5.4.4), converges strongly to a point in F.*

Now, we establish $\triangle$-convergence of the iterative method (5.4.4).

Theorem 5.4.4 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\{T_i : i \in I\}$ be a family of uniformly L-Lipschitzian and generalized asymptotically nonexpansive selfmappings of C, such that $F \neq \phi$. i.e., $d(T_i^n x, T_i^n y) \leq (1 + u_{in}) d(x, y) + c_{in}$ for all $x, y \in C$, where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$, respectively, for each $i \in I$. Then the sequence $\{x_n\}$ in (5.4.4) with $a_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\triangle-$ converges to a common fixed point of $\{T_j : j \in I\}$.*

Proof.

By Lemma 5.4.3, $\{x_n\}$ is bounded and so by Lemma 5.2.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{z_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{z_n\}) = \{z\}$. Also by Lemma 5.4.3, we have $\lim_{n\to\infty} d(z_n, T_j z_n) = 0$, for each $j \in I$.

We claim that z is the common fixed point of $\{T_j : j \in I\}$. To show this, we define a sequence $\{w_k\}$ in C by $w_k = T_j^k z$,

$$
\begin{aligned}
d\left(w_k, z_n\right) &= d\left(T_j^k z, z_n\right) \\
&\leq d\left(T_j^k z, T_j^k z_n\right) + \sum_{i=1}^{k} d\left(T_j^i z_n, T_j^{i-1} z_n\right) \\
&\leq (1+u_{jn}) d\left(z, z_n\right) + c_{jn} + kLd\left(T_j z_n, z_n\right).
\end{aligned}
$$

Taking lim sup,

$$
\limsup_{n\to\infty} d\left(w_k, z_n\right) \leq \limsup_{n\to\infty} d\left(z, z_n\right),
$$

i.e., $r(T_j^k z, z_n) \leq r(z, z_n)$. It follows from Lemma 5.2.2 that $\lim_{k\to\infty} T_j^k z = z$. As T_j is uniformly continuous so we have, $T_j z = T_j\left(\lim_{k\to\infty} T_j^k z\right) = \lim_{k\to\infty} T_j^{k+1} z = z$. Therefore, z is the common fixed point of $\{T_j : j \in I\}$.

Recall that $\lim_{n\to\infty} d(x_n, z)$ exists by Lemma 5.4.3.

Suppose $x \neq z$. By the uniqueness of asymptotic centers, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z),
\end{aligned}
$$

a contradiction. Hence $x = z$. Since $\{z_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{z_n\}) = \{z\}$ for all subsequences $\{z_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle-$converges to a common fixed point of $\{T_j : j \in I\}$.

$\blacksquare$

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

Conclusions

In this thesis, we have set a metric analogue of Berinde and Păcurar Theorem [15] for almost contraction mappings in hyperbolic spaces which gives a positive answer to the problem of having fixed points of some discontinuous mappings on a nonlinear domain.

Nonexpansive mappings are a natural extension of contractive mappings. Goebel and Kirk [41] extended Browder and Göhde Theorem for nonexpansive mappings to the case of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Later on, Kirk [65] substantially weakened the assumption of asymptotic nonexpansiveness to generalize Goebel and Kirk Theorem for non-Lipschitzian mappings of asymptotically nonexpansive type. In this thesis, we have given a metric analogue of Kirk Theorem in uniformly convex hyperbolic spaces.

Recently, a new direction has been discovered for extension of the Banach Contraction Principle to metric spaces endowed with a partial order or a graph. In this thesis, we have obtained an analogue of Browder and Göhde's fixed point theorem for monotone nonexpansive mappings defined on uniformly convex hyperbolic spaces endowed with a graph. This result serves as a bridge between the graph theory and metric fixed point theory. Moreover, some classical fixed point theorems for single-valued nonexpansive mappings have been extended for multivalued mappings. Lami Dozo [79] generalized these results to a Banach space satisfying Opial condition [92]. In this thesis, we have provided an extension of Lami Dozo Theorem in partially ordered Banach spaces.

