Global existence and uniform stability of solutions for a quasilinear viscoelastic problem

Salim A. Messaoudi & Nasser-eddine Tatar
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran 31261, Saudi Arabia.
E-mail : messaoud@kfupm.edu.sa
E-mail : tatarn@kfupm.edu.sa

Abstract

In this paper the nonlinear viscoelastic wave equation in canonical form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{p-2}u$$

with Dirichlet boundary condition is considered. By introducing a new functional and using the potential well method, we show that the damping induced by the viscoelastic term is enough to ensure global existence and uniformly decay of solutions provided that the initial data are in some *stable* set.

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1 Introduction

In elasticity the existing theory accounts for materials which have a capacity to store mechanical energy with no dissipation (of the energy). On the other hand, a Newtonian viscous fluid in a non-hydrostatic stress state has a capacity for dissipation energy without storing it. Materials which are outside the scope of these two theories would be those for which some, but not all, of the work done to deform them, can be recovered. Such materials possess a capacity of storage and dissipation of mechanical energy. This is the case of "viscoelastic" materials.

Polymers, for instance, are viscoelastic materials since they exhibit intermediate position between viscous liquids and elastic solids.

The formulation of Boltzmann's superposition principle leads to a memory term involving a relaxation function of exponential type. But, it has been observed that relaxation functions of some viscoelastic materials are not necessarily of this type.

In this work, we are concerned with the following initial boundary value problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = b|u|^{p-2}u, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0, \ x\in\partial\Omega \ ,t\geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $\gamma \geq 0$, ρ, $b > 0$, $p > 2$ are constants, $\rho \geq 0$, $p \geq 2$ are constants, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive and uniformly decaying function. This type of equations usually appears as a model in nonlinear viscoelasticity (see [1]).

In the case $\rho > 0$ and in the absence of the source term $(b = 0)$, this problem has been studied by Cavalcanti *et al.* in [1]. By assuming $0 < \rho \leq 2/(n-2)$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$, they proved a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$. This decay result was later pushed to a situation where a source term is present $(b > 0)$ by Messaoudi and Tatar [2] .

In the case $\rho = 0$ and in the absence of the dispersion term, problem (1.1) has been extensively studied and several results concerning existence, decay and blow up have been established. In this regard, we mention the work of Cavalcanti *et al.* [3] where the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, \quad \text{in } \Omega\times(0,\infty)$$

has been considered. Here $a : \Omega \to \mathbb{R}^+$, is a function which may vanish outside a subset $\omega \subset \Omega$ of positive measure. Under some geometric restrictions on ω and assuming that

$$\begin{aligned} a(x) &\geq a_0 > 0, \qquad \forall x\in \ \omega, \\ -\xi_1 g(t) &\leq g'(t) \leq -\xi_2 g(t), \ t\geq 0, \end{aligned}$$

for some positive constants ξ_1 and ξ_2, the authors established an exponential decay result. Berrimi and Messaoudi [4] improved Cavalcanti's result by introducing a different functional. This new functional allowed them to weaken the conditions on both a and g. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition is no longer needed. In [5], Cavalcanti *et al.* considered

$$u_{tt} - k_0\Delta u + \int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0.$$

Under similar conditions, as in above, on the relaxation function g and

$$a(x) + b(x) \geq \rho > 0, \quad \forall x\in\Omega,$$

they established an exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Their proof, based on the use of piecewise multipliers, is similar to the one in [3]. Although the results in [4] and [5] improve the earlier one in [3], the approaches and the functionals used are both different. Another problem, where the dissipation induced by the integral term is cooperating with a damping acting on a part of the boundary was also discussed by Cavalcanti *et al.* [6]. A related result is the work of Kawashima [7], in which he considered a one-dimensional model equation for viscoelastic materials of integral type where the memory function is allowed to have an integrable singularity. For small initial data, Muñoz Rivera and Baretto [8] proved that the first and the second-order energies of the solution to a viscoelastic plate, decay exponentially provided that the kernel of the memory decays exponentially. Kirane and Tatar [9] considered a mildly damped wave equation and proved that any small internal dissipation is sufficient to uniformly stabilize the solution by means of a nonlinear feedback of memory type acting on a part of the boundary. This result was established without any restriction on the space dimension or geometrical conditions on the domain or its boundary. Furthermore, Berrimi and Messaoudi [10] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^{p-2}\, u$$

in a bounded domain and $p > 2$. They established a local existence result and showed, under weaker conditions than those in [3] and [5], that the local solution is global and decays uniformly if the initial data are small enough.

Concerning nonexistence, Messaoudi [11] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a\,|u_t|^{\alpha-2}\, u_t = b\,|u|^{p-2}\, u$$

and proved a blow up result for solutions with negative initial energy if $p > \alpha$ and a global result for $p \leq \alpha$. This result has been later improved by Messaoudi [12] to accommodate certain solutions with positive initial energy. By the end it is also worth mentioning the work of Aassila *et al.* [13] in which an asymptotic stability and decay rates, for solutions of the wave equation in star-shaped domains, were established by combination of memory effect and damping mechanism.

