A FAMILY OF NORMALIZED LEAST MEAN FOURTH ALGORITHMS

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ABSTRACT

In this work, a family of normalized least mean fourth algorithms is presented. Unlike the LMF algorithm, the convergence behavior of these algorithms is independent of the input data correlation statistics. The first proposed algorithm uses a simple normalization of the regressor and is called simply the NLMF. The second algorithm consists of a mixed normalized LMF (XE-NLMF) algorithm which is normalized by the mixed signal and error powers. Finally, the third algorithm, called the variable XE-NLMF, is a modified version of the XE-NLMF where the mixed-power parameter is time-varying. An enhancement in performance is obtained through the use of these techniques over the LMF algorithm. Moreover, the simulation results obtained confirm the theoretical predictions on the performance of these normalized LMF algorithms.

1. INTRODUCTION

Adaptive systems are playing a vital role in the development of modern telecommunications. Also, adaptive systems proved to be extremely effective in achieving high efficiency, high quality and high reliability of around-the-world ubiquitous telecommunication services [1].

The key to successful adaptive signal processing is understanding the fundamental properties of adaptive algorithms. These properties are stability, speed of convergence, misadjustement errors, robustness to both additive noise and signal conditioning (spectral coloration), numerical complexity, and round-off error analysis of adaptive algorithms. However, some of these properties are often in direct conflict with each other, since consistently fast converging algorithms tend to be in general more complex and numerically sensitive. Also, the performance of any algorithm with respect to any of these criteria is entirely dependent on the choice of the adaptation update function, that is the cost function used in the minimization process. A compromise must be then reached among these conflicting factors to come up with the appropriate algorithm for the concerned application.

Due to its simplicity, the least mean-square (LMS) algorithm [2] is the most widely used algorithm for adaptive filters in many applications. The least mean-fourth (LMF) [3] algorithm was suggested as a special case of the more general family of steepest descent algorithms with $2k$ error norms, k being a positive integer. But for both of these algorithms, the convergence behavior depends on the condition number, i.e., on the ratio of the maximum to the minimum eigenvalues of the input signal autocorrelation matrix, $\mathbf{R} = E[\mathbf{x}_k\mathbf{x}_k^T]$ where $\mathbf{x}_k$ is the input signal.

2. PROPOSED ALGORITHMS

The LMF algorithm is based on the minimization of the mean-fourth error cost function, that is [3]:

$$J_k = E[e_k^4], \tag{1}$$

where the error $e_k = d_k + \eta_k - \mathbf{x}_k^T\mathbf{w}_k$, with d_k being the desired value, $\mathbf{w}_k$ the filter coefficient of the adaptive filter and η_k the additive noise. In this case, the filter-coefficient vector update of the LMF algorithm is given by [3]:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + 2\mu e_k^3\mathbf{x}_k. \tag{2}$$

The speed of convergence of this algorithm depends on the input signal statistics, or more specifically on the eigenvalue spread of the input autocorrelation matrix, and for convergence, the proper range of the step size μ was shown to be [3]:

$$0 < \mu < \frac{1}{3\sigma_\eta^2\lambda_{max}}, \tag{3}$$

where σ_η^2 and λ_{max} are, respectively, the noise power and the largest eigenvalue of the autocorrelation matrix of the input signal.

2.1. The NLMF algorithm

To overcome the dependency of the LMF on the the condition number, the normalized LMF algorithm is introduced and is interpreted as the solution to the following minimization problem [4]:

$$\min_{\mathbf{w}_k} \left\{ \|d_k + \eta_k - \mathbf{x}_k^T\mathbf{w}_k\|^4 + (\frac{1}{\bar{\mu}} - 1)\|\mathbf{x}_k\|^2\|\mathbf{w}_{k+1} - \mathbf{w}_k\|^2 \right\}, \tag{4}$$

with $\bar{\mu} \in$ [0,1] being the step size. The weight update recursion of the NLMF algorithm is given by the following expression [4]:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + 2\bar{\mu} e_k^3 \frac{\mathbf{x}_k}{\|\mathbf{x}_k\|^2}, \tag{5}$$

where $\|\mathbf{x}_k\|^2$ is the Euclidean norm of the input vector $\mathbf{x}_k$.

