View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Archive Ouverte en Sciences de l'Information et de la Communication

# Finite-time internal stabilization of a linear 1-D transport equation

Christophe Zhang

**To cite this version:**

Christophe Zhang. Finite-time internal stabilization of a linear 1-D transport equation. 2019. hal-01980349v2

**HAL Id: hal-01980349**
**https://hal.archives-ouvertes.fr/hal-01980349v2**

Submitted on 18 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Finite-time internal stabilization of a linear 1-D transport equation


Christophe Zhang

*Laboratoire Jacques-Louis Lions, Sorbonne Université, Boîte courrier 187, 75252 Paris Cedex 05*
*Université Paris-Diderot SPC, CNRS, INRIA équipe Cage, Paris, France.*



## Abstract

We consider a 1-D linear transport equation on the interval $(0, L)$, with an internal scalar control. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized in finite time, and we give an explicit feedback law.



*Keywords:* transport equation, feedback stabilization, internal control, finite-time, backstepping, Fredholm transformations.


## 1. Introduction

We study the linear 1-D hyperbolic equation

$$\begin{cases} y_t + y_x + a(x)y = u(t)\tilde{\varphi}(x), \ x \in [0, L], \\ \qquad y(t, 0) = y(t, L), \ \forall t \geq 0, \end{cases} \tag{1}$$

where $a$ is continuous, real-valued, $\tilde{\varphi}$ is a given real-valued function of space, and at time $t$, $y(t, \cdot)$ is the state and $u(t)$ is the control. As in [40], the system can be transformed into

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), \ x \in [0, L], \\ \qquad \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0, \end{cases} \tag{2}$$

through the state transformation

$$\alpha(t, x) := e^{\int_0^x a(s)ds - \mu x} y(t, x),$$

where $\mu = \displaystyle\int_0^L a(s)ds$, and with

$$\varphi(x) := e^{\int_0^x a(s)ds - \mu x} \tilde{\varphi}(x),$$

so that we focus on systems of the form (2) in this article. Hyperbolic systems with an internal control of this form model a variety of physical systems: let us cite the water tank system (introduced in [16] and further studied in [5, 26]), which is modelled by Saint-Venant equations with boundary conditions analog to our periodic boundary conditions, and the plug-flow reactor system, where the control is the temperature of the reactor, and there is a given input at the boundary (see [25, 27]).

### 1.1. Notations and definitions

We note $\ell^2$ the space of summable square series $\ell^2(\mathbb{Z}, \mathbb{C})$. To simplify the notations, we will note $L^2$ the space $L^2(0, L)$ of complex-valued $L^2$ functions on the interval $(0, L)$, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2, \tag{3}$$

and the associated norm $\| \cdot \|$. Functions of $L^2$ can also be seen as $L$-periodic functions on $\mathbb{R}$, by the usual $L$-periodic extension: in this article, for any $f \in L^2$ we will also note $f$ its $L$-periodic extension on $\mathbb{R}$.

We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}} e^{\frac{2i\pi}{L} nx}, \quad \forall n \in \mathbb{Z}, \tag{4}$$

the usual Hilbert basis for $L^2$. For a function $f \in L^2$, we will note $(f_n)_{n\in\mathbb{Z}} \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n\in\mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n\in\mathbb{Z}} \overline{f_{-n}} e_n,$$

so that, in particular, if $f$ is real-valued:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$

We will use the following definition of the convolution product on $L$-periodic functions:

$$\begin{aligned} f \star g &= \sum_{n\in\mathbb{Z}} f_n g_n e_n \\ &= \int_0^L f(s)g(\cdot - s)ds \in L^2, \quad \forall f, g \in L^2, \end{aligned} \tag{5}$$


*Email address:* christophe.zhang@ljll.math.upmc.fr (Christophe Zhang)

where $g(x-s)$ should be understood as the value taken in $x-s$ by the $L$-periodic extension of $g$.

Let us now note $\mathcal{E}$ the space of finite linear combinations of the $(e_n)_{n\in\mathbb{Z}}$. Then, any sequence $(f_n)_{n\in\mathbb{Z}}$ defines an element $f$ of $\mathcal{E}'$:

$$\langle e_n, f\rangle = \overline{f_n}, \quad \forall n \in \mathbb{Z}. \tag{6}$$

On this space of linear forms, we can extend our previous definition of convolution:

$$\langle e_n, f \star g\rangle = \overline{f_n g_n} \tag{7}$$

derivation can be defined by duality from (6):

$$f' = \left(\frac{2i\pi n}{L} f_n\right)_{n\in\mathbb{Z}}, \quad \forall f \in \mathcal{E}'. \tag{8}$$

We also define the following spaces:

**Definition 1.1.** *Let $m \in \mathbb{N}$. We note $H^m$ the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product*

$$\langle f, g\rangle_m = \int_0^L f\bar{g} + \partial^m f \overline{\partial^m g}, \quad \forall f, g \in H^m,$$

*and the associated norm $\|\cdot\|_m$.*

*For $m \geq 1$ we also define $H^m_{(pw)}$ the space of piecewise $H^m$ functions, that is, $f \in H^m_{(pw)}$ if there exists a finite number $d$ of points $(\sigma_j)_{1\leq j\leq d} \in [0, L]$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, $f$ is $H^m$ on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm*

$$\|f\|_{m,pw} := \sum_{j=0}^{d} \|f_{|[\sigma_j,\sigma_{j+1}]}\|_{H^m(\sigma_j,\sigma_{j+1})}. \tag{9}$$

*For $s > 0$, we also define the periodic Sobolev space $H^s_{per}$ as the subspace of $L^2$ functions $f = \sum_{n\in\mathbb{Z}} f_n e_n$ such that*

$$\sum_{n\in\mathbb{Z}} \left(1 + \left|\frac{2i\pi n}{L}\right|^{2s}\right) |f_n|^2 < \infty.$$

Note that (9) does not depend on the choice of a suitable partition $(\sigma_j)$, so $\|\cdot\|_{m,pw}$ is well-defined.

