ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME FRACTIONAL DIFFERENTIAL PROBLEMS

BY

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TO MY BELOVED PARENTS

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All glory and praise be to the Almighty ALLAH, most gracious, most merciful, who has favored the human beings with wisdom to explore the universe. May peace and blessing be upon Mohammad the last of the Messengers.

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ABSTRACT

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Thesis Title : ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME FRACTIONAL DIFFERENTIAL PROBLEMS

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We consider the initial value problem for a class of nonlinear differential equations that involve Caputo fractional derivative. We obtain some results concerning the asymptotic behavior of solutions of these problems in suitably selected underlying spaces.

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درسنا مسألة القيمة الابتدائية لنوع من المعادلات التفاضلية غير الخطية ذات رتب غير صحيحة تحتوي على مشتقة كابوتو. حصلنا على نتائج تتعلق بالسلوك التقاربي لحلول هذه المعادلات في فضاء تم اختياره بطريقة مناسبة.

INTRODUCTION

The fractional calculus deals with the generalization of differentiation and integration to arbitrary order. Many phenomena in various fields of science and engineering can be described by differential equations of non-integer order. Namely, they arise naturally in viscoelasticity, porous media, electrochemistry, control, electromagnetics, etc. [11, 12, 15, 18, 21, 26, 28, 33]. In fact it has been shown by experiments that derivatives of non-integer order can describe many phenomena better than derivatives of integer order especially hereditary phenomena and processes.

In this thesis, we consider the following fractional differential problems:

$$\begin{cases} ({}^{C}D^{\alpha}_{t_0^+}u)(t) = f\big(t, u(t)\big), & t \geq t_0 \geq 1, \\ u^{(k)}(t_0) = c_k, \end{cases} \tag{0.1}$$

$$\begin{cases} ({}^{C}D^{\alpha}_{t_0^+}u)(t) = f\Big(t, u\big(w(t)\big)\Big), & t \geq t_0 \geq 1, \\ u^{(k)}(t_0) = c_k, \end{cases} \tag{0.2}$$

where $w: [t_0, \infty) \to [t_0, \infty)$ is continuous with $w(t) \leq t$ and $\lim_{t\to\infty} w(t) = \infty$, the function $w(t)$ is called a retarded argument, and

$$\begin{cases} {}^{C}D^{\alpha}_{t_0^+}\big(u(t) + pu(t-\tau)\big) = f\big(t, u(t)\big), & t \geq t_0 \geq 1, \\ u^{(k)}(t_0) + pu^{(k)}(t_0 - \tau) = c_k, \end{cases} \tag{0.3}$$

where $0 \leq p < 1, \tau > 0$.

In problems (0.1), (0.2), and (0.3), ${}^{C}D^{\alpha}_{t_0^+}$ is the fractional derivative in the sense of Caputo of order $\alpha \in (n-1, n), n \geq 2$, $f: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $c_k \in \mathbb{R}$ for $k = 0, 1, \cdots, n-1$, and $c_{n-1} \neq 0$.

Our objective is to investigate the asymptotic behavior of the solutions. We provide reasonable sufficient conditions under which all continuable solutions of these problems behave like polynomials of degree $n-1$ for large values of time.

Fractional derivatives by definition involve all the history of the state through a convolution with a singular kernel. In addition to this singularity, the convolution term is non-local in time. This fact complicates considerably the use of the existing methods in the literature. We intend to overcome these difficulties by some suitable estimations. To achieve this, we first establish the equivalence between (0.1), (0.2), and (0.3) and their corresponding nonlinear Volterra Integral equations in the space of continuous functions and then use suitable conditions on the nonlinearity f.

We mention here that in [2] Baleanu and Mustafa showed that every continuable solution of the initial value problem

$$\begin{cases} ({}^C D_{0^+}^{\alpha} u)(t) = f\big(t, u(t)\big), \qquad 0 < \alpha < 1, \quad t > 0, \\ u(0) = c_0, \end{cases}$$

has the asymptotic behavior $u(t) = o(t^{a\alpha})$ when $t \to \infty$ for $0 < 1 - a < \alpha < 1$.

Also in [19], Medved' showed that every solution of the initial value problem

$$\begin{cases} ({}^C D_{a^+}^{\alpha} u)(t) = f\big(t, u(t)\big), \qquad 1 < \alpha < 2, \quad 1 < a < t, \\ u(a) = c_0, u'(a) = c_1, \end{cases}$$

is asymptotic to $ct + d$ as $t \to \infty$, where c, d are real constants, $c \neq 0$.

In the theory of higher order nonlinear differential equations (say of order n), an interesting topic is the study of the asymptotic behavior of solutions via solutions of the equation $u^{(n)} = 0$. This topic has been extensively investigated during the last four decades for the case of second order nonlinear differential equations; see Cohen [5], Constantine [6, 7], Dzurina [10], Lipovan [16], Meng [20], Y. Rogovchenko [31], S. Rogovchenko and Y. Rogovchenko [29, 30], Y. Rogovchenko and Villari [32], Tong [34], Trench [35] and Yin [37]. Note that papers [5-7, 17, 35] are concerned with differential equations of the form

$$u'' + f(t,u) = 0. \tag{0.4}$$

Equation (0.4) was discussed for the nonlinear case by Cohen [5] and Tong [34], and the linear case was studied by Trench [35]. All the results cited above have been obtained by using the Gronwall-Bellman inequality [3] or its generalizations due to Bihari [4] and Dannan [8]. On the other hand, [24, 29-31] deal with differential equations of the form

$$u'' + f(t,u,u') = 0. \tag{0.5}$$

Conditions presented there to ensure that solutions of (0.4) and (0.5) are asymptotic to linear functions for large values of time.

The autonomous differential equation

$$u'' + f(u,u') = 0, \tag{0.6}$$

has been treated by Y. Rogovchenko and Villari in [32]. The authors transferred equation (0.6) to the following equivalent system

$$\begin{cases} u_1' = u_2, \\ u_2' = f(u_1,u_2), \end{cases} \tag{0.7}$$

and gave conditions to ensure that for every solution (u_1,u_2) of (0.7) there exists $a \in \mathbb{R}$ such that $\lim_{t\to\infty} u_2(t) = a < \infty$ and $\lim_{t\to\infty} u_1(t) = \pm\infty$.

Dzurina [10] studied the neutral differential equation

$$\big(u(t)+pu(t-\tau)\big)''+f\big(t,u(t)\big)=0, \tag{0.8}$$

$0 \le p < 1$, $\tau > 0$, and showed that every nonoscillatory solution u of (0.8) is asymptotic to $at+b$ as $t \to \infty$, a, b are constants and $a \neq 0$.

The above mentioned topic has also been treated for higher order nonlinear differential equations by several researchers; see Akinyele and Dahiya [1] who studied the nth order differential equations with advanced argument

$$u^{(n)}+f\Big(t,u\big(\sigma(t)\big)\Big)=h(t), \qquad n \ge 2, \tag{0.9}$$

$\sigma(t) \ge t \ge 1$, and their main result is concerned with solutions of (0.9) which are asymptotic to the solutions of $u^{(n)}(t)=0$ as $t \to \infty$, and solutions of (0.9) which are asymptotic to $\gamma t^{n-1}, \gamma \neq 0$.

Dahiya and Zafer [9] investigated the nth order differential equations with a retarded argument

$$u^{(n)}+f\Big(t,u(t),u\big(g(t)\big)\Big)=h(t), \qquad n \ge 2, \tag{0.10}$$

$g(t) \le t$, $\lim_{t\to\infty} g(t)=\infty$ and showed that under certain conditions (0.10) has a solution u with the asymptotic property

$$\lim_{t\to\infty}\frac{u(t)}{t^{n-1}}=a, \qquad a \neq 0.$$

We also mention here that Philos, Purnaras, and Tsamatos [27], Trench [36] considered the nth differential equations not involving the lower derivatives of the form

$$u^{(n)}+f(t,u)=0, \qquad n \ge 2. \tag{0.11}$$

In [27], sufficient conditions are established in order that, for any real polynomial of degree at most $m, 1 \leq m \leq n-1$, there exists a solution of (0.11) which is asymptotic at infinity to this polynomial. Also sufficient conditions are given for every solution of (0.11) to be asymptotic at infinity to a real polynomial of degree $n-1$.

In [14], the lower derivatives appear explicitly

$$u^{(n)} + f\left(t, u, u', \cdots, u^{(n-1)}\right) = 0, \qquad n \geq 2. \tag{0.12}$$

Kong showed that under some assumptions every solution u of equation (0.12) satisfies

$\frac{u^{(n-i)}(t)}{t^{i-1}} \to a_i \in \mathbb{R},\ i = 1,2,\cdots,n$ as $t \to \infty$.

Finally, the neutral nth order differential equation

$$\frac{d^n}{dt^n}[u(t) + u(t-\tau)] + \sigma F\left(t, u\big(g(t)\big)\right) = 0, n \geq 2, \tau > 0, \sigma = \pm 1, \tag{0.13}$$

with $\lim_{t\to\infty} g(t) = \infty$, was investigated in [25]. The author presented necessary and sufficient conditions for (0.13) to have the asymptotic property that

$$\lim_{t\to\infty} \frac{u(t)}{t^k}, \qquad 0 \leq k \leq n-1,$$

exists and is a positive value.

This thesis is organized as follows: in chapter 1, we give a historic review. In chapter 2, we introduce some definitions, lemmas, properties and notation needed later in the thesis. Chapter 3 is about the asymptotic behavior of solutions of (0.1). In chapter 4, we study the asymptotic behavior of solutions of (0.2). Chapter 5 is devoted to the asymptotic behavior of solutions of (0.3). We conclude our thesis with some recommendations for future work.

CHAPTER 1

LITERATURE REVIEW

In this chapter, we review some of the works done concerning the asymptotic behavior of solutions of differential equations.

1.1 Second order differential equations

In this section, we present the main theorems concerning the asymptotic behavior of solutions for different forms of second order differential equations.