Ultimately, for the NLMF algorithm to converge in-the-mean, a sufficient condition is that $\bar{\mu}$ be chosen in the following range [4]:

$$0 < \bar{\mu} < \frac{1}{3\sigma_\eta^2 + 1}. \tag{6}$$

2.2. The XE-NLMF algorithm

The power of the LMF lies in its faster initial convergence and lower steady-state error relative to the LMS algorithm. More importantly, its mean fourth error cost function yields a better performance than that of the LMS for noise of sub-Gaussian nature [3], or light-tailed pdf-like noise [5].

However, this higher-order algorithm requires a much smaller step size to ensure stable adaptation [4]. Whereas, the error power three in the LMF gradient vector can cause devastating initial instability. Therefore, it causes unnecessary performance degradation. The solution proposed here is to normalize the step size.

The recursive equation for the XE-NLMF algorithm is defined as follows [6]:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{\gamma_{xe} e_k^3 \mathbf{x}_k}{\delta + (1-\alpha)\|\mathbf{x}_k\|^2 + \alpha\|\mathbf{e}_k\|^2}, \tag{7}$$

where γ_{xe} represents the step size, the error signal, e_k, $\mathbf{w}_k$ is the filter coefficient vector of the adaptive filter, $\mathbf{x}_k$ is the input vector, $\mathbf{e}_k = [e_{k-N+1}, e_{k-N+2}, \ldots, e_k]^T$ is the error vector and N is the length of the filter.

As shown in equation (7), the LMF is normalized by both the signal power and error power, which are balanced by a mixed-power parameter (α). Combining signal power and error power has the advantage that the former normalizes input signal, while the latter can dampen down the outlier estimation errors, thus improving stability while still retaining a fast convergence speed.

Finally, a sufficient condition for convergence in the mean of the XE-NLMF algorithm is given by :

$$0 < \gamma_{xe} < \frac{1}{3\sigma_\eta^2\sigma_x^2}\left[(1-\alpha)\sigma_x^2 + \alpha\sigma_e^2\right], \tag{8}$$

where σ_x^2 is the signal power.

2.3. The variable XE-NLMF algorithm

The mixing power parameter, α_k, is confined to the interval [0,1] and will be recursively adapted to adjust the signal power, $\|\mathbf{x}_k\|^2$, and error power, $\|\mathbf{e}_k\|^2$, for maximum performance. Here, we propose an error square feedback quantity, μ_k, updated according to a variable-step-size-parameter-like [7]:

$$\mu_{k+1} = \nu\mu_k + p_k|e_k e_{k-1}|, \tag{9}$$

where the quantity $e_k e_{k-1}$ determines the distance of $\mathbf{w}_k$ to the optimum weight, ν is a constant and $|\cdot|$ denotes the absolute value operation. The quantity p_k is updated according to the weighted sum of the past three samples of α_k in the following way [6]:

$$p_k = a[\alpha_{k-2} + \alpha_{k-1} + \alpha_k], \tag{10}$$

a is a constant.

With this averaging, the recursion curve of μ_k can be more flexibly controlled. The error power estimate is then used to guide the value of α_k as follows:

$$\alpha_k = erf\{\mu_k\}, \tag{11}$$

where $erf\{r\} = (2/\sqrt{\pi})\int_0^r e^{-y^2}dy$ is the error function with the purpose to confine α_k to the interval [0,1]. Similarly, the parameters ν and p are restricted to the interval [0,1]. Moreover, to avoid zero in the feedback loop, the initial value of p is set at $p_0 = 0.5$.

This scheme provides an automatic adjustment of α_k according to the estimation of the square of the error. When the estimation error is large, α_k will approach unity, thus providing fast adaptation. While the error is small, α_k is adjusted to a smaller value for a lower steady-state error. Based on this motivation, the variable XE-NLMF algorithm is expressed as follows [8]:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{\gamma_{xe} e_k^3 \mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}. \tag{12}$$

Therefore, a sufficient condition for convergence in the mean of the variable XE-NLMF algorithm is given by [8]:

$$0 < \gamma_{xe} < \frac{1}{3\sigma_\eta^2\sigma_x^2}\left[(1-E[\alpha_k])\sigma_x^2 + E[\alpha_k]\sigma_e^2\right], \tag{13}$$

where $E[\alpha_k]$ is the average value of of α_k.

