OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND AND VARIABLE HOLDING COST

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I would like to dedicate this work to my lovely family, the soul of my father, my compassionate mother, my wife and my son.

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All praise and glory goes to Almighty Allah (the powerful and Exalted in Might)who gave me the courage to patience to finish this work. Blessings of Allah be upon his prophet Mohammed Bin Abdullah and Ahl al-Bayt.

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TABLE OF CONTENTS

ACKNOWLEDGMENTS V

TABLE OF CONTENTS VI

LIST OF TABLES IX

LIST OF FIGURES X

THESIS ABSTRACT XI

ملخص الرسالة XII

1 CHAPTER 1 INTRODUCTION 1

1.1 Introduction and Background 1

1.2 Stock Dependent Demand 3

1.3 Holding Cost 3

1.4 Quantity Discount 4

1.5 Thesis Organization 5

CHAPTER 2 LITERATURE REVIEW 7

2.1 EOQ Models with Stock-Dependent Demand 7

2.2 EOQ Models with Variable Holding Cost 8

2.3 EOQ Models with Variable Demand and Variable Holding Cost 9

2.4 EPQ Models with Variable Demand and/or Variable Holding Cost 11

2.5 EPQ/ EPQ-Type Models with Quantity Discounts 11

CHAPTER 3 OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST AND RETROACTIVE HOLDING COST 13

3.1 Assumptions and Notations 13

3.2 The Production-Inventory Model: 15

3.3 Solution Algorithm of Retroactive Holding Cost 21

3.4 Numeric Example.. 22

3.5 Sensitivity Analysis 23

CHAPTER 4 OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST AND INCREMENTAL HOLDING COST.......................... 25

4.1 Production-Inventory Model 25

4.2 Solution Algorithm 28

4.3 Numeric Example 28

4.4 Sensitivity Analysis 31

CHAPTER 5 OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST, RETROACTIVE HOLDING COST AND ALL UNIT DISCOUNT 33

5.1 Production-inventory Model 33

5.2 Solution Algorithm of Retroactive Holding Cost with All Unit Discount 35

5.3 Numeric Example 36

5.4 Sensitivity Analysis 38

CHAPTER 6 OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST, INCREMENTAL HOLDING COST AND ALL UNIT DISCOUNT 40

6.1 Production-Inventory Model 40

6.2 Solution Algorithm 41

6.3 Numeric Example 42

6.4 Sensitivity Analysis 44

CHAPTER 7 CONCLUSION AND CONTRIBUTION 46

7.1 Major Contribution 46

7.2 Further Extension and Research 47

APPENDIX 48

REFERENCES 50

VITAE 56

LIST OF TABLES

Table 1 Sensitivity Analysis Results of Model 1 23
Table 2 Sensitivity Analysis Results of Model 2 31
Table 3 Sensitivity Analysis Results of Model 3 38
Table 4 Sensitivity Analysis Results of Model 4 44

LIST OF FIGURES

Figure 1 Incremental Discount Model 4
Figure 2 All-Unit Discount Model 5
Figure 3 Different attributes used for the four models 6
Figure 4 General EPQ model with stock-dependent demand 16

THESIS ABSTRACT

Full Name : Ahmad Abdulkareem Abdulaziz AlShaer

Thesis Title : Optimal EPQ Model for Stock-Dependent Demand and Variable Holding Cost

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This thesis is structured to develop four different economic production quantity (EPQ) models with stock level dependent demand rate, variable holding cost and quantity discount with total profit per unit time maximization objective function rather than minimizing total cost per unit time and allowing non-zero end inventory for each cycle.

Two types of holding cost are considered during the modeling of the four models: incremental and retroactive holding cost. Furthermore, the first two EPQ models are developed without quantity discount and the remaining two models are developed with the assumption of having all unit discounts to maximize the total profit.

Finally, four mathematical models with optimum solution procedures, including nonlinear programming, and sensitivity analysis are carried out in this thesis work.

Keywords: Inventory models; Stock-dependent demand; Retroactive holding cost; Incremental holding cost; Optimization; Quantity discount.

ملخص الرسالة

الاسم الكامل: أحمد عبدالكريم عبدالعزيز الشاعر

عنوان الرسالة: نظام الإنتاج الأمثل مع كمية إنتاج متغيره مع الطلب المعتمد على مستوى المخزون،و تكلفة التخزين ديناميكية.

التخصص: هندسة النظم الصناعية

تاريخ الدرجة العلمية: ديسمبر كانون الاول ٢٠١٣

تتمحور هذة الرسالة حول تطوير و ايجاد أربعة نماذج من كميات الانتاج الاقتصادية (EPQ) مع فرضية الطلب المعتمد علي مستوى المخزون و تكلفة التخزين المتغيرة و خصم للكميات, حيث يكون الهدف زيادة الربح الاجمالي للوحدة الزمنيه بدلا من تقليل التكلفة الاجمالية للوحدة الزمنية.

تم اعتبار نوعين من تكلفة التخزين خلال وضع النماذج الاربعة و هما تكلفة التخزين الاضافية و الرجعية. علاوة على ذلك تم وضع النموذجين الأوليين بدون خاصية الخصم و من ثم تم تطوير النموذجين الاخريين بإضافة خاصية الخصم للكميات و ذلك للحصول علي أعلى مستوى من الارباح الاجمالية.

الهدف الرئيسي من هذه الرسالة هو ايجاد نماذج رياضية مع طريقة حلها و البرمجة الغير خطيه و تحليل الحساسية لنماذج الانتاج الاربعة.

الكلمات الرئيسية: نماذج الانتاج و التخزين الطلب المعتمد علي مستوي التخزين, تكلفة التخزين المتغيرة, الحل الأمثل خصم للكميات.

CHAPTER 1

INTRODUCTION

1.1 Introduction and Background

Traditional EPQ models are based on the assumptions of constant demand rate, constant inventory carrying cost, and instantaneous order arrival. This thesis work presents a more realistic production-inventory model in which all these assumptions are relaxed. The economic production quantity (EPQ) model relaxes the assumption of instantaneous order arrival by incorporating a gradual order receipt, i.e. a finite production rate. This thesis presents a production-inventory control system with finite production rate, stock-dependent demand rate and variable holding cost.

Demand variability with item availability and price is a frequently observed phenomenon. In stock-dependent demand models, the demand for a given item increases with higher item availability. A bigger item display tends to attract the attention of more customers, leading to increased sales. Moreover, customers may view larger item stock as an indication of the item’s popularity and also as a sign of reliable, continuing supply. Since higher sales are made on the expense of carrying more stock, firms need to find the optimum balance between the extra holding cost and the additional profit resulting from the induced sales. According to Sarkar (2012), “In general, sugar, spices, clothes, gift cards are the real-life examples of the stock-dependent demand". Moreover, Yang et. al. (2013) stated that "The manufacturer should often strike the balance between the production efficiency and market demand rate due to the stock-dependent demand".

In inventory models with variable holding costs, the holding cost is a function of either the storage level or the storage time. Holding cost time variability is used to reflect the fact that longer storage times frequently require higher holding costs. Longer storage periods, especially for perishable products, usually require more expensive specialized storage facilities. Time-dependent holding cost models represent holding cost either as a continuous nonlinear function or as a discontinuous step function of storage time.

The unit cost is usually assumed to be fixed and independent of the order size. Whereas, it is more realistic to consider the case where the unit cost is dependent of order size since the vendor is willing to charge less per unit for large orders. The purpose of the discount is to encourage the customer to buy material in big batches. There are two main type of price discount that are all-units discounts and incremental quantity discounts.

The aim of this thesis is to present a production-inventory model with finite production rate, stock-level dependent demand rate and variable holding cost. In this model, the demand rate is an increasing power function of the instantaneous inventory level, and the holding cost is an increasing step function of the time spent in storage. Two types of holding cost step functions are considered: retroactive holding cost, and incremental holding cost. Whereas quantity discount is considered to the model "all-units discounts" for the two types of holding cost "retroactive holding cost, and incremental holding cost".

1.2 Stock Dependent Demand

The presence of retail inventory is assumed to have an encouraging effect on the customer. Traders usual tend to have mass displays of items inside the stores that are used as ‘‘psychic stock’’ to stimulate sales of some retail items. This phenomenon may also be experienced with products that are generically the same, but are individually slightly different (such as greeting cards); thus, increased inventory levels give the customer a wider selection and increase the probability of making a sale. In these situations, the demand of a given item is not assumed to be an exogenous variable, as with the classical inventory models; instead, it is assumed that the demand rate is endogenous to the firm and is a function of the inventory level. The effect of this dependency is that the retailer has incentive to keep higher levels of inventory— despite increased holding costs—as long as the item is profitable and the demand is an increasing function of the inventory level. This results in additional sales, higher fill rates, and potentially greater profits. The operations management literature has recognized this motivating effect of inventory on demand, and models have been developed that incorporate this relationship.