Also, $H^s_{per}$ is a Hilbert space, equipped with the Hermitian product

$$\langle f, g\rangle_s = \sum_{n\in\mathbb{Z}} \left(1 + \left|\frac{2i\pi n}{L}\right|^{2s}\right) f_n \overline{g_n}, \quad \forall f, g \in H^s_{per},$$

and the associated norm $\|\cdot\|_s$, as well as the Hilbert basis

$$(e^s_n) := \left(\frac{e_n}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}}\right).$$

Finally, for $m \in \mathbb{N}$, $H^m_{per}$ is a closed subspace of $H^m$, with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H^m_{per} = \{f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \\ \forall i \in \{0, \cdots, m-1\}\}. \tag{10}$$

*1.2. Main result*

To stabilize (2), we will be considering linear feedbacks, that is, formally,

$$\langle \alpha(t), F\rangle = \sum_{n\in\mathbb{Z}} \overline{F_n}\alpha_n(t) = \int_0^L \bar{F}(s)\alpha(s)ds$$

where $F \in \mathcal{E}'$ and $(F_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are its Fourier coefficients, and $F$ is "real-valued":

$$F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.$$

In fact, the integral notation will appear as purely formal, as the $(F_n)_{n\in\mathbb{Z}}$ will have a prescribed growth, so that $F \notin L^2$. The associated closed-loop system now writes

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = \langle \alpha(t), F\rangle \varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0. \end{cases} \tag{11}$$

This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. In [40], we proved the following theorem for system (2) when it is controllable:

**Theorem 1.1** (Rapid stabilization in Sobolev norms)**.** *Let $m \geq 1$. Let $\varphi \in H^m_{(pw)} \cap H^{m-1}_{per}$ such that*

$$\frac{c}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \tag{12}$$

*where $c, C > 0$. Then, for every $\lambda \geq 0$ there exists a stationary feedback law $F^\lambda$ such that for all initial data $\alpha^0 \in H^m_{per}$ the closed-loop system (11) has a solution $\alpha(t) \in H^m_{per}$ which satisfies*

$$\|\alpha(t)\|_m \leq \left(\frac{C}{c}\right)^2 e^{(\mu+\lambda)L} e^{-\lambda t} \|\alpha^0\|_m, \quad \forall t \geq 0. \tag{13}$$

The growth condition (12) is equivalent to the exact controllability of (2) (see [30, Equation (2.19) and pages 199-200] where the author uses the moments method). In particular, one can see that if $c$ (resp. $C$) is the largest (resp. smallest) constant such that (12) holds, then the constant in (13), obtained thanks to the constructive proof in [40], can be critical in some cases.

Now, for $\lambda > 0$, the corresponding feedback law obtained in [40, Section 2.3] using the backstepping method is the linear form $F^{\lambda-\mu} \in \mathcal{E}'$ defined by

$$F^{\lambda-\mu}_n := -\frac{K(\lambda - \mu)}{\overline{\varphi_n}}, \quad \forall n \in \mathbb{Z}, \tag{14}$$

where

$$K(\lambda - \mu) := \frac{2}{L} \frac{1 - e^{-(\lambda-\mu)L}}{1 + e^{-(\lambda-\mu)L}} \xrightarrow[\lambda\to\infty]{} \frac{2}{L}, \tag{15}$$

so that

$$F^\lambda \xrightarrow[\lambda\to\infty]{\mathcal{E}'} F^\infty \tag{16}$$

where

$$F_n^{\infty} := -\frac{2}{\overline{\varphi_n} L}, \quad \forall n \in \mathbb{Z}. \tag{17}$$

Moreover, when $\lambda \to \infty$, the stability estimate in Theorem 1.1 becomes, for $t > L$,

$$\|\alpha(t)\|_m = 0.$$

This would suggest that taking the limit feedback $F^{\infty}$ could result in finite-time stabilization of (2). This is indeed the case, and in this article we will prove the following theorem:

**Theorem 1.2** (Finite-time stabilization in Sobolev norms)**.** *Let $m \geq 1$. Let $\varphi \in H^m_{(pw)} \cap H^{m-1}_{per}$, satisfying* (12) *for some $c, C > 0$. Then, if the feedback law is defined by* (17)*, for any initial data $\alpha^0 \in H^m_{per}$ the corresponding closed-loop system* (11) *has a solution $\alpha(t)$ which satisfies*

$$\|\alpha(t)\|_m = 0, \quad \forall t \geq L.$$

*1.3. Related results*

To investigate the stabilization of infinite-dimensional systems, there are three main types of approaches: the Gramian method (see for example [35, 34, 20]), Lyapunov functions (see for example [7], the book [2], and the recent results in [17, 18], which study the boundary stabilization of hyperbolic systems), and the backstepping method. The latter is derived from a method in finite dimension, also called backstepping, used to stabilize stabilizable systems with an added chain of integrators (see [22, 6, 32] for an overview of the finite-dimensional case, and [8] or [24] for applications to partial differential equations). Another way of applying this approach to partial differential equations was then pioneered and developed in [1] and [3]. This new form of backstepping consisted in mapping the system to a stable target system, using a Volterra transformation of the second kind (see [23] for a comprehensive introduction to the method):

$$f(t,x) \mapsto f(t,x) - \int_0^x k(x,y) f(t,y) dy.$$

This was used to prove many results on the boundary stabilization of partial differential equations (see for example [21, 31, 38, 37, 15], and also [2, chapter 7]).

In some cases, the method was used to obtain stabilization with an internal feedback (see [33, 36, 39]). We point out that in the latter reference, a system resembling (1) is studied, which leads to finite-time stabilization. However, several hypotheses are made on the space component of the controller so that a Volterra transformation of the second kind can be successfully applied to the system, whereas in this article and in [40], we simply assume the exact controllability of the system.

Another recent development of the backstepping method is the use of Fredholm transformations:

$$f(t,x) \mapsto \int_0^L k(x,y) f(t,y) dy,$$

to map the control system to a stable target system (see for example [13, 12, 10, 11] for boundary stabilization problems, [9] for an internal stabilization problem). These are more general than Volterra transformation of the second kind, but one has to check that the transformation under consideration is actually invertible, whereas Volterra transformations of the second kind are always invertible if the kernel $k$ has enough regularity.