In this paper, we take $\gamma = 0$ in (1.1). More precisely, we consider

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{p-2}u, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0, \ x\in\partial\Omega\ , t\geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in\Omega. \end{cases} \tag{1.2}$$

In this case the source term competes with the dissipation induced by the viscoelastic term only. As we are in a less favorable situation, it is interesting to study this interaction. We will show that there exists an appropriate set S, called stable, such that if the initial data are in S, the solution continues to live there forever.. Moreover, we will show that the solution goes to zero with an exponential or polynomial rate depending on the decay rate of the relaxation function. To achieve our goal, we combine the potential well method and the perturbation method and use a "new" functional which made our proof easy and allowed us to obtain our result with less requirements on g (see Remark 3.1 below).

The paper is organized as follows. In Section 2, we present some notations and material needed for our work and we state, without a proof, a standard local existence theorem. Section 3 contains the statements and the proofs of the global existence and exponential decay results.

2 Preliminaries

In this section, we present some material needed in the proofs of our results. Namely, we introduce some notations and show the invariance of an appropriately chosen set of initial data.

We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. The symbols ∇ and Δ will stand for the gradient and the Laplacian, respectively. The prime $'$ and the subscript t will denote time differentiation. We will also be using the following Sobolev-Poincaré embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \tag{2.1}$$

so

$$\|v\|_q \leq C_{*q} \|\nabla v\|_2 , \tag{2.2}$$

for $2 \leq q < 2n/(n-2)$ if $n \geq 3$ and $q \geq 2$ if $n = 1, 2$.

For the relaxation function g we assume

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0.$$

(G2) There exists a positive constant ξ such that

$$g'(t) \leq -\xi g^r(t), \qquad 1 \leq r < 3/2, \quad t \geq 0.$$

Proposition 2.1. *Let $u_0, u_1 \in H_0^1(\Omega)$ be given. Assume that g satisfies (G1). Assume further that*

$$\begin{array}{lll} 2 & \leq & p \leq \dfrac{2(n-1)}{n-2}, \qquad n \geq 3 \\ p & \geq & 2, \qquad n = 1,\, 2. \end{array} \tag{2.3}$$

Then problem (1.2) has a unique local solution

$$u,\, u_t \in C\left([0,\ T_m); H_0^1(\Omega)\right), \tag{2.4}$$

for some $T_m > 0$.

Remark 2.1. This theorem can be easily established by combining the arguments of [1] and [10].

Remark 2.2. Condition (2.3) is needed to establish the local existence result (see [10]). In fact under this condition, the nonlinearity is locally Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$.

Remark 2.3. Condition (G1) is necessary to guarantee the hyperbolicity of the system (1.2).

Remark 2.4. Condition $r < 3/2$ is imposed so that $\int_0^\infty g^{2-r}(s)ds < \infty$.

Next, we introduce

$$I(t) = I(u, u_t) := \left(1 - \int_0^t g(s)ds\right) ||\nabla u||_2^2 + ||\nabla u_t||_2^2 - b||u||_p^p,$$

$$J(t) = J(u, u_t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) ||\nabla u(t)||_2^2 + \frac{1}{2}||\nabla u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}||u||_p^p,$$

$$\mathcal{E}(t) = \mathcal{E}(u, u_t) := J(t) + \frac{1}{\rho+2}\int_\Omega |u_t(t)|^{\rho+2}dx, \tag{2.5}$$

where

$$(g^s \circ v)(t) = \int_0^t g^s(t-\tau)||v(t) - v(\tau)||_2^2 d\tau, \quad s \geq 1. \tag{2.6}$$

Lemma 2.2. *Suppose that $v \in L^\infty(0, T; H^1(\Omega))$, g is a continuous function such that*

$$\int_0^\infty g^{1-\theta}(s)ds < \infty\ , \qquad 0 \leq \theta < 1.$$

Then we have

$$(g \circ v)(t) \leq 2\left\{\left(\int_0^t g^{1-\theta}(s)ds\right) ||\nabla v(t)||^2_{L^\infty(0,T;L^2(\Omega))}\right\}^{\frac{r-1}{r-1+\theta}} ((g^r \circ v)(t))^{\frac{\theta}{r-1+\theta}}. \tag{2.7}$$

and

$$(g \circ v)(t) \leq 2\left\{\int_0^t ||\nabla v(s)||_2^2 ds + t||\nabla v(t)||_2^2\right\}^{(r-1)/r} ((g^r \circ v)(t))^{1/r}. \tag{2.8}$$