Definition 1.1

We write $f(t) = o(\varphi(t))$ *if* $\lim_{t\to\infty} \frac{f(t)}{\varphi(t)} = 0$, *and* $f(t) = O(\varphi(t))$ *if* $\overline{\lim_{t\to\infty}} \left|\frac{f(t)}{\varphi(t)}\right| < \infty$, *where* $\overline{\lim_{t\to\infty}} \left|\frac{f(t)}{\varphi(t)}\right| = \limsup_{t\to\infty} \left|\frac{f(t)}{\varphi(t)}\right|$.

Definition 1.2

A solution u of a differential equation is called continuable if u exists for all $t \geq t_0 \geq 1$.

Definition 1.3

We say that a solution u of a differential equation possesses the property (L) if

$u(t) = at + b + o(t)$ *as* $t \to \infty$, *where* a, b *are real constants,* $a \neq 0$.

1.1.1 Differential equations not involving the first derivative

Here we consider the following equation

$$u'' = f(t, u). \tag{1.1}$$

Theorem 1.4

Suppose that f satisfies the following conditions:

(i) f *is continuous in* $D = \{(t, u): t \geq 1, u \in \mathbb{R}\}$;

(ii) *the derivative* f_u *exists in* D *and satisfies* $f_u(t, u) > 0$ *in* D;

(iii) $|f(t, u)| \leq f_u(t, 0)|u|$ *in* D;

(iv) $\int_1^\infty t f_u\,(t, 0) dt < \infty$.

Then any continuable solution of equation (1.1) *possesses the property* (L).

In the proof of Theorem 1.4, Cohen [5] used Bellman's method [3] and Gronwall's inequality. In the next theorem, Tong [34] used the same method as Cohen and Bihari's inequality [4] to generalize Theorem 1.4.

Theorem 1.5

Let f *be continuous in* $D = \{(t, u): t \geq 1, u \in \mathbb{R}\}$. *Assume that there is a nonnegative continuous function* h *defined for* $t \geq 1$, *and a continuous function* g *defined for* $u \geq 0$ *such that*

(i) $|f(t, u)| \leq h(t) g\left(\frac{|u|}{t}\right)$ *in* D;

(ii) $\int_1^\infty h(t) dt < \infty$;

(iii) g *is positive and nondecreasing for* $u > 0$;

(iv) $\int_1^\infty \frac{du}{g(u)} = \infty$.

Then any continuable solution of equation (1.1) *possesses the property* (L).

Remark 1.6

Theorem 1.5 without assumption (iv) becomes false as it has been pointed out by Kong [14] and Meng [20]. They exhibited the following counterexample:

suppose we have the following equation

$$u'' - \frac{2}{t^4}u^2 = 0, \qquad t \geq 1.$$

Let $h(t) = 2t^{-2}$, $g(u) = u^2$, the conditions (i), (ii), and (iii) are satisfied but the previous equation has a solution $u(t) = t^2$ which is not asymptotic to a nontrivial linear function as $t \to \infty$. The assumption (iv) is crucial and it has been added by Constantine [6].

In the next theorem [27], conditions are given which are sufficient for every solution to be asymptotic at infinity to a line.

Theorem 1.7

Assume that

$$|f(t,u)| \leq h_1(t)g\left(\frac{|u|}{t}\right) + h_2(t),$$

is satisfied for all $(t,u) \in [t_0,\infty) \times \mathbb{R}, t \geq t_0 > 0$, *where* h_1*and* h_2 *are nonnegative continuous real-valued functions on* $[t_0,\infty)$ *such that*

$$\int_{t_0}^{\infty} th_1(t)dt < \infty, \quad \int_{t_0}^{\infty} th_2(t)dt < \infty,$$

g is a continuous real valued function on $[0, \infty)$, which is positive, increasing on $(0, \infty)$ and such that $\int_1^{\infty} \frac{du}{g(u)} = \infty$. Then every solution u on the interval $[T, \infty), T \geq t_0$ of the differential equation (1.1) *is asymptotic to a line $c_0 + c_1 t$ for $t \to \infty$, i.e.*

$$u(t) = c_0 + c_1 t + o(1) \text{ for } t \to \infty,$$

and, in addition, we have

$$u'(t) = c_1 + o(1) \text{ for } t \to \infty,$$

where c_0, c_1 are real numbers (depending on the solution u). More precisely every solution u on the interval $[T, \infty), T \geq t_0$, of (1.1) *satisfies*

$$u(t) = C_0 + C_1(t - T) + o(1) \text{ for } t \to \infty,$$

and, in addition,

$$u'(t) = C_1 + o(1) \text{ for } t \to \infty,$$

where $C_0 = u(T) - \int_T^{\infty} (s - T)\, f\big(s, u(s)\big) ds$, and $C_1 = u'(T) - \int_T^{\infty} f\big(s, u(s)\big) ds$.

1.1.2 Second order differential equations involving the first derivative

The asymptotic behavior of solutions of the equation

$$u'' + f(t, u, u') = 0, \qquad t \geq 1, \tag{1.2}$$

was studied by Kong [14] in case f belongs to the following class $\mathcal{F}$.

Definition 1.8

A function g is said to belong to $\mathcal{F}$ if g is positive, nondecreasing, continuous on $\mathbb{R}$, and satisfies

$$\frac{g(u)}{v} \leq g\left(\frac{u}{v}\right), \qquad u \geq 0, v \geq 1. \tag{1.3}$$

It is easy to see that $g \in \mathcal{F}$ implies $\int_1^\infty \frac{du}{g(u)} = \infty$. In fact, from (1.3), letting $u = v \geq 1$, we get $\frac{g(u)}{u} \leq g(1)$, i.e. $g(u) \leq g(1)u$, which implies

$$\int_1^\infty \frac{du}{g(u)} \geq \int_1^\infty \frac{du}{g(1)u} = \infty.$$

In the following we give the main result of [14]. The proof is based on an extension of the basic Bihari's inequality [4].

Theorem 1.9

Assume that

(i) *f is continuous in $D = \{(t,u,v): t \geq 1, u, v \in \mathbb{R}\}$;*

(ii) *there exist nonnegative continuous functions h_1, h_2 defined for $t \geq 1$ such that*

$$|f(t,u,v)| \leq h_1(t) g_1\left(\frac{|u|}{t}\right) + h_2(t) g_2(|v|),$$

where $g_1, g_2 \in \mathcal{F}$, and $\int_1^\infty h_i(t)dt < \infty$, $i = 1,2$. Then every solution u of equation (1.2) satisfies $\frac{u^{(2-i)}(t)}{t^{i-1}} \to a_i \in \mathbb{R}$, $i = 1,2$, as $t \to \infty$, where $u^{(0)} = u$. Furthermore, if f does not change its sign when $u_i > 0$, $i = 1,2$, and $t \geq 1$, then equation (1.2) has solutions such that $a_i > 0$, $i = 1,2$.

Next we present two theorems concerning different forms of the nonlinearity $f(t,u,u')$. Making use of Bihari's inequality [4] and its derivatives due to Dannan [8], S. Rogovchenko and Y. Rogovchenko [29] obtained the following results.

Theorem 1.10

Suppose that

(i) f *is continuous in* $D = \{(t,u,v): t \geq 1, u, v \in \mathbb{R}\}$;

(ii) *there exist nonnegative continuous functions* h_1, h_2, h_3 *defined for* $t \geq 1$, *and continuous functions* g_1, g_2 *defined for* $u, v \geq 0$ *respectively such that*

$$|f(t,u,v)| \leq h_1(t) g_1\left(\frac{|u|}{t}\right) + h_2(t) g_2(|v|) + h_3(t), \qquad t \geq 1,$$

where for $u, v > 0$ *the functions* g_1, g_2 *are positive, nondecreasing,* $\int_1^\infty h_i(t)dt < \infty$, $i = 1,2,3$ *and* $\int_1^\infty \frac{ds}{g_1(s)+g_2(s)} = \infty$. *Then any continuable solution of equation* (1.2) *possesses the property* (L).

Example 1.11

Consider the nonlinear differential equation

$$u'' + t^{-3}(\cos t)\left(\frac{u^2}{u^2+t^2}\right) + t^{-3}(\sin^2 t)\left(\frac{(u')^2}{(u')^2+1}\right) + t^{-2} = 0, t \geq 1. \qquad (1.4)$$

Here we have

$$g_1(u) = \frac{u^2}{u^2+1},\ g_2(v) = \frac{v^2}{v^2+1},\ h_1(t) = h_2(t) = t^{-3},\ h_3(t) = t^{-2}.$$

By Theorem 1.10, all continuable solutions of equation (1.4) have the property (L).

The next theorem gives the desired asymptotic behavior only for continuable solutions with initial data satisfying additional conditions.

Theorem 1.12

Suppose that

(i) f *is continuous in* $D = \{(t, u, v): t \geq 1, u, v \in \mathbb{R}\}$;

(ii) *there exist nonnegative continuous functions* h_1, h_2 *defined for* $t \geq 1$, *and continuous functions* g_1, g_2 *defined for* $u, v \geq 0$ *respectively such that*

$$|f(t, u, v)| \leq h_1(t) g_1\left(\frac{|u|}{t}\right) + h_2(t) g_2(|v|);$$

where for $u, v > 0$ *the functions* g_1, g_2 *are positive and nondecreasing*;

(iii) $g_1(\alpha u) \leq \psi_1(\alpha) g_1(u),\ g_2(\alpha v) \leq \psi_2(\alpha) g_2(v),$ *for* $\alpha \geq 1, u, v \geq 0$, *where the functions* ψ_1, ψ_2 *are continuous for* $\alpha \geq 1$;

(iv) $\int_1^\infty h_i(s) ds = H_i < \infty,\ \ i = 1,2;$

(v) *assume that there exists a constant* $K \geq 1$ *such that*

$$K^{-1}\big(\psi_1(K) + \psi_2(K)\big)(H_1 + H_2) \leq \int_1^\infty \frac{ds}{g_1(s) + g_2(s)}. \tag{1.5}$$

Then any continuable solution u *of equation* (1.2), *with initial data*

$u(1) = c_1, u'(1) = c_2$ *such that* $|c_1| + |c_2| \leq K$, *possesses the property* (L).