3. SIMULATION RESULTS

The performance of these normalized algorithms is compared with that of the LMF and the NLMS algorithms. Experiments are carried out where an unknown system is to be identified under noisy conditions. The unknown system is a non-minimum phase channel. The input signal x_k to the unknown system and to the adaptive filter is obtained by passing a zero mean white Gaussian sequence through a channel used to vary the degree of ill-conditioning of the autocorrelation matrix of the sequence $\{x_k\}$. The additive noise, η_k, is zero-mean. The signal-to-noise-ratio is set to be equal to

20 dB and the performance measure used is the normalized weight error norm $10log_{10}||\mathbf{w}_k - \mathbf{w}_{opt}||^2/||\mathbf{w}_{opt}||^2$. Results are obtained by averaging over 600 independent runs and all the algorithms are obtained for fastest convergence.

3.1. Performance of the NLMF algorithm

Here, three experiments are conducted for different eigenvalue spreads of the autocorrelation matrix of the input signal. The eigenvalue spread ($\lambda_{max}/\lambda_{min}$) is 11.8 for the first experiment, 21 for the second and 68.9 for the third. The insensitivity of the NLMF algorithm to the input data statistics, in contrast to the LMF algorithm, is clearly shown in Fig. 1 for the three eigenvalue spreads.

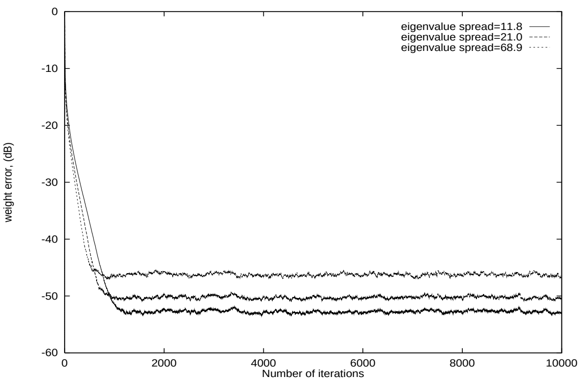

Fig. 1. Convergence characteristics for NLMF algorithm for $\lambda_{max}/\lambda_{min} = 11.8$, $\lambda_{max}/\lambda_{min} = 21$, and $\lambda_{max}/\lambda_{min} = 68.9$.

3.2. Effect of α on the performance of the XE-NLMF algorithm

The convergence characteristics of different α's ($0 \leq \alpha \leq 0.9$) are studied in this section. The results are shown in Figure 2. Typically for $\alpha = 0.1$, the convergence curves are close to the NLMF algorithm, which slows down after the initial fast convergence. Lower weight errors are observed for $\alpha = 0.1$ and $\alpha = 0.5$, but their convergence speeds are slower than that observed when $\alpha = 0.9$. The performance for $\alpha = 0.5$ is in between that of $\alpha = 0.1$ and $\alpha = 0.9$. Therefore, the choice of α value acts as an additional step size control, which provides a compromise between the convergence speed and steady state weight error.

The introduction of the mixed parameter α has increased the flexibility of the XE-NLMF algorithm to achieve faster convergence and lower steady state weight error. The experiment results showed that there is an optimum value for the mixed parameter, α, that will wield an optimum performance. Therefore, the main parameter affecting its performance are the judious choice of α and its consequence on the normalized step size. In the sequel, these two issues are considered for the purpose of motivating the choice of the variable XE-NLMF algorithm.

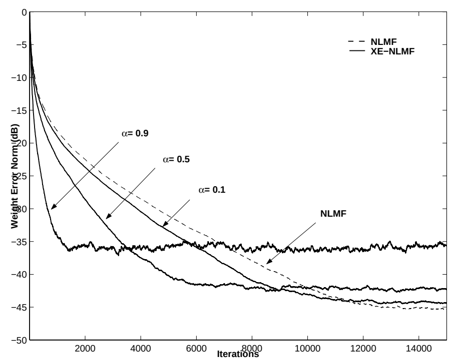

Fig. 2. Effect of α on the convergence performance of the XE-NLMF.

3.3. Performance of varaible XE-NLMF algorithm

Figure 3 depicts the convergence behaviour of the variable XE-NLMF, the XE-NLMF (with $\alpha = 0.5$) and the NLMS with the same convergence rate in an additive white Gaussian noise (AWGN) environment. As can be seen from this figure, the variable XE-NLMF algorithm adapts faster than the XE-NLMF and NLMS algorithms, and at the same time, produces a lower steady-state weight error norm of more than 15 dB. This demonstrates the advantages of incorporating a variable mixed-power parameter in the XE-NLMF algorithm.