Proof. To prove (2.7), it suffices to note that, for $q > 1$, $0 \leq \theta \leq 1$,

$$(g \circ v)(t) = \int_0^t g^{\frac{1-\theta}{q}}(t-s)||\nabla v(t) - \nabla v(s)||_2^{\frac{2}{q}} g^{\frac{q-1+\theta}{q}}(t-s)||\nabla v(t) - \nabla v(s)||_2^{\frac{2(q-1)}{q}} ds.$$

By applying Holder's inequality, we get

$$(g \circ v)(t) \leq \left(\int_0^t g^{\frac{1-\theta}{q}}(t-s)||\nabla v(t) - \nabla v(s)||_2^2 ds\right)^{1/q} \times$$

$$\left(\int_0^t g^{\frac{q-1+\theta}{q-1}}(t-s)||\nabla v(t)-\nabla v(s)||_2^2 ds\right)^{(q-1)/q}$$

By taking $q=(r-1+\theta)/r-1$, we obtain

$$(g\circ v)(t)\leq\left(\int_0^t g^{1-\theta}(t-s)||\nabla v(t)-\nabla v(s)||_2^2 ds\right)^{(r-1)/(r-1+\theta)}\times \tag{2.9}$$
$$\left(\int_0^t g^{r}(t-s)||\nabla v(t)-\nabla v(s)||_2^2 ds\right)^{\theta/(r-1+\theta)}.$$

Estimate (2.7) follows easily for $0\leq\theta<1$.

Finally, by taking $\theta=1$ in (2.9), estimate (2.8) is established.

Lemma 2.3. *If u is the solution of* (1.2) *then the "modified" energy satisfies*

$$\mathcal{E}'(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)||\nabla u(t)||^2\leq 0, \tag{2.10}$$

for almost every t in $[0,T)$, where $(g'\circ\nabla u)(t)$ is defined similarly to (2.6).

Proof. By multiplying the equation in (1.2) by u_t and integrating over Ω, using integration by parts and hypothesis (G2), the assertion of the lemma is established.

Remark 2.5. This means that the energy is uniformly bounded (by $\mathcal{E}(0)$) and is decreasing in t.

Lemma 2.4. *Suppose that (G1), (G2) and the hypotheses on p and ρ hold. Assume further that $u_0,u_1\in H_0^1(\Omega)$ and satisfy*

$$\beta=\frac{b}{l}C_*^p\left(\frac{2p}{(p-2)\,l}\mathcal{E}(u_0,u_1)\right)^{(p-2)/2}<1 \tag{2.11}$$

$$I(u_0,u_1)>0, \tag{2.12}$$

where C_* is the best constant in (2.2) with $q=p$, then $I(u(t),u_t(t))>0$, for each $t\in[0,T_m)$.

Proof. Since $I(u_0,u_1)>0$ then, by continuity, there exists $T_*\leq T_m$ such that $I(u,u_t)\geq 0$ for all $t\in[0,T_*)$. This implies that, for all $t\in[0,T_*)$,

$$\begin{aligned} J(t)&=\tfrac{1}{2}\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2+\tfrac{1}{2}||\nabla u_t||_2^2+\tfrac{1}{2}(g\circ\nabla u)(t)-\tfrac{b}{p}||u(t)||_p^p\\ &\geq\tfrac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2+||\nabla u_t||_2^2+(g\circ\nabla u)(t)\right]+\tfrac{1}{p}I(u,u_t)\\ &\geq\tfrac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2+||\nabla u_t||_2^2+(g\circ\nabla u)(t)\right]. \end{aligned} \tag{2.13}$$

Hence, from (G1), (2.13) and Lemma 2.3, we find

$$\begin{aligned} l||\nabla u(t)||_2^2&\leq\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2\leq\tfrac{2p}{p-2}J(t)\\ &\leq\tfrac{2p}{p-2}\mathcal{E}(t)\leq\tfrac{2p}{p-2}\mathcal{E}(u_0,u_1),\ \forall t\in[0,T_*). \end{aligned} \tag{2.14}$$

By exploiting the embedding relation (2.2), (G1) and the assumption (2.11), we easily arrive at

$$\begin{aligned} b\|u(t)\|_p^p &\le bC_*^p\|\nabla u(t)\|_2^p \le \tfrac{bC_*^p}{l}\|\nabla u(t)\|_2^{p-2} l\|\nabla u(t)\|_2^2 \\ &\le \beta l\|\nabla u(t)\|_2^2 \le \beta\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \\ &< \left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2,\ \forall t\in[0,T_*). \end{aligned} \tag{2.15}$$

Therefore,

$$I(t) = \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 - b\|u(t)\|_p^p + \|\nabla u_t\|_2^2 > 0, \forall t\in[0,T_*).$$

By repeating this procedure and using the fact that

$$\lim_{t\to T_*} \frac{b}{l} C_*^p \left(\frac{2p}{(p-2)\, l}\mathcal{E}(u,u_t)\right)^{(p-2)/2} \le \beta < 1,$$

T_* is extended to T_m.

