Trace class properties of the non homogeneous linear Vlasov-Poisson equation in dimension 1+1

Bruno Després

Trace class properties of the non homogeneous linear Vlasov-Poisson equation in dimension 1+1

Bruno Després

Abstract. We consider the abstract scattering structure of the non homogeneous linearized Vlasov-Poisson equations from the viewpoint of trace class properties which are emblematic of the abstract scattering theory [13, 14, 15, 19]. In dimension 1+1, we derive an original reformulation which is trace class. It yields the existence of the Moller wave operators. The non homogeneous background electric field is periodic with $4+\varepsilon$ bounded derivatives.

Mathematics Subject Classification (2010). Primary: 47A40; Secondary: 35P25.

Keywords. Trace class, linear Vlasov-Poisson, wave operators.

1. Introduction

It has been observed many times in the literature that linear Landau damping for the linear Vlasov-Poisson equation presents many similarities with abstract scattering theory: some references in this direction are [2, 5, 9]. However it seems, let us refer to [17], that it is has rarely been shown that the linear Vlasov-Poisson can be analyzed within abstract scattering theory [13, 14, 15, 19]. A recent attempt is nevertheless [8] with a technique based on a complicated Lipmann-Schwinger equation.

The purpose of this work is precisely to analyze the abstract scattering structure of the non homogeneous linear Vlasov-Poisson equation from the viewpoint of trace class structures which are emblematic of the abstract scattering theory [13, 14]. We will show that the model Vlasov-Poisson equation (rewritten in Vlasov-Ampère form) does not satisfy naturally the trace class property, but an original reformulation (denoted as the reduced equation) satisfies it. It yields the existence of the wave operators by means of the Kato-Birmann theory, and consequently, shows that the absolutely continuous part of the spectrum is unitarily equivalent to the one of free transport. One technical crux of proving the trace class property is the Diperna-Lions Theorem of compactness by integration which provides the required small gain of regularity. In this work, the techniques employed to prove the trace class are strongly restricted to dimension 1+1, that is one in space and one in velocity: in particular the derivation of the reduced Vlasov equation is made possible by means of a specific Ricatti equation which has no simple equivalent in higher dimensions; and also the version of the Diperna-Lions Theorem of compactness used to provide the required small gain of regularity is remark 6 page 741 in [10], which is an one dimensional argument. An explicit form of the wave operators exists [17, 7] in the special case of an homogeneous vanishing electric potential ($\varphi_0 = 0$), even if the notion of wave operators is not explicitly mentioned in these references.

The original results summarized in Theorem 1.1 are general in the sense that, for establishing the unitary equivalence of the absolutely continuous part of the spectrum of the full problem with the one of free transport, no structure on the background electric potential is needed expect that being periodic and having $4+\varepsilon$ bounded derivatives.

General considerations. The starting point is the Vlasov-Poisson equation for one species of negatively charged particle (electrons) in a plasma. For simplicity we consider functions which are 1-periodic in space, that is $g(x+1) = g(x)$ for all quantities $g = f,\ E,\ \rho_{\rm ref},\ \dots$: the torus will be denoted as $\mathbb{T} = [0,1]$. The kinetic density of electrons is $f(t,x,v) \geq 0$. The given stationary fluid density of static ions is $\rho_{\rm ref}(x)$. The total mass of electrons is $\int_{\mathbb{T}\times\mathbb{R}} f_{\rm ini} dxdv = \int_{\mathbb{T}} \rho_{\rm ref}(x)dx$. The electric potential is $\varphi(t,x)$ and the electric field is $E(t,x) = -\partial_x \varphi(t,x)$. As a consequence of a Vlasov equation for electrons and of the Gauss law, one has the fundamental identity $\partial_x \left(\partial_t E - \int_{\mathbb{R}} fvdv\right) = \partial_t \partial_x E - \partial_x \int_{\mathbb{R}} fvdv = -\partial_t \int_{\mathbb{R}} fdv - \partial_x \int_{\mathbb{R}} fvdv = 0$. So one can write the Ampère law under the form $\partial_t E = 1^* \int_{\mathbb{R}} vfdv$ where we will note $1^* : L^2(\mathbb{T}) \to L^2_0(\mathbb{T})$ the usual projection operator such that $1^* g = g - \int_{\mathbb{T}} g(x)dx$. We introduce the operator 1^* in the Vlasov-Ampère system

$$\left\{ \begin{array}{lll} \partial_t f + v\partial_x f - E\partial_v f = 0, & t>0, & (x,v) \in \mathbb{T}\times\mathbb{R}, \\ \partial_t E = 1^* \int_{\mathbb{R}} vfdv, & t>0, & x \in I, \\ \partial_x E = \rho_{\rm ref}(x) - \int_{\mathbb{R}} fdv, & t>0, & x\in I. \end{array} \right. \quad (1)$$

The solutions of this system preserve the physical energy $\frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{R}} fv^2dvdx + \frac{1}{2}\int_{\mathbb{T}} E^2 dx$. They also preserve the mass of electrons $\int_{\mathbb{T}}\int_{\mathbb{R}} f(t,x,v)dxdv =$

$\int_{\mathbb{T}}\int_{\mathbb{R}} f_{\rm ini}(x,v)dxdv$ and the zero mean value of the electric field $\int_{\mathbb{T}} E(t,x)dx = 0$. In this work, the initial data (f,E) will be considered as a small perturbation of a stationary state (f_0,E_0) such that $v\partial_x f_0 - E_0(x)\partial_v f_0 = 0$ with $E_0 = -\varphi_0'$. The natural Boltzmanian hypothesis to represent such stationary states is $f_0(x,v) = \exp\left(-\frac{v^2}{2} + \varphi_0(x)\right)$ where φ_0 is the reference electric potential: the Boltzmanian hypothesis is a strong hypothesis that probably can be modified [3, 4, 11] or relaxed as in [16]: however we keep it in this work for the simplicity of the physical interpretation. Consider a linearization under the form $f(t,x,v) = f_0(x,v) + g(t,x,v)$ and $E(t,x) = E_0(x) + F(t,x)$, inject in (1) and drop the quadratic terms. It yields the system

$$\begin{cases} \partial_t g + v\partial_x g - E_0\partial_v g - F\partial_v f_0 = 0, & t>0, \quad (x,v)\in\mathbb{T}\times\mathbb{R}, \\ \partial_t F = 1^* \int_{\mathbb{R}} vgdv, & t>0, \quad x\in\mathbb{T}, \\ \partial_x F = -\int_{\mathbb{R}} gdv, & t>0, \quad x\in\mathbb{T}. \end{cases}$$

Model problem. Define $M(x,v) = \sqrt{f_0(x,v)} = \exp\left(-\frac{v^2}{4} + \frac{\varphi_0(x)}{2}\right)$ and the function $u = \frac{g}{M}$. Using that $(v\partial_x - E_0\partial_v)M = 0$, one gets the model linear Vlasov-Ampère system studied in this work

$$\begin{cases} \partial_t u + v\partial_x u - E_0\partial_v u &= -vMF, \quad t>0, \quad (x,v)\in\mathbb{T}\times\mathbb{R}, \\ \partial_t F &= 1^*\int_{\mathbb{R}} uvMdv, \quad t>0, \quad x\in\mathbb{T}. \end{cases} \tag{2}$$

The initial data satisfies

$$\int_{\mathbb{T}}\int_{\mathbb{R}} M(x,v)u_{\rm ini}(x,v)dxdv = 0 \text{ and } \int_{\mathbb{T}} F_{\rm ini}(x)dx = 0. \tag{3}$$

The energy is preserved

$$\frac{d}{dt}\left(\int_{\mathbb{T}}\int_{\mathbb{R}} u^2 dvdx + \int_{\mathbb{T}} F^2 dx\right) = 0. \tag{4}$$

The Gauss law

$$\partial_x F = -\int_{\mathbb{R}} uMdv \tag{5}$$

is understood as a constraint satisfied by the initial data and propagated by the equation. It can be characterized in the weak sense

$$\int_{\mathbb{T}\times\mathbb{R}} u(x,v)M(x,v)\varphi(x)dxdv - \int_{\mathbb{T}} F(x)\varphi'(x)dx = 0 \quad \forall\varphi\in H^1(\mathbb{T}). \tag{6}$$

The integrals (3) and the energy (4) are integral invariants of the system (2). The Gauss-law (5)-(6) is an integro-differential invariant.

Functional setting. Define the space $L_0^2(\mathbb{T}) := \left\{ F \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} F(x)dx = 0 \right\}$. Define the space $L_0^2(\mathbb{T}\times\mathbb{R}) := \left\{ u \in L^2(\mathbb{T}\times\mathbb{R}) \mid \int_{\mathbb{T}\times\mathbb{R}} M(x,v)u(x,v)dxdv = 0 \right\}$: it expresses that the physical perturbation has zero mass. Consider the first line of (2) where the operator $vM \in \mathcal{L}\left(L_0^2(\mathbb{T}), L_0^2(\mathbb{T}\times\mathbb{R})\right)$ shows up. One has the identity

$$\int_{\mathbb{T}}\int_{\mathbb{R}} u(vMF)dvdx = \int_{\mathbb{T}} 1^* \left(\int_{\mathbb{R}} uvMdv \right) Fdx$$

which shows that the adjoint operator of vM is

$$(vM)^* : \quad u \mapsto 1^* \int_{\mathbb{R}} vuMdv \in \mathcal{L}\left(L_0^2(\mathbb{T}\times\mathbb{R}), L_0^2(\mathbb{T})\right).$$

Define for convenience the space of complex-valued functions

$$X = L_0^2(\mathbb{T}\times\mathbb{R}) \times L_0^2(\mathbb{T}).$$

The subspace $GL \subset X$ characterizes pairs which satisfy the Gauss law in the weak sense

$$GL = \{(u,F) \in X, \text{ the weak Gauss law (6) holds}\}. \tag{7}$$

We will need two hermitian products. The first one is the classical quadratic product for square integrable functions

$$(u,w) = \int_{\mathbb{T}}\int_{\mathbb{R}} u(x,v)\overline{w(x,v)}dxdv, \quad u,w \in L^2(\mathbb{T}\times\mathbb{R}).$$

The second one is, in view of the energy identity, the natural one for $(u,F) \in X$ and $(w,G) \in X$

$$((u,F),(w,G)) = (u,w) + \int_{\mathbb{T}} F(x)\overline{G(x)}dx. \tag{8}$$

From now on, one systematically introduces the pure imaginary number $i^2 = -1$ to obtain compatibility with more standard notations in scattering theory [13, 15, 19]. One recasts the linear Vlasov-Ampère equations (2) as $\partial_t U(t) + iHU(t) = 0$ where the unknown is $U = \begin{pmatrix} u \\ F \end{pmatrix} \in GL$ and the anti-symmetric operator is formally

$$iH = \left(\begin{array}{c|c} v\partial_x - E_0\partial_v & vM \\ \hline -1^* \int_v vM & 0 \end{array} \right).$$

One has the decomposition of operators $iH = iH_0 + iK$ where

$$iH_0 = \left(\begin{array}{c|c} v\partial_x - E_0\partial_v & 0 \\ \hline 0 & 0 \end{array}\right) \text{ and } iK = \left(\begin{array}{c|c} 0 & vM \\ \hline -1^* \int_v vM & 0 \end{array}\right). \quad (9)$$

In terms of the scattering theory, the main question is to explore the dynamics of the full Hamiltonian e^{-iHt} with respect to the dynamics of the reduced Hamiltonian e^{-iH_0t}. However the dynamics attached to H_0 does not preserve the Gauss law (5) so one can expect troubles with this way of writing the scattering structure in $L^2(\mathbb{T}\times\mathbb{R})$ instead of GL. And indeed, we will show that this decomposition does not have the trace property, even if it almost trace-class (this will be precized).