1.3 Holding Cost

The associated price of storing inventory or assets that remains unsold. Holding costs are a major component of supply chain management, since businesses must determine how much of a product to keep in stock. This represents an opportunity cost, as the presence of the goods means that they are not being sold while that money could be deployed elsewhere. In addition, holding costs include the costs of goods being damaged or spoiled over time and the general costs, such as space, labor and other direct expenses. Inventory is measured by in units rather than dollars; it is more convenient to express the holding cost in terms of dollars per unit per unit time than dollars per unit time. Holding cost h is equal the multiplication of c “the dollar value of one unit of inventory” by I “the annual interest rate”.

There are two main types of holding cost; incremental and retroactive holding cost.

1.4 Quantity Discount

The unit cost c is dependent of the size of the order, so as the size of the order goes up the unit cost will be less as the supplier is willing to charge less for bulk orders. The main goal of the discount is that encouragement of buying larger batches. There are many types of discount exist, but the most popular types are: all unit discount and incremental discount. In each case, we assume there are one or more breakpoints defining changes in the unit cost.

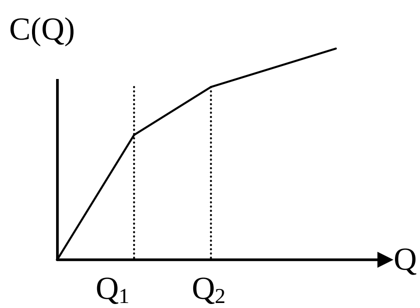

Figure 1: Incremental Discount Model

There are two possibilities: either the discount is applied to all units in an order (all unit), or it is applied only the additional units beyond the break points (incremental). The all-unit case is the more common and it the type that is used for our model here.

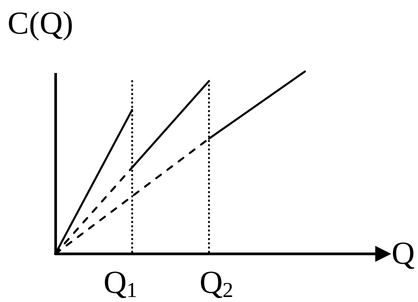

Figure 2: All-Unit Discount Model

1.5 Thesis Organization

The economic production models and problems are of importance on both practical and theoretical. In this thesis, we will focus on modeling four different types of EPQ models to maximize the total profit and finding the optimal solution with the maximum inventory level (Q) and the maximum ending inventory (L). In chapter 2, full literature review along with several categories of recent production-inventory models with variable holding costs and stock-dependent demand rates are presented. Chapter 3 covers the modeling and mathematical formulation of EPQ model with retroactive holding cost and with no quantity discount. Chapter 4 is about EPQ model with incremental holding cost and no discount as well. Chapter 5 is about EPQ model with retroactive holding cost and all unit discounts. In chapter 6, the EPQ model with incremental holding cost and all unit discounts is presented as well. The following figure summarizes the difference between the four models.

	Retroactive Holding Cost	***Incremental holding Cost***
No Discount	Model 1	Model 2
All Unit Discount	Model 3	Model 4

Figure 3: Different attributes used for the four models

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review several categories of recent production-inventory models with variable holding costs and stock-dependent demand rates. Urban (2005) provides a comprehensive review of inventory models with stock-dependent demand rates published up to 2004, classifying work in this area into two main types. In the first type, pioneered by Gupta and Vrat (1986), the demand rate is a function of the initial inventory. In the second type, pioneered by Baker and Urban (1988a, 1988b), the demand rate is a function of the instantaneous inventory. A third type could be considered as models in which the demand has two stages, an initial period of level-dependent demand followed by a period of constant demand.

2.1 EOQ Models with Stock-Dependent Demand

In inventory models with variable demand rate, the demand is either a function of time or a function of the stock level. According to Urban (2005), demand dependence on the stock level has several functional forms, including: linear, power, and posynomial. Several models have been proposed for deteriorating items with stock-level dependent demand rates. Min and Zhou (2009) develop an EOQ model for perishable items with stock-dependent demand rate, where unsatisfied demand is partially backlogged. Sana et al. (2009) present EOQ and EPQ inventory models in which the demand rate depends on three factors: stock level, selling price, and advertising. Both deteriorating and ameliorating items are considered under budget and storage capacity constraints. Yang et al. (2010) propose an EOQ model for deteriorating items with stock-dependent demand rate, partial backlogging, inflation, varying replenishment cycles, and varying shortage intervals. Roy et al. (2009), developed an EOQ model for a deteriorating item with linearly displayed stock dependent demand in imprecise environment, involving both fuzzy and random parameters, under inflation and time value of money. Hsieh and Dye (2010), developed a deterministic EOQ model for deteriorating items with stock-dependent demand and finite shelf/display space, where the shortages are allowed and the unsatisfied demand is partially backlogged at the exponential rate with respect to the waiting time. Das et al. (2010), presented production lot size inventory model in which the production rate constitutes of productions during both regular time and overtime, the demand rate is assumed as stock-dependent. The formulation leads to a single objective optimization problem for maximum average profit evaluation through a real-coded genetic algorithm (GA). Duan et al. (2012), presented inventory models for perishable items with inventory level dependent demand rate where the models with and without backlogging were studied. Recently, Sarkar (2012) considered finite replenishment rate under progressive payment scheme with the production of defective items within the cycle time.

2.2 EOQ Models with Variable Holding Cost

An increasing number of EOQ-type inventory models assume variable and nonlinear holding cost. Muhlemann and Valtis-Spanopoulos (1980) are the first to introduce variable holding costs into the EOQ model. In their EOQ-type model, the holding cost is assumed to be a function of the average inventory value. Their justification is that the cost of financing increases as the amount of investment (value of the inventory) increases. Weiss (1982) develops deterministic and stochastic EOQ models in which the per-unit holding cost is a non-linear function of storage duration. According to Weiss (1982), this assumption is applicable to inventory systems where the value of stored items decreases non-linearly with storage length. Ferguson et al. (2007) present another EOQ-type model in which the holding cost is a nonlinear function of time. Given historical data, regression is used to estimate the parameters of this function for perishable grocery items.

2.3 EOQ Models with Variable Demand and Variable Holding Cost

Goh (1994) presents the first inventory model in which the demand is stock dependent and the holding cost varies with storage duration. Goh considers two types of holding cost variation: (a) a nonlinear function of storage time, and (b) a nonlinear function of storage level. Giri at al. (1996) present a generalized EOQ-type model for deteriorating items, in which the demand rate, the deterioration rate, the ordering cost, and the holding cost are continuous functions of time. Giri and Chaudhuri (1998) develop another EOQ-type model for deteriorating items in which the demand rate is a function of the stock level. Keeping all other costs and parameters constant, the holding cost is assumed a nonlinear function of either the stock level or the storage duration.

Teng and Yang (2007) generalize the EOQ model to consider a time-varying demand rate, assuming that the holding cost includes both size-related and value-related components. Roy (2008) formulates an EOQ model for deteriorating items in which shortages are allowed, where the demand rate is a function of the selling price and the holding cost is a continuous function of time. Gayen and Pal (2009) analyze a two-warehouse inventory model for a deteriorating product where both the demand rate and the holding cost are assumed to be continuous power functions of the current inventory level. Mahata and Goswami (2009) investigate an EOQ model for deteriorating items with stock dependent demand rate, variable holding cost, and fuzzy deterioration rate. The holding cost is considered as a non-linear function of either the length of storage time or the current inventory level.

Alfares (2007) presents a stock-dependent EOQ-type model with two types of holding cost discontinuous step functions. As the storage time extends to the next time period, the new (higher) holding cost can be applied either retroactively to all storage periods, or incrementally to the new period only. Urban (2008) extends Alfares (2007) work by allowing non-zero end inventory for each cycle, and shifting to a maximum-profit objective. Singh, Kumar, and Gaur (2009) present an EOQ model with stock-dependent demand and partial backlogging of unsatisfied demand. They also consider deteriorating items, inflation, and an incremental holding cost function. Pando et al. (2012), studied an EOQ inventory model with demand rate and holding cost rate per unit time, both potentially dependent on the stock level, where the objective is to maximize the average profit per unit time.

2.4 EPQ Models with Variable Demand and/or Variable Holding Cost

Goh (1992) presents three models of stock-dependent demands inventory systems including an EPQ-type model with non-instantaneous receipt of orders. Sarfaraz (2009) develops a modified EPQ model in which the holding cost is composed of two components: an investment cost proportional to the dollar value of inventory, and a capacity cost proportional to the maximum inventory level. As stated earlier, Sana et al. (2009) analyze an EPQ model with stock-dependent demand rate and storage capacity limitations. Tripathy et al. (2010) formulate an EPQ model for deteriorating items, assuming the demand rate is constant and the holding cost is a nonlinear continuous function of storage time. Singh, Singh, and Vaish (2009) develop an EPQ inventory model where demand is a linear function of time, and two cases of holding costs are considered: (i) holding cost is constant, and (ii) holding cost is a continuous function of time. Yang et. al. (2013) consider EOQ modeling for a single-manufacturer and single-retailer where the demand rate at the retailer's end is dependent on the instantaneous stock level.

