INVERSE GRAPHS ASSOCIATED WITH FINITE GROUPS

BY

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عمادة الدراسات العليا
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Dedicated to my father Zakariya Yusuf

May his grave be widen

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Graph Notations	Meanings
G, H	graphs
$V(G)$	vertex set of G
$E(G)$	edge set of G
D	directed graph
$k(G)$	number of connected components in G
$deg(u)$	degree of a vertex u
$G[M]$	induced subgraph of G by M
$diam(G)$	diameter of G
$d(u, v)$	distance between u and v
$g(G)$	girth of G
$\rho(u)$	eccentricity of a vertex u
$rad(G)$	radius of G
K_n	complete graph with n vertices
$K_{n,m}$	complete bipartite graph with n and m vertices in each partition
C_n	cycle of length n
$G_S(\Gamma)$	inverse graph associated with Γ

Group Notations	Meanings
Γ	finite group
S	set of non-self invertible elements of Γ
$\in$	belongs to or is an element of
π	permutation on a set
$C_\Gamma(u)$	centralizer of u in Γ
$Z(\Gamma)$	center of a group Γ
Γ_r	cyclic group of order r
$Aut(G)$	automorphism of graph G

S_n	symmetry group on on n symbols
A_n	alternating group on n symbols
D_{2n}	dihedral group of order $2n$
Q_{2^n}	generalized quaternion group of order 2^n

ABSTRACT

Full Name : Yusuf Feyisara Zakariya

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Let $(\Gamma, *)$ be a finite group and S a possibly empty subset of Γ containing its non-self invertible elements. In this thesis, we introduce the inverse graph associated with Γ whose set of vertices coincides with Γ such that two distinct vertices u and v are adjacent if and only if either $u * v \in S$ or $v * u \in S$. We examine the inter-relatedness between the algebraic properties of Γ and the graph combinatorial properties of $G_S(\Gamma)$. Thereafter, we apply the inverse graphs to characterize some isomorphism problems of groups.

ملخص الرسالة

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لنفرض أن $(\Gamma; *)$ زمرة منتهية و S مجموعة - قد تكون خالية - تحتوي على عناصر غير ذاتية المعكوس. في هذه الرسالة ، ستّعرف الرسم العكسي المرتبط بـ Γ والذي رؤوسه هي عناصر المجموعة Γ، بحيث يرتبط رأسين مختلفين u و v إذا وفقط إذا كان $u * v \in S$ أو $v * u \in S$. أيضا سنختبر العلاقات الداخلية بين الخصائص الجبرية لـ Γ والخصائص الاندماجية لرسم $G_S(\Gamma)$. بعد ذلك سنطبق الرسم العكسي لتشخيص بعض التطابقات لمسائل زمرية.

Chapter 1

Introduction

There has been a close tie between algebraic structures and graphs for more than a century. The robust combinatorial properties of graphs have been employed extensively to investigate the algebraic properties of groups and rings and vice versa. This work of course is not an exception. Recently, associating graphs to rings in order to investigate it seems to be active area and interest catching among the researchers. This is obvious in the amount of publications in this perspective, see [3, 6, 7, 47, 68, 69, 73]. Graphs have been ascribed to commutative ring R with non-zero identity by employing the unit elements of the ring, [7]. Some researchers have used zero-divisors in a ring R, see [3, 6, 47], while others have employed the co-maximal ideals of both commutative and non-commutative rings, see [68, 69] to define graphs on rings. On the other hand, we introduce and investigate *inverse graphs* associated with finite groups.

The significance of graph theory cannot be over-emphasized considering its massive applications in other fields of study apart from mathematics. It is widely utilized in fields like biochemistry, chemistry, sociology, electrical engineering, biology, computer science, operation research, chemical engineering, geosciences and landscape ecology, etc. In group theory, graph combinatorial concepts have been used to solve some eigenstate problems associated with “Lie groups”, see [52]. Several other results in group theory have also been proved easily via associated graphs, [12]. This serves as a motivation for this study. Our main objectives of this thesis are to:

1. Introduce and investigate inverse graphs associated with finite groups.

2. Study the inter-relatedness between the algebraic properties of groups and the combinatorial properties of this graph.

3. Compare this graph with existing graphs associated with finite groups.

4. Apply the inverse graph to some isomorphism problems of groups.

Chapter 2 of this thesis focuses on the required preparations from both the graph and group theories. In Chapter 3, we give a brief account of what researchers have done for over a century in associating graphs to finite groups. We constructed the inverse graphs in Chapter 4 and thereafter investigated some of its properties. In order to justify our claim that this graph is new, we concluded this thesis in Chapter 5 by comparing the inverse graphs with some existing graphs associated with finite groups.

Chapter 2

Preliminaries

2.1 Groups

This section contains fundamental notions and results from group theory used in this thesis. We have employed the standard definitions and results that can easily be found in any good introductory textbook in group theory. For example, see [32, 55, 58].

A *set* is a collection of well defined objects called *elements*. If a set A contains some element a, we write $a \in A$, and A is *non-empty*, otherwise it is *empty*. A *mapping* f from a set A into a set B (denoted by $f : A \to B$) corresponds to a rule that assigns to each element $a \in A$ a unique element $b \in B$. In this case, A is called the *domain* and B the *codomain*. The mapping f is *one-to-one* if it maps distinct elements of A to distinct elements of B. It is *onto* if for every element $b \in B$ there exists at least an element $a \in A$ such that $f(a) = b$. A one-to-one and onto mapping is called a *one-to-one correspondence* or a *bijection*. By a *binary operation* on a non-empty set A we mean a mapping from the *cartesian product* $A \times A$ (defined by $\{(a_1, a_2) : a_1, a_2 \in A\}$) into A. Let A be a non-empty set and let R be a subset of $A \times A$. Then R is called an *equivalence (or an equivalence relation)* on A if the following conditions are satisfied.

1. $(a, a) \in R$ for all $a \in A$ (*reflexive property*).

2. If $(a_1, a_2) \in R$, then $(a_2, a_1) \in R$ (*symmetric property*).

3. If $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$, then $(a_1, a_3) \in R$ (*transitive property*).

Let R be an equivalence relation on a set A. If $a \in A$, we define $aR = \{\omega : \omega \in A$ and $(a, w) \in R\}$. This subset aR of A is called the *R-class* of a, or the *R-equivalence class* of a.

2.1.1 Definitions and examples

Definition 1. *Let * be a binary operation defined on the elements of a non-empty set Γ. Then Γ is a group under the operation * if the following properties hold:*

1. $u * v \in \Gamma$ *for every* $u, v \in \Gamma$ *(closure property).*

2. $(u * v) * w = u * (v * w)$ *for all* $u, v, w \in \Gamma$ *(associativity of *).*

3. *There exists an element* $e \in \Gamma$ *such that* $e * u = u * e = u$ *for all* $u \in \Gamma$ *(existence of identity* e*).*

4. *For every element* $u \in \Gamma$ *there is a unique element* u^{-1} *such that* $u * u^{-1} = u^{-1} * u = e$ *(*u^{-1} *is called the inverse of* u*).*

A *groupoid* is a non-empty set endowed with the closure property of a binary operation. An associative groupoid is a *semigroup*. A semigroup with an identity is called a *monoid*. Hence a group can be regarded as a monoid with each element having a unique inverse. A group is *finite* if it contains a finite number of elements, otherwise it is *infinite*. Throughout this thesis we shall only consider finite groups and for $u, v \in \Gamma$ under the group operation " * ", $u * v$ will be written multiplicatively as uv and will be referred to as the *product* of u and v unless otherwise stated. One method of defining a group is by a *presentation*. One specifies a set C of *generators* so that every element of the group can be written as a product of powers of some of these generators, and a set R of relations among those generators. We then say Γ has a presentation $\langle C \mid R \rangle$.

Definition 2. *A generating set C of a group Γ is a subset of Γ such that every element of Γ can be expressed as a product (under the group operation) of finitely many elements of C and their inverses.*

In other words, if C is a subset of a group Γ, then $\langle C \rangle$, the subgroup[1] generated by C, is the smallest subgroup of Γ containing every element of C, meaning the intersection over all subgroups containing the elements of C. If $\Gamma = \langle C \rangle$, then we say C *generates* Γ; and the elements in C are called *generators or group generators.*

Definition 3. *A group Γ is abelian provided $u * v = v * u$, $\forall\, u, v \in \Gamma$.*

Definition 4. *The number of elements in a group Γ is its order. It is denoted by $|\Gamma|$.*

Definition 5. *The order of an element u, denoted by $|u|$, in a group Γ is the smallest positive integer m such that $u^m = e$, where e is the identity of the group. The element u has infinite order if such integer m does not exist.*

Definition 6. *An element $u \in \Gamma$ is self-invertible if and only if $u = u^{-1}$, otherwise it is non-self invertible. Hence the identity e is a trivial self-invertible element in any group Γ.*

Example 7. The sets $\mathbb{Z}$ of integers, $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers with respect to the usual addition are examples of abelian groups. The sets $\mathbb{Q}^*$ of nonzero rational numbers and $\mathbb{R}^*$ of nonzero real numbers with respect to the usual multiplication of numbers are examples of abelian groups.

Example 8. The set of all n by n invertible matrices with real entries is a group with respect to the usual matrix multiplication. This group is called *general linear group* of n by n matrices in $\mathbb{R}$, denoted by $GL(n, \mathbb{R})$.

[1] See Definition 15.

Example 9. The set of integers modulo n, $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$, under the usual addition modulo n is a group for all $n \in \mathbb{N}$.

Example 10. The set of integers modulo p, $\mathbb{Z}_p = \{1, 2, \cdots, p-1\}$, is a group under the usual multiplication modulo p, where p is prime.

Example 11. Let $\Gamma = \{\omega \in \mathbb{C} : \omega^n = 1\}$ be the set of all n^{th} roots of unity, where $\mathbb{C}$ is the set of complex numbers. Then, Γ is a finite group under the usual multiplication of complex numbers.

Example 12. Suppose $\Gamma = \langle -1, i, j, k |\ (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1\rangle$ which is isomorphic to a certain eight-element subset of the quaternions under multiplication. This is a non-abelian group of eight elements often referred to as *quaternion group*.

Example 13. Let A be a non-empty set. Then, a one-to-one correspondence mapping $\pi : A \to A$ is called a *permutation* on A. To be more specific, if $A = \{1, 2, \cdots, n\}$ then π is called a *permutation of degree* n on A. Suppose π is a permutation of degree n such that $\pi(1) = a_1, \pi(2) = a_2, \cdots, \pi(n) = a_n$ where $a_1, a_2, \cdots, a_n$ is some rearrangement of elements of A. This permutation will be denoted by

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \tag{2.1}$$

The set of all permutations of degree n, denoted by S_n, is a group under the composition of mappings. This group is called a *symmetry group*. In this case, for permutations $\pi_1, \pi_2 \in S_n$, $\pi_1 * \pi_2$ is the *composition of mappings* defined as $(\pi_1 * \pi_2)(a) = \pi_1(\pi_2(a))$, for all $a \in A$. It is not difficult to see that the number of permutations in S_n is $n!$. The *identity permutation*, e, is written as

$$e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \tag{2.2}$$

Example 14. Consider the symmetry group S_3 on the set $\{1, 2, 3\}$.