We have defined the concept of proximally monotone Lipschitzian mappings which reduce to monotone Lipschitzian mappings in the case of self-mappings. As applications, we have studied existence and uniqueness of best proximity points of such mappings in $CAT(0)$ spaces. This work is a continuation of the work of Ran and Reurings [98], Bachar and Khamsi [8] for self-mappings defined on Banach spaces.

In this thesis, we have given a characterization of reflexivity in terms of the P-property. Moreover, we have extended this characterization to hyperbolic spaces. In particular, an example of a nonlinear hyperbolic space where the P-property holds, is given; one may consider any pair of closed, convex and bounded subsets of a $CAT(0)$ space. As an application, we have have generalized Banach Contraction Principle and fundamental fixed point theorem of Takahashi [107] for best proximity points of non-self mappings defined on hyperbolic spaces.

We have extended the Gromov geometric definition of $CAT(0)$ spaces [47] to the case when the comparison triangles lie in a general Banach space. In particular, we study the case of the Banach space is l_p, $p > 2$. In this way, we have obtained a new class of generalized $CAT(0)$ spaces, called $CAT_p(0)$ spaces. Moreover, we have introduced the (CN_p) inequality which coincides with the classical (CN) inequality of Bruhat and Tits [22] when $p = 2$. As a consequence of the (CN_p) inequality, we have shown that any $CAT_p(0)$ space, with $p \geq 2$, is uniformly convex hyperbolic space. Furthermore, we have established fixed point property for nonexpansive mappings in these spaces.

Finally, we have established strong convergence and Δ-convergence theorems of an implicit iteration algorithm associated with a pair of nonexpansive mappings on a nonlinear domain. In particular, we proved that this algorithm converges to a common fixed point of the mappings. In addition, a general viscosity iterative method for a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space is introduced. The special cases of new iterative method are viscosity iterative method of Chang et al [27], analogue of viscosity iterative method of Fukhar-ud-din et al [39]. and an extension of the multistep iterative method of Yildirim and Özdemir [114]. Our results generalize and extend the corresponding known results in uniformly convex Banach spaces and $CAT(0)$

spaces simultaneously.

The following papers based on my Ph.D research work are Published/ Submitted:

(1) A. R. Khan, N. Yasmin, H. Fukhar-Ud-Din and S. A. Shukri, *Viscosity approximation method for generalized asymptotically quasi-nonexpansive mappings in a convex metric space,* Fixed Point Theory Appl., 2015, 2015:196.

(2) A. R. Khan and S. A. Shukri, *Best proximity points in the Hilbert ball,* J. Nonlinear Convex Anal., Accepted.

(3) S. A. Shukri, V. Berinde and A. R. Khan, *Fixed points of discontinuous mappings in uniformly convex metric spaces,* Fixed Point Theory, Accepted.

(4) M. R. Alfuraidan and S. A. Shukri, *Browder & Göhde fixed point theorem for G-monotone nonexpansive mappings,* Submitted.

(5) M. A. Khamsi and S. A. Shukri, *Generalized CAT(0) spaces,* Submitted.

(6) M. A. Khamsi, S. A. Shukri and A. R. Khan, *A metric characterization of reflexivity,* Submitted.

(7) S. A. Shukri & A. R. Khan, *Best proximity points in partially ordered metric spaces,* Submitted.

Future Work

We aim to obtain:

1. A result similar to Kirk's fixed point theorem [64] for monotone nonexpansive mappings.

2. An extension of Goebel and Kirk's fixed point theorem [41] for asymptotically nonexpansive mappings to Banach spaces endowed with a partial order.

3. An extension of Lim [82] result for multivalued nonexpansive mappings to Banach spaces endowed with a partial order.

4. A characterization of the property (R) through the P-property.

5. A characterization of $CAT_p(0)$ spaces in terms of the CN_p inequality.

   We also aim to show that the open unit ball of l_p spaces endowed with the Kobayashi distance [78] is a nonlinear example of $CAT_p(0)$ spaces.

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