2.5 EPQ/ EPQ-Type Models with Quantity Discounts

Regarding inventory models with quantity discounts, most research work has focused on all-units and incremental quantity discounts. An overview of the quantity discounts research is presented by Benton and Park (1996). Munson and Rosenblatt (1998) presented an exploratory study of 39 companies and their different discount strategies in practice. They found that 95% of the companies they studied either offer or receive some type of all-units quantity discounts. In addition, 37% of these firms offer or receive incremental quantity discounts. Hu and Munson (2002) presented a heuristic for incremental quantity discounts with constant demand over a finite horizon. Hu et al. (2004) suggested a modification of the classical Silver-Meal heuristic under the incremental quantity discount case to improve the results presented by Hu and Munson (2002). Mendoza, Ventura (2008) developed an Economic Order Quantity (EOQ) model with two modes of transportation, namely truckload and less than truckload carriers, by introducing all-units and incremental quantity discount structures into the analysis.

CHAPTER 3

OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST AND RETROACTIVE HOLDING COST

The objective of this chapter is to develop mathematical models and find the optimal solution procedures that maximize the total profit of production-inventory systems (EPQ) with stock-dependent demand rate, finite production rate and retroactive holding cost. The demand rate for stock-level dependent demand is affected by the inventory decision made, unlike the traditional inventory models with constant demand. Maximizing profits will lead to a higher demand rate by keeping higher inventory levels. It is more realistic to allow inventory-production system to have positive inventory at the end of the order cycle that will incur higher holding cost but will allow us to have more inventory level, realize a higher subsequent demand rate and potentially receive a greater profit.

3.1 Assumptions and Notations

These are the general assumptions for this problem and the subsequent three models:

1. A single item with an infinite planning horizon is considered.

2. The holding cost is an increasing step function of storage duration ($h_1 < h_2 < \ldots < h_n$).

3. Orders of lot size S are produced gradually in t_1 periods at a constant rate P.

4. The units do not lose value during storage (no item deterioration).

5. Shortages are not allowed.

6. The quality of the prodcution is perferct.

7. The demand rate R is an increasing power function of the inventory level q, given as:

$$R(q) = Dq^{\beta}, \qquad D > 0, \qquad 0 < \beta < 1, \qquad q \geq 0 \tag{1}$$

The used notations to develop the all the four models in this thesis are identified as follows:

D = constant (base) demand rate

P = production rate during the first phase of the cycle ($0 \leq t \leq t_1$)

α $= D/P$

n = number of distinct time periods with different holding cost rates

t = time from the start of the cycle at $t = 0$

t_1 = end time of the first (uptime) phase of the cycle

τ_i = end time of holding-cost interval i, where $i = 1, 2, \ldots n$, $\tau_0 = 0$, and $\tau_n = \infty$

K = ordering cost per order

$h(t)$ =holding cost of the item at time t, $h(t) = h_i$ if $\tau_{i-1} \leq t \leq \tau_i = (ic)$

T = cycle time, i.e. time between producing two consecutive orders of size S

β = demand elasticity rate in relation to the inventory level

Q = maximum inventory level, corresponding to time $t = t_1$

L = ending inventory level, corresponding to time $t = 0$ *and* T

S = production lot size = Pt_1

γ = gross profit per unit of fresh item

η = gross profit per unit of item remaining at the end of the order cycle

$q(t)$ = quantity on-hand (inventory level) at time t

δ = sales prince

γ = gross profit per unit = (sales prince – purchase cost)= (δ-c)

3.2 The Production-Inventory Model:

The inventory level variation over time $q(t)$ during a typical cycle is divided into two phases: uptime phase and downtime phase. During the first (uptime) phase of the cycle, a new order is produced at a constant rate P while the inventory is consumed at the stock-dependent demand rate of Dq^{β}. Hence, the rate of change in the inventory level is expressed as follows:

$$\frac{dq(t)}{dt} = P - D[q(t)]^{\beta} \qquad 0 \le t \le t_1, \qquad P > D \tag{2}$$

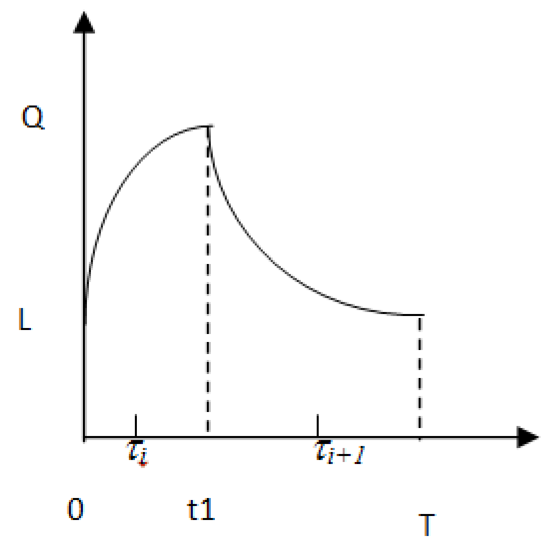

Figure 4: General EPQ model with stock-dependent demand

During the second (downtime) phase of the cycle, the rate of change (decrease) in the inventory level is equal to the demand rate, which is given by:

$$\frac{dq(t)}{dt} = -D[q(t)]^{\beta}\ , \qquad t_1 \le t \le T, \tag{3}$$

<u>During Uptime:</u>

$$\frac{dq(t)}{dt} = P - Dq(t)^{\beta}, \qquad 0 \le t \le t_1, \qquad P > D$$

$$\frac{d^2q(t)}{dt^2} = -\beta Dq(t)^{\beta-1}q'(t) = -\beta Dq(t)^{\beta-1}[P - Dq(t)^{\beta}]$$

Since

$$\alpha Q^{\beta} < 1$$

Where

$$\alpha = D/P$$

$$Q = \max\{q(t)\}$$

Then

$$\frac{D}{P} q(t)^{\beta} < 1$$

or $$P - Dq(t)^{\beta} > 0$$

Therefore, considering $\frac{d^2 q(t)}{dt^2}$ during the uptime phase:

$$\frac{d^2 q(t)}{dt^2} < 0$$ so, $q(t)$ is concave for $0 \le t \le t_1$

<u>During downtime:</u>

$$\frac{dq(t)}{dt} = -D[q(t)]^{\beta} \;, \qquad t_1 \le t \le T,$$

$$\frac{d^2 q(t)}{dt^2} = -Dq(t)^{\beta-1} q'(t) = -\beta D q(t)^{\beta-1}[-Dq(t)^{\beta}]$$
$$= \beta D^2 q(t)^{2\beta-1}$$

Therefore,

$$\frac{d^2 q(t)}{dt^2} > 0$$ so, $q(t)$ is convex for $t_1 \le t \le T,$

The net profit per unit time consists of three components: gross profit for the fresh and older product less the ordering cost and the holding cost. Since one order at a cost K is made in each cycle, the ordering cost per cycle is simply K. The holding cost per cycle is obtained by integrating the product of the holding cost $h(t)$ and the inventory level $q(t)$ for the whole cycle. However, the true objective is to maximize the total profit per unit time π, which is obtained by dividing the total profit per cycle by the cycle time T. Therefore, the total profit per unit time π is given by:

$$\pi = \frac{\gamma[Pt_1]}{T} - \frac{K}{T} - \frac{\int_0^{t_1} h(t)q(t)dt}{T} - \frac{\int_{t_1}^{T} h(t)q(t)dt}{T} \quad (4)$$

During the uptime phase of the cycle ($0 \leq t \leq t_1$), rearranging the ordinary differential equation (2) results in the following:

$$\frac{d\,q(t)}{dt} = P - Dq(t)^{\beta}$$

$$d\,q(t) = Pdt[1 - \frac{D}{P}q(t)^{\beta}], \text{ let } \alpha = D/P$$

$$d\,q(t) = Pdt[1 - \alpha q(t)^{\beta}]$$

$$\frac{1}{(1 - \alpha q(t))} dq = Pdt$$

After integrating the ordinary differential equation we get:

$$\int_0^t \frac{1}{1 - \alpha q(t)^{\beta}} dq = P \int_0^t dt \quad (5)$$

The left-hand side of (5) can be integrated to yield the hyper-geometric function ${}_2F_1$.

$$\int \frac{1}{1 - \alpha q^{\beta}} dq = q \times \left[{}_2F_1(1, \frac{1}{\beta}; 1 + \frac{1}{\beta}; \alpha q^{\beta}) \right]$$