Example 1.13

Consider the nonlinear differential equation

$$u'' + (2t)^{-4}u^2 \cos u + (4t)^{-2}(u')^2 \sin^3 u = 0, \qquad t \geq 1. \tag{1.6}$$

For equation (1.6), we have

$$g_1(u) = u^2,\ g_2(v) = v^2,\ h_1(t) = h_2(t) = (4t)^{-2},\ \psi_1(\alpha) = \psi_2(\alpha) = \alpha^2.$$

To find the value of K we solve the inequality (1.5). The left hand side of (1.5) is equal to

$$K^{-1}(K^2 + K^2)\left(\int_1^\infty (4s)^{-2}ds + \int_1^\infty (4s)^{-2}ds\right) = \frac{K}{4},$$

and the right hand side of (1.5) is equal to

$$\int_1^\infty \frac{ds}{s^2 + s^2} = \frac{1}{2}.$$

Thus we need $\frac{K}{4} \leq \frac{1}{2}$ and this implies that $K \leq 2$. We conclude by Theorem 1.12 that all continuable solutions of equation (1.6) with initial data satisfying $|c_1| + |c_2| \leq 2$ have the property (L).

Remark 1.14

Theorem 1.4 is a special case of Theorem 1.10 because if we take

$$h_1(t) = tf_u(t,0),\ h_2(t) \equiv 0,\ h_3(t) \equiv 0,\ g_1(u) = u,\ g_2(u) \equiv 0,$$

then all the hypothesis of Theorem 1.10 are satisfied. The conclusion of Theorem 1.4 follows from that of Theorem 1.10.

Remark 1.15

Theorem 1.5 is a special case of Theorem 1.10 because if we take

$$h_1(t) = h(t),\ h_2(t) \equiv 0,\ h_3(t) \equiv 0,\ g_1(u) = g(u),\ g_2(u) \equiv 0,$$

then the conditions in Theorem 1.10 are fulfilled. The conclusion of Theorem 1.5 follows from that of Theorem 1.10.

1.1.3 Autonomous differential equations

We consider the autonomous differential equation

$$u'' = f(u,u'). \tag{1.7}$$

The investigation of this type of equations is done by Y. Rogovchenko and Villari [32] in the phase plane. Equation (1.7) is equivalent in the phase plane to the following system

$$\begin{cases} u_1' = u_2, \\ u_2' = f(u_1,u_2), \end{cases} \tag{1.8}$$

and the main result is stated as:

Theorem 1.16

Suppose that for the system (1.8) *the following conditions hold:*

(i) f *is continuous in* $\mathbb{R}^2$;

(ii) *for any* $u_1 \in \mathbb{R}$ *the function* $g(u_1) = f(u_1, 0) \neq 0$;

(iii) *for any fixed* u_1^* *the function* $h(u_2) = f(u_1^*, u_2)$ *is strictly decreasing*;

(iv) *the function* $\varphi(t)$ *which satisfies the equation* $f\big(u_1, \varphi(u_1)\big) = 0$ *is defined for* $u_1 > u_0$ $(u_1 < u_0)$ *and* $\lim_{u_1 \to \infty} \varphi(u_1) = a_+$ $\big(\lim_{u_1 \to -\infty} \varphi(u_1) = a_-\big)$ exists.

Then for every solution (u_1, u_2) there exists $a \in \mathbb{R}$ such that $\lim_{t \to \infty} u_2(t) = a < \infty$ and $\lim_{t \to \infty} u_1(t) = \pm\infty$.

Example 1.17

Consider the system (1.8) with the function

$$f(u_1, u_2) = (u_1^2 + 1)(1 - u_2).$$

Here we have $\varphi(u_1) \equiv 1$, the graph of $\varphi(u_1)$ is actually a trajectory of system (1.8), and the system has a family of solutions $u_1(t) = t + C, u_2(t) = 1$, while all other trajectories in the phase plane tend to the line $u_2(t) = \varphi(u_1) = 1$.

1.1.4 Perturbed differential equations

In this section, we consider the equation

$$u'' + f(t, u) = p(t), \qquad t \geq t_0 \geq 1, \tag{1.9}$$

where the functions $f: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $p: [t_0, \infty) \to \mathbb{R}$ are continuous.

Mustafa [23] established the existence of a global solution u of equation (1.9) that admits the representation $u(t) = P(t) + o(1)$ as $t \to \infty$, where $P''(t) = p(t)$ for $t \geq t_0$.

Theorem 1.18

Assume that the nonlinearity f *in* (1.9) *satisfies the inequality*

$$|f(t,u)| \leq F(t,|u|), \qquad t \geq t_0, \qquad u \in \mathbb{R},$$

where $F: [t_0, \infty) \times \mathbb{R} \to [0, \infty)$ *is a continuous function that is nondecreasing in the second argument. Suppose further that there exists a number* $\varepsilon > 0$ *such that*

$$\int_{t_0}^{\infty} sF(s, |P(s)| + \varepsilon) ds \leq \varepsilon, \tag{1.10}$$

where $P''(t) = p(t)$ *for* $t \geq t_0$. *Then* (1.9) *has a solution* u *defined in* $[t_0, \infty)$ *with the asymptotic representation* $u(t) = P(t) + o(1)$ *as* $t \to \infty$.

Example 1.19

Fix $c > 0$, $\varepsilon \in (0,3]$. Let $p: [t_0, \infty) \to \mathbb{R}$ be a nonnegative continuous function. Introduce P and t_0 by the formulae $P(t) = c + \int_{t_0}^{t} (t-s)p(s)ds$, and $t_0 = \frac{3}{\varepsilon}\left(1 + \frac{\varepsilon}{c}\right)^2 \geq 1$.

The nonlinearity $f(t,u)$ of the Emden-Fowler equation

$$u'' - \frac{2}{t[tP(t)+1]^2} u^2 = p(t), \qquad t \geq t_0, \tag{1.11}$$

satisfies the hypotheses of Theorem 1.18. In fact, condition (1.10) reads as

$$\int_{t_0}^{\infty} \frac{2}{s^2}\left(\frac{P(s)+\varepsilon}{P(s)+s^{-1}}\right)^2 ds \leq \int_{t_0}^{\infty} \frac{2}{s^2}\left(\frac{P(s)+\varepsilon}{P(s)}\right)^2 ds \leq \int_{t_0}^{\infty} \frac{2}{s^2}\left(1 + \frac{\varepsilon}{P(s)}\right)^2 ds$$

$$\leq \int_{t_0}^{\infty} \frac{2}{s^2}\left(1 + \frac{\varepsilon}{c}\right)^2 ds = \frac{2}{t_0}\left(1 + \frac{\varepsilon}{c}\right)^2 < \varepsilon.$$

It is easy to check that equation (1.11) has the exact solution $u(t) = P(t) + t^{-1}$ for $t \geq t_0$.

1.1.5 Quasilinear differential equations

In this section, we consider a class of second order quasilinear ordinary differential equations of the form:

$$(|u'|^{\alpha-1}u')' = p(t)|u|^{\beta-1}u, \tag{1.12}$$

where we assume that $\alpha, \beta > 0$ are constants, and $p: [1, \infty) \to \mathbb{R}$, is continuous and positive on $[1, \infty)$. We call equation (1.12) super-homogeneous if $\alpha < \beta$ and sub-homogeneous if $\alpha > \beta$.

The next theorem due to M. Mizukami, M. Naito and H. Usami [22] provides a necessary and sufficient condition for continuable solutions of (1.12) to have the property (L).

Theorem 1.20

Any continuable solution of (1.12) *has the property* (L) *if and only if* $\int_1^\infty t^\beta p(t)dt < \infty$.

Example 1.21

Let $\alpha \neq \beta$. Consider equation (1.12) with $p(t) = t^\sigma$:

$$(|u'|^{\alpha-1}u')' = t^\sigma |u|^{\beta-1}u, \qquad t \geq 1, \qquad \sigma \in \mathbb{R}. \tag{1.13}$$

Then any continuable solution of (1.13) has the property (L) if and only if $\sigma < -\beta - 1$.

1.1.6 Neutral differential equations

Neutral equations are equations in which the delay appears in the highest derivative. One type of neutral differential equations is the following:

$$\big(u(t) + pu(t-\tau)\big)'' + f\big(t, u(t)\big) = 0, \tag{1.14}$$

where $f:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$, $t_0\geq 1$, is continuous, $0\leq p<1$, and $\tau>0$. Before we mention the main result in [10], we give the definition of oscillatory and nonoscillatory solution.

Definition 1.22

A nontrivial solution is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory.

Using proper integral inequalities, Dzurina [10] proposed the following theorem concerning the asymptotic behavior of solutions of (1.14).

Theorem 1.23

Suppose that $0\leq p<1, \tau>0$ *and:*

(i) f *is continuous in* $D=\{(t,u): t\in[t_0,\infty), u\in\mathbb{R}, t_0\geq 1\}$;

(ii) *there exist a nonnegative continuous function* h *defined for* $t\geq t_0\geq 1$, *and a continuous function* g *defined for* $u\geq 0$ *such that*

$$|f(t,u)|\leq h(t)g\left(\frac{|u|}{t}\right),$$

where for $u>0$, *the function* g *is positive, nondecreasing,* $\int_{t_0}^{\infty}h(t)dt<\infty$, *and* $\int_{t_0}^{\infty}\frac{du}{g(u)}=\infty$. *Then every nonoscillatory solution* u *of* (1.14) *possesses the property* (L).