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ \pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ \pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \ \pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The operation * is the usual composition of mappings. For instance,

$$\pi_2 * \pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \pi_4$$

while,

$$\pi_3 * \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \pi_5$$

Since $\pi_2 * \pi_3 \neq \pi_3 * \pi_2$ the group S_3 is non-abelian.

2.1.2 Cayley table

In some cases when dealing with groups especially finite groups it is helpful to draw the multiplication table. This table is referred to as *Cayley table*. For an arbitrary finite group $\Gamma = \{u_1, u_2, \cdots, u_n\}$ under the operation $*$ its Cayley table is constructed as follows:

Table 2.1: Cayley table of a finite group

$*$	u_1	u_2	$\cdots$	u_j	$\cdots$	u_n
u_1	$u_1 * u_1$				$\cdots$	$u_1 * u_n$
u_2						
$\vdots$	$\cdots$				$\cdots$	$\vdots$
u_i				$u_i * u_j$		$u_i * u_n$
$\vdots$	$\cdots$				$\cdots$	$\vdots$
u_n	$u_n * u_1$				$\cdots$	$u_n * u_n$

For example, the Cayley table of S_3 is as follows:

Table 2.2: Cayley table of S_3

$*$	e	π_1	π_2	π_3	π_4	π_5
e	e	π_1	π_2	π_3	π_4	π_5
π_1	π_1	π_2	e	π_5	π_3	π_4
π_2	π_2	e	π_1	π_4	π_5	π_3
π_3	π_3	π_4	π_5	e	π_1	π_2
π_4	π_4	π_5	π_3	π_2	e	π_1
π_5	π_5	π_3	π_4	π_1	π_2	e

2.1.3 Subgroups and cosets

In some cases a subset of a group can satisfy properties $1-4$ in Definition 1. This motivates the following definition.

Definition 15. *Let γ be a subset of a group Γ. Then, γ is a subgroup if it satisfies properties $1-4$ in Definition 1 under the operation defined on Γ. In other words, γ itself is a group with respect to the operation defined on Γ. If γ is a subgroup of Γ we write $\gamma \leq \Gamma$.*

Remark 16. *Suppose* Γ *is a group under the operation *. A subgroup* γ *is proper if* $\gamma \neq \Gamma$*, otherwise, it is improper.*

Definition 17. *Suppose* Γ *is a group and* γ *a subgroup of* Γ*. Let* $u \in \Gamma$*, then* $\gamma u = \{vu : v \in \gamma\}$ *is called a right coset. Similarly, we can define the left coset* $u\gamma$*.*

Definition 18. *Let* γ *be a subgroup of* Γ*. The index denoted by* $[\Gamma : \gamma]$ *is the number of different left cosets of* γ *in* Γ*.*

Theorem 19. *(Lagrange, see [55]) The order of every subgroup* γ *of a finite group* Γ *divides the order of* Γ*. In other words,* $|\Gamma| = r|\gamma|$ *for some* $r \in \mathbb{N}$*.*

Theorem 20. *(see [55]). Suppose* γ *is a subgroup of* Γ *and* $u \in \Gamma$*. Then, the set* $\chi = u^{-1}\gamma u = \{u^{-1}vu : v \in \gamma\}$ *is a subgroup of* Γ*.*

Definition 21. *A subgroup* γ *of a group* Γ *is normal if and only if* $\gamma = \chi$ *in Theorem 20.*

Definition 22. *A group is simple if its only normal subgroups are the identity subgroup and the group itself.*

Definition 23. *Let* Γ *be a group and define commutator of* $u, v \in \Gamma$ *as* $[u, v] = u^{-1}v^{-1}uv$*. The normal subgroup generated by these commutators is the derived group or the commutator subgroup of* Γ*.*

2.1.4 Cyclic groups

Definition 24. *A group* Γ *is cyclic with a generator* u *if every element of* Γ *is of the form* u^k *for some integer* k*. If the operation is addition we write* ku *instead of* u^k*.*

Remark 25. *The following follows directly from Lagrange's Theorem:*

1. *The order of every element* u *of a finite group* Γ *divides* $|\Gamma|$*.*

2. *Every group of prime order is cyclic. For this group, every element is a generator except the identity.*

3. *For each element $u \in \Gamma$, $u^n = e$, where n is the order of Γ and e the identity.*

Example 26. The set of integers under the usual addition is a cyclic group generated by 1.

Example 27. The finite group described in Example 11 is cyclic. It is generated by any n^{th} root of unity different from 1.

2.1.5 Conjugacy class

Suppose Γ is a group and $u \in \Gamma$. The *centralizer* of u is defined as follows:

$$C_\Gamma(u) = \{v \in \Gamma : uv = vu\}.$$

In other words, the centralizer of u in Γ is this set of elements that commute with u. The set denoted and defined as

$$Z(\Gamma) = \{u \in \Gamma : uv = vu, \, \forall \, v \in \Gamma\}$$

is called the *center* of Γ. Thus, a group Γ is abelian if and only if $\Gamma = Z(\Gamma)$. If $Z(\Gamma) = \{e\}$ then Γ is *centerless* or has a *trivial center*. Otherwise, Γ has *non-trivial center*.

Now, consider a relation " $\sim$ " defined on the group Γ as follows: let $u, v \in \Gamma$ then $u \sim v$ provided there is $\omega \in \Gamma$ such that

$$u = \omega v \omega^{-1} \tag{2.3}$$

Elements u and v satisfying equation 2.3 are called *conjugates*. Also, " $\sim$ " defined in this case is an equivalence relation. The corresponding equivalence classes, $Cl(u)$, are the *conjugacy classes* of the group Γ.

2.1.6 Dihedral group D_{2n}

The *dihedral group* of order $2n$ is denoted and defined as follows:

$$D_{2n} = \langle u, v | v^n = u^2 = e, uvu^{-1} = v^{-1} \rangle,$$

where e is the identity element of D_{2n} and u and v are its generators.

2.1.7 Generalized quaternion groups Q_{2^n}

The *generalized quaternion group* of order 2^n is denoted and defined as follows:

$$Q_{2^n} = \langle u, v | v^4 = u^{2^{n-1}} = e, v^2 = u^{2^{n-2}}, vu = u^{-1}v \rangle$$

Here, e is the identity of Q_{2^n}, $n > 2$ while u and v are its generators. The following Proposition characterizes the generalized quaternion groups.

Proposition 28. *The only involution*[2] *in Q_{2^n} is the element $u^{2^{n-2}}$*

Proof. Since u has order 2^{n-1} its only power with order 2 is $u^{2^{n-2}}$. Every element of Q_{2^n} that is not a power of u has the form $u^m v$ where $m \in \mathbb{N}$ and

$$(u^m v)^2 = u^m (vu^m v^{-1}) v^2 = u^m (vuv^{-1})^m v^2 = u^m u^{-m} v^2 = v^2.$$

Therefore, $u^m v$ has order 4. $\square$

2.1.8 Alternating and solvable groups

Definition 29. *A permutation π on a set $A = \{a_1, a_2, \cdots, a_n\}$ is a cycle of length m if there is a subset $B = \{b_1, b_2, \cdots, b_m\}$ of A, such that $\pi(b_i) = b_{i+1}, 1 \leq i < m$ and $\pi(b_m) = b_1$,*

[2] An element of order 2 in a group is called an *involution*.

while $\pi(a_i) = a_i \, \forall \, a_i \notin B$. *Such a cycle is denoted by* $(b_1, b_2, \cdots, b_m)$.

Facts

1. Every permutation can be written as product of cycles each of length 2 (such cycles are called *transposition*).

2. A permutation that is a product of an even (odd) number of transpositions is called *even (odd) permutations.*

3. The product of two even or two odd permutations is an even permutation.

4. The product of an odd and an even permutations is an odd permutation.

Definition 30. *The group of even permutations of* n *symbols is denoted by* A_n *and is called the alternating group of degree* n.

Definition 31. *We say that a group* Γ *is solvable or soluble if* Γ *has a series of subgroups*

$$\{e\} = \gamma_0 \subset \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_k = \Gamma,$$

where, for each $0 \leq i < k, \gamma_i$ *is normal in* γ_{i+1} *and the derived group* γ_{i+1}/γ_i *is abelian.*

2.1.9 Frobenius groups

A *Frobenius* group is a finite group Γ with a non-trivial normal subgroup γ_1 (called a *Frobenius kernel*) and a non-trivial subgroup γ_2 (called a *Frobenius complement*) such that the orders of γ_1 and of γ_2 are relatively prime and for every $u \in \Gamma \setminus \gamma_1$ there exists a unique $v \in \gamma_1$ with $u \in v\gamma_2 v^{-1}$.

2.1.10 Isomorphism of groups

Definition 32. *Two groups* $(\Gamma_1, *)$ *and* $(\Gamma_2, \bullet)$ *are said to be isomorphic if there exists a mapping*

$$\phi : \Gamma_1 \to \Gamma_2$$

satisfying the following:

1. ϕ *is one-to-one correspondence.*

2. *For* $u, v \in \Gamma_1$*, we have* $\phi(u * v) = \phi(u) \bullet \phi(v)$*.*

The mapping ϕ satisfying 1 and 2 of Definition 32 is called an *isomorphism* from Γ_1 to Γ_2. If such ϕ exists we write $\Gamma_1 \cong \Gamma_2$, otherwise, Γ_1 is not isomorphic to Γ_2 and we write $\Gamma_1 \ncong \Gamma_2$.

Remark 33. *Suppose* ϕ *is an isomorphism from a group* Γ_1 *to a group* Γ_2*. Then, we have the following:*

1. $\phi(e_1) = e_2$*, where* e_1 *is the identity in* Γ_1 *and* e_2 *is identity in* Γ_2*.*

2. *If* $u^n = e_1$ *in* Γ_1*, then* $\phi(u)^n = \phi(u^n) = \phi(e_1) = e_2$ *in* Γ_2*.*

3. *If* $\phi(u) = v, u \in \Gamma_1, v \in \Gamma_2$*, Then,* $\phi(u^{-1}) = v^{-1}$*.*

Example 34. The mapping $\phi : (\mathbb{R}^+, \times) \to (\mathbb{R}, +)$ defined by $\phi(u) = log(u)$, is an isomorphism between the group of positive real numbers under multiplication and the group of real numbers under addition.

2.1.11 Direct product of groups

Definition 35. *The direct product* $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ *of groups* $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ *is the set of* n*-tuples* $(u_1, u_2, \cdots, u_n)$ *where* $u_i \in \Gamma_i$ *with the group operation defined componentwise:*

$$(u_1, u_2, \cdots, u_n)(v_1, v_2, \cdots, v_n) = (u_1v_1, u_2v_2, \cdots, u_nv_n).$$

The group operation in the above definition is written multiplicatively, but in particular examples, whatever is the natural group operation on the Γ_i will be followed. In case $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ are abelian groups written additively, then it is more customary to write the direct product as $\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n$, and refer to this as an *(external) direct sum.*

Example 36. Suppose $\Gamma_i = \mathbb{R}$ for $1 \leq i \leq n$. Then $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ (n-factors) is the ordinary Euclidean n-space $\mathbb{R}^n$ with the usual vector addition:

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

Proposition 37. *If* $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ *are groups, their direct product* Γ *is a group of order* $|\Gamma_1||\Gamma_2|\cdots|\Gamma_n|$*. This means that if any* Γ_i *is infinite, then so is* Γ.