This is why another framework will be introduced in Section 3. The idea is to consider a new kinetic function

$$w(x,v) = u(x,v) + \gamma(x)M(x,v)F(x)$$

and the 1-periodic function γ solution to the Ricati equation

$$\partial_x\gamma + \alpha^2\gamma^2 \exp\varphi_0 = 1, \qquad \alpha = (2\pi)^{\frac{1}{4}}.$$

If the pair (u, F) satisfies (2)-(5), then purely algebraic manipulations show that w is solution to the autonomous equation

$$w'(t) = i\mathcal{H}w(t), \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{K}$$

where $i\mathcal{H}_0 = v\partial_x - E_0\partial_v$ is the transport operator and $i\mathcal{K}$ is integral operator defined by

$$i\mathcal{K}w = \gamma\left(vM\int_{\mathbb{R}} wMdv - M\int_{\mathbb{R}} wvMdv\right) + \gamma M\int_{\mathbb{R}\times\mathbb{T}} wvMdv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right)Mv$$

This is an equivalence, that is there is a bijection between GL (7) and

$$\mathcal{X} := \left\{w \in L^2(\mathbb{T}\times\mathbb{R}),\ (w,\gamma M) = 0\right\} \subset L^2(\mathbb{T}\times\mathbb{R}).$$

The function γM which enters in the definition of the space $\mathcal{X}$ will be shown to be spectral, that is $\mathcal{H}\gamma M = 0$. The main result obtained in Section 4 is that the trace class property holds for this decomposition. It yields the

Theorem 1.1. *Assume the electric potential is smooth $\varphi_0 \in W^{4+\varepsilon,\infty}(\mathbb{T})$. Then the wave operators $\mathcal{W}_\pm(\mathcal{H},\mathcal{H}_0)$ exist and are complete. In particular one has the orthogonal decompositions between spaces associated to absolute continuous, singular continuous and discrete parts of the spectrum*

$$L^2(\mathbb{T}\times\mathbb{R}) = \mathcal{X}_0^{\text{ac}} \oplus \mathcal{X}_0^{\text{sc}} \oplus \mathcal{X}_0^{\text{pp}} = \mathcal{X}^{\text{ac}} \oplus \mathcal{X}^{\text{sc}} \oplus \mathcal{X}^{\text{pp}} \quad (10)$$

and there exists two complete wave operators $\mathcal{W}_\pm$ isometric on $\mathcal{X}_0^{\mathrm{ac}}$.

A direct consequence is that the space X^{ac} (associated to the absolute continuous part of the spectrum of iH) is in bijection with a space isometric to $\mathcal{X}_0^{\mathrm{ac}}$ (associated to the absolute continuous part of the spectrum of the free transport operator $i\mathcal{H}_0$).

This result describes the strong constraints on the absolutely continuous part of the spectrum of the operators. Essentially, they are the same. However more information on the singular continuous part and on the pure point spectrum is needed to describe completely the decomposition (10). As usual for problems which come from classical physics, one can expect that the singular continuous part is empty, that is $\mathcal{X}_0^{\mathrm{sc}} = \mathcal{X}^{\mathrm{sc}} = \emptyset$. Standard explicit representation prove it is indeed the case for some reasonable electric potential φ_0. The pure point spectrum which corresponds to classical eigenvectors of the operators can be studied by direct means which are outside the scope of the present paper.

There are two cases where we know exactly the absolutely continuous part of the spectrum of the different operators. The first one is the homogeneous case E_0, and the spectral decomposition of H_0 and H is explicitly calculated in Section 2, so one knows the absolutely continuous part of the spectrum. The second case is in the recent work [8] where the explicit and complete calculation of the spectral decomposition is performed for a non homogeneous one-bump (in space) background electric field: a phase portrait method gives the spectral decomposition of H_0 and the absolutely continuous part of the spectrum; then a method based on a Lipmann-Schwinger equation gives the the spectral decomposition of H.

Organization. The plan of this paper is detailed below. After providing in Section 2 the reader with elements of abstract scattering theory, we will show in Proposition 2.9 that the operator iK is not a trace class perturbation of iH_0: this will be proved in the homogeneous case ($\varphi_0 = 0$). However, for homogeneous profiles, Fourier decomposition is available, and, Fourier mode per Fourier mode, iK is shown to be a trace class perturbation of iH_0. The conclusion is that, in the general case for which simple Fourier decomposition is not possible, iK is not a trace class perturbation of iH_0. In Section 3, one introduces a further reduction of the Vlasov-Poisson-Ampère model (operators become $i\mathcal{H}$, $i\mathcal{H}_0$ and $i\mathcal{K}$). Then in the next Section, one proves that $i\mathcal{K}$ is a trace class perturbation of $i\mathcal{H}_0$. This reduction is written as a linear Boltzman operator where the integrals are only in the velocity variable. The proof is based on an original reformulation (28) of the equations plus a careful study of the regularity of the operators. A key instrument is the Diperna-Lions compactness by integration Theorem [10]. The main result is Proposition 4.8. The last part of the main Theorem is proved and a classical technical result is provided in the appendix.

2. Elements of Kato-Birman theory

We use the definitions, notations and results from [19, 13, 15]. In this section we work in the space X.

The operators H and H_0 are understood in this section as operators with domain in X and value in X. The transport operator is denoted as $D = v\partial_x - E_0(x)\partial_v$. The domain of $H : X \to X$ and $H_0 : X \to X$ is $\mathcal{D} = \left\{(u, F) \in X,\ Du \in L^2(\mathbb{T}\times\mathbb{R})\right\}$. The condition $Du \in L^2(\mathbb{T}\times\mathbb{R})$ is understood in the sense of distribution.

Lemma 2.1. *The operators H_0 and H are closed.*

Proof. This is immediate by construction for H_0 for which there is no condition on F. Indeed let $(u_n, a_n) \to (u, a)$ in $L^2_0(\mathbb{T}\times\mathbb{R})\times L^2_0(\mathbb{T}\times\mathbb{R})$ be such that $Du_n = a_n$ in the sense of distributions. Passing to the limit in the sense of distributions, one gets that $Du = a$. So H_0 is closed. Concerning H, write $H\begin{pmatrix} u_n \\ F_n \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ in the sense of distributions against a pair (φ, ψ) of smooth test functions with compact support in velocity for ψ. One gets $-i\int_{\mathbb{T}\times\mathbb{R}} u_n (D\varphi + vM\psi) + i\int_{\mathbb{T}} F_n vM\varphi = \int_{\mathbb{T}\times\mathbb{R}} a_n\varphi + \int_I b_n\psi$. Passing to the limit $(u_n, F_n, a_n, b_n) \to (u, F, a, b)$ in $L^2_0(\mathbb{T}\times\mathbb{R})\times L^2_0(\mathbb{T})\times L^2_0(\mathbb{T}\times\mathbb{R})\times L^2_0(\mathbb{T})$, one gets

$$- i\int_{\mathbb{T}\times\mathbb{R}} u\,[D\varphi + vM\psi] + i\int_{\mathbb{T}} FvM\varphi = \int_{\mathbb{T}\times\mathbb{R}} a\varphi + \int_I b\psi \qquad (11)$$

for all admissible test functions. There exists $G \in H^1(\mathbb{T})$ such that $F = -\partial_x G$. One notices the identity $D(MG) = GDM + MDG = 0 - vMF = -vMF$. Plug in (11) and take $\psi = 0$. One gets $-i\int_{\mathbb{T}\times\mathbb{R}} uD\varphi - i\int_{\mathbb{T}} D(MG)\varphi = \int_{\mathbb{T}\times\mathbb{R}} a\varphi$, that is after integration by parts, $-i\int_{\mathbb{T}\times\mathbb{R}}(u - MG)D\varphi = \int_{\mathbb{T}\times\mathbb{R}} a\varphi$. So $D(u - MG) = -ia \in L^2(\mathbb{T}\times\mathbb{R})$ which turns into $Du \in L^2(\mathbb{T}\times\mathbb{R})$. Therefore $(u, F) \in D(H)$ and the proof is ended. □

Lemma 2.2. *The operators H_0 and H are self-adjoint.*

Proof. The proof below uses the Stone's theorem [15, 19] which is convenient for problems coming from transport equations. Since H is a bounded perturbation of H_0, it is sufficient to show that H_0 is self-adjoint. And actually it is sufficient to show that $\mathcal{H}_0 = iD$ with domain $\mathcal{D}(\mathcal{H}_0) = \left\{\psi \in L^2(\mathbb{T}\times\mathbb{R}), D\psi \in L^2(\mathbb{T}\times\mathbb{R})\right\}$ is self-adjoint. The proof will be performed with the regularity assumption $\varphi_0 \in W^{2,\infty}(\mathbb{T})$, which is enough in view of Theorem 1.1.

It is convenient to introduce the semi-group

$$U(t) : L^2(\mathbb{T}\times\mathbb{R}) \longrightarrow L^2(\mathbb{T}\times\mathbb{R})$$

obtained by firstly constructing the characteristic lines and secondly transporting datas at time $t=0$ along the characteristic lines until time $t>0$. The characteristic lines are defined by $t \mapsto (X_t(x,v), V_t(x,v))$ for $t \in \mathbb{R}$ where $\frac{d}{dt}X_t(x,v) = V_t(x,v)$ and $\frac{d}{dt}X_t(x,v) = -E_0(X_t(x,v))$ with initial data $X_0(x,v) = x$ and $V_0(x,v) = v$. Let $\psi \in L^2(\mathbb{T}\times\mathbb{R})$, then the semi-group is defined by

$$U(t)\psi(x,v) = \psi(X_{-t}(x,v), V_{-t}(x,v)).$$

By definition $U(t)^{-1} = U(-t)$. One has the invariance of measure $dX_t dV_t = dxdv$, so by construction, $U(t)$ is unitary. The semi-group is also strongly continuous (it is sufficient to verify this for data with compact support, and this is evident). By the Stone's theorem [15, 19], there exists a self-adjoint operator A such that $U(t) = e^{itA}$. More precisely, see Theorem VIII.7 page 265 in [19], the domain of A (denoted as $\mathcal{D}(A)$) is equal to the set $\psi \in L^2(\mathbb{T}\times\mathbb{R})$ such that $\lim_{t\to 0^+} \frac{U(t)-I}{t}\psi = iA\psi$ exists in $L^2(\mathbb{T}\times\mathbb{R})$ (a similar reasoning is used in [6][Proposition 1, page 219]).