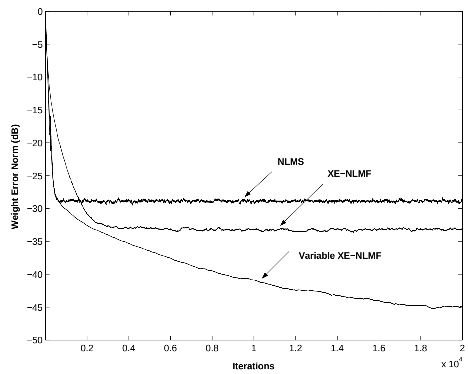

Fig. 3. Convergence performance for the variable XE-NLMF algorithm, the XE-NLMF algorithm (α= 0.5) and the NLMS algorithm in AWGN environment.

In Figure 4, the variable XE-NLMF algorithm converges faster with a lower-steady state error than the NLMS algorithm in a binary (sub-Gaussian) additive noise environment. Here, the difference of 23 dB in weight error norm is more apparent. The LMF-based algorithm performs better in sub-Gaussian noise.

3.4. The Co-Channel Interference (CCI) Effect

In the first experiment, a channel equalizer is used to study the performance of the NLMF algorithm in terms of bit-

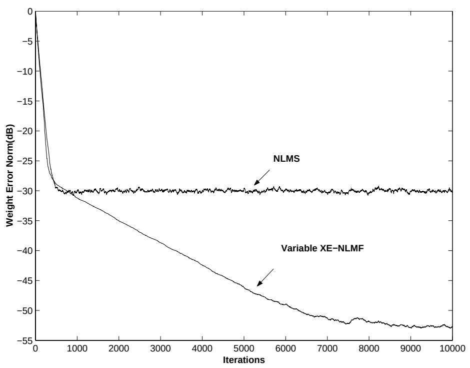

Fig. 4. Convergence performance in Binary additive noise environment.

error-rate (BER). The channel is $h_1(z) = 1 + 0.4z^{-1}$ and the co-channel is $h_2(z) = 1 + 0.2z^{-1}$. The BER results in a co-channel interference scenario are depicted in Figure 5 where the NLMF algorithm outperforms the NLMS algorithm. More than a 1.5 dB improvement over the NLMS algorithm is achieved at a BER of 10^{-5}. It can be seen from these results that the effect of the step size on the performance of the NLMF algorithm is almost negligible.

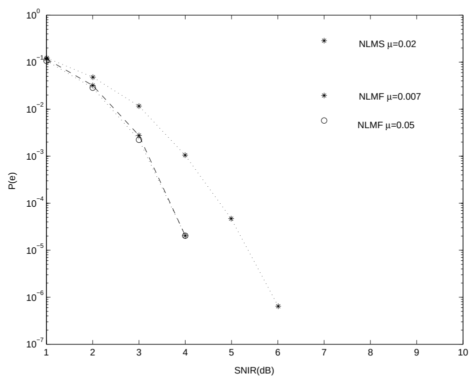

Fig. 5. The BER performance in a co-channel interference environment.

In the second experiment, the BER results in a co-channel interference scenario is depicted in Figure 6 for the variable XE-NLMF algorithm. The channel is $h_1(z)$ and the co-channel is $h_2(z)$. As expected, the variable XE-NLMF algorithm outperforms the rest of the algorithms. A 2 dB improvement over the NLMS algorithm is achieved at a BER of 10^{-6}.

4. CONCLUSION

In this study, newly developed normalized LMF algorithms were investigated and their performance were compared. Among the three versions, the variable XE-NLMF algorithm exhibited the best performance. Moreover, the computational complexity added to these algorithms, and more

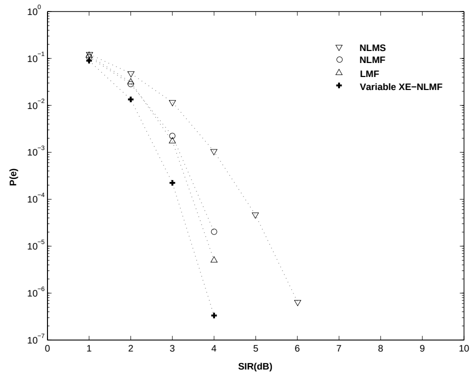

Fig. 6. The BER performance in a co-channel interference environment.

specifically, the variable XE-NLMF algorithm, is minor compared to that of the LMF algorithm. Finally, the variable XE-NLMF can be a potential candidate in the design of new receivers where the noise statistics are not gaussian.

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