Proof. The verification of the group axioms is straightforward from the componentwise definition of the group operation on Γ. We note that the identity of Γ is $e_\Gamma = (e_{\Gamma_1}, e_{\Gamma_2}, \cdots, e_{\Gamma_n})$ and the inverse of $(u_1, u_2, \cdots, u_n)$ is $(u_1^{-1}, u_2^{-1}, \cdots, u_n^{-1})$. The formula for the order of Γ is clear. $\square$

Let p be a prime and let $n \in \mathbb{N}$. Define a group Γ_{p^n} by

$$\Gamma_{p^n} = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \; (n \; factors).$$

The group Γ_{p^n} is an abelian group of order p^n with the property that every nonidentity element has order p. Such a group is said to be an *elementary abelian p-group*. In particular, if $p = 2$, then $\Gamma_{2^n} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors) coincides with a *Boolean group*[3] of order 2^n. Besides, every finite Boolean group Γ is abelian since $uv = u^{-1}v^{-1} = (uv)^{-1} = vv \, \forall \, u, v \in \Gamma$. Hence, we have the following proposition.

Proposition 38. *Every finite group of order* 2^n *whose nonidentity elements have order* 2 *is abelian and it is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ $(n\, factors)$.

Example 39. The Klein four-group $\mathscr{K}_4 = \{e, a, b, ab : e = a^2 = b^2 = (ab)^2\}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ with an explicit isomorphism

$$\phi : \mathscr{K}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$$

such that $\phi(e) = (0,0)$, $\phi(a) = (0,1)$, $\phi(b) = (1,0)$ and $\phi(ab) = (1,1)$.

2.2 Graphs

This section contains the fundamental notions and results from graph theory used in this thesis. We have employed the standard definitions and results that can easily be found in any good introductory textbook in graph theory. For example, see [22, 71].

2.2.1 Definitions and examples

Definition 40. *A graph G consists of a set of points in two dimensional space called vertices, $V(G)$, together with possibly empty set of 2-element subsets of $V(G)$ called edges, $E(G)$.*

The cardinality of the vertex set, $|V(G)|$, is the *order* of G while the number of elements in the edge set, $|E(G)|$, is its *size*. If $\{u, v\} \in E(G)$ we say u and v are *adjacent* i.e., there is

[3] A Boolean group is a group whose every nonidentity element has order 2.

an edge between u and v. In this case, we write $uv \in E(G)$. The number of vertices adjacent to a vertex u is the *degree* of u. A vertex u is *isolated* if it is not adjacent to any other vertex in G. If u is adjacent to only one vertex, we call it a *leaf*. A graph G with vertices each of degree r is called an *r-regular graph*. Two or more edges that join the same pair of distinct vertices are called *parallel edges*. A *loop* is an edge that joins a vertex to itself. A graph is *simple* if it contains neither parallel edges nor loops. We shall consider only simple graphs in this thesis and refer to them simply as graphs. For instance, Figure 2.1 is a graph with set of vertices $V(G) = \{v_1, v_2, v_3, v_4\}$, set of edges $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\}$, order 4 and size 4. The degrees of the vertices v_1, v_2, v_3, v_4 are $2, 2, 3, 1$ respectively. It contains neither parallel edges nor loops. Vertex v_4 is a leaf and G contains no isolated vertex.

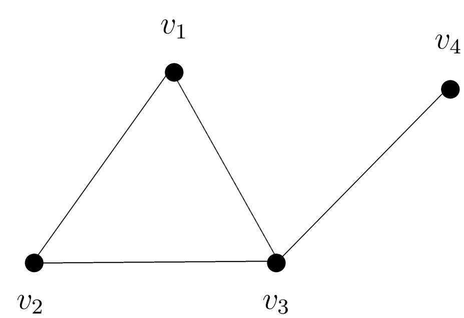

Figure 2.1: A graph of order 4

Definition 41. *A graph G is empty if $E(G)$ is empty and is trivial if $|V(G)| = 1$.*

Definition 42. *A graph of order n is complete, denoted by K_n, if for every distinct pair $u, v \in V(G)$, we have $uv \in E(G)$.*

In a complete graph G of order n the size m of G is always

$$m = \binom{n}{2} = \frac{n(n-1)}{2}.$$

In any graph G, the *maximum* degree is the highest degree among the vertices of $V(G)$, denoted by $\triangle(G)$ while the least degree is the *minimum*, denoted by $\delta(G)$. The following theorem is referred as *The First Theorem of Graph Theory* and its proof can be found in [22].

Theorem 43. *For a graph G of size m we have*

$$\sum_{v \in V(G)} deg(v) = 2m.$$

2.2.2 Subgraphs and induced subgraphs

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in this case we write $H \subseteq G$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. For a non-empty subset M of $V(G)$, $G[M]$ is an *induced subgraph* of G by M if it has M as its vertex set and two vertices $u, v \in M$ are adjacent if and only if they are adjacent in G. This is got by deleting some vertices of G. An induced subgraph of G that is complete is a *clique* in G.

Example 44. In Figure 2.2 , H_1 is a spanning subgraph of G obtained by deleting edges $e_1 = u_2u_3$ and $e_2 = u_3u_4$ i.e., $H_2 = G \setminus \{e_1, e_2\}$. The subgraph, H_2, is a clique in G. It is obtained by deleting vertex u_3 in G.

2.2.3 Graphs isomorphism

A graph G_1 is *isomorphic* to G_2 if there is one-to-one correspondence mapping, ϕ, from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. The mapping ϕ is called an *isomorphism* from G_1 to G_2. If G_1 and G_2 are *isomorphic* we write $G_1 \cong G_2$, otherwise they are *non-isomorphic* graphs. An isomorphism of a graph G to itself is called an *automorphism*. For example, the graphs G_1 and G_2 in Figure 2.3 are isomorphic with an explicit isomorphism ϕ such that $\phi(u_1) = v_1, \phi(u_2) = v_3, \phi(u_3) = v_5, \phi(u_4) = v_2, \phi(u_5) = v_4, \phi(u_6) = v_6$.

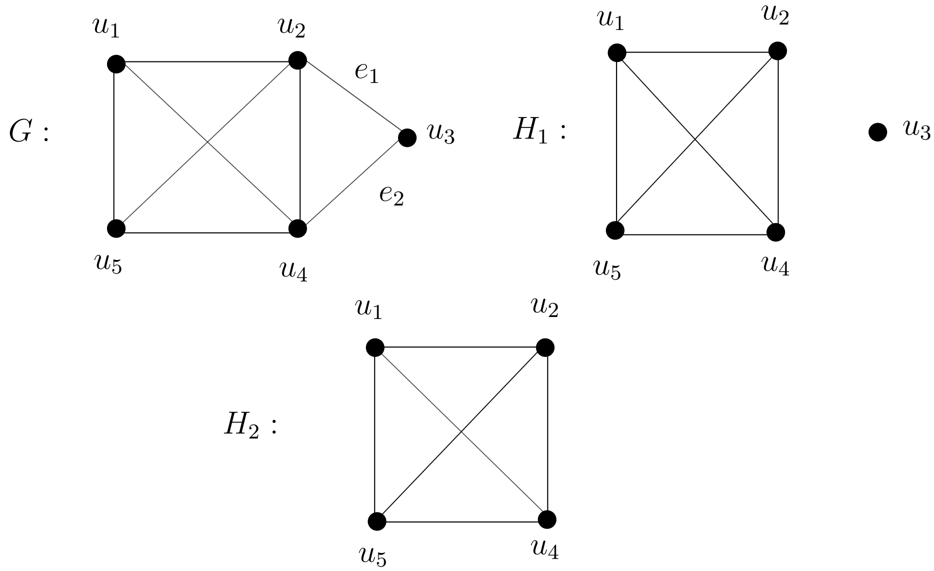

Figure 2.2: Subgraphs of a graph G

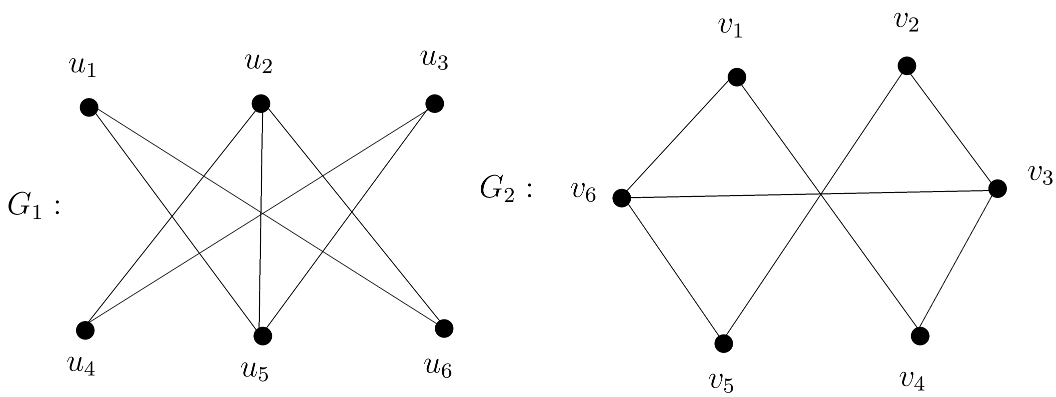

Figure 2.3: Two isomorphic graphs of order 6

Remark 45. *Determining whether two graphs are isomorphic or not requires a great deal of ingenuity. In fact, there is no complete characterization of graph isomorphism problem. Nevertheless, if two graphs G_1 and G_2 are isomorphic each of the following properties should be satisfied:*

- $|V(G_1)| = |V(G_2)|$ *and* $|E(G_1)| = |E(G_2)|$.
- *The degree of each vertex u in G_1 must be equal to the degree of the corresponding vertex $\phi(u)$ in G_2.*

- *Structural properties in G_1 such as the number of cycles, induced subgraphs, girth, eccentricity, diameter, etc must be preserved in G_2.*

Definition 46. *A non-trivial graph G is vertex-transitive if for each pair u, v of distinct vertices in $V(G)$ there is an automorphism Φ on G such that $\Phi(u) = v$.*

2.2.4 Walks, trails, circuits, paths and cycles

For a given graph G with vertex set $V(G) = \{v_0, v_1, \cdots, v_n\}$, a *walk of length $k \leq n$* is a finite sequence

$$v_{i_0} e_{j_1} v_{i_1} e_{j_2} v_{i_2} \cdots v_{i_{k-1}} e_{j_k} v_{i_k}$$

whose terms alternate between vertices and edges such that $v_{i_{t-1}} v_{i_t} = e_{j_t}$ for $1 \leq t \leq k$ and $0 \leq i_k \leq n$. The walk W is *closed* if $v_{i_0} = v_{i_k}$, otherwise it is *open*. A $u - v$ *trail* is a $u - v$ walk starting at u and ending at v in which no edge is repeated. A non-trivial[4] closed trail is called a *circuit*. An open walk in which no vertex is repeated is called a *path*. A closed walk of length $n \geq 3$ with neither vertex nor edge repeated is called a *cycle*, denoted by C_n. A graph is *cyclic* if it contains at least a cycle otherwise it is *acyclic*. The shortest length of a cycle in a non-empty graph G is its *girth*. A non-empty acyclic connected graph is called a *tree*.