Because backstepping provides explicit feedback laws, it has helped prove null-controllability or small-time stabilization (stabilization in an arbitrarily small time) results for some systems: see [14] for the heat equation, and [37] for the Korteweg-de Vries equation. In this article, we use the explicit feedback laws obtained by the backstepping method in [40] to design an explicit stationary feedback law that achieves finite-time stabilization.

*1.4. Structure of the article*

In Section 2, we derive an expression for the exponentially stable semigroup corresponding to the explicit feedback laws obtained in [40] for exponential stabilization. Then, in Section 3, we study the semigroup obtained when $\lambda \to \infty$. In particular, we derive its infinitesimal generator and prove that it corresponds to a closed-loop system which goes to 0 in finite time, which yields a feedback law achieving stabilization in finite time. Finally, Section 4 is devoted to some comments on the result, and on further questions.

## 2. The exponentially stable semigroup

We recall some specifics of Theorem 1.1, which can be found in more detail in [40].

*2.1. Backstepping transformation*

To prove Theorem 1.1, the backstepping method was used. This method consists in mapping our system into a stable target system, here

$$\begin{cases} z_t + z_x + \lambda' z = 0, & x \in (0,L), \\ z(t,0) = z(t,L), & t \geq 0, \end{cases} \tag{18}$$

with $\lambda' > 0$. To find an invertible transformation that does this, the idea is to write it as a Fredholm operator:

$$T : \alpha(t,x) \mapsto \int_0^L k(x,y) \alpha(t,y) dy$$

so that the mapping condition becomes a partial differential equation in $k$ (the kernel equation). This equation contains non-local terms, which are resolved by adding a natural constraint to the kernel equation:

$$\int_0^L k(x,y) \varphi(y) dy = \varphi(x), \quad \forall x \in [0,L], \tag{19}$$

which turns the non-local terms (left hand side) into local terms (right hand side). This constraint is sometimes called the $TB = B$ condition (see [40, 9]).

From this kernel equation, conditions on $F$ for the invertibility of $T$ can be derived. Then, using a weak version of (19) condition, a suitable feedback is computed, so that a candidate for the backstepping transformation can be derived:

$$T^{\lambda}\alpha = \sum_{n\in\mathbb{Z}} \alpha_n \overline{F_n^{\lambda}} \Lambda_{-n}^{\lambda} \star \varphi, \quad \forall \alpha \in H^m_{per}, \tag{20}$$

where $\lambda := \lambda' - \mu$, $\overline{F_n^{\lambda}}$ is defined by (14), and

$$\begin{aligned} \Lambda_n^{\lambda}(x) :&= \frac{\sqrt{L}}{1-e^{-\lambda L}} e^{-\lambda_n x} \\ &= \Lambda(x) e_{-n}(x), \ \forall n \in \mathbb{Z}, \ \forall x \in [0, L), \end{aligned} \tag{21}$$

where

$$\lambda_n = \lambda + \frac{2i\pi n}{L}, \quad \forall n \in \mathbb{Z}, \tag{22}$$

and where $\Lambda$ is the $L$-periodic function defined by

$$\Lambda(x) = \frac{L}{1-e^{-\lambda L}} e^{-\lambda x}, \quad \forall x \in [0, L).$$

*2.2. Well-posedness of the closed-loop system*

Now that a candidate for the backstepping transformation has been determined, it must be proved that it is indeed a backstepping transformation, and that the closed-loop with the feedback defined above is well-posed. We first define the domains

$$\begin{aligned} D_m^{\lambda} := \{\alpha \in & \tau^{\varphi}(H^{m+1}_{(pw)}) \cap H^m_{per}, \\ & -\alpha_x - \mu\alpha + \langle \alpha, F^{\lambda}\rangle \varphi \in H^m_{per}\} \end{aligned} \tag{23}$$

where $\tau^{\varphi}$ is the diagonal operator defined by the eigenvalues

$$\begin{aligned} \tau_n^{\varphi} := \frac{1}{\sqrt{L}} \Bigg( & \sum_{j=1}^{d} e^{-\frac{2i\pi}{L} n\sigma_j} (\partial^{m-1}\varphi(\sigma_j^-) - \partial^{m-1}\varphi(\sigma_j^+)) \\ & + \partial^{m-1}\varphi(L) - \partial^{m-1}\varphi(0) \Bigg), \quad \forall n \in \mathbb{Z}. \end{aligned}$$

In [40], we investigate the regularity of the feedback law, using the controllability condition (12). This helps to prove that the corresponding closed-loop operator

$$A + BK := -\partial_x - \mu I + \langle \cdot, F^{\lambda}\rangle \varphi$$

is densely defined and closed. Finally, to check that the mapping property between systems (11) and (18) is verified, one proves the operator equality

$$\begin{aligned} T^{\lambda}(-\partial_x + \langle \cdot, F^{\lambda}\rangle\varphi)\alpha &= (-\partial_x - \lambda I)T^{\lambda}\alpha \\ &\text{in } H^m_{per}, \quad \forall \alpha \in D_m^{\lambda}. \end{aligned} \tag{24}$$

This operator equality implies that the unbounded operator $A + BK$ is a dense restriction of the infinitesimal generator of an exponentially stable semigroup $S^{\lambda}(t)$.

Now, the basic idea is that for a given initial condition $\alpha^0$, for each feedback $F^{\lambda}$ one has a trajectory of the closed-loop system

$$\alpha_t^{\lambda} + \alpha_x^{\lambda} + \mu\alpha^{\lambda} = \langle \alpha^{\lambda}(t), F^{\lambda}\rangle \varphi(x) \tag{25}$$

and one hopes that the $\alpha^{\lambda}$ converge in some sense towards a trajectory $\alpha^{\infty}$, which should satisfy the closed-loop equation

$$\alpha_t^{\infty} + \alpha_x^{\infty} + \mu\alpha^{\infty} = \langle \alpha^{\infty}(t), F^{\infty}\rangle \varphi(x). \tag{26}$$

However, one can write equation (25) only for $\alpha^0 \in D_m^{\lambda}$, and the to use this equation to study the convergence of the $\alpha^{\lambda}$, one would need $\alpha \in \bigcap_{\lambda>0} D_m^{\lambda}$. This is too restrictive since we would like a statement for $\alpha^0 \in H^m_{per}$. Thus, rather than consider the equations (25), we will work in the more general framework of semigroups.