Example 1.24

Consider the nonlinear neutral differential equation

$$\left(u(t)+\frac{1}{2}u(t-1)\right)''-\left(\frac{2}{t^3}+\frac{1}{(t-1)^3}\right).\left(1+\frac{t^4}{(t^2+1)^2}\right).\frac{u^2(t)}{u^2(t)+t^2}=0, \quad (1.15)$$

where $t \geq 2$. Set $h(t) = 2\left(\frac{2}{t^3}+\frac{1}{(t-1)^3}\right)$ and $g(u) = \frac{u^2}{u^2+1}$. Then applying Theorem 1.23 we deduce that any nonoscillatory solution u of (1.15) possesses the property (L). We observe that

$$u(t) = t + \frac{1}{t}$$

is a solution of (1.15) which is clearly asymptotic to t as $t \to \infty$.

1.2 Higher order differential equations

In this section, we present some results concerning the asymptotic behavior of solutions for different forms of nth order differential equations.

1.2.1 General homogeneous equations

The asymptotic behavior of solutions of the equation

$$u^{(n)} + f\left(t, u, u', \cdots, u^{(n-1)}\right) = 0, \qquad n \geq 2, \quad (1.16)$$

was studied by Kong [14].

Theorem 1.25

Assume that

(i) f *is continuous in* $[1,\infty)\times\mathbb{R}^n$,

(ii) $|f(t,u_0,u_1,\cdots,u_{n-1})| \le \sum_{i=0}^{n-1} h_i(t) g_i\left(\frac{|u_i|}{t^{n-i-1}}\right)$, *for* $t \ge 1$,

where $h_i, i=0,1,\cdots,n-1$, *are nonnegative and continuous on* $[1,\infty)$, $\int_1^\infty h_i(t)dt < \infty$, *and* $g_i \in \mathcal{F}, i=0,1,\cdots,n-1$.

Then every solution u *of equation* (1.16) *satisfies* $\frac{u^{(n-i)}(t)}{t^{i-1}} \to a_i \in \mathbb{R}$, $i=1,2,\cdots,n$ *as* $t\to\infty$, *where* $u^{(0)}=u$. *Furthermore, if* f *does not change its sign when* $u_i>0$, $i=1,2,\cdots,n-1$, *and* $t\ge 1$, *then equation* (1.16) *has solutions such that* $a_i>0$, $i=1,2,\cdots,n$.

Remark 1.26

If $n=2$, then we obtain the result of Theorem 1.9.

1.2.2 Equations not involving the lower order derivatives

The equation

$$u^{(n)}+f(t,u)=0, \qquad n\ge 2, \tag{1.17}$$

which is a special case of (1.16) was studied by Philos, Purnaras, and Tsamatos [27]. In Theorem 1.27, sufficient conditions are given for every solution to be asymptotic at infinity to a real polynomial of degree at most $n-1$.

Theorem 1.27

Assume that

$$|f(t,u)| \leq h_1(t)g\left(\frac{|u|}{t^{n-1}}\right) + h_2(t),$$

is satisfied for all $(t,u) \in [t_0,\infty) \times \mathbb{R}, t \geq t_0 > 0$, *where* h_1 *and* h_2 *are nonnegative continuous real-valued functions on* (t_0,∞) *such that*

$$\int_{t_0}^{\infty} t^{n-1}h_1(t)dt < \infty, \quad \int_{t_0}^{\infty} t^{n-1}h_2(t)dt < \infty,$$

g is a continuous real valued function on $(0,\infty)$, *which is positive, increasing on* $(0,\infty)$ *and such that* $\int_1^{\infty}\frac{du}{g(u)} = \infty$. *Then every solution u on the interval* $[T,\infty), T \geq t_0$ *of the differential equation* (1.17) *is asymptotic to a polynomial* $c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}$ *for* $t \to \infty, i.e.$

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + o \ \text{for} \ t \to \infty,$$

and, in addition, we have

$$u^{(i)}(t) = \sum_{i=j}^{n-1} i(i-1)\cdots(i-j+1)c_i t^{i-j} + o(1),$$

where $c_0, c_1, \ldots, c_{n-1}$ *are real numbers* (*depending on the solution* u). *More precisely every solution u on the interval* $[T,\infty), T \geq t_0$, *of* (1.17) *satisfies*

$$u(t) = C_0 + C_1(t-T) + \cdots + C_{n-1}(t-T)^{n-1} + o(1) \ \text{for} \ t \to \infty,$$

and, in addition,

$$u^{(i)}(t) = \sum_{i=j}^{n-1} i(i-1)\cdots(i-j+1)C_i(t-T)^{i-j} + o(1) \text{ for } t \to \infty,\ j = 1,\cdots,n-1,$$

$$\text{where } C_i = \frac{1}{i!}\left[u^{(i)}(T) + (-1)^{n-1-i}\int_T^\infty \frac{(s-T)^{n-1-i}}{(n-1-i)!} f\big(s,u(s)\big)ds\right],\ i = 1,\cdots,n-1.$$

Remark 1.28

If $n = 2$, then we obtain the result of Theorem 1.6.

1.2.3 Higher order differential equations with retarded argument

Consider the equation

$$u^{(n)} + f\left(t, u(t), u\big(g(t)\big)\right) = h(t), \tag{1.18}$$

where $f: [1,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, g and h are continuous on $[1,\infty)$, $g(t) \le t$, and $\lim_{t\to\infty} g(t) = \infty$.

The following theorem [9] is about the asymptotic behavior of solutions of (1.18).

Theorem 1.29

Assume that φ is a nonnegative continuous function on $[1,\infty)$ and $\omega > 0$ is continuous for $u \ge 0$ and nondecreasing for $u > 0$ such that $|f(t,u,v)| \le \varphi(t)\omega\left(\frac{|v|}{[g(t)]^{n-1}}\right)$, and $\int_1^\infty |h(t)|dt < \infty$. If $\int_1^\infty \varphi(t)dt < \infty$, then equation (1.18) has a solution u with the asymptotic property

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = a, \qquad a \ne 0.$$

Example 1.30

Consider the equation

$$u''' + 8t^{-6}u^3\left(t^{1/2}\right) = 8(t^{-3} + 3t^{-7} + t^{-9}), \qquad t \geq 1, \tag{1.19}$$

so that $h(t) = 8(t^{-3} + 3t^{-7} + t^{-9})$, $g(t) = t^{1/2}$, and $f\left(t, u, u\left(g(t)\right)\right) = 8t^{-6}u^3$. We may take $\omega(u) = u^3$ and $\varphi(t) = 8t^{-3}$. According to Theorem 1.29, equation (1.19) has a solution asymptotic to $\gamma t^2, \gamma \neq 0$. Indeed $u(t) = t^2 + t^{-2}$ is such a solution.

1.2.4 Higher order differential equations with advanced arguments

We consider the equation

$$u^{(n)} + f\left(t, u\left(\sigma(t)\right)\right) = h(t), \tag{1.20}$$

where $f: [1, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, σ and h are continuous on $[1, \infty)$ with $\sigma(t) \geq t \geq 1$.

Theorem 1.31 [1] states that some solutions of (1.20) are asymptotic to polynomials under certain conditions.

Theorem 1.31

Assume that the following hold:

(i) $p(t)$ *is a continuous and nonnegative function on* $[1, \infty)$ *and* $p(t) > 0$ *for* $t \geq 1$;

(ii) $\int_1^\infty \left(\sigma(s)\right)^{\alpha(n-1)} p(s)ds < \infty,\ 0 < \alpha \leq 1;$

(iii) $|f(t, u(\sigma(t))| \leq p(t)\left|u\left(\sigma(t)\right)\right|^\alpha,\ 0 < \alpha \leq 1;$

(iv) $\int^{\infty} |h(s)| ds < \infty$.

Then equation (1.20) *has*

(a) *solutions which are asymptotic to the solutions of* $u^{(n)}(t) = 0$ *as* $t \to \infty$,

(b) *solutions which are asymptotic to* $\gamma t^{n-1}, \gamma \neq 0$ *provided* $\alpha = 1$.

Example 1.32

Consider the third order equation

$$u''' + t^{-5} u^{1/2}(t + \pi) = t^{-4}. \tag{1.21}$$

Here $f\left(t, u\big(\sigma(t)\big)\right) = t^{-5} u^{1/2}(t + \pi)$, so that $p(t) = t^{-5}$, $\sigma(t) = t + \pi$, $h(t) = t^{-4}$ and $\alpha = \frac{1}{2}$. The hypotheses of Theorem 1.31 are satisfied. The conclusion (a) therefore holds. A solution of equation (1.21) is given by $u(t) = (t - \pi)^2$.

Example 1.33

Consider the fourth order equation

$$u^{(4)} + e^{-t}(t + \pi)^{-3} u(t + \pi) = e^{-t}. \tag{1.22}$$

We clearly see that

$$\left| f\left(t, u\big(\sigma(t)\big)\right) \right| = \left| \frac{e^{-t}}{(t + \pi)^3} u(t + \pi) \right| = \frac{e^{-t}}{(t + \pi)^3} |u(t + \pi)|,$$

$p(t) = \frac{e^{-t}}{(t+\pi)^3}, \sigma(t) = t + \pi, h(t) = e^{-t}$ and $\alpha = 1$.

The hypotheses of Theorem 1.31 are satisfied and therefore the conclusion (b) holds.

A solution of equation (1.22) is given by $u(t) = t^3$.

Example 1.34

Consider the nth order equation

$$u^{(n)}(t) + t^{-(n+2)}u^{1/2}(t+\pi) = e^{-t}. \tag{1.23}$$

Here we have $\left|f\left(t, u\big(\sigma(t)\big)\right)\right| < t^{-(n+2)}\left|u^{1/2}(t+\pi)\right|$, so that $p(t) = t^{-(n+2)}$,

$\sigma(t) = t + \pi, \alpha = \frac{1}{2}$ and $h(t) = e^{-t}$. The hypotheses of Theorem 1.31 are satisfied and the conclusion therefore implies that there exist solutions of (1.23) which are asymptotic to the solution of $u^{(n)}(t) = 0$ as $t \to \infty$.