3 Exponential decay

In this section we state and prove our global existence and decay of solutions results.
Theorem 3.1 *Suppose that* (*G*1), (*G*2) *and the hypotheses on* p *and* ρ *hold. If* $u_0, u_1 \in H_0^1(\Omega)$ *and satisfy* (2.11), (2.12), *then the solution of* (1.2) *is bounded and global in time.*
Proof. It suffices to show that $\|\nabla u(t)\|_2^2 + \|\nabla u_t\|_2^2$ is bounded independently of t. To achieve this note that, from (2.5), (2.10), and (2.13), we have, for $t \in [0, T)$

$$\begin{aligned} \mathcal{E}(u_0,u_1) \ge \mathcal{E}(t) &\ge \tfrac{p-2}{2p}\left[l\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + (g\circ\nabla u)(t)\right] + \tfrac{1}{p} I(u,u_t) \\ &\ge \tfrac{p-2}{2p}\left[l\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right] \end{aligned} \tag{3.1}$$

since $I(u,u_t)$ and $(g\circ\nabla u)(t)$ are positive. Therefore,

$$\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 \le C\mathcal{E}(u_0,u_1),$$

where C is positive, depends only on p and l and is independent of t.
Remark 3.1. Observe that in the previous lemmas and Theorem 3.1 we did not use hypothesis (G2). Only the non positivity of g' was needed. This will not be the case in the theorem below. Indeed, we will need g to decrease in an exponential or polynomial rate.
Theorem 3.2. *Suppose that (G1), (G2) and the hypotheses on* p *and* ρ *hold. Assume further that* $p > 2$. *If* $u_0, u_1 \in H_0^1(\Omega)$ *and satisfy* (2.11), (2.12). *Then for each* $t_0 > 0$

there exit positive constants k and K such that the solution of (1.2) *satisfies, for all* $t \geq t_0$,

$$\begin{aligned} \mathcal{E}(t) &\leq Ke^{-kt}, \qquad r = 1 & (3.2)\\ \mathcal{E}(t) &\leq K(1+t)^{-1/(r-1)}, \quad r > 1. & (3.3) \end{aligned}$$

Proof. The proof relies on a suitable modification of the energy. We use the functional

$$\Psi(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho \, u_t u dx + \int_\Omega \nabla u.\nabla u_t dx.$$

A differentiation of $\Psi(t)$ with respect to t along the solution of (1.2) yields

$$\begin{aligned} \Psi'(t) = &-\int_\Omega |\nabla u|^2 \, dx + \int_\Omega \nabla u(t).\int_0^t g(t-s)\nabla u(s) ds dx \\ &+\int_\Omega |\nabla u_t|^2 \, dx + \tfrac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} \, dx + b\int_\Omega |u|^p dx. \end{aligned} \tag{3.4}$$

We now estimate the second term in the right side of (3.4) as follows

$$\int_\Omega \nabla u(t).\int_0^t g(t-\tau)\nabla u(\tau) d\tau dx \leq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(\tau)| d\tau\right)^2 dx$$

$$\leq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau\right)^2 dx.$$

We then use Young's inequality to obtain, for any $\eta > 0$,

$$\begin{aligned} &\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau\right)^2 dx \\ &\leq \int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| d\tau\right)^2 dx + \int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)| d\tau\right)^2 dx \\ &\quad +2\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| d\tau\right)\left(\int_0^t g(t-\tau)|\nabla u(t)| d\tau\right) dx \\ &\leq (1+\eta)\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)| d\tau\right)^2 dx \\ &\quad +(1+\tfrac{1}{\eta})\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| d\tau\right)^2 dx. \end{aligned} \tag{3.5}$$

Simple calculations, using Cauchy-Schwarz inequality, show that

$$\begin{aligned} &\left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| d\tau\right)^2 dx \\ &\leq \int_0^t g^{2-r}(\tau) d\tau \int_\Omega \int_0^t g^r(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2 d\tau dx. \end{aligned}$$

Thus (3.5) and the fact that $\int\limits_0^t g(\tau)d\tau \leq \int\limits_0^\infty g(\tau)d\tau = 1-l$ give

$$\begin{aligned}
&\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)| + |\nabla u(t)|)d\tau \right)^2 dx \\
&\quad \leq (1+\eta) \int\limits_\Omega |\nabla u(t)|^2 \left(\int\limits_0^t g(t-\tau)d\tau \right)^2 dx \\
&\quad +(1+\tfrac{1}{\eta}) \left(\int\limits_0^t g^{2-r}(\tau)d\tau \right) \int\limits_\Omega \int\limits_0^t g^r(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2 d\tau dx \\
&\quad \leq (1+\eta)(1-l)^2 \int\limits_\Omega |\nabla u(t)|^2 dx + (1+\tfrac{1}{\eta}) \left(\int\limits_0^t g^{2-r}(\tau)d\tau \right) (g^r \circ \nabla u)(t).
\end{aligned} \tag{3.6}$$