Where

$$ {}_2F_1(1,\frac{1}{\beta};1+\frac{1}{\beta};\alpha q^{\beta}) = \sum_{n=0}^{\infty}\frac{(1)_n(1/\beta)_n}{(1+1/\beta)_n}\frac{(\alpha q^{\beta})^n}{n!} $$

and

$(a)_0 \quad = 1$

$(a)_n \quad = a(a+1)\,(a+2)\,\ldots\,(a+n-1)$

Then, the right hand side of (5), can be writen as:

$$ q\times {}_2F_1(1,\frac{1}{\beta};1+\frac{1}{\beta};\alpha q^{\beta}) = q\sum_{n=0}^{\infty}\frac{(n!)(\frac{1}{\beta})(\frac{1}{\beta}+1)\ldots(\frac{1}{\beta}+n-1)}{(\frac{1}{\beta}+1)\ldots(\frac{1}{\beta}+n-1)(\frac{1}{\beta}+n)}\frac{(\alpha q^{\beta})^n}{n!} = q\sum_{n=0}^{\infty}\frac{\frac{1}{\beta}(\alpha q^{\beta})^n}{(\frac{1}{\beta}+n)} $$

$$ = q\sum_{n=0}^{\infty}\frac{(\alpha q^{\beta})^n}{(1+n\beta)} = \sum_{n=0}^{\infty}\frac{\alpha^n q^{n\beta+1}}{(1+n\beta)} $$

Therefore, (5) can be written as follows:

$$ \sum_{n=0}^{\infty}\frac{\alpha^n q^{n\beta+1}}{(1+n\beta)}\bigg|_0^t = Pt \qquad (6) $$

Since $q(0)$ = q(T)= L while $q(t_1) = Q$, integrating over the range [0, t_1] gives:

$$ t_1P = \sum_{n=0}^{\infty}\frac{\alpha^n Q^{n\beta+1}}{(n\beta+1)} - \sum_{n=0}^{\infty}\frac{\alpha^n L^{n\beta+1}}{(n\beta+1)} = \sum_{n=0}^{\infty}\frac{\alpha^n(Q^{n\beta+1}-L^{n\beta+1})}{(n\beta+1)}, $$

$$ t_1 = \sum_{n=0}^{\infty}\frac{\alpha^n(Q^{n\beta+1}-L^{n\beta+1})}{P(n\beta+1)} \qquad (7) $$

In order to obtain an expression for $q(t)$ during the uptime phase, (2) is rearranged differently from (5) and integrated as follows:

$$ \int_a^b q(t)dt = \frac{1}{P}\int_a^b \frac{q(t)}{1-\alpha q(t)^{\beta}}dq \qquad (8) $$

The right-hand side of (8) again integrates to the hyper-geometric function ${}_2F_1$.

$$\int \frac{q}{1-\alpha q^{\beta}}dq = \frac{q^2}{2} \times {}_2F_1(1,\frac{2}{\beta};1+\frac{2}{\beta};\alpha q^{\beta})$$

After simplification, "uptime phase"(8) can be written as:

$$\int_a^b q(t)dt = \sum_{n=0}^{\infty} \frac{\alpha^n [q(b)^{n\beta+2} - q(a)^{n\beta+2}]}{P(n\beta+2)} = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{(2+n\beta)} - L^{(2+n\beta)})}{P(2+n\beta)} \tag{9}$$

During the downtime phase of the cycle ($t_1 \le t \le T$):

$$\int_a^b q(t)dt = \frac{q(a)^{2-\beta} - q(b)^{2-\beta}}{D(2-\beta)} = \frac{Q^{2-\beta} - L^{2-\beta}}{D(2-\beta)} \tag{10}$$

$$T = t_1 + \frac{Q^{1-\beta} - L^{1-\beta}}{D(1-\beta)} \tag{11}$$

To add a constraint that governs the relation between Q and L from (7):

$$t_1 = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+1} - L^{n\beta+1})}{P(n\beta+1)} > 0,$$

$\sum_{n=0}^{\infty} \alpha^n \left(Q^{n\beta+1} - L^{n\beta+1}\right) > 0$, after simplification:

$$\sum_{n=0}^{\infty} \alpha^n \left(Q^{n\beta+1}\right) > \sum_{n=0}^{\infty} \alpha^n \left(L^{n\beta+1}\right) \tag{12}$$

$$\alpha Q^{\beta} < 1,\ \ Q \ge 0, L \ge 0 \tag{13}$$

From the optimum values of *Q and L*, the following quantities can be calculated:

$$\int_0^T q(t)dt = \frac{Q^{2-\beta} - L^{2-\beta}}{D(2-\beta)} + \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+2} - L^{n\beta+2})}{P(n\beta+2)} \tag{14}$$

$$S = Pt_1 = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+1} - L^{n\beta+1})}{(n\beta + 1)} \qquad (15)$$

3.3 Solution Algorithm of Retroactive Holding Cost

Assuming a retroactive holding cost, only the holding cost of the last storage interval is used retroactively for *all* earlier intervals. Assuming that the cycle ends in interval *e*, ($\tau_{e-1} \leq T \leq \tau_e$), then the rate h_e is applied to holding cost intervals 1, 2, ..., *e*. Therefore, the total profit per unit time π is given by:

$$\pi = \frac{\gamma(Pt_1)}{T} - \frac{K}{T} - \frac{h_i}{T}\int_0^T q(t)dt \qquad \tau_{i-1} \leq T \leq \tau_i \qquad (16)$$

The optimum solution can be determined by using the following steps:

1. Starting with the lowest holding cost h_1, solve NLP1 (with equation (16) as objective function, (12- 13) as constraints) with *Q* and *L as* decision variable, and then solve equation (7) to determine the value of t_1, and finally solve equation (11) to find the value of T for each h_i until Q and L are realizable (i.e., $\tau_{i-1} \leq T \leq \tau_i$). Call these values T_R , L_R and Q_R.
2. Calculate all break-point values of Q_i *and* L_i, by adding "$T = \tau_I$ "that is in (11) to NLP1 as an additional constraint and then solve for *Q and L*.
3. For all Q_R, Q_i , L_R and L_i , use (14) to calculate $\int_0^T q(t)dt$, and then substitute the result using the appropriate h_i into (16) to calculate the total profit π.
4. Choose the value of *Q* that gives the maximum total profit, and then use (15) to calculate the corresponding production lot size *S*.

3.4 Numeric Example

Given the following parameters:

D = 400 units per year, P = 1,000 units per year, K = \$300 per order

$\beta = 0.1$, h_1 = \$6/unit/year, h_2 = \$8/unit/year, h_3 = \$10/unit/year

τ_1 = 0.3 years, τ_2 = 0.6 years, $\tau_3 = \infty$, $\gamma = \$20/unit$,

The solution algorithm is implemented in the following steps.

<u>Step 1</u>

Substituting $h_1 = 6$ (for $0 < T \leq 0.3$), solving (NLP1) gives:

Q = 425.321 and L = 103.567, $t1$ = 0.545162, T= 1.00925, π= 12,102 (not realizable).

Substituting $h_2 = 8$ (for $0.3 < T \leq 0.6$), solving (NLP1) gives:

Q = 332.684 and L = 66.1937, t1=0.8238, T = 1.21996, with π= 11,637.1 (not realizable).

Substituting $h_3 = 10$ (for $T > 0.6$), solving (NLP1) gives:

Q = 276.12 and L = 46.100, t1=0.679742, T = 1.002964, with π= 11,273.4 (realizable)

<u>Step 2</u>

Setting $T = \tau_1 = 0.3$ with h=6, solving the NLP1 model gives: Q_1 = 12.231 and L_1=0, π=3,228.21.

Setting $T = \tau_2 = 0.6$ with h=8, solving the NLP1 model gives: Q_2 = 17.2986 and L_2= 0.02, π= 5,291.36.

Step 3

Q_R = 276.12, L_R=46.1 and h_3 = 10: π= $ 11,273.4

Q_1 = 12.231, L_1=0 and h_1 = 6: π=3,228.21

Q_2 = 17.298, L_2= 0.02 and h_2 = 8: π= 5,291.36

Step 4

The maximum profit (π= 11,273.4/year) is obtained with Q = 276.12. Using (15), the corresponding lot size is calculated as: S = 338. Therefore, the optimum inventory policy is obtained with cycle time T = 1.002964 year and lot size S = 679.742.

3.5 Sensitivity Analysis

The sensitivity analysis is carried out for this model by increasing the value of each parameter by 20% and resolving the model and then decreasing the value by 20%.

Table 1 Sensitivity Analysis Results of Model 1

Parameter	*Original Value*	*New Values*	*Q**	*L**	*T*	*t1*	*Π*
D	400	480	291.150	73.584	1.571	1.130	14,019.5
		320	247.605	27.062	0.8862	0.45830	8,629.9
		1200	293.748	40.816	0.85456	0.47056	11,184.2

P	1,000	800	242.55	58.13	1.603	1.3215	11,428.3
K	300	360	287.38	42.65	1.09778	0.7260	11,217
		240	263.54	50.28	0.9524	0.6276	11,333.9
β	0.1	0.12	323.97	83.39	1.3075	0.9869	12,768.8
		0.08	233.28	22.64	0.8695	0.5076	10,041.46
γ	20	24	310.04	66.13	0.7289	0.3645	13,937.3
		16	242.47	28.32	0.9429	0.6103	8,658
$h_{1,2 \text{ and } 3}$	6,8 and 10	7.2, 9.6 & 12	237.78	33.93	0.898	0.5822	10,972.95
		4.8, 6.4 & 8	332.68	66.19	1.2199	0.82382	11,637.1

From the above table, we can observe the effect of the demand while the remaining parameters have less effect to the overall total profit.