2.2.5 Reachability and connectivity

Let G be a non-empty graph. A vertex u of G is said to be *reachable* if there is walk starting from or ending at u from every other vertex in G. A graph in which all vertices are reachable is *connected*, otherwise it is *disconnected*. A connected graph G is said to be k-connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed. A *component* of a graph G is a connected subgraph of G that is not properly contained in

[4] Involving more than one vertex.

any other connected subgraph of G. The number of such connected components is denoted by $k(G)$. Hence, a graph is connected if and only if $k(G) = 1$. The graph G of Figure 2.2 is connected while H_1 is disconnected with 2 components.

2.2.6 Distance, diameter, radius and eccentricity of a graph

Definition 47. *In a connected graph G, the distance between two vertices $u, v \in V(G)$, is the length of the shortest path between u and v. It is denoted by $d(u, v)$.*

If G is a disconnected graph, then we define the distance between vertices in the same component of G. If u and v are vertices in distinct components of G, then $d(u, v) = \infty$. *Eccentricity* $\rho(u)$ of a vertex u in a connected graph G is defined as

$$\rho(u) = Max\{d(u, v) : \forall\, v \in V(G)\}.$$

The minimum eccentricity in a graph G is the *radius*, rad (G), while the maximum eccentricity is the *diameter*, diam(G).

2.3 Special graphs

In what follows are the definitions of some special types of graphs and examples that are of interest in this thesis. The reader may see [22] for the proofs of some stated results.

2.3.1 Bipartite graphs

Definition 48. *A graph G is bipartite if $V(G)$ can be partitioned into 2 disjoint sets V_1 and V_2 such that $uv \in E(G)$ implies either $u \in V_1$ and $v \in V_2$ or $v \in V_1$ and $u \in V_2$.*

If each vertex in V_1 is adjacent to each vertex in V_2, then G is a *complete bipartite graph.* If $|V_1| = r$ and $|V_2| = s$, then this complete bipartite graph, denoted by $K_{s,r}$, has order $s + r$

and size sr . The special case when $s = 1$ i.e., $K_{1,r}$ is called a *star*. Figure 2.4 illustrates a bipartite graph G, a star and a complete bipartite graph of order 5.

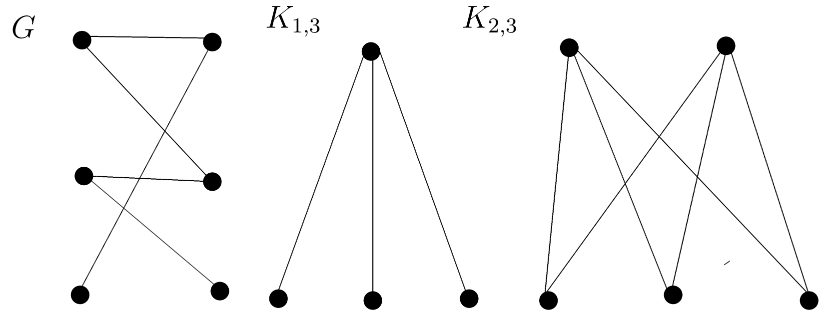

Figure 2.4: Bipartite, star and complete bipartite graphs

2.3.2 Planar graphs

Definition 49. *A graph is planar provided it can be represented on a plane such that there is no crossing of its edges except at its vertices.*

2.3.3 Eulerian and Hamiltonian cycle

An *Euler circuit* in a graph is a circuit that utilizes every edge of the graph exactly once. An open trail in a graph that contains every edge of the graph exactly once is called an *Euler trail.*

Definition 50. *A connected graph G is Eulerian if it contains an Eulerian circuit.*

A path in a graph G that visits every vertex of G is called a *Hamiltonian path* of G. As for a cycle in G that visits every vertex of G it is called a *Hamiltonian cycle* of G.

Definition 51. *A graph is Hamiltonian if it contains a Hamiltonian cycle.*

Unlike the Euler circuit problem, finding Hamilton circuits is very difficult. There is no simple set of necessary and sufficient conditions, and no simple algorithm. Instead, we have the following sufficient conditions.

Theorem 52. *(Dirac's Theorem)*

If G is a graph with $n \geq 3$ vertices such that $deg(v) \geq n/2$ for each vertex $v \in V(G)$, then G is Hamiltonian.

2.3.4 Directed graphs

Definition 53. *A directed graph or digraph D is a finite non-empty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of D called arcs or directed edges.*

Chapter 3

Review of related literature

In this chapter, we present chronological accounts of what researchers have done for over a century in associating graphs to finite groups. Definitions and some important results are stated with references.

3.1 Cayley graphs, 1878.

It was Arthur Cayley in 1878 who was considered the first to associate graphs called the *Cayley graph* to finite groups, [20]. Cayley graph is known to have originated from *Cayley color diagram* which is a graphical representation of groups introduced by Cayley. This color diagram is a directed graph with edges colored while the underlying uncolored or undirected graph is the Cayley graph. For interesting applications of Cayley graphs in algebra, computer science, biological Sciences, etc, see [9, 41, 45].

Definition 54. *Given a finite group Γ and a non-empty subset, $C \subseteq \Gamma$ that generates Γ, the Cayley graph denoted by $Cay(\Gamma, C)$ is defined as follows: each vertex is an element of Γ, and two vertices $u, v \in V(Cay(\Gamma, C))$ are adjacent if either $uv^{-1} \in C$ or $vu^{-1} \in C$, [61].*

3.1.1 Basic properties

Deep investigations and various characterizations have been carried out on Cayley graphs for over a century. In what follows are the basic properties of Cayley graphs.

Theorem 55. *(see [44])*

Suppose C is a set of generators of a finite group Γ. The Cayley graph $Cay(\Gamma, C)$ is:

1. *Connected.*

2. *$|C|$-regular, where $|C|$ is the cardinality of C.*

3. *Vertex-transitive.*

Remark 56. *It is worthwhile to note that every Cayley graph is vertex-transitive but not every vertex-transitive graph is Cayley. An interesting example is the Petersen graph which is vertex-transitive but not Cayley.*

Theorem 57. *(see [57])*

Let Γ be an abelian group generated by two elements i.e $|C| = 2$. If k is the diameter of $Cay(\Gamma, C)$ and m its size then,

$$m \leq 2k^2 + 2k + 1.$$

Theorem 58. *(see [8])*

Every subgroup of a planar group[1] is planar

Corollary 59. *(see [19])*

Suppose $Cay(\Gamma, C)$ is planar and $\gamma \leq \Gamma$, then there exists a generating set A of γ with $\gamma \cap C \subseteq A$ such that $Cay(\gamma, A)$ is planar.

[1] A group Γ is planar provided its Cayley graph, $Cay(\Gamma, C)$, is planar and C is called planar generating set.

3.1.2 Hamiltonicity of Cayley graphs

Hamiltonian characterizations of Cayley graphs have been a very hot research area for many decades. The classical conjecture of Lovász[2] in 1969 coupled with huge applications of Hamiltonian graphs have attracted the interests of researchers for the past 45 years. Up till today, several researches are ongoing to completely characterize the Hamiltonian cycles in Cayley graphs. In what follows are some established results on this topic with references.

Theorem 60. *(see [54]) Suppose* Γ *is a finite group with order* $n \geq 3$ *and* C *a generating set such that* $|C| \leq log_2 n$. *Then, the associated Cayley graph* $Cay(\Gamma, C)$ *contains a Hamiltonian cycle.*

Conjecture 61. *(Lovász, see [24])*

Every finite connected vertex-transitive graph has a Hamiltonian path.

Conjecture 62. *(see [4]) There is a Hamiltonian cycle in every connected Cayley graph of size* $n \geq 3$.

Theorem 63. *(see [39]) Almost all Cayley graphs are Hamiltonian*

Many other results abound in the literature regarding the Hamiltonian cycles in Cayley graphs. For additional results and problems on Cayley graphs one can consider the comprehensive surveys [43, 44, 58, 72].

3.2 Commuting graphs, 1955

The *commuting graph*, $G(\Gamma; P)$, where Γ is a group and P a subset of Γ, has P as its vertex set with two distinct elements $u, v \in P$ joined by an edge whenever $uv = vu$ in Γ. The associated graph, $G(\Gamma; P)$, where P contains involutions of Γ is called *commuting involution*

[2]See Conjecture 61.

graph. Everett in [30] asserted that commuting graphs came to prominence in the ground breaking paper of Brauer and Fowler [16]. Bertram in [12] also proved three fundamental and non-trivial theorems on finite groups using combinatorial properties of commuting graphs. For some applications of commuting graphs on bounded linear operators of Hilbert spaces see [5].

3.2.1 Basic properties of commuting graphs

Ever since the introduction of commuting graphs several investigations have been carried out. We present below some important results for commuting graphs associated with finite groups.

Theorem 64. *(see [51]) Suppose* Γ *is a non-abelian finite group. If* $P = \Gamma \setminus \{e\}$ *then the commuting graph* $G(\Gamma, P)$ *has the following properties:*

1. *The diameter of every connected component of* $G(\Gamma, P)$ *is bounded above by 10.*

2. *For a connected* $G(\Gamma, P)$*, the diameter* $diam[G(\Gamma, P)] \leq 10$.

Theorem 65. *(see [49]) Suppose* Γ *is a finite group and* $P = \Gamma \setminus Z(\Gamma)$*. Then the commuting graph* $G(\Gamma, P)$ *is a union of complete subgraphs,* K_2*, if and only if* $\Gamma \cong D_8$ *or* Q_8*.* D_8 *is the dihedral group of order 8 and* Q_8 *the quaternion group.*

Theorem 66. *(see [49]) Suppose the order of* Γ *is greater than 2 and* $P = \Gamma \setminus Z(\Gamma)$*. Then, the automorphism group of the associated commuting graph,* $Aut(G(\Gamma, P))$ *is a non-abelian group.*

3.2.2 Commuting graphs on symmetry, dihedral and generalized quaternion groups

Some authors have singled out the study of the commuting graphs associated with symmetry, dihedral and generalized quaternion groups. We shall present only two results here and the interested reader may see [10, 17] for more results.

Theorem 67. *(see [23]) Let $n \geq 3$ be any integer. There exists no subset P of D_{2n} such that the associated commuting graph, $G(D_{2n}, P)$ is :*

1. *n-regular.*
2. *a cycle of length 4.*

Theorem 68. *(see [65]) Suppose Q_{2^n} is a generalized quaternion group and $Z(Q_{2^n})$ its center. Then, the associated commuting graph, $G(Q_{2^n}, Q_{2^n} \setminus Z(Q_{2^n})) = G(Q_{2^n})$ for short, has the following properties:*

1. *$\omega_1(G(Q_{2^n})) = 2^{n-1} - 2$,*
2. *$\omega_2(G(Q_{2^n})) = 2^{n-2} + 1$,*

where ω_1 and ω_2 are clique number and independence number of $G(Q_{2^n})$ respectively.