1.2.5 Higher order neutral differential equations

The neutral differential equation

$$\frac{d^n}{dt^n}[u(t) + u(t-\tau)] + \sigma F\left(t, u\big(g(t)\big)\right) = 0, \tag{1.24}$$

is considered under the following conditions: $n \geq 2, \tau > 0, \sigma = \pm 1, F(t,u)$ is nonnegative on $[t_0, \infty) \times (0, \infty)$ nondecreasing in $u \in (0, \infty)$, and $\lim_{t\to\infty} g(t) = \infty$. In [25], the author gives a sufficient and necessary condition for (1.24) to have the asymptotic property $\lim_{t\to\infty} \frac{u(t)}{t^k}, 0 \leq k \leq n-1$, exists and is a positive value.

Theorem 1.35

Let k be an integer with $0 \leq k \leq n-1$. Then equation (1.24) *has a solution u such that $\lim_{t\to\infty}\frac{u(t)}{t^k}$ exists and is a positive value if and only if*

$$\int_{t_0}^{\infty} t^{n-k-1}\, F(t, c[g(t)]^k)dt < \infty$$

for some $c > 0$.

1.2.6 Differential equations involving disconjugate differential operators

We consider the equation

$$L_n u + f(t,u) = r(t), \tag{1.25}$$

where $n \geq 2$ and L_n denotes the disconjugate differential operators

$$L_n = \frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\cdot}{p_0(t)} .$$

We assume that $p_i, r: [1,\infty) \to \mathbb{R}$ and $f: [1,\infty)\times\mathbb{R} \to \mathbb{R}$ are continuous with

$p_i(t) > 0, 0 \leq i \leq n$. Put $L_0 u(t) = \frac{u(t)}{p_0(t)}, L_i u(t) = \frac{1}{p_i(t)}\frac{d}{dt}L_{i-1}u(t), 1 \leq i \leq n$, and let $i_k \in \{1,2,\dots,n-1\}, t, s \in [1,\infty)$ and $I_0 = 1$,

$$I_k = \left(t, s: p_{i_k}, \dots, p_{i_1}\right) = \int_s^t p_{i_1}(r) I_{k-1}(t, r; p_{i_k}, \dots, p_{i_2})dr.$$

For convenience of notation we let

$$J_{n-1}(t,s) = p_0(t) I_{n-1}(t,s;p_1,\cdots,p_{n-1}),\ J_{n-1}(t) = J_{n-1}(t,0).$$

In [20], Meng proposed the following theorem about the asymptotic behavior of solutions of (1.25).

Theorem 1.36

Suppose that $\int_1^\infty p_i(t)dt = \infty, 1 \leq i \leq n-1$, *and that there is a nonnegative continuous function* h, *and* $t \geq 1$, *and a continuous function* g *defined for* $u \geq 0$ *such that*

(i) $\int_1^\infty p_n(t)h(t)dt < \infty,\ \int_1^\infty p_n(t)|r(t)|dt < \infty$;

(ii) *for* $u > 0, g$ *is positive nondecreasing and* $\int_1^\infty \frac{du}{g(u)} = \infty$;

(iii) $|f(t,u)| \leq h(t)g\left(\frac{|u|}{J_{n-1}(t)}\right)$ *for* $t \geq 1, u \in \mathbb{R}$.

Then every solution u *of* (1.25) *satisfies* $u(t) = O\big(J_{n-1}(t)\big)$ *as* $t \to \infty$, *and* $L_{n-1}u(t) = O(1)$ *as* $t \to \infty$.

Remark 1.37

If $n = 2$ and $p_i(t) = 1$, for $i = 0, 1, 2$, $r(t) = 0$ then equation (1.25) reduces to (1.1) and we get the result of Theorem 1.5.

CHAPTER 2

PRELIMINARIES

In this chapter, we present some definitions, lemmas, properties and notation which will be used in our theorems later.

2.1 Spaces of Integrable and Continuous Functions

In this section, we present the definition of the space of p-integrable functions, and the space of continuous functions. We also give a characterization of the latter space.

Definition 2.1 [13]

We denote by $L_p(a,b), 1 \leq p \leq \infty$, the space of Lebesgue real-valued measurable functions u on (a,b) for which $\|u\|_p < \infty$ where

$$\|u\|_p = \left(\int_a^b |u(t)|^p dt \right)^{\frac{1}{p}}, 1 \leq p < \infty.$$

Definition 2.2 [13]

Let $[a,b]$ be a bounded interval and let $n = \{0,1,\dots\}$. We denote by $C^n[a,b]$ a space of functions which are m times continuously differentiable on $[a,b]$ with the norm

$$\|u\|_{C^n} = \sum_{k=0}^{n} \left\| u^{(k)} \right\|_C = \sum_{k=0}^{n} \max_{t \in [a,b]} \left| u^{(k)}(t) \right|, n = \{0,1,\dots\}.$$

In particular, for $n = 0$, $C^0[a,b] = C[a,b]$ is the space of continuously functions u on $[a,b]$ with the norm

$$\|u\|_C = \max_{t \in [a,b]} |u(t)|.$$

Lemma 2.3 [13]

The space $C^n[a,b]$ *consists of those and only those functions* u *which can be represented in the form*

$$u(t) = (I_{a+}^n \varphi)(t) + \sum_{k=0}^{n-1} c_k (t-a)^k , \tag{2.1}$$

where $\varphi \in C[a,b],\ c_k, k = 0,1,\cdots,n-1,$ *are arbitrary constants, and*

$$(I_{a+}^n \varphi)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \varphi(s) ds.$$

It follows from (2.1) *that*

$$\varphi = u^{(n)},\ c_k = \frac{u^{(k)}(a)}{k!},\ k = 0,1,\cdots,n-1.$$

2.2 Riemann-Liouville Fractional Integral and Fractional Derivative

In this section, we introduce the definition of the Riemann-Liouville fractional integral and fractional derivative on a finite interval of the real line.

Definition 2.4 [13]

The Gamma function $\Gamma(z)$ *is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad z > 0, \tag{2.2}$$

where $t^{z-1} = e^{(z-1)\ln t}$.

Definition 2.5 [13]

The Riemann-Liouville fractional integral $I_{a^+}^{\alpha}u$ *of order* $\alpha > 0$ *is defined by*

$$\left(I_{a^+}^{\alpha}u\right)(t) \coloneqq \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\frac{u(s)ds}{(t-s)^{1-\alpha}}, \quad t > a,$$

provided that the right hand side exists. It is called the left-sided Riemann-Liouville fractional integral.

Definition 2.6 [13]

The Riemann-Liouville fractional derivative $D_{a^+}^{\alpha}u$ *of order* $\alpha \geq 0$ *is defined by*

$$\left(D_{a^+}^{\alpha}u\right)(t) \coloneqq \left(\frac{d}{dt}\right)^{n}\left(I_{a^+}^{n-\alpha}u\right)(t)$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}\frac{u(s)ds}{(t-s)^{\alpha-n+1}}, n = [\alpha]+1; t > a,$$

provided that the right hand side exists. If $\alpha = m \in \mathbb{N}_0$, *then*

$$(D_{+}^{0}u)(t) = u(t),\ (D_{+}^{m}u)(t) = u^{(m)}(t).$$

Property 2.7 [13]

If $\alpha > \beta > 0$, *then for* $u \in L_p(a,b)$ $(1 \leq p \leq \infty)$, *the relation*

$$\left(D_{a^+}^{\beta}I_{a^+}^{\alpha}u\right)(t) = I_{a^+}^{\alpha-\beta}u(t), \tag{2.3}$$

hold almost everywhere on $[a,b]$.

2.3 Caputo Fractional Derivative

In this section, we present the definition of Caputo fractional derivative.

Definition 2.8 [13]

The fractional derivative ${}^{C}D_{a^+}^{\alpha}u$ *of order* $\alpha > 0$ *on* $[a,b]$ *is defined via the above Riemann-Liouville fractional derivative by*

$$({}^{C}D_{a^+}^{\alpha}u)(t) \coloneqq \left(D_{a^+}^{\alpha}\left[u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right]\right)(t),$$

where $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}_0, n = \alpha$ *for* $\alpha \in \mathbb{N}_0$.

Theorem 2.9 [13]

Let $\alpha > 0$ *and* $n = [\alpha] + 1$. *If* $u \in C^n[a,b]$, *then the Caputo fractional derivative* $({}^{C}D_{a^+}^{\alpha}u)(t)$ *exists almost everywhere on* $[a,b]$ *and is represented by*

$$({}^{C}D_{a^+}^{\alpha}u)(t) =: \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{u^{(n)}(s)dt}{(t-s)^{\alpha-n+1}} = \left(I_{a^+}^{n-\alpha}D^n u\right)(t).$$

The next lemma provides a formula for the composition of the fractional differentiation operator with the fractional integration operator. It shows that fractional differentiation is not the right inverse operator of the fractional integral in general.

Lemma 2.10 [13]

Let $\alpha > 0$ *and* $n = [\alpha] + 1$. *If* $u \in C^n[a,b]$, *then*

$$\left(I_{a^+}^{\alpha}\, {}^{C}D_{a^+}^{\alpha} u \right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}\,(t-a)^k. \tag{2.4}$$

2.4 Two Important Lemmas

In this section, we present the two lemmas which are very useful in our proofs.