CHAPTER 4

OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST AND INCREMENTAL HOLDING COST

In this chapter, a mathematical models is developed to find the optimal solution that maximize the total profit of production-inventory systems (EPQ) with stock-dependent demand rate, finite production rate with incremental holding cost. The main difference between this model and the pervious one is the nature of holding cost as there different holding cost used for each storage interval un-like the retroactive holding cost.

4.1 Production-Inventory Model

An incrementally increasing holding cost means that a different holding cost h_i applies to each storage interval i ($\tau_{i-1} \leq t \leq \tau_i$). The objective, which is to maximize the total profit per unit time, is expressed as follows:

$$\text{Max } \pi = \frac{\gamma(Pt_1)}{T} - \frac{K}{T} - \frac{1}{T}\sum_{i=1}^{e} h_i \int_{\tau_{i-1}}^{\tau_i} q(t)dt \qquad (17)$$

A given interval $[\tau_{i-1}, \tau_i]$ may fall entirely in the first phase (before t_1), entirely in the second phase(after t_1), or straddle both phases. Therefore, depending on the relationship between end point $[\tau_{i-1}, \tau_i]$ and t_1, there are three possibile procedures for calculating the integrals in (17). If the interval falls entirely in the first phase ($\tau_{i-1} < \tau_i \le t_1$), then we use (9), replacing a and b by τ_{i-1} and τ_i. If it falls entirely in the second phase ($t_1 \le \tau_{i-1} < \tau_i$), then we use (10), replacing a and b by τ_{i-1} and τ_i. However, if the interval straddles both phases ($\tau_{i-1} \le t_1 < \tau_i$), then we have to evaluate the integral over $[\tau_{i-1}, \tau_i]$ using a combination of (9) and (10) as follows:

$$\int_{\tau_{i-1}}^{\tau_i} q(t)dt = \sum_{n=0}^{\infty}\frac{\alpha^n[Q^{n\beta+2} - q(\tau_{i-1})^{n\beta+2}]}{P(n\beta+2)} + \frac{Q^{2-\beta} - q(\tau_i)^{2-\beta}}{D(2-\beta)}, \quad \tau_{i-1} \le t_1 < \tau_I \qquad (18)$$

The inventory level $q(t)$ is determined from (2) during the uptime phase and from (3) during the downtime phase. Therefore, the functions $q(\tau_i)$ and $q(\tau_{i-1})$ are determined from two different expressions in the two phases.

$\underline{0 \le t \le t_1}$

$$\frac{dq(t)}{dt} = P - D[q(t)]^{\beta}, \qquad D > 0, \qquad 0 \le t \le t_1, \qquad 0 < \beta < 1$$

Therefore, after integraing the above equation and simplyfied , it can be written as follows:

$$\sum_{n=0}^{\infty}\frac{\alpha^n q^{n\beta+1}}{(1+n\beta)}\bigg|_0^t = Pt$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n q(t)^{n\beta+1}}{(1+n\beta)} - \sum_{n=0}^{\infty}\frac{\alpha^n q(0)^{n\beta+1}}{(1+n\beta)} = Pt$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n q(t)^{n\beta+1}}{(1+n\beta)}-\sum_{n=0}^{\infty}\frac{\alpha^n L^{n\beta+1}}{(1+n\beta)}=Pt$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n\left(q(t)^{n\beta+1}-L^{n\beta+1}\right)}{(n\beta+1)}=Pt\,, \qquad 0 \le \tau_i \le t_1 \qquad (19)$$

$t_1 \le t \le T$

The on-hand inventory level at time t, $q(t)$, can be evaluated by solving (3):

$$q^{-\beta}\,dq = -D\,dt$$

by integrating both sides:

$$\int_{t_1}^{t} q^{-\beta}\,dq = \int_{t_1}^{t} -D\,dt\,, \text{ where } t_1 \le t \le T$$

$$\left.\frac{q^{1-\beta}}{(1-\beta)}\right|_{t_1}^{t} = -D(t-t_1)$$

$$q^{1-\beta}(t) - q^{1-\beta}(t_1) = -D(1-\beta)(t-t_1)$$

$$q^{1-\beta}(t) = q^{1-\beta}(t_1) - D(1-\beta)(t-t_1)$$

However, since $q(t_1) = Q$, *then*

$$q^{1-\beta}(t_1) = Q^{1-\beta}$$

Thus;

$$q^{1-\beta}(t) = Q^{1-\beta} - D(1-\beta)(t-t_1)$$

$$q(t) = [Q^{1-\beta} - D(1-\beta)(t-t_1)]^{1/(1-\beta)} \qquad t_1 \le \tau_i \le T \qquad (20)$$

If the current interval i is the last interval in the cycle (i.e. $i = e$), then the interval's end point $\tau_i = \tau_e$ has to be replaced by the cycle time T. Therefore, it should be assumed that τ_i

$= T$ and $q(\tau_i) = L$ for the last interval i in (17). Furthermore, the cycle time T must fall in the last interval e.

$$\tau_{e-1} \leq T \leq \tau_e$$

4.2 Solution Algorithm

Since the objective function (17) is constrained, direct optimization by differential calculus is not feasible. For the incremental holding cost case, the optimum solution is obtained by nonlinear programming according to the following steps:

1. Substitute the minimum and maximum values of h_i into NLP1 to determine the range of values for Q and L, and then use (7) and (11) to determine the corresponding range of t_1 and T.
2. For each possible combination of t_1 and T, formulate a nonlinear programming (NLP2) model whose objective function is (17), decision variable is Q, L *and* T, and constraints are (7), (11), and (19)-(20). In each NLP model, use the applicable terms for each interval $[\tau_{i-1}, \tau_i]$ in the objective and constraints.
3. For each combination of t_1 and T, solve the corresponding NLP model to find the optimum solution.
4. Choose the feasible solution with the maximum total profit π.

4.3 Numeric Example

Resolve Example 1 assuming the holding cost increases incrementally. The solution algorithm is implemented in the following steps.

Step 1

For h_1 = 6 ($0 < T \leq 0.3$), (12), (7), and (11) give Q = 425.321, L= 103.567, t_1 = 0.545162, and T = 1.00925.

For h_3 = 10 ($T > 0.6$), (12), (7), and (11) give Q = 276.12, L= 46.100, t_1 = 0.679742, and T = 1.002964.

Clearly T falls in the third interval. However, the results indicate that t_1 may fall either in the second or the third interval.

Step 2 (a)

Assuming t_1 is located in the second interval while T is located in the third interval, then e =3. The NLP model is formulated as follows:

$$Max\pi = \frac{\gamma(Pt_1)}{T} - \frac{K}{T} - \frac{h_1}{T}\sum_{n=0}^{\infty}\frac{\alpha^n\left[q(0.3)^{n\beta+2} - q(0)^{n\beta+2}\right]}{P(n\beta+2)}$$
$$- \frac{h_2}{T}\left(\sum_{n=0}^{\infty}\frac{\alpha^n[Q^{n\beta+2} - q(0.3)^{n\beta+2}]}{P(n\beta+2)} + \frac{Q^{2-\beta} - q(0.6)^{2-\beta}}{D(2-\beta)}\right) - \frac{h_3}{T}\left(\frac{q(0.6)^{2-\beta} - q(T)^{2-\beta}}{D(2-\beta)}\right)$$

Subject to:

$$\sum_{n=0}^{\infty}\frac{\alpha^n\left(q(0.3)^{n\beta+1} - L^{n\beta+1}\right)}{(n\beta+1)} = P(0.3), \qquad 0 \leq \tau_i \leq t_1$$

$$t_1 = \sum_{n=0}^{\infty}\frac{\alpha^n\left[Q^{n\beta+1} - L^{n\beta+1}\right]}{P(n\beta+1)}$$

$$T = t_1 + \frac{Q^{1-\beta} - L^{1-\beta}}{D(1-\beta)}$$

$$q(0.6) = [Q^{1-\beta} - D(1-\beta)(0.6 - t_1)]^{\frac{1}{1-\beta}} \quad , \qquad 0 \le \tau_i \le t_1$$

$$T > 0.6$$

Step 2 (b)