3.2.3 Commuting involution graphs

Several researches have been devoted to the study of special commuting graphs $G(\Gamma, P)$, where P is a conjugacy class of involutions in Γ. This graph is called *commuting involution graph.* According to Bates *et al* in [11], commuting involution graphs originated from the investigations carried out by Fisher on the "3-transposition" groups in 1971. Ever since then, studies have been directed to this approach. This paper [11], contains a very deep analysis and comprehensive study of the commuting involution graph, $G(\Gamma, P)$, where Γ is a symmetry group of order n.

3.3 Intersection graphs, 1969

Zelinka in [75] asserted that it was Csákány who introduced the *intersection graphs* on finite groups, $G(\Gamma)$, in 1969. The vertices of this graph are the non-trivial subgroups of the group Γ and there is an edge between two vertices γ_1, γ_2, provided $\gamma_1 \setminus \{e\} \cap \gamma_2 \setminus \{e\} \neq \emptyset$, where e is the identity element in Γ. Literature reveals that Bosák in 1964 (see [15]) had previously defined the intersection graphs on semigroups but intersection graph on finite groups was credited to Csákány.

3.3.1 Basic properties of intersection graphs

Ever since the introduction of intersection graphs several investigations have been carried out. We present below some few results on intersection graphs associated with finite groups. For a deep investigation of a special intersection graph where only the normal subgroups are considered, see [38].

Theorem 69. *(see [60]) Suppose Γ is a finite group. Then, its intersection graph, $G(\Gamma)$, is disconnected if Γ is :*

1. *$\mathbb{Z}_r \times \mathbb{Z}_s$ where r and s are prime numbers.*
2. *a Frobenius group whose complement is a group of prime order and the kernel is a minimal normal subgroup*

Proposition 70. *(see [40]) Suppose Γ is a finite abelian group and denote the cyclic group of order r by Γ_r. Then, Γ is planar[3] if and only if it is isomorphic to one of the following groups:*

1. *Γ_{n^i} where i is between 0 and 5 inclusive,*

[3] A group is planar if its associated intersection graph is planar.

2. $\Gamma_{n^2} \times \Gamma_m$,

3. $\Gamma_n \times \Gamma_m$,

4. $\Gamma_4 \times \Gamma_2$,

5. $\Gamma_n \times \Gamma_n$,

6. $\Gamma_l \times \Gamma_m \times \Gamma_n$

7. $\Gamma_2 \times \Gamma_2 \times \Gamma_n$, $n \neq 2$,

where l, m, n are distinct prime numbers.

Lemma 71. *(see [40]). Let Γ be a non-abelian group and $|\Gamma| = p^4$ where p is prime. Then, Γ is non-planar.*

Proposition 72. *(see [75]) Let Γ be a finite abelian group. The cardinality of the largest independence set[4] of the graph $G(\Gamma)$ equals the maximal number of prime order subgroups of Γ.*

3.4 Prime graphs, 1970

Gruber *et al* in [33] claimed that *prime graphs*, $G(\Gamma)$, came into existence as a by-product of some cohomological questions raised by K.W. Gruenberg in the 1970s. Let $\Pi(\Gamma)$, the set of primes dividing the order of Γ, be the vertices of the prime graph $G(\Gamma)$. There is an edge between two vertices r_1 and r_2 provided there exists an element $u \in \Gamma$ of order $r_1 r_2$. Vasil'ev in [66] investigated the relationship between finite groups and their prime graphs. For excellent works on generalization of prime graphs and full classifications of prime graphs see [2, 35, 67, 70, 76].

[4]Independence set of a graph G consists of vertices of G such that no two vertices in the set are adjacent. The cardinality of the largest independence set in a graph is the *independence number*.

3.4.1 Basic properties of prime graphs

Theorem 73. *(see [64]) Suppose Γ is a group of order n. Then, the prime graph $G(\Gamma)$ is 3-regular if and only if it is K_4.*

Lemma 74. *(see [64]) Let Γ be a finite group with k-regular prime graph, $G(\Gamma)$, where $0 \leq k \leq 2$.*

1. *If $k = 0$, then $G(\Gamma)$ is an empty graph of at most order 3.*

2. *If $k = 1$, then $G(\Gamma) \cong K_2$.*

3. *If $k = 2$, then $G(\Gamma)$ is either a triangle[5] or a square[6]. Besides, if $G(\Gamma)$ is a square, then Γ is a solvable group.*

The following conjecture is due to Hung in [64] regarding the k-regular prime graphs.

Conjecture 75. *Suppose Γ is a finite group with k-regular prime graph, $G(\Gamma), k \geq 2$. Then,*

1. *If k is an odd number greater than 4, then, $G(\Gamma) = K_{k+1}$*

2. *If k is an even number greater than 3, then, either $G(\Gamma) = K_{k+1}$ or a k-regular graph of order $k + 2$.*

3. *If $|\Pi(\Gamma)| = k + 2$, then G is solvable.*

Theorem 76. *(see [63]) Suppose Γ is a group whose prime graph $G(\Gamma)$ contains no triangle. Then, the order of $G(\Gamma)$ is bounded above by 5. In particular, if $G(\Gamma)$ is a cycle or a tree then the order of $G(\Gamma)$ is bounded above by 4.*

[5] A cycle of length 3.
[6] A cycle of length 4.

3.5 Non-commuting graphs, 1975

Neumann in 1975 asserted that Erdös in the same year introduced *non-commuting graph*, $G(\Gamma)$, associated with a group Γ. In this graph, the vertices are the elements of Γ and there is an edge between two vertices $u, v \in G(\Gamma)$ if and only if $uv \neq vu$ i.e the commutator $u^{-1}v^{-1}uv$ is not the identity, [53]. If Γ is non-abelian and $Z(\Gamma)$ is the center of Γ then the non-commuting graph, $G(\Gamma)$, is also defined as follows: $\Gamma \setminus Z(\Gamma)$ as vertex set and there is an edge between $u, v \in \Gamma \setminus Z(\Gamma)$ provided $uv \neq vu$, [1, 50].

In 2013, Erfanian and Tolue introduced the *relative* n^{th} *non-commuting graph*, $G^n_\gamma(\Gamma)$, where γ is a non-abelian subgroup of Γ for a fixed $n \in \mathbb{Z}^+$. The vertex set is $\Gamma \setminus S^n_{\gamma,\Gamma}$ where $S^n_{\gamma,\Gamma} = \{u \in \Gamma : [u, v^n] = 1 \text{ and } [u^n, v] = 1 \text{ for all } v \in \gamma\}$. Further there is an edge between u and v if either u or $v \in \gamma$ and $uv^n \neq v^n u$ or $u^n v \neq vu^n$, [29]. Some characterizations of this graph were also investigated. For nice discussions on non-commuting graphs of dihedral groups see [62].

3.5.1 Basic properties of non-commuting graphs

Theorem 77. *(see [1])*

Suppose Γ is a non-abelian group. Then, the associated non-commuting graph has the following properties:

1. *It has a diameter of 2.*
2. *It is connected.*
3. *It has a girth of 3.*
4. *It is Hamiltonian.*
5. *It is planar if and only if Γ is isomorphic to either Q_8, D_8 or S_3.*

Lemma 78. *(see [1]). Let* Γ *be a non-abelian group such that* $|\{deg(u) : u \in V(G(\Gamma))\}| = 2$*. Then,* Γ *is solvable.*

Theorem 79. *(see [28]). Suppose* Γ *is a non-abelian group and* m *the size of the associated non-commuting graph,* $G(\Gamma)$ *. Then* $G(\Gamma)$ *has the following properties:*

1. $G(\Gamma)$ *is not a complete graph.*

2. $G(\Gamma)$ *is not bipartite.*

3. $m \geq \frac{3}{2}|\Gamma|$*. Equality holds provided that* $\Gamma \cong Q_8, D_8$ *or* S_3

4. $m \neq 2|\Gamma|$*.*

5. *If* $|\Gamma| > \frac{16n}{3}$*, where* $n \in \mathbb{N}$*, then* $m > n|\Gamma|$*.*

6. *If* $m \leq 31|\Gamma|$ *and* Γ *is simple, then* $\Gamma \cong A_5$*.*

3.5.2 Conjectures and further results

The following conjectures are due to Abdollahi *et al* in [1].

Conjecture 80. *Suppose* Γ_1 *and* Γ_2 *are two non-abelian groups. If* $G(\Gamma_1) \cong G(\Gamma_2)$*, then* $|\Gamma_1| = |\Gamma_2|$ *where* $G(\Gamma_1)$ *and* $G(\Gamma_2)$ *are non-commuting graphs associated with* Γ_1 *and* Γ_2 *respectively.*

Conjecture 81. *Suppose* Γ_1 *and* Γ_2 *are two non-abelian groups and either* Γ_1 *or* Γ_2 *is a simple group. If the non-commuting graphs* $G(\Gamma_1) \cong G(\Gamma_2)$*, then* $\Gamma_1 \cong \Gamma_2$*.*

Theorem 82. *(see [27]). Let* Γ_1 *be a simple group for which the Thompson's conjecture*[7] *holds and* Γ_2 *a finite group. If* $G(\Gamma_1) \cong G(\Gamma_2)$*, then* $\Gamma_1 \cong \Gamma_2$*.*

[7] See [34] for the verification of Thompson's conjecture on simple groups.

3.6 Conjugacy class graphs, 1990

Bertram in 1990 introduced the *conjugacy class graph* associated with a finite group Γ as follows: The vertices of $G(\Gamma)$ are the *non-central conjugacy classes*[8] of Γ and two vertices $Cl(u)$ and $Cl(v)$ are adjacent provided $(|Cl(u)|, |Cl(v)|) \neq 1$, i.e. $|Cl(u)|$ and $|Cl(v)|$ are not co-prime, see [13]. You *et al* in an unpublished article, [74], proposed a modified definition of conjugacy class graphs as follows: The vertices are the non-central conjugacy classes of Γ and two vertices $Cl(u)$ and $Cl(v)$ are adjacent provided $(|u|, |v|) > 1$. Furthermore, Lu and Zhang in [48] devoted a full length article to the study of conjugacy class graphs of p-regular conjugacy classes[9] in a group. For a comprehensive account of interesting results on conjugacy class graphs, see [46]. Bianchi *et al* in [14] gave excellent applications of conjugacy class graphs to proving some results in group theory. The following are some of the characterizations of conjugacy class graphs associated with finite groups the reader may see the corresponding references for the proofs.

3.6.1 Basic properties of conjugacy class graphs

Theorem 83. *(see [13]). Suppose Γ is a finite group. Then, the associated conjugacy class graph, $G(\Gamma)$, has the following properties:*

1. *Number of connected components $k(G(\Gamma)) \leq 2$. Equality holds if and only if Γ is quasi-Frobenius*[10] *with commutative kernel and complement.*
2. *If $k(G(\Gamma)) = 1$, then $diam(G(\Gamma)) \leq 4$.*
3. *If $G(\Gamma)$ is an empty graph, then $\Gamma \cong S_3$.*
4. *It is a complete graph if Γ is a non-abelian simple group*

[8] Conjugacy classes containing more than one element.