*2.3. Expression of the semigroup*

In [40], an expression of the semigroup $S^{\lambda}$ is given using the transformation $T^{\lambda}$. Here, to study what happens when $\lambda \to \infty$, we need to expand that expression.

First, we derive from (20) and (21) the following expression for the backstepping transformation:

$$T^{\lambda}\alpha = \varphi \star \left(\Lambda(\alpha \star \widetilde{F}^{\lambda})\right), \quad \forall \alpha \in H^m_{per}, \tag{27}$$

where $\widetilde{F}^{\lambda} \in \mathcal{E}'$ is defined by:

$$\langle e_n, \widetilde{F}^{\lambda}\rangle = F_n^{\lambda}, \quad n \in \mathbb{Z}.$$

Now, define the following operators:

$$\begin{aligned} &C_{\varphi} f = \varphi \star f \in H^m_{per}, \quad \forall f \in L^2, \\ &C_{\widetilde{F}^{\lambda}} f = \widetilde{F}^{\lambda} \star f \in L^2, \quad \forall f \in H^m_{per}, \\ &M_{\Lambda} f = \Lambda f, \quad \forall f \in L^2. \end{aligned}$$

Then, by definition of $F^{\lambda}$, it follows that

$$\begin{aligned} &C_{\varphi} \circ C_{\widetilde{F}^{\lambda}} = -K(\lambda) Id_{H^m_{per}}, \\ &C_{\widetilde{F}^{\lambda}} \circ C_{\varphi} = -K(\lambda) Id_{L^2}, \\ &M_{\Lambda} \circ M_{\frac{1}{\Lambda}} = Id_{L^2}, \end{aligned} \tag{28}$$

where $K$ is defined by (15). Moreover, with these notations, we have

$$T^{\lambda} = C_{\varphi} \circ M_{\Lambda} \circ C_{\widetilde{F}^{\lambda}} \tag{29}$$

hence

$$(T^{\lambda})^{-1} = \frac{1}{K(\lambda)^2} C_{\varphi} \circ M_{\frac{1}{\Lambda}} \circ C_{\widetilde{F}^{\lambda}}, \tag{30}$$

i.e.

$$(T^{\lambda})^{-1}\alpha = \frac{1}{K(\lambda)^2} \varphi \star \left(\frac{1}{\Lambda}(\alpha \star \widetilde{F}^{\lambda})\right), \ \forall \alpha \in H^m_{per}. \tag{31}$$

Now, recall that for all initial data $z_0 \in H^m_{per}$, the solution of system (18) can be written

$$z(t,x) = e^{-\lambda' t} z_0(x-t), \quad \forall (t,x) \in \mathbb{R}^+ \times (0,L). \quad (32)$$

Thus, by the expression of the semigroup (see [40, Subsection 3.2]), for all initial data $\alpha^0 \in D^\lambda_m$, the solution of system (11) can be written:

$$\begin{aligned} \alpha(t,x) = (T^\lambda)^{-1} e^{-\lambda' t} (T^\lambda \alpha^0)(x-t), \\ \forall (t,x) \in \mathbb{R}^+ \times (0,L). \end{aligned} \quad (33)$$

Now, notice that convolution and translation commute, so for $(t,x) \in [0,L] \times (0,L)$, we get, using (27) and (31),

$$\begin{aligned} \alpha(t,x) &= \frac{1}{K(\lambda)^2} \varphi \star \left( \frac{1}{\Lambda} \left( \widetilde{F}^\lambda \star \right.\right. \\ &\qquad \left.\left. \left( e^{-\lambda' t} \left( \varphi \star \left( \Lambda(\alpha^0 \star \widetilde{F}^\lambda) \right) (\cdot - t) \right) \right) \right) \right) \\ &= -\frac{e^{-\lambda' t}}{K(\lambda)} \varphi \star \left( \frac{1}{\Lambda} \left( \left( \Lambda(\alpha^0 \star \widetilde{F}^\lambda) \right) (\cdot - t) \right) \right) \\ &= -\frac{e^{-\lambda' t}}{K(\lambda)} \varphi \star \left( \chi_{[0,t]} e^{\lambda(t-L)} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t + L) \right. \\ &\qquad \left. + \chi_{[t,L]} e^{\lambda t} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t) \right) \\ &= -\frac{e^{-\mu t}}{K(\lambda)} \varphi \star \left( \chi_{[0,t]} e^{-\lambda L} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t + L) \right. \\ &\qquad \left. + \chi_{[t,L]} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t) \right). \end{aligned}$$

This expression is derived for $\alpha^0 \in D^\lambda_m$, but it is actually well-defined on all of $H^m_{per}$, as $\alpha^0 \star \widetilde{F}^\lambda \in L^2$ when $\alpha^0 \in H^m_{per}$. This gives us an expression for $S^\lambda(t)$ on all of $H^m_{per}$:

$$\begin{aligned} S^\lambda(t)\alpha^0 = \frac{-e^{-\mu t}}{K(\lambda)} \varphi \star \left( \chi_{[t,L]} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t) \right. \\ \left. + \chi_{[0,t]} e^{-\lambda L} \alpha^0 \star \widetilde{F}^\lambda(\cdot - t + L) \right), \\ \forall t \in [0,L], \ \forall \alpha^0 \in H^m_{per}, \end{aligned} \quad (34)$$

which defines $S^\lambda(t)$ for $t \geq 0$ by the semigroup property.