Assuming both t_1 and T is in the third interval, then e = 3. The NLP model is formulated as follows:

$$Max\pi = \frac{\gamma(Pt_1)}{T} - \frac{K}{T} - \frac{h_1}{T}\sum_{n=0}^{\infty}\frac{\alpha^n\left[q(0.3)^{n\beta+2} - q(0)^{n\beta+2}\right]}{P(n\beta+2)}$$
$$- \frac{h_2}{T}\left(\sum_{n=0}^{\infty}\frac{\alpha^n[q(0.6)^{n\beta+2} - q(0.3)^{n\beta+2}]}{P(n\beta+2)}\right) - \frac{h_3}{T}\left(\sum_{n=0}^{\infty}\frac{\alpha^n[q(t_1)^{n\beta+2} - q(0.6)^{n\beta+2}]}{P(n\beta+2)} + \frac{q(t_1)^{2-\beta} - q(T)^{2-\beta}}{D(2-\beta)}\right)$$

This objective is maximized subject to constraints:

$$\sum_{n=0}^{\infty}\frac{\alpha^n\left(q(0.6)^{n\beta+1} - L^{n\beta+1}\right)}{(n\beta+1)} = P(0.6), \qquad 0 \le \tau_i \le t_1$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n\left(q(0.3)^{n\beta+1} - L^{n\beta+1}\right)}{(n\beta+1)} = P(0.3), \qquad 0 \le \tau_i \le t_1$$

$$t_1 = \sum_{n=0}^{\infty}\frac{\alpha^n\left[Q^{n\beta+1} - L^{n\beta+1}\right]}{P(n\beta+1)}$$

$$T = t_1 + \frac{Q^{1-\beta} - L^{1-\beta}}{D(1-\beta)}$$

$$T > 0.6$$

Step 3

The optimum solution of the NLP in step 2(a) is:

Q = 243.482, L= 86.4519, t_1 = 0.468867, T = 0.70513, π = \$11,277.9

The optimum solution of the NLP in step 2(b) is:

Q = 286.174, L= 90.3833, t_1 = 0.6, T = 0.890842, π = \$11,586.0*.

Step 4

The optimum solution of Example 2 for incremental holding cost is:

Q = 286.174, L= 90.3833, t_1 = 0.6, T = 0.890842, π = \$11,586.0*.

4.4 Sensitivity Analysis

The same concept that was carried out to make the sensitivity analysis is used for this model by increasing and decreasing the parameters by 20% and then re-solving the model.

Table 2 Sensitivity Analysis Results of Model 2

Parameter	***Original Value***	***New Values***	***Q****	***L****	***T***	***t1***	***π***
D	*400*	*480*	268.14	158.74	0.733	0.6	14,308.29
		320	314.126	32.712	1.133	0.6	8,829.41
P	*1,000*	*1200*	366.42	53.168	1.0645	0.6	11,455.87
		800	Not feasible				
K	*300*	*360*	285.416	89.429	0.89	0.6	11,518.61
		240	286.93	91.3399	0.890	0.6	11,653.32
		0.12	Not feasible				

β	*0.1*	*0.08*	Not feasible				
γ	*20*	*24*	Not feasible				
		16	258.2899	54.38	0.9	0.6	8,919.76
$h_{1,2\ and\ 3}$	*6,8 and 10*	*7.2, 9.6 & 12*	208.362	67.167	0.623	0.41	10,988.11
		4.8, 6.4 & 8	326.096	139.389	0.87	0.6	11,926.57

We can observe the scenarios where there is no feasible solution as the constraints are not met and the total profit per unit time function is not continuous as the holding cost is incremental.

CHAPTER 5

OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST, RETROACTIVE HOLDING COST AND ALL UNIT DISCOUNT

In this chapter, we introduce the concept of all unit discounts where the sellers tend to give to encourage buying in big batches. The main objective of this chapter is to develop a mathematical model with optimization solution procedure of EPQ with stock-dependent demand rate, finite production rate and variable holding cost retroactive holding cost with all-units discounts.

5.1 Production-inventory Model

The inventory level variation over time $q(t)$ during a typical cycle is divided into two phases as well: uptime and downtime. During the first (uptime) phase of the cycle, a new order is produced at a constant rate P while the inventory is consumed at the stock-dependent demand rate of Dq^{β}. During the second (downtime) phase of the cycle, the rate of change (decrease) in the inventory level is equal to the demand rate.

The net profit per unit time consists of four components: gross profit for the fresh product less the procurement cost, ordering cost and the holding cost. In this model, the holding cost is defined as the production of interest rate by the purchase cost (i*c) and the gross profit per unit γ equals to (δ-c) which is (sales prince – purchase cost).

The total profit per unit time is as follows:

$$\pi = \frac{(\delta - c_j)[Pt_1]}{T} - \frac{K}{T} - \frac{ic_j \int_0^{t_1} q(t)dt}{T} - \frac{ic_j \int_{t_1}^{T} q(t)dt}{T} \tag{21}$$

The same concept and derivation used in section (3.2) is utilized to develop t1, $\int_a^b q(t)$ uptime, $\int_a^b q(t)$ down time and T, consequently.

$$t_1 = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+1} - L^{n\beta+1})}{P(n\beta+1)} \tag{22}$$

$$\int_a^b q(t)dt = \sum_{n=0}^{\infty} \frac{\alpha^n [q(b)^{n\beta+2} - q(a)^{n\beta+2}]}{P(n\beta+2)} = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{(2+n\beta)} - L^{(2+n\beta)})}{P(2+n\beta)} \tag{23}$$

$$\int_a^b q(t)dt = \frac{q(a)^{2-\beta} - q(b)^{2-\beta}}{D(2-\beta)} = \frac{Q^{2-\beta} - L^{2-\beta}}{D(2-\beta)} \tag{24}$$

$$T = t_1 + \frac{Q^{1-\beta} - L^{1-\beta}}{D(1-\beta)} \tag{25}$$

From the optimum value of Q, the following quantities can be calculated:

$$\int_0^T q(t)dt = \frac{Q^{2-\beta} - L^{2-\beta}}{D(2-\beta)} + \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+2} - L^{n\beta+2})}{P(n\beta+2)} \tag{26}$$

$$S = Pt_1 = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{n\beta+1} - L^{n\beta+1})}{(n\beta+1)} \tag{27}$$

The relationship between Q and L is described in the following constraint:

$$\sum_{n=0}^{\infty} \alpha^n \left(Q^{n\beta+1}\right) > \sum_{n=0}^{\infty} \alpha^n \left(L^{n\beta+1}\right) \qquad (28)$$

5.2 Solution Algorithm of Retroactive Holding Cost with All Unit Discount

Assuming a retroactive holding cost, only the holding cost of the last storage interval is used retroactively for *all* earlier intervals. Assuming that the cycle ends in interval *e*, ($\tau_{e-1} \leq T \leq \tau_e$), then the rate h_e is applied to holding cost intervals 1, 2, ..., *e*. Furthermore, the discount is applied to all units in an order (all unit discounts). The optimum solution is obtained by nonlinear programming according to the following steps:

1. Starting with the lowest holding cost*;* (*i*c*), use NLP3 (with equation (21) as objective function, (28 and non-negativity) as constraints) to determine *Q* and L, and use (25) to determine *T* for each *possible combination of* i_j *and* c_j until *Q and T* are realizable (i.e., $\tau_{j-1} \leq T \leq \tau_j$) and (i.e., $\sigma_{j-1} \leq Q \leq \sigma_j$). Call these values Q_R , L_R *and* T_R.
2. Staring with the realizable points from step 1, calculate all break-point values of *Q and L*, by setting $Q_i = Q(\tau_i)$ and solve NLP3 for L and then find the value of T from (25). Then, set the border value of T and then re-solve NLP3 for Q and L.
3. For all Q_R and Q_i, use (26) to calculate $\int_0^T q(t)dt$, and then substitute the result using the appropriate i_j *and* c_j into (21) to calculate the total profit π.

4. Choose the value of Q that gives the maximum total profit, and then use (27) to calculate the corresponding production lot size S.

5.3 Numeric Example

Given the following parameters:

D = 400 units per year, P = 1,000 units per year, K = \$300 per order

$\beta = 0.1$, $i_1 = 12\%$, $i_2 = 16\%$, $i_3 = 20\%$, $c_1 = 50$, $c_2 = 45$, $c_3 = 40$

h_1 = 0.12(50) = \$6/unit/year, h_2 = 0.16(45) = \$8/unit/year, h_3 = 0.20(40) = \$10/unit/year

τ_1 = 0.3 years, τ_2 = 0.6 years, $\tau_3 = \infty$

$\sigma_1 = 300$, $\sigma_2 = 500$ and $\delta = \$70/unit$,

The solution procedure is implemented in the following steps.