To identify this limit, one can perform the following calculation. Let us take two smooth functions ψ and φ. One has

$$\left(\frac{U(t)\psi-\psi}{t},\varphi\right) = \frac{1}{t}\int_{\mathbb{T}}\int_{\mathbb{R}} \left(\psi(X_{-t}(x,v),V_{-t}(x,v)) - \psi(x,v)\right)\varphi(x,v)dxdv$$

$$= -\frac{1}{t}\int_{-t}^{0}\int_{\mathbb{T}}\int_{\mathbb{R}} (D\psi)\,(X_{-s}(x,v),V_{-s}(x,v)\varphi(x,v)dxdvds.$$

By the invariance of measure, one gets the identity

$$\left(\frac{U(t)\psi-\psi}{t},\varphi\right) = -\frac{1}{t}\int_{-t}^{0}\int_{\mathbb{T}}\int_{\mathbb{R}} (D\psi)\,(x,v)\varphi(X_s(x,v),V_s(x,v))dxdvds$$

$$= -\left(D\psi, \frac{1}{t}\int_{-t}^{0} U(-s)\varphi ds\right) = -(D\psi,\varphi) + \left(D\psi, \frac{1}{t}\int_{-t}^{0} (\varphi - U(-s)\varphi)ds\right). \tag{12}$$

• Let us now take $\psi \in \mathcal{D}(A)$, φ a smooth test function with compact support and do an integration by parts in the right hand side of (12)

$$\left(\frac{U(t)\psi-\psi}{t},\varphi\right) = (\psi, D\varphi) - \left(\psi, \frac{1}{t}\int_{-t}^{0}(D\varphi - DU(-s)\varphi)ds\right).$$

One can pass to the limit $t \to 0^+$. Since $E_0 \in W^{1,\infty}(\mathbb{T})$, one can show that $\|D\varphi - DU(-s)\varphi\|_{L^2(\mathbb{T}\times\mathbb{R})} \le C(\varphi)s$. It shows that $\lim_{t\to 0^+} \frac{U(t)\psi-\psi}{t} = -D\psi$ in the sense of distributions. Therefore, in the sense of distributions, $iA = -D$ that is $A = \mathcal{H}_0$, and also $\mathcal{D}(A) \subset \mathcal{D}(\mathcal{H}_0)$.
• Reciprocally, take $\psi \in \mathcal{D}(\mathcal{H}_0)$. From the identity (12), one derives the estimate $\left|\left(\frac{U(t)\psi-\psi}{t},\varphi\right) + (D\psi,\varphi)\right| \le \|D\psi\|\frac{1}{t}\int_{-t}^{0}\|\varphi - U(-s)\varphi\|ds$ which makes sense for $\varphi \in L^2(\mathbb{T}\times\mathbb{R})$. Since the semi-group is strongly continuous, one gets that $\frac{U(t)\psi-\psi}{t} \to D\psi$ in $L^2(\mathbb{T}\times\mathbb{R})$. Therefore $\psi \in \mathcal{D}(A)$.
• Therefore $\mathcal{D}(A) = \mathcal{D}(\mathcal{H}_0)$ and the operators coincide in the sense of distribution. So $\mathcal{H}_0 = A$ is a self-adjoint operator.
• Finally H_0 and H are self-adjoint operators. □

Therefore, [15, 13, 19], the spectrum of the operators can be decomposed in terms of theory of measure. Below, we characterize the results for the operator H_0. There is a orthogonal decomposition of the Hilbert space X into the orthogonal sum of invariant subspaces of the operator H_0

$$X = X_0^{\rm ac} \oplus X_0^{\rm sc} \oplus X_0^{\rm pp} \tag{13}$$

where $X_0^{\rm ac}$ (resp. $X_0^{\rm sc}$, resp. $X_0^{\rm pp}$) corresponds to the absolutely continuous (resp. singular continuous, resp. pure point) part of the spectrum. The pure point subspace $X_0^{\rm pp}$ is spanned by the eigenvectors

$$X_0^{\rm pp} = \text{Span } \{\varphi \in X,\ \varphi = \lambda\varphi \text{ for some } \lambda \in \mathbb{R}\}.$$

The subspace $X_0^{\rm ac}$ is characterized by the existence of a dense set $\mathcal{D}^{\rm ac} \subset X^{\rm ac}$ such that

$$\varphi \in \mathcal{D}^{\rm ac} \Longrightarrow \|(H_0 - \lambda \pm i\varepsilon)^{-1}\varphi\| \le \frac{C_{\varphi,\lambda}}{\sqrt{\varepsilon}} \text{ for } \lambda \in \mathbb{R} \text{ and } 0 < \varepsilon \le 1. \tag{14}$$

This result is proved in [12](see also [13]). Mutatis mutandis, this characterization hold also for H.

Let P_0 be the projection operator onto the $X_0^{\rm ac}$.

Definition 2.3. The limit $W_\pm = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} P_0$ (if it exists) is called the wave operator. If $W_\pm$ exists, it is isometric on $X_0^{\rm ac}$.

Definition 2.4. If ran $W_\pm = X^{\rm ac}$, then $W_\pm$ is said to be complete.

If the wave operators exist and are complete, then H_0 and H are unitarily equivalent and their absolutely continuous spectrum is the same (as a subset of $\mathbb{R}$). In the context of this study, it has the following interesting consequence. Let $f \in X^{\rm ac}$, so there exists $g_\pm \in X$ such that $W_\pm g_\pm = f$. It yields that $f = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} P_0 g_\pm$ which is rewritten as $\lim_{t\to\pm\infty} \left\| e^{-itH} f - e^{-itH_0} P_0 g_\pm \right\| = 0$. In other words, the long time dynamics of the full Hamiltonian applied to $f \in X^{\rm ac}$ is identical to the long time dynamics of the simplified Hamiltonian for $g_\pm$. Linear Landau damping is just a corollary of this property for initial data in the absolutely continuous part of the spectrum.

However it is necessary to prove the existence and completeness of the wave operator. The celebrated Kato-Birman theory brings a criterion, see Theorem 2.5 below. It is based the trace class criterion for a compact operator T

$$\|T\|_1 = \sum \lambda_j (T^*T)^{\frac{1}{2}} < \infty \tag{15}$$

where the $(\lambda_j)_{j\in\mathbb{N}}$ is the sequence of all non negative eigenvalues of the compact operator T^*T. Actually $\|\cdot\|_1$ is a norm, refer to section X.1.3 in [13]).

Theorem 2.5 (Kato-Birman). *Suppose that $(H-z)^{-n} - (H_0 - z)^{-n}$ is trace-class for some $n \in \mathbb{N}^*$ and all z with Im $z \neq 0$. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

Let us now derive a partially negative result which concerns the homogeneous case, that is $E_0 := 0$ and $M := \exp(-v^2/4)$. In this case one can split the operators along a Fourier decomposition (mode $k \in \mathbb{Z}$) so as to define

$$H_0^k = \left(\begin{array}{c|c} vk & 0 \\ \hline 0 & 0 \end{array} \right) \text{ and } H^k = H_0^k - i\delta_k \left(\begin{array}{c|c} 0 & ve^{-\frac{v^2}{4}} \\ \hline -\int ve^{-\frac{v^2}{4}} \cdot dv & 0 \end{array} \right) \tag{16}$$

where $\delta_0 = 0$ and $\delta_k = 1$ for $k \neq 0$. In this case the orthogonal decompositions are easy to determine.

Lemma 2.6 (Operator $H_0^0 = H^0 = 0$, that is $k = 0$). *The orthogonal decomposition of the space $L_0^2(\mathbb{R}) \times \{0\}$ is given by $(X_0^0)^{\rm ac} = (X_0^0)^{\rm sc} = (X^0)^{\rm ac} = (X^0)^{\rm sc} = \emptyset$ and $(X_0^0)^{\rm pp} = (X^0)^{\rm pp} = L_0^2(\mathbb{R}) \times \{0\}$.*

Lemma 2.7 (Operator H_0^k for $k \neq 0$). *The orthogonal decomposition of the space $L^2(\mathbb{R}) \times \mathbb{C}$ is given by $(X_0^k)^{\rm ac} = L^2(\mathbb{R})$, $(X_0^k)^{\rm sc} = \emptyset$ and $(X_0^k)^{\rm pp} = Ker(H_0^k) = \{0\} \times \mathbb{C}$.*

Proof. Characterization of the kernel $\mathrm{Ker}(H_0^k) = \{0\} \times \mathbb{C}$ is immediate. Next $u \in Y = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ wich is a dense subset in $L^2(\mathbb{R})$. Then the criterion (14) holds with $\mathcal{D}^{\rm ac} = Y \times \{0\}$ because

$$\left\|(H_0^k - \lambda + i\varepsilon)^{-1}(u,0)\right\|^2 = \int_{\mathbb{R}} \frac{|u(v)|^2}{(vk-\lambda)^2+\varepsilon^2} dv$$

$$\leq \int_{\mathbb{R}} \frac{\|u\|^2_{L^\infty(\mathbb{R})}}{(vk-\lambda)^2+\varepsilon^2} dv = \frac{\|u\|^2_{L^\infty(\mathbb{R})}}{|k|\varepsilon}.$$

Since $L^2(\mathbb{R}) \times \{0\} \oplus \mathrm{Ker}(H_0^k) = L^2(\mathbb{R}) \times \mathbb{C}$ is the full space, it is the orthogonal decomposition of the claim. In passing it shows there is no singular continuous part of the spectrum. □

Lemma 2.8 (Operator H^k for $k \neq 0$). *The orthogonal decomposition of the space is given by $(X^k)^{\rm ac} = \left((X^k)^{\rm pp}\right)^T$ and $(X^k)^{\rm sc} = \emptyset$ where*

$$(X^k)^{\rm pp} = Ker(H_0^k) = Span(u_k) \text{ where } u_k = (i\exp(-v^2/4), k).$$

Proof. Note $u_k = (a_k, F_k)$. The equation $H_k u_k = 0$ reduces to $vka_k - iv\exp(-v^2/4)F_k = 0$ and $\int v \exp\left(-\frac{v^2}{4}\right) a_k(v)dv = 0$, which yields $ka_k = i\exp(-v^2/4)F$. One obtains the eigenvector

$$u_k = (i\exp(-v^2/4), k)^t. \tag{17}$$

Therefore $\mathrm{Ker}(H_0^k) = \mathrm{Span}(u_k)$.

Next take $(u, F) \in \mathrm{Ker}(H_0^k)^\perp$ in the orthogonal of the Kernel

$$-i \int_{\mathbb{T}\times\mathbb{R}} u(x,v)\exp(-v^2/4)dxdv + Fk = 0. \tag{18}$$

This identity is the translation in Fourier that the pair (u, F) satisfies the Gauss law. Let us consider a pair $(g, h) \in \mathrm{Ker}(H_0^k)^\perp$ and a pair (u, F)

(necessarily in $\mathrm{Ker}(H_0^k)^\perp$) such that $(H_0^k - \lambda - i\varepsilon)(u, F) = (g, h)$. We try to verify the criterion (14). Note that

$$(vk - \lambda - i\varepsilon)u(v) - iFve^{-\frac{w^2}{4}} = g$$

and that F can be determined with the Gauss law under the form (18).