## 3. The limit semigroup

Now, notice that, from (15) and (17),

$$F^\lambda = \frac{LK(\lambda)}{2} F^\infty,$$

so that we can write

$$\begin{aligned} S^\lambda(t)\alpha = -\frac{Le^{-\mu t}}{2} \varphi \star \left( \chi_{[0,t]} e^{-\lambda L} \alpha \star \widetilde{F}^\infty(\cdot - t + L) \right. \\ \left. + \chi_{[t,L]} \alpha \star \widetilde{F}^\infty(\cdot - t) \right), \quad \forall t \in [0,L], \ \forall \alpha \in H^m_{per}. \end{aligned}$$

Then it is clear that

$$\chi_{[0,t]} e^{-\lambda L} \alpha \star \widetilde{F}^\infty(\cdot - t + L) \xrightarrow[\lambda \to \infty]{L^2} 0, \quad \forall t \in [0,L], \ \forall \alpha \in H^m_{per},$$

so that, after convolution with $\varphi$,

$$S^\lambda(t)\alpha \xrightarrow[\lambda \to \infty]{H^m} S^\infty(t)\alpha, \quad \forall t \in [0,L], \ \forall \alpha \in H^m_{per},$$

where

$$\begin{aligned} S^\infty(t)\alpha := -\frac{Le^{-\mu t}}{2} \varphi \star \left( \chi_{[t,L]} \alpha \star \widetilde{F}^\infty(\cdot - t) \right), \\ \forall t \geq 0, \ \forall \alpha \in H^m_{per}, \end{aligned}$$

with the convention that $\chi_{[t,L]} \equiv 0$ when $t \geq L$. Hence we have defined a new semigroup $S^\infty(t)$ on $H^m_{per}$, which we now study in order to establish Theorem 1.2.

### 3.1. A useful semigroup

Consider the semigroup given by

$$S_0(t)\alpha = e^{-\mu t} \chi_{[t,L]} \alpha(\cdot - t), \quad \forall t \geq 0, \ \forall \alpha \in L^2.$$

This is actually a contraction semigroup, the infinitesimal generator of which is given by

$$\begin{aligned} D(A_0) = \left\{ \alpha \in H^1, \quad \alpha(0) = 0 \right\} \\ A_0 = -\partial_x - \mu I \end{aligned} \quad (35)$$

where the derivative is to be understood as the usual derivative of a Sobolev function, not as the derivative in $\mathcal{E}'$. Note that this semigroup is associated to the following transport equation:

$$\begin{cases} y_t + y_x + \mu y = 0, \ x \in [0,L], \\ \quad y(t,0) = 0, \ \forall t \geq 0, \end{cases} \quad (36)$$

and that in particular

$$S_0(t)\alpha = 0, \quad \forall t \geq L, \ \forall \alpha \in L^2. \quad (37)$$

### 3.2. Infinitesimal generator

Now let us compute the infinitesimal generator of $S^\infty$. First, notice that

$$S^\infty(t)\alpha = -\frac{L}{2} \varphi \star S_0(t) \left( \alpha \star \widetilde{F}^\infty \right). \quad (38)$$

Now, let us define the following domain, in the same spirit as in section 2.1:

$$\begin{aligned} D^\infty_m := \{ \alpha \in \tau^\varphi(H^{m+1}_{(pw)}) \cap H^m_{per}, \\ -\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi \in H^m_{per} \}. \end{aligned} \quad (39)$$

This domain is dense in $H^m_{per}$, as it contains the following dense subspace (see [40, Proposition 3.1]):

$$\left\{ \alpha \in H^{m+1}_{per}, \ \langle \alpha, F \rangle = 0 \right\}.$$

Let us now prove that on this domain, $S^\infty$ has an infinitesimal generator. For $\alpha \in D^\infty_m$, we have

$$r := \widetilde{F}^\infty \star (-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi) \in L^2. \quad (40)$$

Thus, taking the Fourier coefficients, we get:

$$\alpha_n\overline{F_n^\infty} = -\frac{r_n + \mu\alpha_n\overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)} + i\frac{\langle\alpha, F^\infty\rangle}{\pi n}, \quad \forall n \neq 0.$$

Now, note that

$$\sum_{n\in\mathbb{Z}^*} i\frac{\langle\alpha, F^\infty\rangle}{\pi n}e_n(x) = \frac{2}{L}\langle\alpha, F^\infty\rangle\left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2}\right),$$

so that

$$\alpha \star \widetilde{F}^\infty = \tilde{r} + \frac{2}{L}\langle\alpha, F^\infty\rangle\left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2}\right), \tag{41}$$

where

$$\tilde{r} = \frac{\alpha_0 F_0^\infty}{\sqrt{L}} - \sum_{n\in\mathbb{Z}^*}\frac{r_n + \mu\alpha_n\overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)}e_n \in H^1_{per}. \tag{42}$$

Hence, $\alpha \star \widetilde{F}^\infty \in H^1$, and, from (41) and (42) we get

$$(\alpha \star \widetilde{F}^\infty)_x = -\left(r - \frac{r_0}{\sqrt{L}} + \mu\alpha \star \widetilde{F}^\infty - \mu\frac{\alpha_0 F_0^\infty}{\sqrt{L}}\right) + \frac{2}{L\sqrt{L}}\langle\alpha, F^\infty\rangle.$$

Now, by (40), (17) and by definition of the convolution product,

$$r_0 = -\mu F_0^\infty\alpha_0 - \frac{2\langle\alpha, F^\infty\rangle}{L},$$

so that, again by (40),

$$\begin{aligned}(\alpha \star \widetilde{F}^\infty)_x &= -r + \mu\alpha \star \widetilde{F}^\infty \\ &= -\widetilde{F}^\infty \star (-\alpha_x + \langle\alpha, F^\infty\rangle\varphi) \quad \text{in } L^2.\end{aligned} \tag{43}$$

On the other hand, we know, by the Dirichlet convergence theorem (see [19]) applied to $\alpha \star \widetilde{F}^\infty \in H^1$ at point 0, that

$$\frac{\alpha \star \widetilde{F}^\infty(0) + \alpha \star \widetilde{F}^\infty(L)}{2} = \sum_{n\in\mathbb{Z}}\frac{\alpha_n\overline{F_n^\infty}}{\sqrt{L}} = \frac{\langle\alpha, F^\infty\rangle}{\sqrt{L}}.$$

On the other hand, by (41),

$$\begin{aligned}(\alpha \star \widetilde{F}^\infty - \tilde{r})(0) &= -\frac{\langle\alpha, F^\infty\rangle}{\sqrt{L}} \\ &= -(\alpha \star \widetilde{F}^\infty - \tilde{r})(L),\end{aligned} \tag{44}$$

thus, as $\tilde{r}$ is periodic,

$$\tilde{r}(0) = \frac{\alpha \star \widetilde{F}^\infty(0) + \alpha \star \widetilde{F}^\infty(L)}{2} = \frac{\langle\alpha, F^\infty\rangle}{\sqrt{L}}. \tag{45}$$

From (44) and (45), we get

$$\alpha \star \widetilde{F}^\infty(0) = \tilde{r}(0) - \frac{\langle\alpha, F^\infty\rangle}{\sqrt{L}} = 0,$$

so that $\alpha \star \widetilde{F}^\infty \in D(A_0)$.