Lemma 2.11 [17]

If $\lambda, v, \omega > 0$, then for any $t > 0$ we have

$$t^{1-v} \int_0^t (t-s)^{v-1} s^{\lambda-1} e^{-\omega s} ds \ \le L, \tag{2.5}$$

for some positive constant L, independent of t, given by

$$L = \max\{1, 2^{1-v}\}\, \Gamma(\lambda) \left(1 + \frac{\lambda}{v}\right) \omega^{-\lambda}. \tag{2.6}$$

Lemma 2.12 [17]

Let $\alpha \in [0,1)$ and $\beta \in \mathbb{R}$. Then there exists a positive constant $C = C(\alpha, \beta)$ such that

$$\int_0^t s^{-\alpha} e^{\beta s} ds \le \begin{cases} Ce^{\beta t} & \text{if } \beta > 0, \\ C(t+1) & \text{if } \beta = 0, \\ C, & \text{if } \beta < 0. \end{cases} \tag{2.7}$$

CHAPTER 3

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A CAUCHY TYPE PROBLEM

In this chapter, we consider the following problem:

$$\begin{cases} ({}^{C}D^{\alpha}_{t_0^+}u)(t) = f\big(t, u(t)\big), \quad t \geq t_0 \geq 1, \\ u^{(k)}(t_0) = c_k, \end{cases} \tag{3.1}$$

where ${}^{C}D^{\alpha}_{t_0^+}$ is the fractional derivative in the sense of Caputo of order

$\alpha \in (n-1, n), n \geq 2$, where $f: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $c_k \in \mathbb{R}$ for

$k = 0, 1, \cdots, n-1$, and $c_{n-1} \neq 0$.

In Theorem 3.1, we assume that $u \in C^n[t_0, \infty)$.

Theorem 3.1

Suppose that

(i) f *is continuous in* $D = \{(t, u): t \geq t_0 \geq 1, u \in \mathbb{R}\}$;

(ii) *there exists a nonnegative continuous function* h *defined for* $t \geq t_0 \geq 1$, *and a continuous function* g *defined for* $u \geq 0$ *such that*

$$|f(t,u)| \leq h(t) g\left(\frac{|u|}{t^{n-1}}\right) \text{ in } D,$$

where for $u > 0$, *the function* g *is positive, nondecreasing,* $\int_1^{\infty} \frac{ds}{g(u)} = \infty$, *and*

$h(t) = O\big(e^{-\omega t} t^{-\lambda}\big)$, $\omega > 0, 0 < \lambda < 1$. *Then any continuable solution* u *of* (3.1) *satisfies*

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = \frac{c_{n-1}}{(n-1)!}.$$

Proof:

We split our proof into two steps.

Step 1:

In this step, we want to show that there exists $H_0 > 0$ such that $\frac{|u(t)|}{t^{n-1}} \leq H_0$ for any continuable solution u of (3.1).

By applying $I^{\alpha}_{t_0^+}$ to both sides of the equation in (3.1) we get

$$(I^{\alpha}_{t_0^+} {}^{C}D^{\alpha}_{t_0^+} u)(t) = I^{\alpha}_{t_0^+} f\left(t, u(t)\right).$$

Since $u \in C^n[t_0, \infty)$, then from Lemma 2.10 and Definition 2.5 we obtain the Volterra integral equation associated to (3.1):

$$u(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(t - t_0)^k + I^{\alpha}_{t_0^+} f\left(t, u(t)\right). \tag{3.2}$$

It follows that

$$|u(t)| \leq \sum_{k=0}^{n-1} \frac{|c_k|}{k!} t^k + I^{\alpha}_{t_0^+} \left|f\left(t, u(t)\right)\right|.$$

Making use of the assumption on f in Theorem 3.1 and the fact that $\alpha > 1$, we get

$$|u(t)| \leq \sum_{k=0}^{n-1} \frac{|c_k|}{k!} t^k + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{|u(s)|}{s^{n-1}}\right) ds.$$

Thus we see that

$$|u(t)| \le \left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) t^{n-1} + \frac{t^{n-1}}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{|u(s)|}{s^{n-1}}\right) ds,$$

$$\le t^{n-1} \left[\left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)\, g\left(\frac{|u(s)|}{s^{n-1}}\right) ds\right]. \tag{3.3}$$

Let

$$H(t) := \sum_{k=0}^{n-1} \frac{|c_k|}{k!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{|u(s)|}{s^{n-1}}\right) ds. \tag{3.4}$$

Then

$$|u(t)| \le t^{n-1}\, H(t), \qquad t \ge 1. \tag{3.5}$$

Differentiating both sides of (3.4) yields

$$H'(t) = \frac{1}{\Gamma(\alpha)} h(t) g\left(\frac{|u(t)|}{t^{n-1}}\right).$$

By (3.5) and the assumption that g is nondecreasing we have

$$H'(t) \le \frac{1}{\Gamma(\alpha)} h(t) g\big(H(t)\big)$$

or

$$\frac{H'(t)}{g\big(H(t)\big)} \le \frac{1}{\Gamma(\alpha)} h(t), \qquad t \ge 1. \tag{3.6}$$

We integrate both sides of (3.6) and obtain

$$\int_{H(t_0)}^{H(t)} \frac{ds}{g(s)} \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)ds. \tag{3.7}$$

Denoting by $G(t)$ an antiderivative of $\frac{1}{g(t)}$ it appears from (3.7) that

$$G\big(H(t)\big) - G\big(H(t_0)\big) \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)ds.$$

Note that $G^{-1}(t)$ exists and is monotone increasing because G is monotone increasing. Therefore

$$H(t) \leq G^{-1}\left(G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)ds \right)$$

$$\leq G^{-1}\left(G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\infty} h(s)ds \right) = H_0, \tag{3.8}$$

where H_0 is a positive real number.

Since $\int_{t_0}^{\infty} h(s)ds$ is bounded, $G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\infty} h(s)ds$ is in the image of G. The right hand side of (3.8) is well defined and

$$H(t) \leq H_0.$$

Therefore in view of (3.3) the constant H_0 is also a bound of $\frac{|u(t)|}{t^{n-1}}$, i.e.

$$\frac{|u(t)|}{t^{n-1}} \leq H_0. \tag{3.9}$$

Thus

$$\left|f\big(t,u(t)\big)\right| \le h(t)g\left(\frac{|u(t)|}{t^{n-1}}\right) \le h(t)g(H_0). \tag{3.10}$$

<u>Step 2:</u>

In this step, we want to show that $u^{(r)}(t)$ converges to infinity as $t \to \infty$,

$r = 0,1,\dots,n-2$. By differentiating (3.2) we get

$$u^{(r)}(t) = \sum_{k=r}^{n-1} \frac{c_k}{(k-r)!}(t-t_0)^{k-r} + I_{t_0^+}^{\alpha-r} f\left(t,u(t)\right). \tag{3.11}$$

For the first term in the right hand side of (3.11) we see that

$$\sum_{k=r}^{n-1} \frac{c_k}{(k-r)!}(t-t_0)^{k-r} \to \infty \text{ as } t \to \infty.$$

For the second term we have

$$\left|I_{t_0^+}^{\alpha-r} f\left(t,u(t)\right)\right| \le I_{t_0^+}^{\alpha-r}\left|f\left(t,u(t)\right)\right| \le g(H_0) I_{t_0^+}^{\alpha-r} h(t) \le g(H_0) K I_{t_0^+}^{\alpha-r}(e^{-\omega t} t^{-\lambda})$$

$$\le \frac{g(H_0)KL}{\Gamma(\alpha-r)}\, t^{\alpha-r-1} \to \infty \text{ as } t \to \infty,$$

where $L = \max\{1, 2^{1-\alpha+r}\}\, \Gamma(-\lambda+1)\left(1+\frac{-\lambda+1}{\alpha-r}\right)\omega^{\lambda-1}$ is the constant defined by (2.6) with υ replaced by $\alpha - r$ and λ replaced by $-\lambda + 1$.

Therefore, $u^{(r)}(t) \to \infty$ as $t \to \infty, r = 0,1,\dots,n-2$.

Now, we want to prove that $u^{(n-1)}(t)$ converges to a finite limit (different from zero) as $t \to \infty$.

We have from (2.3) and (3.2)

$$u^{(n-1)}(t) = c_{n-1} + I_{t_0{}^+}^{\alpha-n+1} f\left(t, u(t)\right). \tag{3.12}$$

We want to show that the limit of the integral in (3.12) is equal to zero.

For this integral we have

$$\left|I_{t_0{}^+}^{\alpha-n+1} f\left(t, u(t)\right)\right| \leq I_{t_0{}^+}^{\alpha-n+1}\left|f\left(t, u(t)\right)\right| \leq g(H_0) I_{t_0{}^+}^{\alpha-n+1} h(t)$$

$$\leq g(H_0) K I_{t_0{}^+}^{\alpha-n+1}(e^{-\omega t} t^{-\lambda}) \leq g(H_0) K L t^{\alpha-n} \to 0,$$

where $L = \max\{1, 2^{n-\alpha}\} \, \Gamma(-\lambda+1) \left(1 + \frac{-\lambda+1}{\alpha-n+1}\right) \omega^{\lambda-1}$ is the constant defined by (2.6), with υ replaced by $\alpha - n + 1$ and λ replaced by $-\lambda + 1$.

Thus

$$\lim_{t\to\infty} I_{t_0{}^+}^{\alpha-n+1} f\left(t, u(t)\right).$$

Hence, from (3.12) we see that

$$\lim_{t\to\infty} u^{(n-1)}(t) = c_{n-1} \neq 0.$$

By (3.9) and L′ Hopital’s rule, it follows that

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = \lim_{t\to\infty} \frac{u^{(n-1)}(t)}{(n-1)!} = \frac{c_{n-1}}{(n-1)!}.$$

The proof is complete.