To check the criterion (14), one can use a method originally due to [18] which has been adapted in [7] to the quadratic framework. It corresponds to the definition of the operator L_k

$$(L_k u)(v) = (k^2 + q(v))u(v) - ve^{-\frac{v^2}{4}} P.V. \int_{\mathbb{R}} \frac{1}{w - v} u(w) e^{-\frac{w^2}{4}} dw$$

where $q(v) = P.V. \int_{\mathbb{R}} \frac{w}{w-v} e^{-\frac{w^2}{2}} dw$. For $k \neq 0$, the operator L_k is bounded in $H^s(\mathbb{R}) \to H^s(\mathbb{R})$ for all $s \geq 0$, is invertible with inverse $(L_k)^{-1}$ also bounded in $H^s(\mathbb{R}) \to H^s(\mathbb{R})$ for all $s \geq 0$. Let $z = L_k u$. With [7][page 2566-2567], it is an algebraic exercise to show the identity $(vk - \lambda - i\varepsilon)z = L_k g$. Take $g \in H^1(\mathbb{R})$, then $L_k g \in H^1(\mathbb{R})$. Since $H^1(\mathbb{R}) \subset Y$ (defined in the proof of Lemma 2.7), then $\|z\|_{L^2(\mathbb{R})} \leq \frac{C}{\sqrt{\varepsilon}}$. Invertibility of L_k and the Gauss law show that a similar bound is satisfied by (u, F) which is the searched criterion (14). It shows that $\mathrm{Ker}(H_0^k)^\perp \subset (X^k)^{\mathrm{ac}}$ which ends the proof. □

The negative result is the following.

Proposition 2.9. *For all $n \in \mathbb{N}^*$, the operator $(H - z)^{-n} - (H_0 - z)^{-n}$ is not trace class.*

Proof. If the the trace class property is proved for one z with $\mathrm{Im}(z) \neq 0$, then it holds for every non real z, see Lemma 4.11 page 547 in [13]. For practical convenience one can take $z = i\beta$ with $\beta \in \mathbb{R}^*$, and note $T^k = (H^k - z)^{-n} - (H_0^k - z)^{-n}$. Its maximal singular value $\mu_k = \sqrt{\lambda_k}$ is given by

$$\lambda^k = \sup_{w \neq 0} \frac{\|T^k w\|^2}{\|w\|^2} \geq \frac{\|T^k u_k\|^2}{\|u_k\|^2}, \qquad u_k \text{ given in (17).}$$

One has $\|T^k u_k\|^2 = \int |(-z)^{-n} - (vk - z)^{-n}|^2 e^{-\frac{v^2}{2}} dv$ and $\|u_k\|^2 = \int e^{-\frac{v^2}{2}} dv + k^2$. Since $z = i\eta$ with $\beta \in \mathbb{R}^*$, one gets $\lim_{|k| \to \infty} \left(k^2 |z|^{2n} \frac{\|T^k u_k\|^2}{\|u_k\|^2} \right) = \sqrt{2\pi}$. By definition (15) of the trace, one gets $\|T\|_1 \gtrsim \frac{(2\pi)^{\frac{1}{4}}}{|z|^n} \sum_{k \neq 0} \frac{1}{|k|} = +\infty$ due to the logarithmic divergence, so the proof is ended. □

The conclusion is that the T is not globally trace-class because of the logarithmic divergence. However, Fourier mode per Fourier mode, the operators $(T^k)_{k\in\mathbb{Z}}$ are trace-class individually because the extra-diagonal perturbation in (16) is of finite rank [7]. In this sense, T is "almost" trace-class. Inspection of the structure (16) of $H^k - z$ shows that the lack of a differential operator on the bottom right part of the matrix is involved in the fact that the trace class property does not hold. This is also related to the fact that the Gauss law is not preserved by the dynamics of H_0, that is $H_0(GL) \not\subset GL$.

3. The reduced Vlasov-Poisson-Ampère model

The reduced Vlasov-Poisson-Ampère model (also called the reduced equation) is obtained after a purely algebraic manipulation. Technically it allows to hide the electric field F into a new kinetic function w (19) so the compatibility issue with the Gauss law emphasized above is no more an issue.

3.1. Reduction. Let $(u, F) \in X$. The idea is to find a weight $x \mapsto \gamma(x) > 0$ and a new function w

$$w(x, v) = u(x, v) + \gamma(x)M(x, v)F(x) \tag{19}$$

such that

$$\int_{\mathbb{T}}\int_{\mathbb{R}} w^2 dvdx = \int_{\mathbb{T}}\int_{\mathbb{R}} u^2 dvdx + \int_{\mathbb{T}} F^2. \tag{20}$$

Lemma 3.1. *Assume the 1-periodic function γ is solution to the equation*

$$\partial_x\gamma + \alpha^2\gamma^2 \exp\varphi_0 = 1, \qquad \alpha = (2\pi)^{\frac{1}{4}}. \tag{21}$$

Assume $(u, F) \in GL$ satisfies the Gauss law. Then the identity (20) holds.

Proof. One has $M^2 = \exp\left(-\frac{v^2}{2} + \varphi_0(x)\right)$, so $\int_{\mathbb{R}} M^2(x, v)dv = \alpha^2 \exp(\varphi_0(x))$. Therefore

$$\int_{\mathbb{T}}\int_{\mathbb{R}} w^2 dvdx = \int_{\mathbb{T}}\int_{\mathbb{R}} u^2 dvdx + 2\int \gamma F \int uM dvdx + \alpha^2 \int \gamma^2 F^2 \exp\varphi_0(x)dx$$

$$= \int_{\mathbb{T}}\int_{\mathbb{R}} u^2 dvdx + 2\int \gamma F(-\partial_x F)dx + \alpha^2 \int \gamma^2 F^2 \exp\varphi_0(x)dx$$

$$= \int_{\mathbb{T}} \int_{\mathbb{R}} u^2 dv dx + \int F^2 \partial_x \gamma dx + \alpha^2 \int \gamma^2 F^2 \exp \varphi_0(x) dx,$$

from which the result proceeds. □

Let $\gamma > 0$ be a solution to (21) (we do not know yet if it exists). Set $g = \frac{1}{\gamma} > 0$ which is a formal solution of the equivalent Ricati equation $g'(x) + g(x)^2 = \alpha^2 \exp(\varphi_0(x))$.

Lemma 3.2. *There exists a 1-periodic positive solution $g = \frac{1}{\gamma} > 0$ of the Ricati equation.*

Proof. The proof of the existence is by a shooting method. Consider the solution given by the Cauchy-Lipshitz theorem of the equation

$$\begin{cases} g_a'(x) = \alpha^2 \exp(\varphi_0(x)) - g_a(x)^2, & 0 < x \le 1, \\ g_a(0) = a \end{cases}$$

and define the function $Z(a) = g_a(1)$. The objective is to find a solution to the equation $Z(a) = a$. Take $K > 0$ sufficiently large. For $a \in [0, K]$ then $g_a(x) \in [0, K]$ for all x: this is evident since if $g_a(x) = 0$ then $g_a'(x) > 0$, and if $g_a(x) = K$ then $g_a'(x) < 0$ (because K is large). Therefore $Z[0, K] \subset [0, K]$. Since Z is a continuous function, there exists $b \in [0, K]$ such that $Z(b) = b$. Therefore the trajectory such that $g_b(0) = b$ is periodic. Of course b cannot be equal to zero, nor to K. The trajectory is globally positive (it can be proved it is unique). □

Note that $\|\gamma M\|_{L^2(\mathbb{T}\times\mathbb{R})} = 1$: indeed

$$\|\gamma M\|^2_{L^2(\mathbb{T}\times\mathbb{R})} = \int_{\mathbb{T}} \int_{\mathbb{R}} \gamma(x)^2 M(x,v)^2 dv dx = \int_{\mathbb{T}} \gamma(x)^2 \alpha^2 e^{\varphi_0(x)} dx = \int_{\mathbb{T}} (1 - \partial_x \gamma) dx = 1.$$

Assume the Gauss law (19) and take $w = \Lambda(u, F)$. One has $\int_{\mathbb{R}} w M dv = \int_{\mathbb{R}} u M dv + \gamma \alpha^2 e^{\varphi_0} F = -\partial_x F + \frac{1}{\gamma}(1 - \partial_x \gamma) F$. So one can write

$$\gamma \int_{\mathbb{R}} w M dv = -\partial_x(\gamma F) + F. \tag{22}$$

Using that F has zero mean value and is periodic, one gets the orthogonality identity $\int_{\mathbb{T}} \int_{\mathbb{R}} w \gamma M dx dv = 0$ by integration with respect to x. It shows that $\Lambda(GL) \subset \mathcal{X} := \{ w \in L^2(\mathbb{T} \times \mathbb{R}), \ (w, \gamma M) = 0 \}$ where

$$\begin{array}{llll} \Lambda : & GL & \longrightarrow & \mathcal{X} \\ & (u, F) & \mapsto & w \text{ defined by (19)}. \end{array}$$

One has more precisely the following.

Lemma 3.3. *Λ is a bijective isometry from GL into $\mathcal{X}$. An explicit form of the inverse given in (23-24).*

Proof. One has $\|\Lambda(u,F)\|_{\mathcal{X}} = \|(u,F)\|_X$ in view of (20). It remains to prove that Λ is onto: that is one takes $w \in \mathcal{X}$ and tries to solve the equation (19) where u and F are the unknowns. One has

$$\int_{\mathbb{R}} wM(x,v)dv = \int_{\mathbb{R}} uM(x,v)dv + \gamma\alpha^2 \exp\left(\varphi_0(x)\right) F$$

$$= -\partial_x F + \alpha^2 \gamma(x) \exp\left(\varphi_0(x)\right) F = -\partial_x F + \frac{1}{\gamma}(1-\partial_x\gamma)F$$

where one recognizes (22). One obtains an equation for F which is $\partial_x(\gamma F) - F = -\gamma \int_{\mathbb{R}} wM(x,v)dv$. This equation is solvable for $F \in L^2_0(\mathbb{T})$ due the hypothesis $\int_{\mathbb{T}} \left(\gamma \int_{\mathbb{R}} wM(x,v)dv\right) = 0$. Set for convenience $G = \gamma F$ which is introduced to write explicitly the form of the solution F. One has $G - \frac{1}{\gamma}\partial_x G = -\gamma(x) \int_{\mathbb{R}} wM(x,v)dv$ and

$$\partial_x \left(\exp\left(-\int_0^x \frac{dy}{\gamma(y)} \right) G \right) = -\exp\left(-\int_0^x \frac{dy}{\gamma(y)} \right) \gamma(x) \int_{\mathbb{R}} wM(x,v)dv.$$

One gets

$$\left(e^{-\int_0^1 \frac{dy}{\gamma(y)}} - 1 \right) G(0) = -\int_0^1 \left(\exp\left(-\int_0^x \frac{dy}{\gamma(y)} \right) \gamma(x) \int_{\mathbb{R}} wM(x,v)dv \right) dx.$$

Since $\exp\left(-\int_0^1 \frac{dy}{\gamma(y)}\right) < 1$ as a consequence of $\gamma > 0$, the Cauchy data $G(0)$ is uniquely determined, which is enough to construct G. So F can be written as

$$F(x) = \int_{\mathbb{T}} K(x,y) \left(\int_{\mathbb{R}} wM(y,v)dv \right) dy \tag{23}$$

where the kernel $K \in L^\infty(\mathbb{T}\times\mathbb{R})$ is explicitly calculable. After that u is recovered as

$$u = w - \gamma(x)M(x,v)F. \tag{24}$$

By construction, it satisfies $\int uMdv = \int wMdv - \alpha^2\gamma e^{\varphi_0}F = -\partial_x F + \frac{1}{\gamma}(1-\partial_x\gamma)F - \alpha^2\gamma e^{\varphi_0}F = -\partial_x F$, and $u \in L^2_0(\mathbb{T}\times\mathbb{R})$. The proof is ended. □

3.2. Construction of an autonomous equation for $w = \Lambda(u,F)$. Since the system (2) preserves the quadratic norm of the pair (u,F) (which is equal to the quadratic norm of w), it is not surprising w is the solution of an autonomous equation where the operator is closed, self-adjoint and can be decomposed in two operators which are also closed and self-adjoint over $\mathcal{X}$. The construction of the corresponding operator is performed in several steps.