<u>Step 1</u>

	c1=50 (0<=Q<300)	c2=45 (300<=Q<500)	c3=40 (Q>=500)
i1= 0.12 (0<=T<0.3)	Q=424.134, L=103.2, T=1.52015 t1=1.0571 (not realizable)	Q=544.754, L=177.112, T=1.82335 t1=1.31063 (not realizable)	Q=695.79, L=278.215, T=2.18229 t1=1.61803 (not realizable)
i2= 0.16 (0.3<=T<0.6)	Q=332.151, L=66.0562, T=1.2179 t1=0.822303 (not realizable)	Q=422.611, L=116.908, T= 1.4528 t1=1.01332 (not realizable)	Q=536.212, L=187.607, T=1.73003 t1=1.24428 (not realizable)
i3= 0.20 (T>=0.6)	Q=275.833, L=46.0374, T=1.02853 t1=0.678933	Q=348.165, L=83.7566, T=1.22194 t1=0.832718	Q=439.118, L=136.912, T=1.44951 t1=1.01828

	π= $ 11,272.3 (realizable)	π= $ 14,816.4 (realizable)	(not realizable)

Step 2

Break points:

	c1=50 (0<=Q<300)	c2=45 (300<=Q<500)	c3=40 (Q>=500)
i1= 0.12 (0<=T<0.3)	T=0.3 "as constraint" π=11,395.6, Q=262.901, L= 198.557	T=0.3"as constraint" π=15,071.8, Q=361.213, L= 299.863	
i2= 0.16 (0.3<=T<0.6)	T=0.6 "as constraint" π=11,477.9, Q=241.305, L=108.525,	T=0.6 "as constraint" π=15,050.7, Q= 308.961, L =181.172,	
i3= 0.20 , (T>=0.6)	Q=300, C=45, i=0.2 "as constraint" π=14,795.65 L = 81.64, T=0.995	Q=500, C=40,i=0.2, "as constraint", π=18,511.96, L = 134.193, T=1.776	

Step 3

Q_R = 275.833, L_R=46.0374 , i = 0.2, c=50 with π_R= $ 11,272.3

Q_R = 348.165, L_R=83.7566 , i = 0.2, c=45 with π_R= $ 14,816.4

Q_1=262.901, L_1= 198.557, π_1=11,395.6

Q_2=361.213, L_2= 299.863, π_2=15,071.8

Q_3=241.305, L_3=108.525, π_3=11,477.9

Q_4= 308.961, L_4 =181.172, π_4=15,050.7

Q_5=300.000, L_5=81.64, π_5=14,795.65

Q_6=500.000,L_6=134.193, π_6= 18,511.96

Step 4

The maximum profit (π= 18,511.96/year) is obtained with Q = 500. Therefore, the optimum inventory policy is:

Cycle time T = 1.776 year

5.4 Sensitivity Analysis

The same concept that was carried out to make the sensitivity analysis in the previous models, is used for this model by increasing and decreasing the parameters by 20% and then re-solving the model.

Table 3 Sensitivity Analysis Results of Model 3

Parameter	***Original Value***	***New Values***	***Q****	***L****	***T***	***π***
D	*400*	*480*	500	176.25	2.696	22,693.2
		320	500	82.748	1.697	14,267.6
P	*1,000*	*1200*	500	125.54	1.291	18,452.0
		800	500	119.29	4.908	18,361.5
K	*300*	*360*	500	129.27	1.799	18,478.4
		240	500	139.56	1.754	18,546.0
β	*0.1*	*0.12*	500	220.25	1.826	21,145.1
		0.08	500	66.055	1.853	16,250.8
δ	*70*	*84*	589.83	250.564	1.726	28,568.7
		56	500	28.433	2.226	8,717.16

$i_{1,2 \text{ and } 3}$	*12, 16 and 20*	*14.4, 19.2 and 24*	500	95.253	1.947	18,011.6
		9.6, 12.8 and 16	536.21	187.607	1.730	19,051.4
$c_{1,2 \text{ and } 3}$	*50, 45 and 40*	*60, 54 and 48*	500	43.995	2.163	12,425.2
		40, 36 and 32	644.45	270.688	1.931	24,875.9

Most of the optimal cases reported in this table are located at the break-points. If we compare the results in this model and the one reported in section 3.5 where there is no quantity discount, we will find that quantity discount will have positive impact to the total profile and will almost double the profit as that will encourage the buyer to raise the demand.

CHAPTER 6

OPTIMAL EPQ MODEL FOR STOCK-DEPENDENT DEMAND, VARIABLE HOLDING COST, INCREMENTAL HOLDING COST AND ALL UNIT DISCOUNT

In this chapter, all unit discounts along with incremental holding cost is introduced in developing the mathematical model that maximize the total profit of EPQ. The main difference between this model and the pervious one is the nature of holding cost as there different holding cost used for each storage interval un-like the retroactive holding cost.

6.1 Production-Inventory Model

The objective, which is to maximize the total profit per unit time, is expressed as follows:

$$\text{Max } \pi = \frac{(\delta - c_j)(Pt_1)}{T} - \frac{K}{T} - \frac{c_j}{T}\sum_{j=1}^{e} i_j \int_{\tau_{j-1}}^{\tau_j} q(t)dt \tag{29}$$

calculating the integrals in (29) may have three possibile procedures for soltuion as an interval [τ_{i-1}, τ_i] may fall entirely in the first phase (before t_1), entirely in the second phase(after t_1), or straddle both phases. If the interval falls entirely in the first phase (τ_{i-1} < $\tau_i \le t_1$), then we use follwing equation:

$$\int_{\tau_{j-1}}^{\tau_j} q(t)dt = \sum_{n=0}^{\infty} \frac{\alpha^n (Q^{(2+n\beta)} - L^{(2+n\beta)})}{P(2+n\beta)} \tag{30}$$

If it falls entirely in the second phase ($t_1 \le \tau_{i-1} < \tau_i$), then we use the follwing equation:

$$\int_{\tau_{j-1}}^{\tau_j} q(t)dt = \frac{Q^{2-\beta} - L^{2-\beta}}{D(2-\beta)} \tag{31}$$

However, if the interval straddles both phases ($\tau_{i-1} \le t_1 < \tau_i$), then we have to evaluate the integral over [τ_{i-1}, τ_i] using a combination of (30) and (31) as follows:

$$\int_{\tau_{i-1}}^{\tau_i} q(t)dt = \sum_{n=0}^{\infty} \frac{\alpha^n [Q^{n\beta+2} - q(\tau_{i-1})^{n\beta+2}]}{P(n\beta+2)} + \frac{Q^{2-\beta} - q(\tau_i)^{2-\beta}}{D(2-\beta)}, \quad \tau_{i-1} \le t_1 < \tau_I \tag{32}$$

By using the same concept and mathematical derivation descripted in section (4.1), the following two equations are used to find the q at the break points:

$$\sum_{n=0}^{\infty} \frac{\alpha^n \left(q(t)^{n\beta+1} - L^{n\beta+1}\right)}{(n\beta+1)} = Pt, \qquad 0 \le \tau_i \le t_1 \tag{33}$$

$$q(t) = [Q^{1-\beta} - D(1-\beta)(t-t_1)]^{\frac{1}{1-\beta}}, \qquad t_1 \le \tau_i \le T \tag{34}$$

$$Where,\ \tau_{e-1} \le T \le \tau_e \tag{35}$$

6.2 Solution Algorithm

Direct optimization of the objective function (29) by differential calculus is not feasible. For this case, the optimum solution is obtained by nonlinear programming according to the following steps:

1. Substitute the minimum and maximum values of the product *(i*c)* into (NLP3) to determine the range of values for Q and L, and then use (22) and (25) to determine the corresponding range of t_1 and T.
2. For each possible combination of t_1 and T, formulate a nonlinear programming (NLP4) model whose objective function is (29), decision variable is Q and L, and constraints are (22), (25), and (33)-(35). In each NLP model, use the applicable terms for each interval $[\tau_{i-1}, \tau_i]$ in the objective and constraints.
3. For each combination of t_1 and T, solve the corresponding NLP model to find the optimum solution.
4. Choose the feasible solution with the maximum total profit π.

6.3 Numeric Example

Resolve the same example in section (5.3) with the assumption that the holding cost increases incrementally.

The solution procedure is implemented in the following steps.

Step 1

For i_1=0.12, c_3 = 40 and ($0 < T \leq 0.3$), Q=695.79, L=278.215, T=2.18229, t1=1.61803

For i_3=0.20, c_1 = 50 and ($T > 0.6$), Q=275.833, L=46.0374, T=1.02853, t1=0.678933

Clearly *T and t_1 fall* in the third interval.