Taking $\eta = \frac{l}{1-l}$, we find

$$\begin{aligned}
&\int\limits_\Omega \nabla u(t) \int\limits_0^t g(t-s)\nabla u(s) ds dx \\
&\quad \leq \tfrac{2-l}{2} \int\limits_\Omega |\nabla u|^2 dx + \tfrac{1}{2l} \left(\int\limits_0^t g^{2-r}(\tau)d\tau \right) (g^r \circ \nabla u)(t).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Psi'(t) \leq & -\tfrac{l}{2} \int\limits_\Omega |\nabla u|^2 dx + \tfrac{1}{2l} \left(\int\limits_0^t g^{2-r}(\tau)d\tau \right) (g^r \circ \nabla u)(t) \\
& + \int\limits_\Omega |\nabla u_t|^2 dx + \tfrac{1}{\rho+1} \int\limits_\Omega |u_t|^{\rho+2} dx + b \int\limits_\Omega |u|^p dx.
\end{aligned} \tag{3.7}$$

The second functional we introduce is

$$\chi(t) := \int\limits_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1} \right) \int\limits_0^t g(t-s)\left(u(t)-u(s)\right) ds dx. \tag{3.8}$$

Differentiating (3.8) with respect to t and exploiting the equation in (1.2), we obtain

$$\begin{aligned}
\chi'(t) = & \int\limits_\Omega \nabla u(t). \int\limits_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds dx \\
& + \int\limits_\Omega \left(\int\limits_0^t g(t-s)\Delta u(s) ds \right) \left(\int\limits_0^t g(t-s)\left(u(t)-u(s)\right) ds \right) dx \\
& - \left(\int\limits_0^t g(s) ds \right) \int\limits_\Omega |\nabla u_t|^2 dx - \int\limits_\Omega \nabla u_t(t). \int\limits_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds dx \\
& - \tfrac{1}{\rho+1} \int\limits_\Omega |u_t|^\rho u_t \int\limits_0^t g'(t-s)\left(u(t)-u(s)\right) ds dx - \tfrac{1}{\rho+1} \left(\int\limits_0^t g(s) ds \right) \int\limits_\Omega |u_t|^{\rho+2} dx \\
& - b \int\limits_\Omega |u|^{p-2} u \left(\int\limits_0^t g(t-s)\left(u(t)-u(s)\right) ds \right) dx.
\end{aligned} \tag{3.9}$$

Similarly to (3.4), we proceed to estimate each term in the right-hand side of relation (3.9) separately. To this end, we shall use repeatedly Cauchy-Schwarz inequality, Hölder's inequality and Young's inequality.

The first term in the right-hand side of (3.9) may be estimated as follows

$$\begin{aligned} &\int_\Omega \nabla u(t).\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx \\ &\leq \delta\int_\Omega |\nabla u|^2\,dx+\frac{1}{4\delta}\int_\Omega\left[\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right]^2 dx \\ &\leq \delta\int_\Omega |\nabla u|^2 dx+\frac{1}{4\delta}\left[\int_0^t g^{2-r}(\tau)d\tau\right](g^r\circ\nabla u)(t), \qquad \forall\delta>0. \end{aligned} \tag{3.10}$$

As for the second term in (3.9) we have, by (3.6) with $\eta=1$,

$$\begin{aligned} &\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right).\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx \\ &\leq \delta\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)ds\right|^2 dx+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right|^2 dx \\ &\leq \left(2\delta+\frac{1}{4\delta}\right)\left(\int_0^t g^{2-r}(\tau)d\tau\right)(g^r\circ\nabla u)(t)+2\delta(1-l)^2\int_\Omega|\nabla u|^2\,dx. \end{aligned} \tag{3.11}$$

For the fourth term, it is easy to see that

$$\begin{aligned} &\int_\Omega \nabla u_t(t).\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx \\ &\leq \delta\int_\Omega|\nabla u_t|^2\,dx+\frac{g(0)}{4\delta}\int_\Omega\int_0^t -g'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx,\ \delta>0. \end{aligned} \tag{3.12}$$

The fifth term in the right-hand side of (3.9) may be handled similarly,

$$\begin{aligned} &\int_\Omega |u_t|^\rho\, u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)dsdx \\ &\leq \delta\int_\Omega|u_t|^{2(\rho+1)}\,dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\right)^2 dx \\ &\leq \delta\int_\Omega|u_t|^{2(\rho+1)}\,dx+\frac{g(0)}{4\delta}C_p\int_\Omega\int_0^t -g'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx, \end{aligned} \tag{3.13}$$

for any $\delta>0$ and C_p is the Poincaré constant. Next, by the Sobolev embedding

$$H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega) \text{ for } 0<\rho\leq 2/(n-2) \text{ if } n\geq 3 \text{ and } \rho>0 \text{ if } n=1,2$$