We can now compute the infinitesimal generator of $S^\infty$: let $\alpha \in D_m^\infty$. Then, thanks to the above, $\alpha \star \widetilde{F}^\infty \in D(A_0)$, which means in particular that

$$\frac{S_0(t)(\alpha \star \widetilde{F}^\infty) - (\alpha \star \widetilde{F}^\infty)}{t} \xrightarrow[t\to 0^+]{L^2} -(\alpha\star\widetilde{F}^\infty)_x - \mu(\alpha\star\widetilde{F}^\infty).$$

This, together with (38) and (12), implies that

$$\begin{aligned}\frac{S^\infty(t)\alpha - \alpha}{t} &= -\frac{L}{2}\varphi \star \left(\frac{S_0(t)(\alpha \star \widetilde{F}^\infty) - (\alpha \star \widetilde{F}^\infty)}{t}\right) \\ &\xrightarrow[t\to 0^+]{H^m} \frac{L}{2}\varphi \star \left((\alpha \star \widetilde{F}^\infty)_x + \mu(\alpha \star \widetilde{F}^\infty)\right).\end{aligned}$$

By (43), we have

$$\varphi \star \left((\alpha \star \widetilde{F}^\infty)_x + \mu\,(\alpha \star \widetilde{F}^\infty)\right) = \frac{2}{L}\left(-\alpha_x - \mu\alpha + \langle\alpha, F^\infty\rangle\varphi\right)$$

so that, finally,

$$\frac{S^\infty(t)\alpha - \alpha}{t} \xrightarrow[t\to 0^+]{H^m} -\alpha_x - \mu\alpha + \langle\alpha, F^\infty\rangle\varphi.$$

This, together with (39), means that the infinitesimal generator of $S^\infty(t)$ can be given by the domain $D_m^\infty$ and the unbounded operator $-\partial_x - \mu I + \langle\cdot, F^\infty\rangle\varphi$. Hence, $S^\infty(t)$ corresponds to the closed loop system

$$\begin{cases}\alpha_t + \alpha_x + \mu\alpha = \langle\alpha(t), F^\infty\rangle\varphi(x),\ x \in [0, L],\\ \alpha(t, 0) = \alpha(t, L),\ \forall t \geq 0,\end{cases} \tag{46}$$

which is well-posed. Moreover, by (37) and (38),

$$S^\infty(t)\alpha^0 = 0, \quad \forall t \geq L, \forall\alpha^0 \in H^m_{per}, \tag{47}$$

which proves Theorem 1.2.

## 4. An explicit example

Consider the control system

$$\begin{cases}\alpha_t + \alpha_x = u(t)(L - x),\ x \in [0, L],\\ \alpha(t, 0) = \alpha(t, L),\ \forall t \geq 0.\end{cases} \tag{48}$$

In this case, $\varphi(x) = L - x$, so $\varphi \in H^1$ and the Fourier coefficients of the controller are

$$\begin{aligned}\varphi_n &= -\frac{iL^{\frac{3}{2}}}{2\pi n}, \quad \forall n \in \mathbb{Z}^*,\\ \varphi_0 &= \frac{L^{\frac{3}{2}}}{2}.\end{aligned} \tag{49}$$

so that (12) is clearly satisfied for $m = 1$. Now, from (49) and (17) we get

$$\begin{aligned}F_n^\infty &= \frac{2}{L}\frac{2i\pi n}{L\sqrt{L}}, \quad \forall n \in \mathbb{Z}^*,\\ F_0^\infty &= -\frac{4}{L^{\frac{5}{2}}}.\end{aligned} \tag{50}$$

Now, using the Dirichlet convergence theorem, we have for $\alpha \in H^2 \cap H^1_{per}$,

$$\sum_{n=-N}^{N} \overline{F_n^\infty} \alpha_n = -\frac{2}{L} \sum_{n=-N}^{N} \frac{2i\pi n}{L} \alpha_n \frac{1}{\sqrt{L}} - \frac{4}{L^{\frac{5}{2}}} \alpha_0 \\ \xrightarrow[N\to\infty]{} -\frac{\alpha_x(0)+\alpha_x(L)}{L} - \frac{4}{L^{\frac{5}{2}}}\alpha_0, \tag{51}$$

so that

$$\langle \alpha, F^\infty \rangle = -\frac{\alpha_x(0)+\alpha_x(L)}{L} - \frac{4}{L^{\frac{5}{2}}}\alpha_0, \\ \forall \alpha \in H^2 \cap H^1_{per}. \tag{52}$$

One can see from the above expression that even though our method defines $F^\infty$ by its Fourier coefficients, with some controllers the feedback law can be expressed quite simply.

Now, let us consider solutions of the closed-loop system (48) with $u(t) = \langle \alpha(t), F^\infty \rangle$, with initial conditions in the domain

$$D_1^\infty = \Big\{ \alpha \in H^2 \cap H^1_{per}, \\ -\alpha_x + \langle \alpha, F^\infty \rangle (L-x) \in H^1_{per} \Big\}, \tag{53}$$

which can be rewritten as

$$D_1^\infty = \Big\{ \alpha \in H^2 \cap H^1_{per}, \\ \alpha_x(0) = -\frac{2}{L\sqrt{L}}\alpha_0 \in H^1_{per} \Big\}. \tag{54}$$

Indeed, $-\alpha_x + \langle \alpha, F^\infty \rangle (L-x) \in H^1$, so the above condition simply corresponds to its being periodic in addition.