By symmetry with the usual notation in scattering, let us note $i\mathcal{H}_0 = D = v\partial_x - E_0(x)\partial_v$ and let us define the operator $\mathcal{H}_{\rm trunc}$ by

$$i\mathcal{H}_{\rm trunc} w = i\mathcal{H}_0 w + \gamma \left(vM1^* \int wMdv - M \int wvMdv \right).$$

The subscript $\cdot_{\rm trunc}$ indicates that the operator is truncated with respect to the complete definition (25).

Lemma 3.4. *Assume $(u,F) \in GL$ satisfies (2). Then $w = \Lambda(u,F)$ satisfies $\partial_t w + i\mathcal{H}_{\rm trunc} w = 0$.*

Proof. Indeed

$$\partial_t w + Dw = \partial_t u + Du + \gamma M(x,v)1^* \int uvMdv + D(\gamma(x)FM(x,v))$$

$$= -vMF + \gamma(x)M(x,v)1^* \int uvMdv + MD(\gamma(x)F)$$

$$= -vMF + \gamma M(x,v)1^* \int wvMdv - \gamma M(x,v)1^* \int (\gamma FM)vMdv + Mv\partial_x(\gamma F).$$

One has that $\int_{\mathbb{R}} vM^2(x,v)dv = 0$. So

$$\partial_t w + Dw = -vMF + \gamma M1^* \int wvMdv + Mv\gamma\partial_x F + MvF\partial_x\gamma$$

$$= -vMF + \gamma M1^* \int wvMdv - Mv\gamma \int uMdv + MvF(1 - \alpha^2\gamma^2 \exp(\varphi_0))$$

$$= \gamma M(x,v)1^* \int wvMdv - Mv\gamma \int uMdv - MvF\alpha^2\gamma^2 \exp(\varphi_0)$$

$$= \gamma M(x,v)1^* \int wvMdv - \gamma vM \int wMdv$$

$$+\gamma vM \int (\gamma MF)Mdv - MvF\alpha^2\gamma^2 \exp(\varphi_0) = \gamma M(x,v)1^* \int wvMdv - \gamma vM \int wMdv,$$

which yields the claim. □

Lemma 3.5. *Consider that w satisfies $\partial_t w + i\mathcal{H}_{\rm trunc} w = 0$. One has the identity $\frac{d}{dt}(w(t), \gamma M) = 0$.*

Proof. It is sufficient to show that $\int_{\mathbb{T}\times\mathbb{R}} \gamma M(\mathcal{H}_{\rm trunc} w) dv dx = 0$.

One has

$$\begin{array}{rcl} \int_{\mathbb{T}\times\mathbb{R}} \gamma M D w dv dx &=& \int_{\mathbb{T}\times\mathbb{R}} \gamma D(Mw) dv dx \\ &=& -\int_{\mathbb{T}\times\mathbb{R}} M w D\gamma dv dx \\ &=& -\int_{\mathbb{T}\times\mathbb{R}} M w v \partial_x \gamma dv dx \\ &=& -\int_I \partial_x \gamma \left(1^* \int_{\mathbb{R}} M w v\right) dv dx. \end{array}$$

A trivial identity is $\int_{\mathbb{T}\times\mathbb{R}} \gamma M \left(\gamma \left(vM \int wM\right)\right) dv dx = \int_{\mathbb{T}} \gamma^2 \left(\int wM\right)\left(\int M^2 v\right) dx = 0$. A third identity is $\int_{\mathbb{T}\times\mathbb{R}} \gamma M \left(\gamma \left(M 1^* \int w v M\right)\right) dv dx = \int_{\mathbb{T}} \gamma^2 \alpha^2 e^{\varphi_0(x)} \left(1^* \int w v M\right) dx$. So by summation

$$\begin{array}{rcl} \int_{\mathbb{T}\times\mathbb{R}} \gamma M(i\mathcal{H}_{\rm trunc} w) dv dx &=& -\int_I \partial_x \gamma \left(1^* \int_{\mathbb{R}} M w v\right) dv dx \\ && -\int_{\mathbb{T}} \gamma^2 \alpha^2 e^{\varphi_0(x)} \left(1^* \int w v M\right) dx \\ &=& -\int_{\mathbb{T}} \left(1^* \int w v M\right) dx \\ &=& 0, \end{array}$$

which ends the proof. □

Define for convenience the bounded operator

$$i\mathcal{K}_{\rm trunc} w = \gamma \left(vM \int wM dv - M 1^* \int w v M dv \right)$$

so that one has the decomposition $\mathcal{H}_{\rm trunc} = \mathcal{H}_0 + \mathcal{K}_{\rm trunc}$.

Lemma 3.6. *The operator $\mathcal{K}_{\rm trunc}$ is closed and self adjoint in $\mathcal{X}$.*

Proof. The closedness of the bouded operator $\mathcal{K}_{\rm trunc}$ is immediate. It is sufficient to check its symmetry. For $w, z \in \mathcal{X}$, one has

$$\begin{array}{rcl} (\mathcal{K}_{\rm trunc} w, z) &=& -i \int_{\mathbb{T}} \left(\int_{\mathbb{R}} \gamma w M dv\right) \left(\overline{\int z v M dv}\right) dx \\ && +i \int_{\mathbb{T}} \left(1^* \int_{\mathbb{R}} w v M dv\right) \left(\overline{\int \gamma z M dv}\right) dx \\ &=& -i \int_{\mathbb{T}} \left(\int_{\mathbb{R}} \gamma w M dv\right) \left(\overline{\int z v M dv}\right) dx \\ && +i \int_{\mathbb{T}} \left(\int_{\mathbb{R}} w v M dv\right) \left(\overline{\int \gamma z M dv}\right) dx \\ &=& \int_{\mathbb{T}} \left(\int_{\mathbb{R}} \gamma w M dv\right) \left(\overline{i 1^* \int z v M dv}\right) dx \\ && -\int_{\mathbb{T}} \left(\int_{\mathbb{R}} w v M dv\right) \left(\overline{i \int \gamma z M dv}\right) dx \\ &=& (w, \mathcal{K}_{\rm trunc} z). \end{array}$$

It yields the claim. □

A simple transformation of $\mathcal{K}_{\rm trunc}$ yields a formulation which is fully symmetric in $L^2(\mathbb{T}\times\mathbb{R})$ (not only in $\mathcal{X}$) , it reveals more convenient for further manipulations. Set

$$i\mathcal{K}w = i\mathcal{K}_{\rm trunc}w - (i\mathcal{H}_0 + i\mathcal{K}_{\rm trunc})\left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right)\gamma M \tag{25}$$

and define two bounded and symmetric integral operators:

$$i\mathcal{K}_1 w = \gamma\left(vM\int_{\mathbb{R}} wM dv - M\int_{\mathbb{R}} wvM dv\right) \tag{26}$$

and

$$i\mathcal{K}_2 w = \gamma M\int_{\mathbb{R}\times\mathbb{T}} wvM dv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right) Mv. \tag{27}$$

Lemma 3.7. *One has the identity $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ and $\mathcal{K}$ is self-adjoint in $L^2(\mathbb{T}\times\mathbb{R})$.*

Proof. One has the identity $(i\mathcal{H}_0 + i\mathcal{K}_{\rm trunc})\gamma M = Mv$ which comes from $i\mathcal{H}_0\gamma M = iM\mathcal{H}_0\gamma = Mv\partial_x\gamma$ and $i\mathcal{K}_{\rm trunc}\gamma M = \gamma vM\int\gamma M^2 = Mv\gamma^2\alpha^2 e^{\varphi_0}$. So

$$i\mathcal{K}w = i\mathcal{K}_{\rm trunc}w - i(\mathcal{H}_0 + \mathcal{K}_{\rm trunc})\left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right)\gamma M = i\mathcal{K}_{\rm trunc}w - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right) Mv$$

$$= \gamma\left(vM\int_{\mathbb{R}} wM dv - M1^*\int_{\mathbb{R}} wvM dv\right) - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right) Mv$$

$$= \gamma\left(vM\int_{\mathbb{R}} wM dv - M\int_{\mathbb{R}} wvM dv\right) + \gamma M\int_{\mathbb{T}\times\mathbb{R}} wvM dv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\right) Mv$$

which yield the claim. □

Define $i\mathcal{H} : L^2(\mathbb{T}\times\mathbb{R}) \to L^2(\mathbb{T}\times\mathbb{R})$ with $i\mathcal{H} = i\mathcal{H}_0 + i\mathcal{K}$. For $w = \Lambda(u,F) \in GL$ which satisfies (2), one finally obtains the formulation

$$\partial_t w + i\mathcal{H}w = 0. \tag{28}$$

The interesting point is that $\mathcal{K}$ is bounded and self-adjoint operator in $L^2(\mathbb{T}\times\mathbb{R})$. Of course only physically sound initial conditions $w_0 \in \mathcal{X}$ make sense. Since the equation propagates $w(t) \in \mathcal{X}$, we can work with the operators $\mathcal{H}$, $\mathcal{H}_0$ and $\mathcal{K}$ viewed as unbounded or bounded operators defined either in the space $\mathcal{X}$ or in the space $L^2(\mathbb{T}\times\mathbb{R})$.

Two technical properties which will be used later follow.

Lemma 3.8. *One has* $H = \Lambda^{-1}\mathcal{H}\Lambda : GL \to GL$.

Proof. Since $\Lambda(GL) = \mathcal{X}$, it is sufficient to interpret the equality to understand $\mathcal{H}$ as an operator in $\mathcal{X}$. The equation (28) is rewritten as $\Lambda\partial_t(u,F) + i\mathcal{H}\Lambda(u,F) = 0$ for all $(u,F) \in GL$ which satisfy $\partial_t(u,F) + iH(u,F) = 0$. So $\left(\Lambda^{-1}\mathcal{H}\Lambda - H\right)(u,F) = 0$ for all $(u,F) \in GL$. That is $\left(\Lambda^{-1}\mathcal{H}\Lambda - H\right)\Lambda^{-1} = 0$ from which is the claim. □

Lemma 3.9. *One has* $\mathcal{H}(\gamma M) = 0$.

Proof. The proof can be deduced from the previous material, but a more direct path is possible. Set $w(x,v) = \gamma(x)M(x,v)$. One has

$$\left\{\begin{array}{l} i\mathcal{K}_1 w = \gamma\left(vM\int \gamma M^2 dv - 0\right) = vM\gamma^2 e^{\varphi_0}\alpha^2, \\ i\mathcal{K}_1 w = 0 - vM\int_{\mathbb{T}\times\mathbb{R}}(\gamma M)^2 = -vM, \\ i\mathcal{H}_0 w = (v\partial_x - E_0\partial_v)(\gamma M) = M(v\partial_x - E_0\partial_v)\gamma = vM\partial_x\gamma. \end{array}\right.$$

Using (21), the sum of these three term cancels. □

4. Trace class properties of $(\mathcal{H}, \mathcal{H}_0)$

Now that the Gauss law is hidden inside the definition of $\mathcal{K}$, one is free to study trace class properties of the pair $(\mathcal{H}, \mathcal{H}_0)$ understood as unbounded operators in the space $L^2(\mathbb{T}\times\mathbb{R})$, by means of standard techniques.