and the fact that $\mathcal{E}(t)\leq\mathcal{E}(0)$, $\forall t\geq 0$, we get

$$\int_\Omega |u_t|^{2(\rho+1)}\,dx\leq C_s\left(2\mathcal{E}(0)\right)^\rho\int_\Omega|\nabla u_t|^2\,dx, \tag{3.14}$$

where C_s is the embedding constant. From (3.13) and (3.14), we obtain

$$\begin{aligned} &\frac{1}{\rho+1}\int_\Omega|u_t|^\rho\, u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)dsdx \\ &\leq \delta\frac{C_s}{\rho+1}\left(2\mathcal{E}(0)\right)^\rho\int_\Omega|\nabla u_t|^2\,dx+\frac{g(0)C_p}{4\delta(\rho+1)}(-g'\circ\nabla u)(t). \end{aligned} \tag{3.15}$$

For the seventh term, it is easy to see that

$$\begin{aligned}
&-b\int_\Omega |u|^{p-2}u\left(\int_0^t g(t-s)\,(u(t)-u(s))\,ds\right)dx \\
&\leq \delta b\int_\Omega |u|^{2p-2}dx+\frac{b}{4\delta}\int_\Omega\left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 dx \\
&\leq \delta b\int_\Omega |u|^{2p-2}dx+\frac{C_p^2}{4\delta}b\left(\int_0^t g^{2-r}(\tau)d\tau\right)(g^r\circ\nabla u)(t).
\end{aligned} \tag{3.16}$$

By using (2.2), (2.10) and (2.14) we have the following

$$\begin{aligned}
b\int_\Omega |u|^{2p-2}dx &\leq bC_*^{2p-2}||\nabla u||_2^{2(p-1)} = bC_*^{2p-2}||\nabla u||_2^{2(p-2)}||\nabla u||_2^2 \\
&\leq bC_*^{2p-2}\left(\frac{2p}{(p-2)l}\mathcal{E}(0)\right)^{p-2}||\nabla u||_2^2 =: C_1||\nabla u||_2^2.
\end{aligned}$$

Hence (3.16) becomes

$$\begin{aligned}
&-b\int_\Omega |u|^{p-2}u\left(\int_0^t g(t-s)\,(u(t)-u(s))\,ds\right)dx \\
&\leq \delta C_1||\nabla u||_2^2+\frac{C_p^2}{4\delta}b\left(\int_0^t g^{2-r}(\tau)d\tau\right)(g^r\circ\nabla u)(t).
\end{aligned} \tag{3.17}$$

Taking into account the estimates (3.10)-(3.17) we infer from (3.9) that

$$\begin{aligned}
\chi'(t) &\leq \left(1+2(1-l)^2+C_1\right)\delta\int_\Omega |\nabla u|^2\,dx-\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\int_\Omega |u_t|^{\rho+2}\,dx \\
&+\left(2\delta+\frac{1}{2\delta}+\frac{C_p^2}{4\delta}b\right)\left(\int_0^t g^{2-r}(s)ds\right)(g^r\circ\nabla u)(t)+\frac{g(0)}{4\delta}\left(1+\frac{C_p}{\rho+1}\right)(-g'\circ\nabla u)(t). \\
&+\left[\delta+\frac{C_s\delta}{\rho+1}\left(2\mathcal{E}(0)\right)^\rho-\int_0^t g(s)ds\right]\int_\Omega |\nabla u_t|^2\,dx.
\end{aligned} \tag{3.18}$$

Now we consider the functional

$$\mathcal{L}(t)=M\mathcal{E}(t)+\varepsilon\Psi(t)+\chi(t)$$

where M and ε are to be precised later on. From the relations (2.10), (3.7), and (3.18) we have, for $t\geq t_0$,

$$\begin{aligned}
\mathcal{L}'(t) &\leq \left[\frac{M}{2}-\frac{g(0)}{4\delta}\left(1+\frac{C_p}{\rho+1}\right)\right](g'\circ\nabla u)(t)-\frac{g_0-\varepsilon}{\rho+1}\int_\Omega |u_t|^{\rho+2}\,dx \\
&-\left[\frac{\varepsilon l}{2}-\left(1+2(1-l)^2+C_1\right)\delta\right]\int_\Omega |\nabla u|^2\,dx+\varepsilon b\int_\Omega |u|^p\,dx \\
&-\left[(g_0-\varepsilon)-\delta\left(1+\frac{C_s}{\rho+1}\left(2\mathcal{E}(0)\right)^\rho\right)\right]\int_\Omega |\nabla u_t|^2\,dx \\
&+\left(\frac{\varepsilon}{2l}+2\delta+\frac{1}{2\delta}+\frac{C_*^2}{4\delta}b\right)\left(\int_0^t g^{2-r}(s)ds\right)(g^r\circ\nabla u)(t),
\end{aligned} \tag{3.19}$$

where $g_0 = \int\limits_0^{t_0} g(s)ds$. At this point, we choose $\varepsilon < g_0$ and δ small enough so that

$$\delta < \min\left\{ \frac{\varepsilon l}{2\left(1+2(1-l)^2+C_1\right)}, \frac{g_0-\varepsilon}{1+\frac{C_s}{\rho+1}\left(2\mathcal{E}(0)\right)^\rho} \right\}.$$