Let $\alpha^0 \in D_1^\infty$, and note $\alpha(t)$ the corresponding solution of (48). We can make the following computations for $t \geq 0$, using (50) for the first, the periodicity of $\alpha$, and differentiating the first equation of (48) in space for the second, and (54) for the third:

$$\begin{aligned}
\alpha(t) \star \widetilde{F}^\infty &= -\frac{2}{L\sqrt{L}}\alpha_x - \frac{4}{L^3}\alpha_0, \\
(\alpha \star \widetilde{F}^\infty)_t &= -\frac{2}{L\sqrt{L}}\alpha_{xt} - \frac{4}{L^3}(\alpha_0)_t \\
&= -\frac{2}{L\sqrt{L}}\alpha_{tx} - \frac{4}{L^3\sqrt{L}}\int_0^L \alpha_t \\
&= \frac{2}{L\sqrt{L}}((\alpha_x - \langle \alpha, F^\infty \rangle (L-x))_x) \\
&\quad - \frac{4}{L^3\sqrt{L}}\int_0^L \alpha_t \\
&= \frac{2}{L\sqrt{L}}(\alpha_{xx} + \langle \alpha, F^\infty \rangle) \\
&\quad - \frac{4}{L^3\sqrt{L}}\int_0^L \langle \alpha, F^\infty \rangle (L-x) - \alpha_x \\
&= \frac{2}{L\sqrt{L}}(\alpha_{xx} + \langle \alpha, F^\infty \rangle) \\
&\quad - \frac{4}{L^3\sqrt{L}}\langle \alpha, F^\infty \rangle \int_0^L (L-x)dx \\
&= \frac{2}{L\sqrt{L}}\alpha_{xx} \\
&= -(\alpha \star \widetilde{F}^\infty)_x, \\
\alpha(t) \star \widetilde{F}^\infty(0) &= -\frac{2}{L\sqrt{L}}\alpha_x(0) - \frac{4}{L^3}\alpha_0 \\
&= 0.
\end{aligned} \tag{55}$$

So in particular we can see quite clearly how $\alpha \star \widetilde{F}^\infty$ satisfies the equation (36) with $\mu = 0$. In particular,

$$\alpha(t) \star \widetilde{F}^\infty = 0, \quad \forall t \geq L, \tag{56}$$

which implies, using the first equation of (55), that $\alpha_x(t)$ is a constant function of space, i.e. $\alpha(t)$ is an affine function of space. However, it is also periodic, so we get

$$\alpha(t) = 0, \quad \forall t \geq L. \tag{57}$$

## 5. Comments and further questions

### *5.1. Backstepping and finite-time stabilization*

As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide explicit feedback laws for exponential stabilization. This allows the construction of explicit controls for null controllability ([38, 14]) as well as time-varying feedbacks that stabilize the system in finite time $T > 0$ ([37, 14]).

The general strategy in these articles is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$ on which the feedback corresponding to some $\lambda_n > 0$ is applied. The idea is then to chose the $t_n$ so that the length of the intervals $[t_n, t_{n+1}]$ tends to 0 fast enough to compensate the growth of the norm of the feedback law as $\lambda_n \to \infty$. Building from this, the authors design a time-varying feedback law that stabilizes the system in finite-time.

Here, the feedback is stationary, and we do not need to define it piecewise: indeed, the norm of the feedback law $F^\lambda$ is bounded when $\lambda \to \infty$. This comes from the fact that we used a special type of convergence to define the feedback law, using a weak version of (19). Indeed, in [40], we set

$$\varphi^{(N)} := \sum_{n=-N}^{N} \varphi_n e_n \in H^m_{per}, \quad \forall N \in \mathbb{N}.$$

Then,

$$\begin{aligned} T^\lambda \varphi^{(N)} &= \sum_{n=-N}^{N} -\varphi_n \overline{F^\lambda_n} \Lambda^\lambda_{-n} \star \varphi \\ &= \sum_{n=-N}^{N} \sum_{p\in\mathbb{Z}} \frac{-\varphi_n \overline{F_n} \varphi_p}{\lambda_{-n+p}} e_p \\ &= \sum_{p\in\mathbb{Z}} \varphi_p \left( \sum_{n=-N}^{N} \frac{-\varphi_n \overline{F_n}}{\lambda_{-n+p}} \right) e_p. \end{aligned}$$

and $F^\lambda$ is defined by

$$\frac{1}{-\varphi_n \overline{F^\lambda_n}} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\lambda_{-n+p}}$$

in order to have

$$\langle T^\lambda \varphi^{(N)}, e_n \rangle \xrightarrow[N\to\infty]{} \varphi_n, \quad n \in \mathbb{Z},$$

which is the weak version of (19).

Now, if the convergence of the right-hand side had been absolute, the limit would have gone to 0 when $\lambda \to \infty$. However, here the sum converges in a special way due to the Dirichlet convergence theorem (see for example [19]), which is why it remains positive (and thus $F^\lambda$ remains bounded) when $\lambda \to \infty$.

Hence, a weaker $TB = B$ condition seems to allow for better behavior of the feedback law when $\lambda \to \infty$.

### *5.2. Regularity of the feedback law*

A remarkable point of this application of the backstepping method, both for rapid and finite-time stabilization, is that the feedback law is not regular on the state space: indeed, it is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$.

On the other hand, it seems that a continuous feedback law would have a more restricted action on the eigenvalues of the system. Indeed, in [30] it is proved that if the sequence of complex numbers $(\rho_n)_{n\in\mathbb{Z}}$ satisfies

$$\left( \left| \frac{\rho_n - \frac{2i\pi n}{L}}{\varphi_n} \right| \right) \in \ell^2, \tag{58}$$

then there exists a *bounded* feedback law such that the resulting closed-loop system has eigenvalues $(\rho_n)_{n\in\mathbb{Z}}$. It is clear that (58) does not allow for a uniform pole-shifting as we have done in [40]. But even though (58) is not a necessary condition, subsequent works such as [29, 4, 28] turn to unbounded feedback laws, as they are proved to allow for more eigenvalue displacement, and in particular uniform pole-shifting. *A fortiori*, the stronger notion of finite-time stabilization, in which case the operator associated to the closed-loop system has an empty spectrum (see for example [29, Theorem 3 and comments]), probably requires an unbounded feedback law.


## Acknowledgements

This work is partially supported by ANR project Finite4SoS (ANR-15-CE23-0007), and the French Corps des Mines.