Step 2

When both t_1 and T are in the third interval, then e = 3. The NLP model is formulated as follows:

$$\pi = \frac{(\delta - c_3)(Pt_1)}{T} - \frac{K}{T}$$

$$- \frac{c_3}{T}\left(i_1 \left(\sum_{n=0}^{\infty} \frac{\alpha^n \left[q(0.3)^{n\beta+2} - q(0)^{n\beta+2} \right]}{P(n\beta + 2)} \right) \right.$$

$$+ i_2 \left(\sum_{n=0}^{\infty} \frac{\alpha^n \left[q(0.6)^{n\beta+2} - q(0.3)^{n\beta+2} \right]}{P(n\beta + 2)} \right)$$

$$\left. + i_3 \left(\sum_{n=0}^{\infty} \frac{\alpha^n \left[q(t_1)^{n\beta+2} - q(0.6)^{n\beta+2} \right]}{P(n\beta + 2)} + \frac{q(t_1)^{2-\beta} - q(T)^{2-\beta}}{D(2-\beta)} \right) \right)$$

This objective is maximized and subject to the follwing constraints:

$$\sum_{n=0}^{\infty} \frac{\alpha^n \left(q(0.6)^{n\beta+1} - L^{n\beta+1} \right)}{(n\beta + 1)} = P(0.6), \qquad 0 \le \tau_i \le t_1$$

$$\sum_{n=0}^{\infty} \frac{\alpha^n \left(q(0.3)^{n\beta+1} - L^{n\beta+1} \right)}{(n\beta + 1)} = P(0.3), \qquad 0 \le \tau_i \le t_1$$

$$t_1 = \sum_{n=0}^{\infty} \frac{\alpha^n \left[Q^{n\beta+1} - L^{n\beta+1} \right]}{P(n\beta + 1)}$$

$$T = t_1 + \frac{Q^{1-\beta} - L^{1-\beta}}{D(1-\beta)}$$

$$T > 0.6$$

Step 3

The optimum solution of the NLP in step 2 is:

Q = 447.794, L=280.305, t_1 = 0.599998, T = 0.830678, π = \$18,991.1.

<u>Step 4</u>

There is only one feasible solution and it is the optimum solution which is:

Q = 447.794, L=280.305, t_1 = 0.599998, T = 0.830678, π = \$18,991.1.

6.4 Sensitivity Analysis

The sensitivity analysis is carried out by increasing and decreasing the parameters by 20% and then re-solving the model.

Table 4 Sensitivity Analysis Results of Model 4

Parameter	***Original Value***	***New Values***	***Q****	***L****	***T***	***t1***	***π***
D	*400*	*480*	*Not Feasible*				
		320	407.79	143.69	1.072	0.6	14,637.6
P	*1,000*	*1200*	415.03	220.4	0.671	0.397	18,564.2
		800	339.45	280.61	0.682	0.6	18,807.8
K	*300*	*360*	446.47	278.81	0.831	0.6	18,918.9
		240	449.12	281.79	0.830	0.6	19,063.4
β	*0.1*	*0.12*	476.35	243.70	1.480	1.213	21,716.2
		0.08	361.772	134.07	0.967	0.6	16,652.4

δ	*70*	*84*	*Not feasible*				
		56	285.22	89.190	0.891	***0.6***	9,201.43
$i_{1,2 \text{ and } 3}$	*12, 16 and 20*	*14.4, 19.2 and 24*	385.06	208.77	0.848	0.6	18,571.8
		9.6, 12.8 and 16	*Not feasible*				
$c_{1,2 \text{ and } 3}$	*50, 45 and 40*	*60, 54 and 48*	267.93	109.82	0.720	0.486	12,685.4
		40, 36 and 32	705.19	562.19	0.783	0.6	25,548.9

There are two observations with this table, as few scenarios with no feasible solution and the total profit function is not continues, and the second observation is that the quantity discount encourage the buyers raise the demand and increase the total profit.

CHAPTER 7

CONCLUSION AND CONTRIBUTION

In this thesis, four different economic production quantity (EPQ) models with stock level dependent demand rate, variable holding cost “the new (higher) holding cost can be applied either retroactively to all storage periods, or incrementally to the new period only.” and quantity discount “All-unit quantity dicount” were developed with mathematical formulation and NLP programming with the results of one example that is used in the literature. Furthermore, sensitivity analyses of all the parameters in the EPQ models were conducted for the four models and extensive study on the effect of increasing and decreasing the value of the parameters by 20 percent over the total profit per unit time were conducted.

7.1 Major Contribution

After completing this thesis work, the major contributions added to the literature are as follows:

- Extended Al-Fares (2012) models “with single decision variable Q to multiple decision variables” with maximization objective function rather than minimizing total cost per unit time and allowing non-zero end inventory for each cycle.
- Introduced the “all unit quantity” discount to the last two EPQ models, and redeveloped the EPQ models with incremental and retroactive holding cost.

7.2 Further Extension and Research

There are several directions for future research can be considered. Some of these are:

- Assume the quality as not perfect, and introduce type I and II error.
- Introducing the incremental discount rather than the all unit discount.
- Introduce quality and inspection and figure out the effect of the overall supply chain.
- Allow shortages, where all the above models assume shortages are not allowed.
- Variable ordering cost
- Variable production rate P.

Appendix

The Mathimatica codes that have been used to find the solution of the NLP problem in example 1 are listed to this appendix.

The first model:

```
d=400;
P=1000;
β=0.1;
h=6;
K=300;
γ=20;
η=0;
α=d/P;
```

$$TP=(\gamma*(P*\sum_{n=0}^{\infty}\frac{\alpha^n*(Q^{n*\beta+1}-L^{n*\beta+1})}{P*(1+n*\beta)})-K-h*(\sum_{n=0}^{\infty}\frac{\alpha^n*(Q^{2+n*\beta}-L^{2+n*\beta})}{P*(2+n*\beta)}+(Q^{2-\beta}-L^{2-\beta})/(d*(2-\beta))))/(\sum_{n=0}^{\infty}\frac{\alpha^n*(Q^{n*\beta+1}-L^{n*\beta+1})}{P*(1+n*\beta)}+(Q^{1-\beta}-L^{1-\beta})/(d*(1-\beta)));\text{(*Total profit equation*)}$$

$$sol=NMaximize[\{TP,\sum_{n=0}^{\infty}\alpha^n*Q^{n*\beta+1}>=\sum_{n=0}^{\infty}\alpha^n*L^{n*\beta+1}\text{(*derived from t1*)},\alpha*Q^{\beta}<1,Q>=0,L>=0\},\{Q,L\}]$$

```
Q=sol[[2,1,2]];L=sol[[2,2,2]];
```

$$solve[t1==\sum_{n=0}^{\infty}\frac{\alpha^n*(Q^{n*\beta+1}-L^{n*\beta+1})}{P*(1+n*\beta)},t1]$$

$$solve[T==t1+(Q^{1-\beta}-L^{1-\beta})/(d*(1-\beta)),T]$$

```
{12102.3,{Q→425.321,L→103.567}}
solve[t1==1.06053,t1]
solve[T==0.464092_+t1,T]
```

The second model:

d=400;
P=1000;
β=0.1;
h=6;
K=300;
γ=20;
η=0;
α=d/P;

$$TP=\left(\gamma*\left(P*\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{n*\beta+1}-L^{n*\beta+1}\right)}{P*(1+n*\beta)}\right)-K-h*\left(\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{2+n*\beta}-L^{2+n*\beta}\right)}{P*(2+n*\beta)}+\frac{Q^{2-\beta}-L^{2-\beta}}{d*(2-\beta)}\right)\right)\Big/\left(\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{n*\beta+1}-L^{n*\beta+1}\right)}{P*(1+n*\beta)}+\frac{Q^{1-\beta}-L^{1-\beta}}{d*(1-\beta)}\right);\text{(*Total profit equation*)}$$

$$sol=NMaximize\Big[\Big\{\left(\gamma*\left(P*\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{n*\beta+1}-L^{n*\beta+1}\right)}{P*(1+n*\beta)}\right)-K-h*\left(\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{2+n*\beta}-L^{2+n*\beta}\right)}{P*(2+n*\beta)}+\frac{Q^{2-\beta}-L^{2-\beta}}{d*(2-\beta)}\right)\right)\Big/\left(\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{n*\beta+1}-L^{n*\beta+1}\right)}{P*(1+n*\beta)}+\frac{Q^{1-\beta}-L^{1-\beta}}{d*(1-\beta)}\right),\ \sum_{n=0}^{\infty}\alpha^n*Q^{n*\beta+1}>=\sum_{n=0}^{\infty}\alpha^n*L^{n*\beta+1}\text{(*derived from t1 *)},\ \alpha*Q^{\beta}<1,\ \sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{2+n*\beta}-L^{2+n*\beta}\right)}{P*(2+n*\beta)}+\frac{Q^{2-\beta}-L^{2-\beta}}{d*(2-\beta)}==0.6\text{(*additional condition*)},\ Q>=0,\ L>=0\Big\},\{Q,L\}\Big]$$

Q=sol[[2,1,2]];L=sol[[2,2,2]];

$$Solve\left[t1==\sum_{n=0}^{\infty}\frac{\alpha^n*\left(Q^{n*\beta+1}-L^{n*\beta+1}\right)}{P*(1+n*\beta)},t1\right]$$

$$Solve\left[T==t1+\frac{Q^{1-\beta}-L^{1-\beta}}{d*(1-\beta)},T\right]$$

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- March 2008 to Present: Engineer at Saudi Aramco, Project and Strategic Purchasing Department.
    - Worked as material standardization engineer, quality management system inspector, local and Middles East manufactures pre-qualification coordinator.
- January 2007 to February 2008: Measurement While Drilling Engineer at Baker Hughes Inc.