Once ε and δ are fixed, we pick M sufficiently large so that

$$\xi\left[\frac{M}{2} - \frac{g(0)}{4\delta}\left(1+\frac{C_p}{\rho+1}\right)\right] - \left(\int\limits_0^\infty g^{2-r}(s)ds\right)\left(\frac{\varepsilon}{2l} + 2\delta + \frac{1}{2\delta} + \frac{C_*^2}{4\delta}b\right) > 0.$$

Therefore, using the assumption $g'(t) \leq -\xi g^r(t)$ in (G2), (3.19) takes the form

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left\{\xi\left[\tfrac{M}{2} - \tfrac{g(0)}{4\delta}\left(1+\tfrac{C_p}{\rho+1}\right)\right] - \left(\tfrac{\varepsilon}{2l} + 2\delta + \tfrac{1}{2\delta} + \tfrac{C_{*2}^2}{4\delta}b\right)\left(\int\limits_0^t g^{2-r}(s)ds\right)\right\} \\ & \times(g^r \circ \nabla u)(t) - \tfrac{g_0-\varepsilon}{\rho+1}\int\limits_\Omega |u_t|^{\rho+2}dx - \left[\tfrac{\varepsilon l}{2} - \left(1+2(1-l)^2+C_1\right)\delta\right]\int\limits_\Omega |\nabla u|^2 dx \\ & - \left[(g_0-\varepsilon) - \delta\left(1+\tfrac{C_s}{\rho+1}\left(2\mathcal{E}(0)\right)^\rho\right)\right]\int\limits_\Omega |\nabla u_t|^2 dx + \varepsilon b\int\limits_\Omega |u|^p dx. \end{aligned} \tag{3.20}$$

Case 1. $r = 1$:

By virtue of the choice of ε, δ, and M, estimate (3.20) yields, for some constant $\alpha > 0$,

$$\mathcal{L}'(t) \leq -\alpha\mathcal{E}(t), \qquad \forall t \geq t_0. \tag{3.21}$$

On the other hand, proceeding similarly to [10] and using the following estimates

$$\int\limits_\Omega |u_t|^\rho u_t u dx \leq \delta_1 C_p \int\limits_\Omega |\nabla u|^2 dx + \frac{C_s}{4\delta_1}\left(2\mathcal{E}(0)\right)^\rho \int\limits_\Omega |\nabla u_t|^2 dx,\ \delta_1 > 0,$$

$$\begin{aligned} & \int\limits_\Omega |u_t|^\rho u_t \int\limits_0^t g(t-s)\left(u(t)-u(s)\right)dsdx \\ & \leq \delta_2 C_s\left(2\mathcal{E}(0)\right)^\rho \int\limits_\Omega |\nabla u_t|^2 dx + \tfrac{1}{4\delta_2}\tilde{C}_p\left(\int\limits_0^t g^{2-r}(s)ds\right)(g^r \circ \nabla u)(t),\ \delta_2 > 0, \end{aligned}$$

it is easy to show that there exist positive numbers β_1 and β_2 such that

$$\beta_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2\mathcal{E}(t). \tag{3.22}$$

Hence, combining (3.21) and (3.22), we find

$$\mathcal{L}'(t) \leq -\frac{\alpha}{\beta_2}\mathcal{L}(t), \qquad \forall t \geq t_0. \tag{3.23}$$

A simple integration of (3.23) over (t_0, t) leads to

$$\mathcal{L}(t) \leq \mathcal{L}(t_0)e^{-\frac{\alpha}{\beta_2}(t-t_0)}, \qquad \forall t \geq t_0.$$

Therefore (3.2) is established again by virtue of (3.22).

Case 2. $1 < r < 3/2.$:

By using (G1) and (G2) we easily see that

$$\int_0^\infty g^{1-\theta}(\tau)d\tau < \infty, \quad 0 \le \theta < 2-r,$$

so (2.7), (3.1), and Lemma 2.3 yield, for some constant $C_2 > 0$,

$$\begin{aligned}(g\circ\nabla u)(t) &\le C_2\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\sup\mathcal{E}(t)\right\}^{(r-1)/(r-1+\theta)}\{(g^r\circ\nabla u)(t)\}^{\theta/(r-1+\theta)}.\\ &\le C_2\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\mathcal{E}(0)\right\}^{(r-1)/(r-1+\theta)}\{(g^r\circ\nabla u)(t)\}^{\theta/(r-1+\theta)}\end{aligned} \quad (3.24)$$