Take $z \in \mathbb{C}$ with $\mathrm{Im}(z) \neq 0$. Let us study the trace of $(\mathcal{H}-z)^{-1} - (\mathcal{H}_0 - z)^{-1}$. The complex number $z \in \mathbb{C}$ is arbitrary: that is if the the trace class property is proved for one z with $\mathrm{Im}(z) \neq 0$, then it holds for every non real z, see Lemma 4.11 page 547 in [13]. For practical convenience we will take $z = i\beta$ with $\beta \in \mathbb{R}^*$ and $|\beta|$ large enough, typically

$$|\beta| > C\left(\|E_0\|_{W^{2,\infty}(\mathbb{T})} + 1\right) \text{ for some } C > 0, \tag{29}$$

to have the benefit of the elementary regularity result of Lemma 4.5.

One has the identity

$$\begin{aligned} (\mathcal{H}-z)^{-1} - (\mathcal{H}_0-z)^{-1} &= (\mathcal{H}_0-z)^{-1}(\mathcal{H}_0-\mathcal{H})(\mathcal{H}-z)^{-1} \\ &= -(\mathcal{H}_0-z)^{-1}\mathcal{K}(\mathcal{H}-z)^{-1} \\ &= -(\mathcal{H}_0-z)^{-1}\mathcal{K}(\mathcal{H}_0-z)^{-1}\left((\mathcal{H}-z)-\mathcal{K}\right)(\mathcal{H}-z)^{-1} \\ &= -(\mathcal{H}_0-z)^{-1}\mathcal{K}(\mathcal{H}_0-z)^{-1}\left(I-\mathcal{K}(\mathcal{H}-z)^{-1}\right) \\ &= -(\mathcal{T}+\mathcal{S})\mathcal{C} \end{aligned}$$

where

$$\begin{cases} \mathcal{T} = (\mathcal{H}_0 - z)^{-1}\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}, \\ \mathcal{S} = (\mathcal{H}_0 - z)^{-1}\mathcal{K}_2(\mathcal{H}_0 - z)^{-1}, \\ \mathcal{C} = I - \mathcal{K}(\mathcal{H} - z)^{-1}. \end{cases} \tag{30}$$

The operator $\mathcal{C}$ is bounded in $\mathcal{L}(\mathcal{X})$, since $\|I - \mathcal{K}(\mathcal{H} - z)^{-1}\| \le 1 + \|\mathcal{K}\|/|\mathrm{Im}(z)|$. So by inequalities (1.17) and (1.20) from section X.1.3 in [13], one gets

$$\|(\mathcal{T} + \mathcal{S})\mathcal{C}\|_1 \le \|\mathcal{T}\|_1\|\|\mathcal{C}\| + \|\mathcal{S}\|_1\|\mathcal{C}\|.$$

Therefore it is sufficient to study the trace of $\mathcal{T}$ and $\mathcal{S}$ separately to obtain the trace class property of the pair $(\mathcal{H}, \mathcal{H}_0)$. By definition (27) of $\mathcal{K}_2$, the operator $\mathcal{S}$ has finite rank so has a finite trace, Lemma 4.9. All efforts must concentrate on the analysis of $\mathcal{T}$, and the main result will be that $\mathcal{T}$ is trace class, see Proposition 4.8. It will prove Theorem 1.1.

4.1. Operator $\mathcal{T}$ in (30). The trace class property of $\mathcal{T}$ is proved in this section, after a careful study the kernel of $\mathcal{T}$ and the construction of an original integral equation. For technical convenience, we will consider the family of Hermite functions $(\psi_n)_{n\in\mathbb{N}}$, refer to [1, 7]. The family is orthonormal and complete in $L^2(\mathbb{R})$: $u(v) = \sum_{n\ge 0} a_n\psi_n(v)$ and $\|u\|^2_{L^2(\mathbb{R})} = \sum_{n\ge 0} |a_n|^2$. In particular, one has $\psi_0(v) = \exp(-v^2/4)/\alpha$, $\psi_1(v) = v\exp(-v^2/4)/\alpha$ and $\psi_n \in H^1(\mathbb{R})$ for all n. One can rewrite $\mathcal{K}_1$ as

$$\mathcal{K}_1 w = -i\alpha^2\gamma e^{\varphi_0(x)}\left(\psi_1(v)\int_{\mathbb{R}}\psi_0(v)w(x,v)dv - \psi_0(v)\int_{\mathbb{R}}\psi_1(v)w(x,v)dv\right).$$

4.1.1. The kernel of $\mathcal{T}$. Eigenvectors of $\mathcal{T}^*\mathcal{T}$ associated to non zero eigenvalues are the ones involved in the calculation of the trace of T. Since $\mathcal{T}^*\mathcal{T}$ is a symmetric operator, such eigenvectors belong to $\mathrm{Ker}(\mathcal{T}^*\mathcal{T})^\perp$.

Lemma 4.1. *One has* $Ker(\mathcal{T}^*\mathcal{T}) = Span_{n\ge 2}\left\{(\mathcal{H}_0 - z)a_n\psi_n, a_n \in H^1(\mathbb{T})\right\}$.

Proof. In this notation, $a_n\psi_n$ means the function $(x, v) \mapsto a_n(x)\psi_n(v)$. Take $n \ge 2$ and $a_n \in H^1(\mathbb{T})$. One has

$$(\mathcal{H}_0 - z)a_n\psi_n = (v\partial_x - E_0\partial_v)(a_n\psi_n) - za_n\psi_n \in L^2(\mathbb{T}\times\mathbb{R}).$$

One has $\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}\left((\mathcal{H}_0 - z)a_n\psi_n\right) = \mathcal{K}_1 a_n\psi_n = 0$ in view of definition (26). So $(\mathcal{H}_0 - z)a_n\psi_n \in \mathrm{Ker}\left(\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}\right) \subset \mathrm{Ker}(T)$.

Reciprocally take $w \in \mathrm{Ker}(\mathcal{T}) = \mathrm{Ker}(\mathcal{K}_1(\mathcal{H}_0 - z)^{-1})$. Using (26) one gets for almost all $x \in I$

$$\int_{\mathbb{R}} \left((\mathcal{H}_0 - z)^{-1})w\right)(x,v)\psi_0(v) = \int_{\mathbb{R}} \left((\mathcal{H}_0 - z)^{-1})w\right)(x,v)\psi_1(v) = 0.$$

So $\left((\mathcal{H}_0 - z)^{-1}w\right)(x,v) \in \mathrm{Span}\left\{\psi_0(v), \psi_1(v)\right\}^{\perp}$ for almost all x. One gets the representation $\left((\mathcal{H}_0 - z)^{-1}w\right)(x,v) = \sum_{n\geq 2} a_n(x)\psi_n(v)$, that is

$$w \in \mathrm{Span}_{n\geq 2}\left\{(\mathcal{H}_0 - z)a_n\psi_n, a_n \in H^1(\mathbb{T})\right\}$$

and the proof is ended. □

4.1.2. An integral equation. A technical lemma is the following.

Lemma 4.2. *Assume $w \in \mathcal{X}$ is even (resp. odd) w.r.t. the velocity v. Then $(\mathcal{H}_0^2 + |z|^2)^{-1}w$ is even (resp. odd) w.r.t. the velocity v.*

Proof. The operator $\mathcal{H}_0 = v\partial_x - E_0(x)\partial_v$ is odd w.r.t. v. So its square is an even operator which preserves the parity w.r.t. v. It ends the proof. □

Lemma 4.3. *Let $\lambda \in \mathbb{R}^*$. The equation $(\mathcal{T}^*\mathcal{T})w = \lambda w$ for $w \neq 0$ is equivalent to two decoupled integral equations*

$$\begin{pmatrix} \gamma e^{\varphi_0}\mathcal{T}_2\gamma e^{\varphi_0}\mathcal{T}_1 a \\ \gamma e^{\varphi_0}\mathcal{T}_1\gamma e^{\varphi_0}\mathcal{T}_2 b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \quad (a,b) \neq (0,0), \tag{31}$$

where $\mathcal{T}_1, \mathcal{T}_2 : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ are integral operators

$$\left\{ \begin{array}{l} \mathcal{T}_1 a(x) = \alpha^2 \int_{\mathbb{R}} \psi_0(v)\left[(\mathcal{H}_0^2 + |z|^2)^{-1}(a\psi_0)\right](y,v)dv, \\ \mathcal{T}_2 b(x) = \alpha^2 \int_{\mathbb{R}} \psi_1(v)\left[(\mathcal{H}_0^2 + |z|^2)^{-1}(b\psi_1)\right](y,v)dv. \end{array}\right. \tag{32}$$

Proof. Take $w = (\mathcal{H}_0 - \overline{z})^{-1}\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right)$, so that $w \in \mathrm{Ker}(\mathcal{T}^*\mathcal{T})^{\perp}$ in view of Lemma 4.1. Then this representation shows that

$$\mathcal{T}^*\mathcal{T}w = (\mathcal{H}_0 - \overline{z})^{-1}\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right)$$

and that $\lambda w = \lambda(\mathcal{H}_0 - \overline{z})^{-1}\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right)$. So the spectral problem $\mathcal{T}^*\mathcal{T}w = \lambda w$ is equivalent to

$$\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right) \tag{33}$$

$$= \lambda\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right).$$

Take $b=0$ and $d(x,v)=a(x)\psi_0(v)$. By construction one has

$$\begin{array}{rcl}\mathcal{K}_1(\mathcal{H}_0^2+|z|^2)^{-1}d &=& i\alpha^2\psi_1(v)\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_0(\mathcal{H}_0^2+|z|^2)^{-1}a\psi_0\\ &-& i\alpha^2\psi_0(v)\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_1(\mathcal{H}_0^2+|z|^2)^{-1}a\psi_0\\ &=& i\alpha^2\psi_1(v)\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_0(\mathcal{H}_0^2+|z|^2)^{-1}a\psi_0\\ &=& i\gamma e^{\varphi_0}\psi_1\mathcal{T}_1 a\end{array}$$

because the term $\int_{\mathbb{R}}\psi_1(\mathcal{H}_0^2+|z|^2)^{-1}a\psi_0$ vanishes due to the parity of Lemma 4.2. A similar annulation argument shows that

$$\begin{array}{rcl}\left(K_1(\mathcal{H}_0^2+|z|^2)^{-1}\right)^2 d &=& -\psi_1\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_0(\mathcal{H}_0^2+|z|^2)^{-1}(\mathcal{T}_1 a)\psi_1\\ && +\psi_0\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_1(\mathcal{H}_0^2+|z|^2)^{-1}(\mathcal{T}_1 a)\psi_1\\ &=& \psi_0\gamma e^{\varphi_0}\int_{\mathbb{R}}\psi_1(\mathcal{H}_0^2+|z|^2)^{-1}(\mathcal{T}_1 a)\psi_1\\ &=& \psi_0\gamma e^{\varphi_0}\mathcal{T}_2(\gamma e^{\varphi_0}\mathcal{T}_1 a).\end{array}$$