[9] See [48] for the meaning of this.

[10] Γ is defined to be quasi-Frobenius provided that $\Gamma \setminus Z(\Gamma)$ is Frobenius.

Theorem 84. *(see [31]). Suppose* Γ *is a non-abelian finite group. Then, the conjugacy class graph* $G(\Gamma)$ *contains no triangle if and only if there is an isomorphism between* Γ *and any of the following groups:*

1. S_3.

2. D_{12}.

3. A_4.

4. $\Gamma = \langle u, v \mid u^6 = e, v^2 = u^3, vu = u^{-1}v \rangle$.

5. $\Gamma = \langle u, v \mid u^3 = v^7 = 1, vu = uv^2 \rangle$.

Theorem 85. *(see [31]). Suppose the conjugacy class graph* $G(\Gamma)$ *is connected with no triangle and distinct* $Cl(u)$ *and* $Cl(v)$ *non-central conjugacy classes. Then,* $|Cl(u)| \neq |Cl(v)|$*.*

Theorem 86. *see ([31]). Suppose* Γ *is a non-solvable finite group. Then, the associated conjugacy class graph* $G(\Gamma)$ *contains* C_3*.*

3.7 Power graphs, 2000

Directed power graph, $\vec{G}(\Gamma)$, was introduced by Kelarev in 2000 as a tool to studying the combinatorial properties of groups with infinite sequences. This graph is defined on a finite group Γ with its elements as the vertices and there is a directed edge from u to $v \in \vec{G}(\Gamma)$ provided $v = u^t, u \neq v$ and $t \in \mathbb{N}$, see [42]. In this article, Kelarev was able to establish a new combinatorial property of groups via the directed power graphs. This was the first application of directed power graph in group theory. In 2007, Imani *et al* [36] applied some properties of power graphs to resource placement in networks. For a recent and excellent survey on power graphs see [37]. The underlying graph, $G(\Gamma)$, when all the directions are removed is the *power graph* first considered by Chakrabarty in 2009, see [21].

3.7.1 Basic Properties of Power graphs

Theorem 87. *(see [25]). For all finite groups of a given order, the cyclic group of that order has the maximum number of edges in its power graph.*

Theorem 88. *(see [56]). Let Γ be a p-group*[11]*. Then $G(\Gamma)$ is 2-connected if and only if Γ is a cyclic group or Q_n.*

Theorem 89. *(see [18]). Suppose Γ_1 and Γ_2 are finite abelian groups. If $G(\Gamma_1) \cong G(\Gamma_2)$ then $\Gamma_1 \cong \Gamma_2$.*

Conjecture 90. *(see [18]). Suppose Γ_1 and Γ_2 are finite groups. If $\vec{G}(\Gamma_1) \cong \vec{G}(\Gamma_2)$ then Γ_1 and Γ_2 have equal numbers of elements of each order.*

Theorem 91. *(see [18]). The only finite group Γ satisfying $Aut(\Gamma) = Aut(G(\Gamma))$ is the Klein four group.*

3.7.2 Eulerian and Hamiltonian cycles in power graphs

Theorem 92. *(see [56]). Suppose Γ is any finite group. The power graph $G(\Gamma)$ is Eulerian if and only if the order of Γ is odd.*

Theorem 93. *(see [56]). Let Γ be a p-group. The power graph $G(\Gamma)$ is Hamiltonian if and only if Γ is cyclic and its order is not 2.*

[11] A group Γ is called a p-group if it has order a power of the prime p.

Chapter 4

Construction and characterizations of inverse graphs

4.1 Definition of inverse graph and examples

Definition 94. *Let $(\Gamma, *)$ be a finite group and $S = \{u \in \Gamma \mid u \neq u^{-1}\}$. We define the inverse graph $G_S(\Gamma)$ associated with Γ as the graph whose set of vertices coincides with Γ such that two distinct vertices u and v are adjacent if and only if either $u * v \in S$ or $v * u \in S$.*

Remark 95. *It is worthwhile to observe the following:*

1. *Clearly, the identity e is a trivial self-invertible element in any finite group Γ. Hence $e \notin S$. Consequently, the cardinality of S is strictly less than the cardinality of Γ. In particular, if Γ contains no self-invertible element other than the identity then $|S| = |\Gamma| - 1$.*

2. *As S has always an even number of elements, then $|S| = |\Gamma| - 1$ if Γ contains an odd number of elements.*

3. *In any inverse graph $deg(e) = |S|$.*

Example 96. The graphs in Figure 4.1 are the inverse graphs of the groups $(\mathbb{Z}_3, +)$ and $(\mathbb{Z}_5 \setminus \{0\}, \cdot)$ respectively.

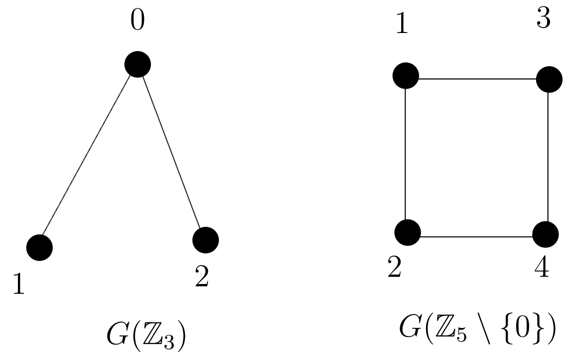

Figure 4.1: Inverse graphs of $(\mathbb{Z}_3, +)$ and $(\mathbb{Z}_5 \setminus \{0\}, \cdot)$.

Example 97. Consider $S_3 = \{e, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$ the symmetry group[1] of order 6. $S = \{\pi_1, \pi_2\}$ the set of non-self invertible elements in S_3. The inverse graph $G_S(S_3)$ is as follows:

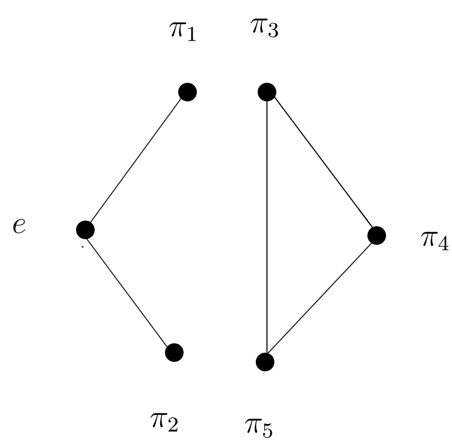

Figure 4.2: Inverse graph of S_3.

Example 98. Consider the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with the set of non-self invertible elements $S = \{i, -i, j, -j, k, -k\}$. The inverse graph $G_S(Q_8)$ is as follows:

[1]See Example 14.

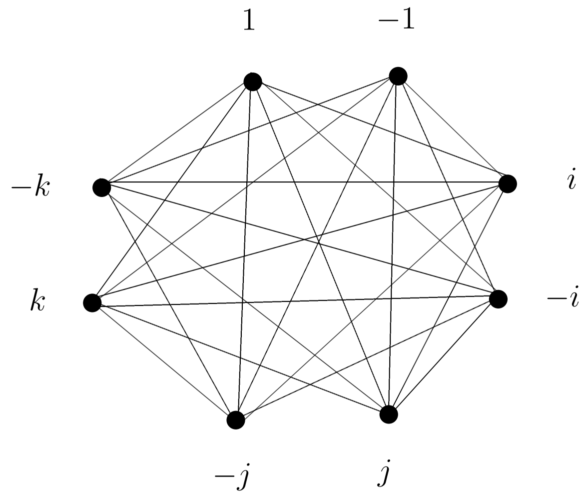

Figure 4.3: Inverse graph of Q_8.

4.2 Basic properties of the inverse graphs

In this section, we study some basic properties of inverse graphs associated with finite groups.

4.2.1 Trivial/Empty inverse graphs

Proposition 99. *For any finite group Γ, the inverse graph $G_S(\Gamma)$ is empty if and only if $|\Gamma| = 1$ or $|S| = 0$.*

Proof. Suppose $G_S(\Gamma)$ is empty. Then by Definition 41 either $|\Gamma| = 1$ or for any $u, v \in \Gamma$ with $u \neq v$, we have $u * v \notin S$. Hence S is empty i.e., $|S| = 0$. The converse is obvious. $\square$

Remark 100. *Proposition 99 characterizes the inverse graphs of finite groups all of whose elements are self-invertible. In particular, the inverse graphs associated with the following groups are empty graphs:*

1. *Groups consisting of two elements.*
2. $\Gamma_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$, *with usual addition of matrices.*

3. $\Gamma_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$, *with usual addition of matrices.*

4. $\Gamma_3 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, *with usual addition of* $\mathbb{Z}_2^2$.

On the other extreme, it is natural to ask when will the inverse graphs be complete? Unfortunately, such graphs never exist. Recall that a graph G is complete provided it is not empty and there is an edge between every distinct pair of its vertices. In a complete graph G with $|G| = n$, we have $deg(v) = n - 1$ for any vertex v of G.

Theorem 101. *There is no inverse graph that is complete for any finite group* Γ.

Proof. Suppose on the contrary that there exists an inverse graph $G_S(\Gamma)$ that is complete. Then for each vertex $v \in V(G_S(\Gamma))$, $deg(v) = n - 1$, where $n = |\Gamma|$.

Case 1: n is even. Since $deg(e) = n - 1$, by Remark 95-(3) we have $|S| = n - 1$ which is not possible as $|S|$ is always even.

Case 2: n is odd. Let $u \neq v \in \Gamma$ such that $u = v^{-1}$. Since there is an edge between every two distinct vertices, we have $e = u * v \in S$ which is not possible, see Remark 95-(1). □

4.2.2 Connectedness of inverse graphs

Theorem 102. *For any finite group* Γ *with at least three elements and a non-empty subset* S *of non-self invertible elements, the graph* $G_S(\Gamma)$ *has no isolated vertex.*

Proof. Suppose by contradiction that there exists an isolated vertex v in $G_S(\Gamma)$. Then we have the following two cases.

Case 1 : $v \in S$. But this is not possible for if $e * v = v \in S$, where e is the identity element of Γ, then v is connected to e in $G_S(\Gamma)$.