Therefore we get, for $\gamma > 1$,

$$\begin{aligned}\mathcal{E}^\gamma(t) &\le C_3\left[\mathcal{E}^{\gamma-1}(0)\left\{\int_\Omega |u_t|^{\rho+2}dx + ||\nabla u_t||_2^2 + ||\nabla u||_2^2 - \frac{2b}{p}||u||_p^p\right\} + \{(g\circ\nabla u)(t)\}^\gamma\right]\\ &\le C_3\mathcal{E}^{\gamma-1}(0)\left\{\int_\Omega |u_t|^{\rho+2}dx + ||\nabla u_t||_2^2 + ||\nabla u||_2^2 - \frac{2b}{p}||u||_p^p\right\} \\ &\quad + C_4\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\mathcal{E}(0)\right\}^{\gamma(r-1)/(r-1+\theta)}\{(g^r\circ\nabla u)(t)\}^{\gamma\theta/(r-1+\theta)},\end{aligned} \quad (3.25)$$

for some $C_3, C_4 > 0$. By choosing $\theta = \frac{1}{2}$ and $\gamma = 2r-1$ (hence $\gamma\theta/(r-1+\theta) = 1$), estimate (3.25) gives

$$\mathcal{E}^\gamma(t) \le C_5\left\{\int_\Omega |u_t|^{\rho+2}dx + ||\nabla u_t||_2^2 + ||\nabla u||_2^2 - \frac{2b}{p}||u||_p^p + (g^r\circ\nabla u)(t)\right\}, \quad (3.26)$$

for some $C_5 > 0$. A combination of (3.20), (3.22) and (3.26) then leads to

$$\mathcal{L}'(t) \le -\frac{\beta}{C_5}\mathcal{E}^\gamma(t) \le -\frac{\beta}{C_5}(\beta_2)^{-\gamma}\mathcal{L}^\gamma(t), \qquad \forall t \ge t_0, \quad (3.27)$$

for some $\beta > 0$. A simple integration of (3.27) over $(t_0,\, t)$ yields

$$\mathcal{L}(t) \le C(1+t)^{-1/(\gamma-1)}, \qquad \forall t \ge t_0. \quad (3.28)$$

As a consequence of (3.28), we have

$$\int_0^\infty \mathcal{L}(t)dt + \sup_{t\ge 0} t\mathcal{L}(t) < \infty.$$

Therefore, by using (2.8) we have

$$g\circ\nabla u \le \left[\int_0^t \|\nabla u(s)\|_2^2 ds + t\|\nabla u(t)\|_2^2\right]^{(r-1)/r}(g^r\circ\nabla u)^{1/r}$$

$$\leq C_6 \left[\int_0^t \mathcal{E}(s)ds + t\mathcal{E}(t)\right]^{(r-1)/r} (g^r \circ \nabla u)^{1/r}$$

$$\leq \frac{C_6}{\beta_1}\left[\int_0^t \mathcal{L}(s)ds + t\mathcal{L}(t)\right]^{(r-1)/r} (g^r \circ \nabla u)^{1/r} \leq C_7(g^r \circ \nabla u)^{1/r},$$

which implies that

$$g^r \circ \nabla u \geq C_8(g \circ \nabla u)^r. \tag{3.29}$$

Consequently, a combination of (3.20) and (3.29) yields

$$\mathcal{L}'(t) \leq -C_9 \left[\int_\Omega |u_t|^{\rho+2}dx + \int_\Omega |\nabla u|^2 dx - \frac{2b}{p}||u||_p^p \right. \tag{3.30}$$

$$\left. + \int_\Omega |\nabla u_t|^2 dx + (g \circ \nabla u)^r\right], \qquad \forall\, t \geq t_0.$$

On the other hand, we have similarly to (3.25),

$$\mathcal{E}^r(t) \leq C_{10} \left[\int_\Omega |u_t|^{\rho+2}dx + \int_\Omega |\nabla u|^2 dx - \frac{2b}{p}||u||_p^p \right. \tag{3.31}$$

$$\left. + \int_\Omega |\nabla u_t|^2 dx + (g \circ \nabla u)^r\right], \qquad \forall\, t \geq t_0.$$

Combining the last two inequalities, we obtain

$$\mathcal{L}'(t) \leq -C_{11}\mathcal{L}^r(t), \quad t \geq t_0. \tag{3.32}$$

A simple integration of (3.32) over (t_0, t) gives

$$\mathcal{L}(t) \leq K(1+t)^{-1/(r-1)}, \quad t \geq t_0. \tag{3.33}$$

Therefore (3.3) is obtained by virtue of (3.22). This completes the proof.

Remark 3.1 Note that our result is proved without any condition on g'' and g''' unlike what was assumed in (2.4) of [5]. We only need g to be differentiable and satisfying (G1) and (G2).

Remark 3.2 By using the fact that $\mathcal{E}$ is bounded on $[0, t_0]$, we can easily show that estimates (3.2), (3.3) hold for $t \geq 0$.

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