Example 3.2

Suppose that we have the following fractional differential problem:

$$\begin{cases} \left({}^{C}D_{t_0^+}^{\frac{3}{2}} u \right)(t) = 3e^{-4t}\, t^{\frac{-3}{2}} u \sin^3 u, \qquad t \geq t_0 \geq 1, \\ u(t_0) = c_0, u'(t_0) = c_1, \end{cases} \tag{3.13}$$

where $c_0, c_1 \in \mathbb{R}$, and $c_1 \neq 0$. For problem (3.13), we have

$$g(u) = u, h(t) = 3e^{-4t} t^{\frac{-1}{2}}.$$

Therefore all conditions of Theorem 3.1 are satisfied so any continuable solution u of (3.13) satisfies the asymptotic property

$$\lim_{t \to \infty} \frac{u(t)}{t} = c_1.$$

CHAPTER 4

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A RETARDED CAUCHY TYPE PROBLEM

In this chapter, we are concerned with the following problem:

$$\begin{cases} \left({}^{C}D_{t_0^+}^{\alpha}u\right)(t) = f\left(t, u\big(w(t)\big)\right), & t \geq t_0 \geq 1, \\ u^{(k)}(t_0) = c_k, \end{cases} \tag{4.1}$$

where ${}^{C}D_{t_0^+}^{\alpha}$ is the fractional derivative in the sense of Caputo of order

$\alpha \in (n-1, n), n \geq 2$, and $f: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$, $w: [t_0, \infty) \to [t_0, \infty)$ are continuous with $w(t) \leq t$ and $\lim_{t\to\infty} w(t) = \infty$, $c_k \in \mathbb{R}$ for $k = 0, 1, \cdots, n-1$, and $c_{n-1} \neq 0$. The function $w(t)$ is called a retarded argument.

In Theorem 4.1, we assume that $u \in C^n[t_0, \infty)$.

Theorem 4.1:

Suppose that

(i) *f is continuous in $D = \{(t, u): t \geq t_0 \geq 1, u \in \mathbb{R}\}$;*

(ii) *there exists a nonnegative continuous function h defined for $t \geq t_0 \geq 1$, and a continuous function g defined for $u \geq 0$ such that*

$$\left|f\left(t, u\big(w(t)\big)\right)\right| \leq h(t) g\left(\frac{|u|}{\big(w(t)\big)^{n-1}}\right) \text{ in } D,$$

where for $u > 0$, the function g is positive, nondecreasing, $\int_1^\infty \frac{ds}{g(u)} = \infty$, and

$h(t) = O\left(e^{-\omega t} t^{-\lambda}\right)$, $\omega > 0, 0 < \lambda < 1$. Then any continuable solution u of (4.1) *satisfies*

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = \frac{c_{n-1}}{(n-1)!}.$$

Proof:

By applying $I^{\alpha}_{t_0^+}$ to both sides of the equation in (4.1) we get

$$(I^{\alpha}_{t_0^+}\,{}^{C}D^{\alpha}_{t_0^+}u)(t) = I^{\alpha}_{t_0^+} f\left(t, u\big(w(t)\big)\right),$$

and since $u \in C^n[t_0, \infty)$, then the last equation may be written as

$$u(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(t - t_0)^k + I^{\alpha}_{t_0^+} f\left(t, u\big(w(t)\big)\right). \tag{4.2}$$

It follows that

$$|u(t)| \leq \sum_{k=0}^{n-1} \frac{|c_k|}{k!} t^k + I^{\alpha}_{t_0^+} \left|f(t, u\big(w(t)\big)\right|.$$

From the assumption on f and the fact that $\alpha > 1$ we obtain

$$|u(t)| \leq \sum_{k=0}^{n-1} \frac{|c_k|}{k!} t^k + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{\left|u\big(w(s)\big)\right|}{\big(w(s)\big)^{n-1}}\right) ds,$$

and for $t > t_0 \geq 1$ we see that

$$|u(t)| \leq \left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) t^{n-1} + \frac{t^{n-1}}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{\left|u\big(w(s)\big)\right|}{\big(w(s)\big)^{n-1}}\right) ds,$$

or

$$|u(t)| \leq t^{n-1} \left[\sum_{k=0}^{n-1} \frac{|c_k|}{k!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)\, g\left(\frac{\left|u\big(w(s)\big)\right|}{\big(w(s)\big)^{n-1}}\right) ds\right]. \tag{4.3}$$

Let

$$H(t) := \sum_{k=0}^{n-1} \frac{|c_k|}{k!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)\, g\left(\frac{|u(w(s))|}{(w(s))^{n-1}}\right) ds. \tag{4.4}$$

We see from (4.3) that

$$|u(t)| \leq t^{n-1} H(t), \qquad t \geq 1. \tag{4.5}$$

If we choose $t_1 \geq t_0$ so large that $w(t) \geq t_0$ for $t \geq t_1$ then it follows from (4.5) and the increasing nature of $H(t)$ that

$$|u(w(t))| \leq (w(t))^{n-1} H(t), \qquad t \geq t_1.$$

Differentiating both sides of (4.4) yields

$$H'(t) = \frac{1}{\Gamma(\alpha)} h(t) g\left(\frac{|u(w(t))|}{(w(t))^{n-1}}\right).$$

Since g is nondecreasing we have for $t \geq t_1$

$$H'(t) \leq \frac{1}{\Gamma(\alpha)} h(t) g(H(t))$$

or

$$\frac{H'(t)}{g(H(t))} \leq \frac{1}{\Gamma(\alpha)}\, h(t), \qquad t \geq t_1.$$

If $t_1 > 1$ then we see that

$$\int_{H(t_1)}^{H(t)} \frac{ds}{g(s)} \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} h(s) ds. \tag{4.6}$$

Denoting by $G(t)$ an antiderivative of $\frac{1}{g(t)}$ it appears from (4.5) that

$$G\big(H(t)\big) - G\big(H(t_1)\big) \le \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} h(s)ds.$$

Note that G^{-1} exists and is monotone increasing because G is monotone increasing. Therefore

$$H(t) \le G^{-1}\left((G(H(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} h(s)ds \right)$$

$$\le G^{-1}\left((G(H(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\infty} h(s)ds \right) = H_0. \tag{4.7}$$

where H_0 is a positive real number.

Notice that the range of G is open and $\int_{t_1}^{\infty} h(s)ds$ can be made arbitrarily small by increasing the value t_1. Thus $G\big(H(t_1)\big) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\infty} h(s)ds$ is in the image of G for t_1 sufficiently large and thus the right hand side of (4.7) is well defined. Therefore $H(t) \le H_0,\ t \ge t_1$ and in view of (4.5) the constant H_0 is also a bound for $\frac{|u(t)|}{t^{n-1}}$ i.e.

$$\frac{|u(t)|}{t^{n-1}} \le H_0, \qquad t \ge t_1. \tag{4.8}$$

Now, we have by (4.2) and (2.3)

$$u^{(n-1)}(t) = c_{n-1} + I_{t_0^+}^{\alpha-n+1} f\left(t, u\big(w(t)\big)\right).$$

We can show that $u^{(r)}(t)$ converges to infinity as $t \to \infty, r = 0,1, \dots, n-2$, $u^{(n-1)}(t)$ converges to a finite limit (different from zero) as $t \to \infty$,

and $\lim_{t\to\infty} I_{t_0^+}^{\alpha-n+1} f\left(t, u\big(w(t)\big)\right) = 0$ with the same argument as in Theorem 3.1, so we have

$$\lim_{t\to\infty} u^{(n-1)}(t) = c_{n-1} \neq 0.$$

By (4.8) and L′Hopital’s Rule, it follows that

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = \lim_{t\to\infty} \frac{u^{(n-1)}(t)}{(n-1)!} = \frac{c_{n-1}}{(n-1)!}.$$

The proof is complete.

Example 4.2

Suppose that we have the following fractional differential problem:

$$\begin{cases} \left({}^{C}D_{t_0^+}^{\frac{7}{3}} u\right)(t) = 5e^{-6t}\, t^{\frac{-1}{4}} \dfrac{u^2\big(\sqrt{t}\big)}{u^2\big(\sqrt{t}\big) + t^2}, & t \geq t_0 \geq 1, \\ u(t_0) = c_0, u'(t_0) = c_1, u''(t_0) = c_2, \end{cases} \tag{4.9}$$

where $c_0, c_1, c_2 \in \mathbb{R}$, and $c_2 \neq 0$. For problem (4.9), we have

$$g(u) = \frac{u^2}{u^2+1}, h(t) = 5e^{-6t} t^{\frac{-1}{4}}, w(t) = \sqrt{t}.$$

Therefore all conditions of Theorem 4.1 are satisfied so any continuable solution u of (4.9) satisfies the asymptotic property

$$\lim_{t\to\infty} \frac{u(t)}{t^2} = \frac{c_2}{2}.$$

CHAPTER 5

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NEUTRAL CAUCHY TYPE PROBLEM

In this chapter, we are interested in the following problem:

$$\begin{cases} {}^{C}D^{\alpha}_{t_0^+}\left(u(t)+pu(t-\tau)\right)=f\left(t,u(t)\right), \quad t\geq t_0\geq 1, \\ u^{(k)}(t_0)+pu^{(k)}(t_0-\tau)=c_k, \end{cases} \tag{5.1}$$

where ${}^{C}D^{\alpha}_{t_0^+}$ is the fractional derivative in the sense of Caputo of order

$\alpha\in(n-1,n), n\geq 2$, $0\leq p<1, \tau>0$, where $f:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous,

$c_k\in\mathbb{R}$ for $k=0,1,\cdots,n-1$, and $c_{n-1}\neq 0$. In Theorem 5.3, we assume that

$u\in C^n[t_0,\infty)$.

In what follows, we shall use the following lemma, which gives us useful information about properties of nonoscillatory solutions of (5.1).

Lemma 5.1 [10]

Let $y(t)>0$ $(or\ y(t)<0)$ *eventually and define*

$$\omega(t)=y(t)+p\frac{(t-\tau)^{n-1}}{t^{n-1}}y(t-\tau), \qquad n\geq 2, 0\leq p<1, \tau>0. \tag{5.2}$$

If $\lim_{t\to\infty}\omega(t)=c$, *then* $\lim_{t\to\infty}y(t)=\frac{c}{1+p}$.