Plugging in (33) and simplification by $\psi_0(v)$ yield the reduced eigenvalue equation $\gamma e^{\varphi_0}\mathcal{T}_2\gamma e^{\varphi_0}\mathcal{T}_1 a=\lambda a$. A similar algebra holds starting from $d(x,v)=b(x)\psi_1(v)$, but the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ commute in the result. The proof is ended. □

Lemma 4.4. *The operators $\mathcal{T}_{1,2}: L^2(\mathbb{T})\to L^2(\mathbb{T})$ are self adjoint, bounded, positive and injective.*

Proof. We perform the proof for $\mathcal{T}_2$ only (it is the same for $\mathcal{T}_1$). The operator $\mathcal{T}_2$ is symmetric since

$$\begin{array}{rcl}(a,\mathcal{T}_2 b) &=& \int_{\mathbb{T}\times\mathbb{R}}(a\psi_1)(y,v)\overline{(\mathcal{H}_0^2+|z|^2)^{-1}(b\psi_1)}(y,v)dydv\\ &=& \int_{\mathbb{T}\times\mathbb{R}}\left[(\mathcal{H}_0-z)^{-1}(a\psi_1)\right](y,v)\overline{(\mathcal{H}_0-z)^{-1}(b\psi_1)}(y,v)dydv\\ &=& \int_{\mathbb{T}\times\mathbb{R}}(\mathcal{H}_0^2+|z|^2)^{-1}(a\psi_1)(y,v)\overline{(b\psi_1)}(y,v)dydv\\ &=& (a,\mathcal{T}_2 b) \qquad \forall\ a,b\in L^2(\mathbb{T}).\end{array}$$

The operator T_2 is also bounded with $\|T_2\|\le\frac{1}{|z|^2}$. So it is self-adjoint. It is injective because $(a,\mathcal{T}_2 a)=\left\|(\mathcal{H}_0-z)^{-1}(a\psi_1)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}>0$ for $a\neq 0$. It ends the proof. □

We will use the following elementary result.

Lemma 4.5. *Assume $E_0\in W^{s,\infty}(\mathbb{T})$ for $s\ge 0$. Take $z=i\beta$ with $|\beta|$ large enough (for example $|\beta|>\frac{\|E_0\|_{W^{1,\infty}(\mathbb{T})}+1}{2}$). Take $d\in H^s(\mathbb{T})$ and a smooth function φ exponentially decreasing at infinity and denote $u(x,v)=d(x)\varphi(v)$. Then $(\mathcal{H}_0\pm i\beta)^{-1}$ preserve the regularity with respect to x: $\left\|(\mathcal{H}_0\pm i\beta)^{-1}u\right\|_{H^s(\mathbb{T}\times\mathbb{R})}\le C_s^\varphi\|d\|_{H^s(\mathbb{T})}$.*

Proof. Consider $(\mathcal{H}_0 \pm i\beta)u(x,v) = d(x)\varphi(v)$. One gets by differentiation

$$\begin{array}{lcl} (\mathcal{H}_0 \pm i\beta)(\partial_x u) \quad +iE_0'(x)\partial_v u & = & d'(x)\varphi(v), \\ (\mathcal{H}_0 \pm i\beta)(\partial_v u) \qquad -i\partial_x u & = & d(x)\varphi'(v). \end{array}$$

Multiply the first equation by $\overline{\partial_x u}$, the second equation by $\overline{\partial_v u}$ and integrate in space-velocity. One gets the natural bound

$$|\alpha|\,|u|^2_{H^1(\mathbb{T}\times\mathbb{R})} \le \|d\varphi\|_{H^1(\mathbb{T}\times\mathbb{R})}\,|u|_{H^1(\mathbb{T}\times\mathbb{R})} + \frac{\|E_0\|_{W^{1,\infty}(\mathbb{T})}+1}{2}\,|u|^2_{H^1(\mathbb{T}\times\mathbb{R})}$$

where $|u|_{H^1(\mathbb{T}\times\mathbb{R})}$ is the H^1 semi-norm. The hypothesis yields the control of the semi-norm and ends the claim for $s=1$. For higher derivatives $s=n\in\mathbb{N}$, $s\ge 2$, the result is proved after successive derivation. For $s>0$, interpolation ends the proof. □

Lemma 4.6. *Assume (29) and $E_0 \in W^{s+1,\infty}(\mathbb{T})$ for $s\ge 0$. Then there exists $C_s>0$ such that $\|\mathcal{T}_2 b\|_{H^{s+2}(\mathbb{T})} \le C_s\|b\|_{H^s(\mathbb{T})}$.*

Proof. We notice to $\psi_1(v) = ve^{-\frac{v^2}{4}}$ contains the monomial v and that this function enters twice in the definition of T_2. The key remark is that it is possible to convert v into a gain of regularity of one order. Since v is present twice, it explains the gain of 2 orders of regularity.

Let $b\in H^s(\mathbb{T})$. The definition (32) recasts as $\mathcal{T}_2 b = \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v(D-\beta)^{-1}(D+\beta)^{-1} v b(x) e^{-\frac{v^2}{4}}\,dv$. One has

$$v\partial_x(D-\beta)^{-1} \;=\; 1+\beta(D-\beta)^{-1} + E_0(x)\partial_v(D-\beta)^{-1}.$$

So one deduces

$$\begin{array}{rcl} \partial_x(\mathcal{T}_2 b) & = & \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left(1+\beta(D-\beta)^{-1}\right)(D+\beta)^{-1} v b(x) e^{-\frac{v^2}{4}}\,dv \\ & + & \int_{\mathbb{R}} e^{-\frac{v^2}{4}} E_0(x)\partial_v(D-\beta)^{-1}(D+\beta)^{-1} v b(x) e^{-\frac{v^2}{4}}\,dv \\ & = & \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left(1+\beta(D-\beta)^{-1}\right)(D+\beta)^{-1} v b(x) e^{-\frac{v^2}{4}}\,dv \\ & + & \int_{\mathbb{R}} \frac{v}{2} e^{-\frac{v^2}{4}} E_0(x)(D-\beta)^{-1}(D+\beta)^{-1} v b(x) e^{-\frac{v^2}{4}}\,dv \\ & = & \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left[1+\beta(D-\beta)^{-1}+\frac{v}{2}E_0(x)(D-\beta)^{-1}\right](D+\beta)^{-1} v b e^{-\frac{v^2}{4}}\,dv \end{array}$$

where an integration by part w.r.t. v is used to transform the integral which contains a ∂_v. One notices that the right hand side of this identity belongs to $H^s(\mathbb{T})$ for for $b\in H^s(\mathbb{T})$, so one has already $\mathcal{T}_2 b \in H^{s+1}(\mathbb{T})$.

To gain one more order of integration, decompose $b = c+\partial_x d$ with c the mean value of b and $d\in H^{s+1}(\mathbb{T})$ with zero mean value: $|c| + \|d\|_{H^{s+1}(\mathbb{T})} \le$

$\|b\|_{H^s(\mathbb{T})}$. One has the identity which is somehow dual of the previous one

$$\begin{aligned}(D+\beta)^{-1}vbe^{-\frac{v^2}{4}} &= (D+\beta)^{-1}vce^{-\frac{v^2}{4}} \\ &+ de^{-\frac{v^2}{4}} - (D+\beta)^{-1}\beta de^{-\frac{v^2}{4}} - (D+\beta)^{-1}E_0(x)d\tfrac{v}{2}e^{-\frac{v^2}{4}}.\end{aligned}$$

One gets another identity

$$\begin{aligned}\partial_x(\mathcal{T}_2 b)(x) &= \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left[1+\beta(D-\beta)^{-1}+\tfrac{v}{2}E_0(x)(D-\beta)^{-1}\right](D+\beta)^{-1}vce^{-\frac{v^2}{4}}dv \\ &+ \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left[1+\beta(D-\beta)^{-1}+\tfrac{v}{2}E_0(x)(D-\beta)^{-1}\right]\left[1-\beta(D+\beta)^{-1}\right]de^{-\frac{v^2}{4}}dv \\ &- \int_{\mathbb{R}} e^{-\frac{v^2}{4}}\left[1+\beta(D-\beta)^{-1}+\tfrac{v}{2}E_0(x)(D-\beta)^{-1}\right](D+\beta)^{-1}E_0(x)d\tfrac{v}{2}e^{-\frac{v^2}{4}}dv.\end{aligned}$$

We notice that: in these integrals the last terms $vce^{-\frac{v^2}{4}}$, $de^{-\frac{v^2}{4}}$ and $E_0(x)d\frac{v}{2}e^{-\frac{v^2}{4}}$ are H^{s+1} with respect to x (the hypothesis $E_0 \in W^{s+1,\infty}(\mathbb{T})$ is used); the operators $(D\pm\beta)^{-1}$ preserve the regularity the H^s regularity with respect to x (cf. Lemma 4.5). Therefore $\|\partial_x(\mathcal{T}_2 b)\|_{H^{s+1}(\mathbb{T})} \le C\|b\|_{H^s(\mathbb{T})}$ which ends the proof of the claim. □

Lemma 4.7. *Under the same assumptions, there exists a constant $C_s > 0$ such that $\|\mathcal{T}_1 a\|_{H^{s+\frac14}(\mathbb{T})} \le C_s\|a\|_{H^s(\mathbb{T})}$.*

Proof. One has $\mathcal{T}_1 a = \int_{\mathbb{R}} \psi_0(v)u(x,v)dv$ with

$$(D-\beta)u = g = -(D+\beta)^{-1}(a\psi_0)(x,v).$$

By construction $\|g\|_{H^s(\mathbb{T}\times\mathbb{R})} + \|u\|_{H^s(\mathbb{T}\times\mathbb{R})} \le C\|a\|_{H^s(\mathbb{T})}$. Therefore one can write

$$v\partial_x u = \underbrace{g+\beta u}_{=g_1} + \partial_v(\underbrace{E_0(x)u}_{=g_2}).$$

Note that $\|g_1\|_{H^s(\mathbb{T}\times\mathbb{R})} \le C_s^1\|a\|_{H^s(\mathbb{T})}$ and $\|g_2\|_{H^s(\mathbb{T}\times\mathbb{R})} \le C_s^2\|a\|_{H^s(\mathbb{T})}$. For $s=0$, the Diperna-Lions Theorem of compactness by integration [10][remark 6 page 741] yields the claim. For higher s, the result holds after differentiation w.r.t. x and the regularity of E_0. □

Proposition 4.8. *Assume $E_0 \in W^{3+\varepsilon,\infty}(\mathbb{R})$ with $\varepsilon > 0$. Then $\mathcal{T}$ is trace class.*

Proof. We study the eigenvalues $\neq 0$ of the operators in (31). Set $\mathcal{T}_3$ the multiplication operator by γe^{φ_0}. By definition and hypothesis $\mathcal{T}_3 : H^s(\mathbb{T}) \to H^s(\mathbb{T})$ for $0 \le s \le 4+\varepsilon$. The two problems (31) can be reduced to the same eigenproblem

$$\left(\mathcal{T}_1^{\frac12}\mathcal{T}_3\mathcal{T}_2\mathcal{T}_3\mathcal{T}_1^{\frac12}\right)c_n = \lambda_n c_n, \quad 0 \neq c_n \in L_0^2(\mathbb{T}),\ \lambda_n > 0.$$