Case 2 : $v \notin S$, then either v is the identity or v is a non-trivial self-invertible element of Γ. It follows from Remark 95-(3), that v cannot be e. Hence v is a non-trivial self-invertible element. Let $\omega \in S$. If v is the only non-trivial self-invertible element of Γ and as v is an isolated vertex, then $\omega * v = e$ which implies $\omega = v^{-1}$, a contradiction. Hence v is not the only element of $\Gamma \setminus \{S \cup \{e\}\}$. Thus there exists $v' \neq v \in \Gamma \setminus \{S \cup \{e\}\}$ such that $v * \omega = v'$. Hence $\omega = v * v' \in S$ i.e., there is an edge between v and v', a contradiction. $\square$

Theorem 103. *For any finite abelian group Γ with at least three elements and a non-empty subset S of non-self invertible elements, the graph $G_S(\Gamma)$ is connected.*

Proof. By Remark 95-(3) the identity e is adjacent to every element of S. We are left to show that every element of $\Gamma \setminus \{S \cup \{e\}\}$ is adjacent to each element of S. For this, consider the product $u * v$ where $u \in S$, $v \in \Gamma \setminus \{S \cup \{e\}\}$ and * is the operation defined on Γ. Then, $(u * v)^{-1} = u^{-1} * v^{-1} = u^{-1} * v \neq u * v$ since $u \in S$. So $u * v \in S$. Since both u and v are arbitrarily chosen we have every element of $\Gamma \setminus \{S \cup \{e\}\}$ to be adjacent to each element of S. Therefore, each vertex of $G_S(\Gamma)$ is reachable and therefore connected. $\square$

Remark 104. *Note that the commutativity of Γ in Theorem 103 cannot be dropped for the conclusion to hold. An example of a non-abelian group with a disconnected inverse graph is the symmetry group S_3 of order 6, see Example 97. On the other hand, it is not true that the inverse graph of every non-abelian group is disconnected. A counterexample is the inverse graph associated with the quaternion group Q_8, see Example 98.*

4.2.3 Diameter of inverse graph

Theorem 105. *The diameter of a connected inverse graph is two.*

Proof. Let $G_S(\Gamma)$ be a connected inverse graph associated with a group Γ. We consider the following vertex partition: $V(G_S(\Gamma)) = \{e\} \cup S \cup S'$, where S is the set of all non-self invertible elements and S' is the set of all non-trivial self-invertible elements. Since every element in S is adjacent to e and there is no edge between e and the elements of S', $\rho(e)$, the eccentricity of e, is 2 . Also, for each element $u \in S$, $\rho(u) = 2$ as u is not adjacent to its inverse but adjacent to e and every element in S'. Now take an arbitrary element $v \in S'$, it follows from the construction and connectedness of $G_S(\Gamma)$ that v is not adjacent to e but adjacent to every vertex in S. Hence $\rho(v) = 2$. □

4.2.4 Bounds on sum of degrees and size of the inverse graphs

Lemma 106. *In a non-empty inverse graph $G_S(\Gamma)$ of order n, the sum of the degrees is bounded above by $n(n-1) - |S|$.*

Proof. Let $V(G_S(\Gamma)) = \{v_1, v_2, \cdots, v_n\}$. By the first theorem of graph theory, we have

$$\sum_{i=1}^{n} deg(v_i) = 2|E(G_S(\Gamma))|.$$

By Theorem 101, $G_S(\Gamma)$ cannot be complete. Hence

$$\sum_{i=1}^{n} deg(v_i) = 2|E(G_S(\Gamma))| < 2 \cdot \frac{n(n-1)}{2} = n(n-1).$$

Since $G_S(\Gamma)$ is a non-empty graph, by Proposition 99, $S \neq \emptyset$. As any pair $u, v \in S$ with $u = v^{-1}$ has no edge in $G_S(\Gamma)$, such a pair contributes -2 to the total degrees of the vertices of $G_S(\Gamma)$. Hence a total of $-|S|$ degrees is contributed by the elements of S. Therefore,

$$\sum_{i=1}^{n} deg(v_i) \leq n(n-1) - |S|.$$

□

The following corollary elicits the fact that the inequality in Lemma 106 cannot be improved.

Corollary 107. *Let Γ be a finite group with no element of order 2. Then*

$$\sum_{i=1}^{n} deg(v_i) = n(n-1) - |S|.$$

Proof. Let $V(G_S(\Gamma)) = \{v_1, v_2, \cdots, v_n\}$. Without loss of generality, let $v_1 = e$, the identity element of Γ. By Remark 95-(1), we have $|S| = |\Gamma| - 1 = n - 1$, i.e., $S = \{v_2, \ldots, v_n\}$. Hence, $deg(v_1) = n - 1$. Also, $deg(v_i) = n - 2$ for all $v_i \in S$ since each v_i is not adjacent to itself and its inverse. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} deg(v_i) &= deg(v_1) + \sum_{i=2}^{n} deg(v_i)\\ &= n - 1 + (n-1)(n-2)\\ &= n(n-1) - (n-1)\\ &= n(n-1) - |S|.\end{aligned}$$

□

Corollary 108. *Suppose Γ is a cyclic group of prime order $n > 2$. Then for all $v_i \in V(G_S(\Gamma))$ we have $\sum_{i=1}^{n} deg(v_i) = (n-1)^2$ and size $m = \frac{(n-1)^2}{2}$.*

The following example illustrates Corollary 108.

Example 109. Let $\Gamma = \{z \in \mathbb{C} : z^5 = 1\}$ be the group of fifth roots of unity under multiplication. As Γ is generated by one of its non-trivial elements say ω, then we can write $\Gamma =< \omega >= \{\omega, \omega^2, \cdots, \omega^5\}$. Hence $S = \{\omega, \omega^2, \omega^3, \omega^4\}$. $G_S(\Gamma)$ is shown in Figure 4.4 .
It is obvious that $\sum\limits_{i=1}^{5} deg(\omega^i) = 5(5-1) - 4 = 16$.

Proposition 110. *Let Γ be a finite group of an odd order n. Then the size of $G_S(\Gamma)$ is bounded above by the size of the inverse graph $G_S(\mathbb{Z}_n)$.*

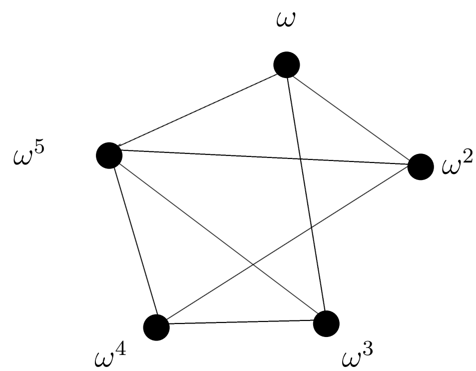

Figure 4.4: Inverse graph, $G_S(\Gamma)$, of the group Γ

Proof. Let m be the size of $G_S(\Gamma)$. By Lemma 106 and the first theorem of graph theory, we have $m \leq \frac{n(n-1)-|S|}{2}$. Now, suppose that $\mathbb{Z}_n = \langle \omega \rangle = \langle e, \omega, \ldots, \omega^{n-1} \rangle$ is a cyclic group of order n. As n is odd, $\omega^i \in \mathbb{Z}_n$ is non-self invertible since $(\omega^i)^{-1} = \omega^j$ where $i + j = n$. It follows from Corollary 107 that the size of $G(\mathbb{Z}_n)$ is equal to $\frac{n(n-1)-|S|}{2}$. $\square$

4.2.5 Regular inverse graphs

Theorem 111. *Let Γ be a finite group and the set S of non-self invertible elements be non-empty. The associated inverse graph $G_S(\Gamma)$ is*

1. *2-regular if Γ is a group of four elements.*

2. *$(2^n - 2)$-regular if Γ is a generalized quaternion group of order 2^n.*

Proof. (1) Let $\Gamma = \{e, v_1, v_2, v_3\}$. Since S is non-empty, without loss of generality, $S = \{v_1, v_2\}$. By construction $G_S(\Gamma)$ is the cycle C_4. Hence it is 2-regular.
(2) Let $\Gamma = \langle u, v : v^4 = u^{2^{n-1}} = e, v^2 = u^{2^{n-2}}, vu = u^{-1}v \rangle$ be the generalized quaternion group with $n > 2$. Since the only non-trivial self-invertible element in Γ is $u^{2^{n-2}}$, we have $S = \Gamma \setminus \{e, u^{2^{n-2}}\}$. Consequently, $deg(e) = deg(u^{2^{n-2}}) = 2^n - 2$. Now, for each element $v \in S$, we have $deg(v) = 2^n - 2$ as v is adjacent to all vertices except itself and its inverse.

$\square$

Corollary 112. *Let G be a connected $(2^n - 2)$-regular graph, where $n > 2$. Then there exists a generalized quaternion group of order 2^n whose inverse graph is isomorphic to G.*

4.3 Hamiltonian cycles in inverse graphs

Recall that for a given graph G and vertex set $V(G) = \{v_0, v_1, \cdots, v_n\}$ a walk of length $k \leq n$ is a finite sequence

$$v_{i_0} e_{j_1} v_{i_1} e_{j_2} v_{i_2} \cdots v_{i_{k-1}} e_{j_k} v_{i_k}$$

whose terms alternate between vertices and edges such that $v_{i_{t-1}} v_{i_t} = e_{j_t}$ for $1 \leq t \leq k$ and $0 \leq i_k \leq n$. The walk W is closed if $v_{i_0} = v_{i_k}$, otherwise it is open. An open walk in which no vertex is repeated is called a path. A closed walk of length $l \geq 3$ with neither vertex nor edge repeated is called a cycle.

4.3.1 Inverse graph as a cycle

Lemma 113. *Let Γ be a group of order four and $G_S(\Gamma)$ be a non-empty graph. Then $\Gamma \cong \mathbb{Z}_4$.*

Proof. Let Γ be a group of four elements and S be the set of non-self invertible elements of Γ. Since non-self invertible elements occur in pairs, $|S|$ is equal to $0, 2$, or 4. $|S|$ cannot be zero because this would imply that $G_S(\Gamma)$ is empty. Also $|S| \neq 4$ since $e \notin S$. Hence $|S| = 2$ and Γ has exactly one non-trivial self-invertible element.

□

Theorem 114. *Let Γ be a group of order four. Then $G_S(\Gamma)$ is the cycle C_4 if and only if $\Gamma \cong \mathbb{Z}_4$.*

Proof. Suppose that $G_S(\Gamma) = C_4$. Then $|\Gamma| = 4$ and so by Lemma 113, we have $\Gamma \cong \mathbb{Z}_4$. The converse follows immediately by constructing $G_S(\mathbb{Z}_4)$.

□

4.3.2 Hamiltonian characterization of inverse graphs

Recall that a graph G is Hamiltonian provided G contains a cycle that visits all its vertices. It follows immediately that every cycle C_n is Hamiltonian and hence $G_S(\mathbb{Z}_4)$ is Hamiltonian. The following theorems characterize the Hamiltonicity of the inverse graphs.

Theorem 115. *Let Γ be a finite abelian group of order n greater than three with at most two self-invertible elements. Then $G_S(\Gamma)$ is Hamiltonian.*

Proof. Suppose Γ has exactly one self-invertible element which is the identity e of the group. We have $deg(v_i) = n - 2$ for all $v_i \neq e \in G_S(\Gamma)$, see the proof of Corollary 107. By Remark 95-(3), $deg(e) = n - 1 > n - 2 \geq \frac{n}{2}$, $\forall n \ \geq 4$. Hence $deg(v) \geq \frac{n}{2}$ for all vertices of $G_S(\Gamma)$. Thus by Dirac's Theorem, $G_S(\Gamma)$ is Hamiltonian. On the other hand, if Γ has exactly two self-invertible elements then it contains only one non-trivial self-invertible element, say u with $deg(u) = deg(e) = n - 2$, see the proof of Theorem 103. For each element $v \in S$, we have

$$deg(v) = \begin{cases} n-3, & \text{if } v \neq v^{-1}u \in S \text{ such that } v * v^{-1}u = u; \\ n-2, & \text{if } v = v^{-1}u. \end{cases}$$

Therefore,

$$deg(e) = deg(u) = n - 2 > n - 3 \geq \tfrac{n}{2},$$

for all $n \geq 6$. Thus by Dirac's Theorem the graph $G_S(\Gamma)$ is Hamiltonian. Observe that when $n = 4$, $G_S(\Gamma)$ is a cycle of length 4 and when $n = 5$ such group Γ does not exist since the order of u which is 2 must divide 5 by the Larange's Theorem.