I would like to thank Jean-Michel Coron, for bringing the problem to my attention, for his patience, and his valuable remarks, as well as Amaury Hayat and Shengquan Xiang for discussions on this problem.


## References


[1] Andras Balogh and Miroslav Krstić. Infinite Dimensional Backstepping-Style Feedback Transformations for a Heat Equation with an Arbitrary Level of Instability. *European Journal of Control*, 8(2):165–175, 2002.

[2] Georges Bastin and Jean-Michel Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*, volume 88 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.

[3] Dejan M. Bosković, Andras Balogh, and Miroslav Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. *Math. Control Signals Systems*, 16(1):44–75, 2003.

[4] B. M. N. Clarke and D. Williamson. Control canonical forms and eigenvalue assignment by feedback for a class of linear hyperbolic systems. *SIAM J. Control Optim.*, 19(6):711–729, 1981.

[5] Jean-Michel Coron. Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. *ESAIM Control Optim. Calc. Var.*, 8:513–554, 2002. A tribute to J. L. Lions.

[6] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[7] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM J. Control Optim.*, 47(3):1460–1498, 2008.

[8] Jean-Michel Coron and Brigitte d'Andréa Novel. Stabilization of a rotating body beam without damping. *IEEE Trans. Automat. Control*, 43(5):608–618, 1998.

[9] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 115:24–73, 2018.

[10] Jean-Michel Coron, Long Hu, and Guillaume Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *J. Funct. Anal.*, 271(12):3554–3587, 2016.

[11] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Automatica J. IFAC*, 84:95–100, 2017.

[12] Jean-Michel Coron and Qi Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. *J. Math. Pures Appl. (9)*, 102(6):1080–1120, 2014.

[13] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *J. Differential Equations*, 259(8):3683–3729, 2015.

[14] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3):993–1023, 2017.

[15] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstić, and Georges Bastin. Local exponential $H^2$ stabilization of a $2 \times 2$ quasilinear hyperbolic system using backstepping. *SIAM J. Control Optim.*, 51(3):2005–2035, 2013.

[16] François Dubois, Nicolas Petit, and Pierre Rouchon. Motion planning and nonlinear simulations for a tank containing a fluid. In *1999 European Control Conference (ECC)*, pages 3232–3237. IEEE, 1999.

[17] Amaury Hayat. Exponential stability of general 1-D quasilinear systems with source terms for the C 1 norm under boundary conditions. preprint, October 2017.

[18] Amaury Hayat. On boundary stability of inhomogeneous 2 x 2 1-D hyperbolic systems for the $C^1$ norm. preprint, July 2018.

[19] Jean-Pierre Kahane and Pierre-Gilles Lemarié-Rieusset. *Séries de Fourier et ondelettes*. Cassini, 1998.

[20] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. *SIAM J. Control Optim.*, 35(5):1591–1613, 1997.

[21] Miroslav Krstić, Bao-Zhu Guo, Andras Balogh, and Andrey Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica J. IFAC*, 44(1):63–74, 2008.

[22] Miroslav Krstic, Petar V. Kokotovic, and Ioannis Kanellakopoulos. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1995.

[23] Miroslav Krstić and Andrey Smyshlyaev. *Boundary control of PDEs*, volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. A course on backstepping designs.

[24] Wei Jiu Liu and Miroslav Krstić. Backstepping boundary control of Burgers' equation with actuator dynamics. *Systems Control Lett.*, 41(4):291–303, 2000.

[25] Yuri Orlov and Denis Dochain. Discontinuous feedback stabilization of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor. *IEEE Trans. Automat. Control*, 47(8):1293–1304, 2002.

[26] Nicolas Petit and Pierre Rouchon. Dynamics and solutions to some control problems for water-tank systems. *IEEE Trans. Automat. Control*, 47(4):594–609, 2002.

[27] J.L. Pitarch, Mohsen Rakhshan, Mohammad Mardani, Mokhtar Sadeghi, and C Prada. Distributed nonlinear control of a plug-flow reactor under saturation. *IFAC-PapersOnLine*, 49:87–92, 12 2016.

[28] Richard Rebarber. Spectral assignability for distributed parameter systems with unbounded scalar control. *SIAM J. Control Optim.*, 27(1):148–169, 1989.

[29] David L. Russell. Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory. *J. Math. Anal. Appl.*, 40:336–368, 1972.

[30] David L. Russell. Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems. *J. Math. Anal. Appl.*, 62(1):186–225, 1978.

[31] Andrey Smyshlyaev, Eduardo Cerpa, and Miroslav Krstić. Boundary stabilization of a 1-D wave equation with in-domain antidamping. *SIAM J. Control Optim.*, 48(6):4014–4031, 2010.

[32] Eduardo D. Sontag. *Mathematical control theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998. Deterministic finite-dimensional systems.

[33] Daisuke Tsubakino, Miroslav Krstić, and Shinji Hara. Backstepping control for parabolic pdes with in-domain actuation. *2012 American Control Conference (ACC)*, pages 2226–2231, 2012.

[34] Jose Manuel Urquiza. Rapid exponential feedback stabilization with unbounded control operators. *SIAM J. Control Optim.*, 43(6):2233–2244, 2005.

[35] Ambroise Vest. Rapid stabilization in a semigroup framework. *SIAM J. Control Optim.*, 51(5):4169–4188, 2013.

[36] Frank Woittennek, Siqian Wang, and Torsten Knüppel. Backstepping design for parabolic systems with in-domain actuation and Robin boundary conditions. *IFAC Proceedings Volumes*, 47(3):5175–5180, 2014. 19th IFAC World Congress.

[37] Shengquan Xiang. Small-time local stabilization for a Korteweg-de Vries equation. *Systems & Control Letters*, 111:64 – 69, 2018.

[38] Shengquan Xiang. Null controllability of a linearized Korteweg-de Vries equation by backstepping approach. *SIAM J. Control Optim.*, 57(2):1493–1515, 2019.

[39] Xin Yu, Chao Xu, Huacheng Jiang, Arthi Ganesan, and Guojie Zheng. Backstepping synthesis for feedback control of first-order hyperbolic PDEs with spatial-temporal actuation. *Abstr. Appl. Anal.*, pages Art. ID 643640, 13, 2014.

[40] Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. preprint, October 2018.