It is better for technical reasons to consider the square of this operator

$$\mathcal{T}_1^{\frac{1}{2}}\mathcal{T}_3\left(\mathcal{T}_2\mathcal{T}_3\mathcal{T}_1\mathcal{T}_3\mathcal{T}_2\right)\mathcal{T}_3\mathcal{T}_1^{\frac{1}{2}}c_n=\mu_n c_n, \quad \mu_n=\lambda_n^2.$$

Using Lemmas 4.6-4.7, the operator $\mathcal{R}=\mathcal{T}_2\mathcal{T}_3\mathcal{T}_1\mathcal{T}_3\mathcal{T}_2$ is self-adjoint and compact from $L^2(\mathbb{T})$ into $H^{4+\varepsilon}(\mathbb{T})$ for some $\varepsilon>0$

$$\mathcal{R}: L^2(\mathbb{T}) \xrightarrow{\mathcal{T}_2} H^2(\mathbb{T}) \xrightarrow{\mathcal{T}_3} H^2(\mathbb{T}) \xrightarrow{\mathcal{T}_1} H^{2+\varepsilon}(\mathbb{T}) \xrightarrow{\mathcal{T}_3} H^{2+\varepsilon}(\mathbb{T}) \xrightarrow{\mathcal{T}_2} H^{4+\varepsilon}(\mathbb{T}).$$

For $n\in\mathbb{N}^*$, we consider $V_n\subset L_0^2(\mathbb{T})$ any subspace such that $\dim(V_n)=n$ and note $V_n^*=V_n-\{0\}$. The min-max principle [13, 19] yields

$$\mu_n=\max_{V_n}\min_{c\in V_n^*}\frac{(S^*\mathcal{R}Sc,c)}{\|c\|^2}, \quad S=\mathcal{T}_3\mathcal{T}_1^{\frac{1}{2}}.$$

Since S is a bounded operator, one gets

$$\frac{(S^*\mathcal{R}Sc,c)}{\|c\|^2}=\frac{\|Sc\|^2}{\|c\|^2}\frac{(\mathcal{R}(Sc),Sc)}{\|Sc\|^2}\le\|S\|^2\frac{(\mathcal{R}d,d)}{\|d\|^2}.$$

Since S is injective, then $d=Sc\neq 0$. So $\mu_n\le\|S\|^2\max_{V_n}\min_{d\in V_n^*}\frac{(\mathcal{R}d,d)}{\|d\|^2}$, that is $\mu_n\le\|S\|^2\sigma_n$ where $\sigma_n>0$ is the n-th eigenvalue (counted in decreasing order). Since $\mathcal{R}=\mathcal{R}^*>0$ is a compact hermitian operator from $L_0^2(\mathbb{T})$ into $H^{4+\varepsilon}(\mathbb{T})$, the technical Lemme A.1 yields $0<\mu_n\le C\frac{1}{n^{4+\varepsilon}}$ for $n\in\mathbb{N}$ and $\varepsilon>0$. Finally $\sum_n\lambda_n^{\frac{1}{2}}=\sum_n\mu_n^{\frac{1}{4}}\le\widetilde{C}\sum_n\frac{1}{n^{1+\varepsilon/4}}<\infty$, so the trace class estimate holds. □

4.2. Operator $\mathcal{S}$ in (30). The analysis of $\mathcal{S}$ is much simpler.

Lemma 4.9. *One has (evident)*

$$\begin{aligned} Ker(\mathcal{S}^*\mathcal{S}) &= \left\{w\in L^2(\mathbb{T}\times\mathbb{R}), \int_{\mathbb{T}\times\mathbb{R}} wvM\,dxdv=\int_{\mathbb{T}\times\mathbb{R}} w\gamma M\,dxdv=0\right\} \\ &= Span\left\{vM,\gamma M\right\}^{\perp}. \end{aligned}$$

So the spectral problem $\mathcal{S}^*\mathcal{S}w=\lambda w$ can be reduced by looking only at $w=\alpha vM+\beta\gamma M$. It is a finite dimensional spectral problem, so the full operator $\mathcal{S}^*\mathcal{S}$ is trace class and one obtains Theorem 1.1 about the existence of wave operators.

5. Last part of the Theorem 1.1

One easily transfers Theorem 1.1 to the original Vlasov-Poisson equation. One starts from the decomposition, where the orthogonal product (8) is $GL = X^{\mathrm{ac}} \oplus X^{\mathrm{sc}} \oplus X^{\mathrm{pp}}$. The operator Λ is an isometric bijection from GL onto $\mathcal{X} = L^2(\mathbb{T}\times\mathbb{R}) \cap \{(w,\gamma M) = 0\} = (\gamma M)^{\perp}$. Since γM is in the kernel of $\mathcal{H}$ (Lemma 3.9), one gets the decomposition $\mathcal{X} = \mathcal{X}^{\mathrm{ac}} \oplus \mathcal{X}^{\mathrm{sc}} \oplus \left(\mathcal{X}^{\mathrm{pp}} \cap (\gamma M)^{\perp}\right)$. Lemma 3.8 yields that $\Lambda(X^{\mathrm{ac}}) = \mathcal{X}^{\mathrm{ac}}$, $\Lambda(X^{\mathrm{sc}}) = \mathcal{X}^{\mathrm{sc}}$ and $\Lambda(X^{\mathrm{pp}}) = \mathcal{X}^{\mathrm{pp}} \cap (\gamma M)^{\perp}$. Since $\mathcal{X}^{\mathrm{ac}}$ is isometric to $\mathcal{X}_0^{\mathrm{ac}}$ due to the existence of the wave operators, then X^{ac} is in bijection with a space isometric to $\mathcal{X}_0^{\mathrm{ac}}$. It ends the proof of the last part.

A. A technical lemma

Lemma A.1. *Let $R = R^* > 0$ be a positive hermitian operator defined in $L^2(\mathbb{T})$. Assume R is also bounded as an operator from $L^2(\mathbb{T})$ into $H^\alpha(\mathbb{T})$, $\alpha > 0$. Then there exists a constant $C > 0$ such that the eigenvalues of R counted in decreasing order satisfy $0 < \sigma_{n+1} \le \sigma_n \le \frac{C}{n^\alpha+1}$ for $n \in \mathbb{N}$.*

Proof. Define the operator hermitian $A : L^2(\mathbb{T}) \longrightarrow H^\alpha(\mathbb{T})$ by

$$Av(x) = \sum_{n\in\mathbb{Z}} \frac{1}{|n|^\alpha + 1}(v, e_n)e_n(x)$$

where $e_n(x) = e^{2i\pi nx}$ is the Fourier mode and $(v, e_n) = \int_0^1 v(x)\overline{e_n(x)}dx$ is the corresponding coefficient of the Fourier decomposition $v = \sum_{n\in\mathbb{Z}}(v, e_n)e_n$. The inverse operator $A^{-1} : H^\alpha(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ is defined by $A^{-1}w(x) = \sum_{n\in\mathbb{Z}}(|n|^\alpha + 1)(w, e_n)e_n(x)$. By construction $\|Av\|_{H^\alpha(\mathbb{T})} \le \|v\|_{L^2(\mathbb{T})}$ and $\|A^{-1}w\|_{L^2(\mathbb{T})} \le \|w\|_{H^\alpha(\mathbb{T})}$. Take $v \in L^2(\mathbb{T})$ and $u \in L^2(\mathbb{T})$. One has $\left|(Av, A^{-1}Ru)\right| \le C\|Av\|_{L^2(\mathbb{T})}\|u\|_{L^2(\mathbb{T})} \Longrightarrow |(Rv, u)| \le C\|Av\|_{L^2(\mathbb{T})}\|u\|_{L^2(\mathbb{T})}$. Since this inequality holds for all $u \in L^2(\mathbb{T})$, it yields $\|Rv\|_{L^2(\mathbb{T})} \le C\|Av\|_{L^2(\mathbb{T})}$. The min-max principle immediately gives an estimate of the eigenvalues σ_n of R with respect to the eigenvalues τ_n of A, that is $\sigma_n \le C\tau_n$ for $n \in \mathbb{N}$. Since $\tau_{2n+1} = \tau_{2n+2} = \frac{1}{n^\alpha+1}$ for $n \ge 0$, it proves the claim. □

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York: Dover Publications, 1972.

[2] C. Bardos and P. Degond, Existence globale et comportement asymptotique de la solution de l'équation de Vlasov-Poisson, C.R. Acad. Sc. Paris, t. 297, 321–324, 1983.

[3] J. Barré, A. Olivetti and Y.Y. Yamaguchi, Dynamics of perturbations around inhomogeneous backgrounds in the HMF model, Journal of Statistical Mechanics: Theory and Experiment, Volume 2010, 2010.

[4] J. Barré, A. Olivetti and Y.Y. Yamaguchi, Landau damping and inhomogeneous reference states, Comptes Rendus Physique, Volume 16, Issue 8, Pages 723-728, 2015.

[5] K.M. Case, Plasma oscillations, Annals of Physics, 349-364, 1959.

[6] R. Dautray and J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Volume 6 Evolution problems II, Springer, 1991.

[7] B. Després, Symmetrization of Vlasov-Poisson equations. SIAM J. Math. Anal. 46 (2014), no. 4, 2554-2580.

[8] B. Després, Scattering Structure and Landau Damping for Linearized Vlasov Equations with Inhomogeneous Boltzmannian States. Ann. Henri Poincaré 20 (2019), no. 8, 2767-2818.

[9] P. Degond, Spectral theory of the linearized Vlasov-Poisson equation, Trans. of the AMS, 294, 1986, 435-453.

[10] R.J. Diperna and P.L. Lions, Global weak solutions of Vlasov-Maxwell systems, Comm. on Pure and App. Math., Vol. XLII, 729-757, 1989.

[11] E. Faou and F. Rousset, Landau Damping in Sobolev Spaces for the Vlasov-HMF Model, Archive for Rational Mechanics and Analysis 2016, Volume 219, Issue 2, pp 887-902. 2016.

[12] K. Gustafson and J. Guy, On the absolutely continuous subspace of a self-adjoint operator, Helvetica Physica Acta, Vol. 47, pp 163-166, 1974.

[13] T. Kato, Wave operator and unitary equivalence, Pacific journal of mathematics, 15, 1, 1965.

[14] P.D. Lax, Functional analysis, Wiley-Interscience, 2002.

[15] P.D. Lax and R.S. Phillips, Scattering Theory, Academic Press, 1967.

[16] M. Lemou, F. Mehats, P. Raphael, A new variational approach to the stability of gravitational systems, Comm. Math. Phys. 302 (2011).

[17] P. J. Morrison, private communication at the *Collisionless Boltzmann (Vlasov) Equations and Modeling of Self- Gravitating Systems and Plasmas* confernece, Lumigny, 2017.

[18] P. J. Morrison, Hamiltonian Description of Vlasov Dynamics: Action-Angle Variables for the Continuous Spectrum, Transport theory and statistical physics, 29(3-5), 397-414, 2000.

[19] M. Reed and B. Simon, Methods of modern mathematical physics: III Scattering theory, Academic Press, 1979.

Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

e-mail: despres@ann.jussieu.fr