□

Theorem 116. *Let Γ be a finite abelian group of order $n > 3$. If $n = 2|S|$, then $G_S(\Gamma)$ is Hamiltonian.*

Proof. Let $\Gamma = \{e\} \cup T \cup S$, where T is the set of all non-trivial self-invertible elements of Γ and S is the set of non-self invertible elements in Γ. We have $|T| = \left(\frac{n}{2} - 1\right)$ and $|S| = \frac{n}{2}$. By Remark 95-(3) $deg(e) = \frac{n}{2}$. For $v_i, v_k \in T$, we have $(v_i * v_k)^2 = v_i^2 * v_k^2 = e * e = e$. Thus, there is no edge between any pair of non-trivial self-invertible elements in $G_S(\Gamma)$.

However, for $v_i \in T$ and $v_j \in S$, we must have $v_i * v_j \in S$. Otherwise, there exists an element $v_k \in T$ such that $v_i * v_j = v_k$ which implies that $v_j = v_i * v_k \in T$, a contradiction. Therefore, $deg(v) = |S| = \frac{n}{2}$ for each element $v \in T$. For each $v_j \in S$, the elements of T contribute $\left(\frac{n}{2} - 1\right)$ to its degree and the identity contributes 1. So we have $deg(v_j) \geq \frac{n}{2}$. Thus by Dirac's Theorem $G_S(\Gamma)$ is Hamiltonian. $\square$

4.4 Application of inverse graphs to the isomorphism problem of groups

Recall that two groups or graphs are isomorphic if there is an isomorphism between them. The concept of isomorphism is of prime importance in both group and graph theories. Among the three crucial problems for groups raised by Max Dehn in 1911 the isomorphism problem is the most difficult (the word and conjugacy problems constitute the other two: see [26, 59]). The isomorphism problem has to do with determining whether two groups that appear different are actually isomorphic. In fact, Dahmani and Guirardel [26] asserted that, the isomorphism problem is unsolvable for some classes of groups.

Informally speaking, isomorphic groups are really the same groups except for the notations used. Besides, if Γ_1 and Γ_2 are two isomorphic groups then they share the same group properties such as being abelian, cyclic, having equal elements of finite order, the same number of involutions, etc. It is clear from the definition of inverse graphs that if $\Gamma_1 \cong \Gamma_2$ then $G_{S_1}(\Gamma_1) \cong G_{S_2}(\Gamma_2)$ where S_1 and S_2 are the sets of non-self invertible elements in Γ_1 and Γ_2 respectively.

Corollary 117. *Let Γ_1 and Γ_2 be finite groups with $G_{S_1}(\Gamma_1) \cong G_{S_2}(\Gamma_2)$. Suppose Γ_1 is a group whose non-trivial elements are involutions. Then $\Gamma_1 \cong \Gamma_2$.*

Proof. Since $G_{S_1}(\Gamma_1) \cong G_{S_2}(\Gamma_2)$ and Γ_1 is a group whose non-trivial elements are involutions, it follows from Proposition 99 that $G_{S_2}(\Gamma_2)$ is an empty graph. Hence, the set S_2 of the non-self invertible elements in Γ_2 is empty. Then Γ_2 is a group whose non-trivial elements are also involutions. Since $|\Gamma_1| = |\Gamma_2|$ by the isomorphism between their corresponding inverse graphs it follows from Proposition 38 that $\Gamma_1 \cong \Gamma_2$. $\square$

Chapter 5

Comparison between inverse graph and some other graphs associated with finite groups

In this chapter, we illustrate by examples how our new inverse graphs are different from some known graphs associated with groups.

5.1 Cayley graph

Recall that for a finite group Γ and a non-empty subset, $C \subseteq \Gamma$ that generates Γ, the Cayley graph denoted by $\mathrm{Cay}(\Gamma, C)$ is defined as follows: each vertex is an element of Γ, and two vertices $u, v \in V(Cay(\Gamma, C))$ are adjacent if either $uv^{-1} \in C$ or $vu^{-1} \in C$. Consider the Klein four-group $\mathscr{K}_4 = \{e, a, b, ab : e = a^2 = b^2 = (ab)^2\}$. The only possible generating sets of $\mathscr{K}_4$ are $C_1 = \{a, b\}, C_2 = \{a, ab\}, C_3 = \{b, ab\}, C_4 = \{a, b, ab\}$ and $\mathscr{K}_4$ itself. Figure 5.1 reveals that each of the Cayley graphs $\mathrm{Cay}(\mathscr{K}_4, C_1)$, $\mathrm{Cay}(\mathscr{K}_4, C_2)$ and $\mathrm{Cay}(\mathscr{K}_4, C_3)$ is a cycle of length four while $\mathrm{Cay}(\mathscr{K}_4, C_4)$ and $\mathrm{Cay}(\mathscr{K}_4, \mathscr{K}_4)$ are complete graphs of four vertices. On

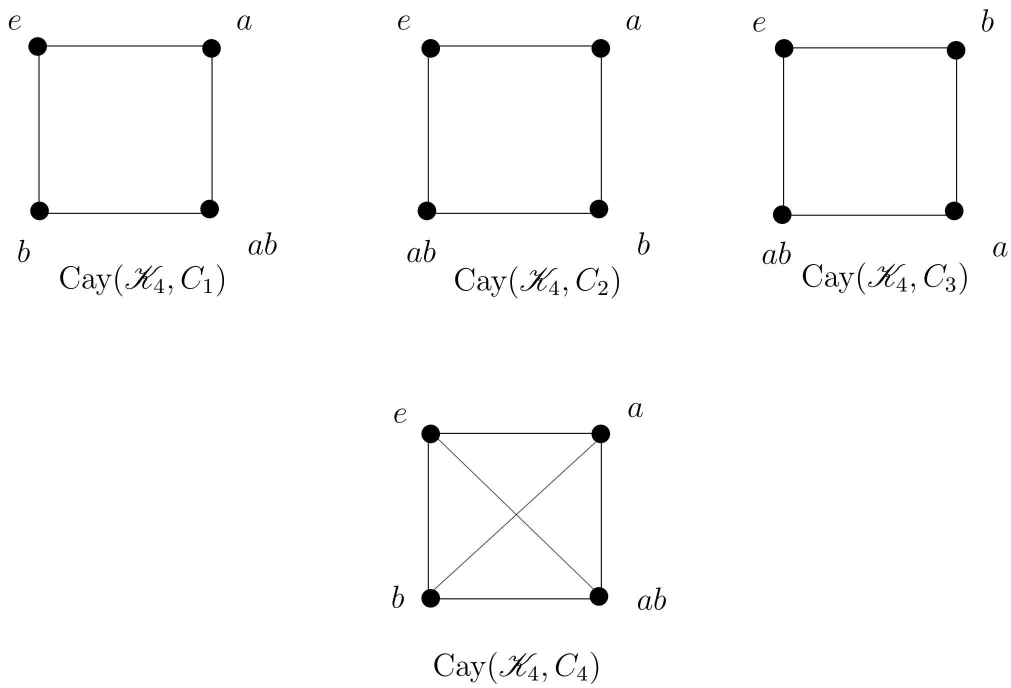

Figure 5.1: Cayley graphs of Klein four group

the other hand, since $S = \emptyset$, $G_S(\mathscr{K}_4)$ is empty by Proposition 99.

5.2 Commuting graphs

Recall that the commuting graph $G(\Gamma; P)$, where Γ is a group and P a subset of Γ, has P as its vertex set with two distinct elements $u, v \in P$ joined by an edge whenever $uv = vu$ in Γ. Notice that for any abelian group Γ and $P = \Gamma$, the associated commuting graph $G(\Gamma; P)$ is complete. However, by Theorem 101 there is no inverse graph that is complete.

5.3 Intersection graphs

Recall that the intersection graph $G(\Gamma)$, has the non-trivial subgroups of Γ as vertices and two vertices γ_1, γ_2 are adjacent if and only if $|\gamma_1 \cap \gamma_2| > 1$, where e is the identity element of Γ. Consider the group of integers modulo p, $(\mathbb{Z}_p, +)$, where p is prime. By Lagrange's

Theorem, the only non-trivial subgroup of $\mathbb{Z}_p$ is itself. Hence the intersection graph $G(\mathbb{Z}_p)$ is a trivial graph. On the other hand, the inverse graph of $(\mathbb{Z}_p, +)$ is non-trivial since $S \neq \emptyset$, see Proposition 99.

5.4 Prime graphs

Recall that the prime graph $G(\Gamma)$, has the primes dividing the order of Γ as its vertices and two vertices r_1 and r_2 are adjacent if and only if there exists an element $u \in \Gamma$ of order $r_1 r_2$. Again consider the group of integers modulo p, $(\mathbb{Z}_p, +)$, where p is prime. The prime graph $G(\mathbb{Z}_p)$ is a trivial graph. On the other hand, the inverse graph of $(\mathbb{Z}_p, +)$ is non-trivial since $S \neq \emptyset$, see Proposition 99.

5.5 Non-commuting graphs

Recall that the non-commuting graph $G(\Gamma)$, has the elements of Γ as its vertices and two vertices u, v are adjacent if and only if $uv \neq vu$. If Γ is non-abelian and $Z(\Gamma)$ is the center, then the non-commuting graph $G(\Gamma)$ is also defined on the set $\Gamma \setminus Z(\Gamma)$ as well. Consider the symmetry group $S_3 = \langle a, b | a^2 = b^2 = (ab)^2 \rangle$. The non-commuting graph $G(S_3)$ has the identity as an isolated vertex which is not possible in any inverse graph of a group with more than two vertices, see Theorem 102. Moreover, $G(S_3 \setminus \{e\})$ has five vertices while the inverse graph of S_3 contains six vertices.

5.6 Conjugacy class graphs

Recall that the conjugacy class graph $G(\Gamma)$, has non-central conjugacy classes of Γ as its vertices and two vertices U and V are adjacent if and only if $|U|$ and $|V|$ are not coprime. Again consider the symmetry group S_3. Since S_3 has two non-central conjugacy classes, then its conjugacy class graph contains only two vertices while its inverse graph contains six vertices.

5.7 Power graphs

Recall that the directed power graph $G(\Gamma)$, has the elements of Γ as its vertices and there is a directed edge from u to v if and only if $v = u^t, u \neq v$ and $t \in \mathbb{N}$. The power graph associated with Γ is defined to be the underlying graph of $G(\Gamma)$. Consider the cyclic group $\Gamma = \{e, w, w^2\}$. Then the power graph $G(\Gamma)$ is the complete graph K_3 while its inverse graph is a path P_3 of length two.

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