Proof:

Suppose that $y(t)>0$. Then $c\geq 0$ and we see that

$$\limsup_{t\to\infty} y(t)\geq\frac{c}{1+p}$$

and

$$\liminf_{t\to\infty} y(t)\leq\frac{c}{1+p}.$$

Assume that

$$\limsup_{t\to\infty} y(t) = \lim_{n\to\infty} y(\overline{t_n}) = \frac{c+q_1}{1+p}$$

and

$$\liminf_{t\to\infty} y(t) = \lim_{n\to\infty} y\left(\underline{t_n}\right) = \frac{c-q_2}{1+p},$$

where $q_1 \geq 0, q_2 \geq 0$. We shall prove that $q_1 = q_2 = 0$.

(a) Suppose that $q_1 \geq q_2 \geq 0$ and $q_1 \geq 0$. It follows from (5.2) that for any $\varepsilon > 0$

$$\omega(t) \geq y(t) + p\frac{(t-\tau)^{n-1}}{t^{n-1}}\frac{c-q_2-\varepsilon}{1+p}.$$

Taking $t = \overline{t_n}$ and letting $n \to \infty$, we get

$$c \geq \frac{c+q_1}{1+p} + p\frac{c-q_2-\varepsilon}{1+p}.$$

That is

$$q_1 \leq q_2 p + p\varepsilon.$$

Setting $\varepsilon = \frac{(1-p)q_2}{2p}$ we are led to $q_1 \leq p(2q_2 - q_1) \leq pq_2 < q_2$. This is a contradiction.

(b) Suppose that $q_2 \geq q_1 \geq 0$ and $q_2 \geq 0$. Then (5.2) implies

$$\omega(t) \leq y(t) + p\frac{c+q_1+\varepsilon}{1+p}, \varepsilon > 0.$$

Putting $t = \underline{t_n}$ and letting $n \to \infty$, we get

$$c \leq \frac{c-q_2}{1+p} + p\frac{c+q_1+\varepsilon}{1+p}.$$

That is

$$q_2 \leq q_1 p + p\varepsilon.$$

Setting $\varepsilon = \frac{(1-p)q_2}{2p}$ we are led to $q_2 \leq p(2q_1 - q_2) \leq pq_1 < q_1$. This is a contradiction.

The proof is complete.

Remark 5.2

All inequalities in the proof of Theorem 5.3 are assumed to hold eventually, i.e. they are satisfied for all sufficiently large t.

Theorem 5.3

Suppose that

(i) f *is continuous in* $D = \{(t,u):\ t \geq t_0 \geq 1, u \in \mathbb{R}\}$;

(ii) *there exists a nonnegative continuous function* h *defined for* $t \geq t_0 \geq 1$, *and a continuous function* g *defined for* $u \geq 0$ *such that*

$$|f(t,u)| \leq h(t)g\left(\frac{|u|}{t^{n-1}}\right) \text{ in } D,$$

where for $u > 0$, *the function* g *is positive, nondecreasing,* $\int_1^\infty \frac{ds}{g(s)} = \infty$, *and*

$h(t) = O\left(e^{-\omega t}t^{-\lambda}\right), \omega > 0, 0 < \lambda < 1$. *Then any nonoscillatory continuable solution* u *of* (5.1) *satisfies*

$$\lim_{t\to\infty} \frac{u(t)}{t^{n-1}} = \frac{c_{n-1}}{(1+p)(n-1)!}.$$

Proof:

By applying $I^{\alpha}_{t_0^+}$ to both sides of the equation in (5.1) we get

$$I^{\alpha}_{t_0^+}\,{}^{C}D^{\alpha}_{t_0^+}\left(u(t) + pu(t-\tau)\right) = I^{\alpha}_{t_0^+} f\left(t, u(t)\right).$$

and since $u \in C^n[t_0, \infty)$, then the Volterra integral equation associated to (5.1) is:

$$u(t) + p\big(u(t-\tau)\big) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(t-t_0)^k + I_{t_0^+}^{\alpha} f\ \big(t, u(t)\big). \tag{5.3}$$

Let

$$z(t) = u(t) + pu(t-\tau), \tag{5.4}$$

and since $u(t)$ is a nonoscillatory solution of (5.3) thus we have

$$|z(t)| > |u(t)|,$$

and (5.3) becomes

$$z(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(t-t_0)^k + I_{t_0^+}^{\alpha} f\ \big(t, u(t)\big).$$

It follows that

$$|z(t)| \leq \sum_{k=0}^{n-1} \frac{|c_k|}{k!}(t-t_0)^k + I_{t_0^+}^{\alpha} \big|f\ \big(t, u(t)\big)\big|.$$

From the assumption on f and the fact that $\alpha > 1$ we obtain

$$\begin{aligned} |z(t)| &\leq \left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) t^{n-1} + \frac{t^{n-1}}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{|u(s)|}{s^{n-1}}\right) ds, \\ &\leq t^{n-1} \left[\left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)\, g\left(\frac{|u(s)|}{s^{n-1}}\right) ds\right], \qquad (5.5) \\ &\leq t^{n-1} \left[\left(\sum_{k=0}^{n-1} \frac{|c_k|}{k!}\right) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s)\, g\left(\frac{|z(s)|}{s^{n-1}}\right) ds\right]. \end{aligned}$$

Let

$$H(t) := \sum_{k=0}^{n-1} \frac{|c_k|}{k!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) g\left(\frac{|z(s)|}{s^{n-1}}\right) ds. \tag{5.6}$$

We see from (5.5) that

$$|z(t)| \le t^{n-1} H(t), \qquad t \ge 1. \tag{5.7}$$

Differentiating both sides of (5.6) gives

$$H'(t) = \frac{1}{\Gamma(\alpha)} h(t) g\left(\frac{|z(t)|}{t^{n-1}}\right).$$

Since g is nondecreasing we have for $t \ge 1$

$$H'(t) \le \frac{1}{\Gamma(\alpha)} h(t) g\big(H(t)\big),$$

or

$$\frac{H'(t)}{g\big(H(t)\big)} \le \frac{1}{\Gamma(\alpha)}\, h(t), \qquad t \ge 1.$$

Then we see that

$$\int_{H(t_0)}^{H(t)} \frac{ds}{g(s)} \le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) ds. \tag{5.8}$$

Denoting by $G(t)$ an antiderivative of $\frac{1}{g(t)}$ it appears from (5.8) that

$$G\big(H(t)\big) - G\big(H(t_0)\big) \le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} h(s) ds.$$

Note that G^{-1} exists and is monotone increasing because G is monotone increasing.

Therefore

$$H(t) \le G^{-1}\left(G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} h(s)ds\right)$$

$$\le G^{-1}\left(G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{\infty} h(s)ds\right) = H_0, \tag{5.9}$$

where H_0 is a positive real number.

Since $\int_{t_0}^{\infty} h(s)ds$ is bounded, $G\big(H(t_0)\big) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{\infty} h(s)ds$ is in the image of G. The right hand side of (5.9) is well defined. Therefore $H(t) \le H_0, t \ge t_0$ and in view of (5.7) the constant H_0 is also a bound for $\frac{|z(t)|}{t^{n-1}}$ i.e.

$$\frac{|z(t)|}{t^{n-1}} \le H_0. \tag{5.10}$$

We have from (2.3) and (5.3)

$$z^{(n-1)}(t) = c_{n-1} + I_{t_0^+}^{\alpha-n+1} f\left(t, u(t)\right).$$

Now, we can show that $z^{(r)}(t)$ converges to infinity as $t \to \infty, r = 0,1, \dots, n-2$, $z^{(n-1)}(t)$ converges to a finite limit (different from zero) as $t \to \infty$, and $\lim_{t\to\infty} I_{t_0^+}^{\alpha-n+1} f\left(t, u(t)\right) = 0$ in exactly the same way we have done in the previous proofs, so we find

$$\lim_{t\to\infty} z^{(n-1)}(t) = c_{n-1} \ne 0.$$

By (5.10) and L′Hopital’s Rule, it follows that

$$\lim_{t\to\infty}\frac{z(t)}{t^{n-1}} = \lim_{t\to\infty}\frac{z^{(n-1)}(t)}{(n-1)!} = \frac{c_{n-1}}{(n-1)!}.$$

Now we put $\omega(t) = \frac{z(t)}{t^{n-1}}$, then (5.2) implies

$$\omega(t) = y(t) + p\frac{(t-\tau)^{n-1}}{t^{n-1}}y(t-\tau),$$

where $y(t) = \frac{u(t)}{t^{n-1}}$. Lemma 5.1 insures that

$$\lim_{t\to\infty}\frac{u(t)}{t^{n-1}} = \frac{c_{n-1}}{(1+p)(n-1)!}.$$

The proof is complete.

Example 5.4

Suppose that we have the following fractional differential problem:

$$\begin{cases} {}^{C}D_{t_0^+}^{\sqrt{2}}\left(u(t) + \frac{1}{4}u(t-2)\right) = 12e^{-2t}\, t^{\frac{-1}{5}}\frac{u^4}{u^4+t^4}, \qquad t \geq t_0 \geq 1, \\ u(t_0) = c_0, u'(t_0) = c_1, \end{cases} \tag{5.11}$$

where $c_0, c_1 \in \mathbb{R}$, and $c_1 \neq 0$. For problem (5.11), we have

$$g(u) = \frac{u^4}{u^4+1}, h(t) = 12e^{-2t}t^{\frac{-1}{5}},$$

therefore all conditions of Theorem 5.3 are satisfied so any nonoscillatory continuable solution u of (5.11) satisfies the asymptotic property

$$\lim_{t\to\infty}\frac{u(t)}{t} = \frac{4}{5}c_1.$$

FUTURE WORK

In the thesis, we have restricted ourselves to Caputo fractional derivative. It will be interesting to consider other types of fractional derivatives (Riemann-Liouville, Hilfer, Hilfer-Hadamard,…etc). Also it is important to consider different conditions on nonlinearities other than the ones we have considered. Further studies can be carried out on different forms of equations (Laplacian, autonomous, perturbed,…etc).

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