The spectrum of a Schrödinger operator in a wire-like domain with a purely imaginary degenerate potential in the semiclassical limit

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Abstract

Consider a two-dimensional domain shaped like a wire, not necessarily of uniform cross section. Let V denote an electric potential driven by a voltage drop between the conducting surfaces of the wire. We consider the operator $\mathcal{A}_h = -h^2\Delta + iV$ in the semi-classical limit $h \to 0$. We obtain both the asymptotic behaviour of the left margin of the spectrum, as well as resolvent estimates on the left side of this margin. We extend here previous results obtained for potentials for which the set where the current (or ∇V) is normal to the boundary is discrete, in contrast with the present case where V is constant along the conducting surfaces.

1 Introduction

1.1 Main assumptions

We consider the operator

$$\mathcal{A}_h = -h^2\Delta + i\,V\,, \tag{1.1a}$$

defined on

$$D(\mathcal{A}_h) = \{\, u \in H^2(\Omega, \mathbb{C}) \,|\, u|_{\partial\Omega_D} = 0\,;\, \partial u/\partial\nu|_{\partial\Omega_N} = 0 \,\}\,. \tag{1.1b}$$

In the above, $\Omega \subset \mathbb{R}^2$ denotes a bounded, simply connected domain which has the same characteristics as in [8, 5]. In particular its boundary $\partial\Omega$ contains two disjoint open subsets $\partial\Omega_D$ and $\partial\Omega_N$ such that

$$\overline{\partial\Omega_D} \cup \overline{\partial\Omega_N} = \partial\Omega\,,$$

where $\partial\Omega_D$ is a union of two disjoint smooth interfaces on which we prescribe a Dirichlet boundary condition, and $\partial\Omega_N$ is a union of two disjoint smooth interfaces on which we prescribe a Neumann boundary condition. Hence $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ consists of four points which will be called corners. The analysis can be extended to domains, where $\partial\Omega_D$ (and $\partial\Omega_N$) consists of a greater number of disjoint components. In the interest of simplicity we shall confine ourselves to the simplest possible case.

In the context of superconductivity we may say that $\partial\Omega_D$ and $\partial\Omega_N$, are respectively adjacent either to a normal metal or to an insulator. We denote each connected component of $\partial\Omega_\#$ ($\# \in \{D, N\}$) by a superscript $i \in \{1, 2\}$, i.e.,

$$\partial\Omega_\# = \partial\Omega^1_\# \cup \partial\Omega^2_\#\,, \quad \# \in \{D, N\}\,,\ i \in \{1, 2\}\,.$$

We say that $\partial\Omega$ is of class $C^{n,+}$ for some $n \in \mathbb{N}$, if there exists $\check{\beta} > 0$ such that $\partial\Omega$ is of class $C^{n,\check{\beta}}$. As in [3, 8, 5] we make the following assumptions on $\partial\Omega$

$$(R1)\ \begin{cases} (a)\ \overline{\partial\Omega_\#} \text{ is of class } C^{n,+} \text{ for } \# \in \{D, N\}\,; \\ (b) \text{ near each corner, } \overline{\partial\Omega_D} \text{ and } \overline{\partial\Omega_N} \text{ meet with an angle of } \frac{\pi}{2}\,. \end{cases} \tag{1.2}$$

We define n for each result separately (but always have $n \geq 2$). We occasionally use the notation $(R1(n))$ to specify n in the assumption.

Near the corners, we assume in addition that there exists a smooth tranformation, mapping the vicinity of the corner onto a vicinity of rectangular corner. More precisely

$$(R2)\ \begin{cases} \text{For each corner } \mathfrak{c}\,, \text{ there exist } R > 0 \text{ and an invertible holomorphic function} \\ \Phi \text{ in } B(\mathfrak{c}, R) \cap \Omega, \text{ which is in addition in } C^{n,+}(\bar{\Omega} \cap B(\mathfrak{c}, R)), \\ \text{such that } \Phi(\mathfrak{c}) = 0\,,\ \Phi(B(\mathfrak{c}, R) \cap \Omega) \subset Q := \mathbb{R}_+ \times \mathbb{R}_+\,, \\ \text{and } \Phi(\partial\Omega \cap B(\mathfrak{c}, R)) \subset (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+) \cup \{0\}\,. \end{cases} \tag{1.3}$$

Again we use $(R2(n))$ to specify n in the assumption.

We consider potentials $V \in H^2(\Omega)$ satisfying

$$\begin{cases} \Delta V = 0 \quad \text{in } \Omega\,, \\ V = C_i \quad \text{on } \partial\Omega^i_D \text{ for } i = 1, 2\,, \\ \frac{\partial V}{\partial\nu} = 0 \quad \text{on } \partial\Omega_N\,, \end{cases} \tag{1.4}$$

describing a potential drop along a wire.
Assumptions $(R1(n))$ and $(R2(n))$ imply that $V \in C^{n,+}(\bar{\Omega})$. Away from the corners, we may rely on Schauder estimates to establish the desired regularity. In the neighborhood of a corner, we may use the conformal map given by Assumption $(R2(n))$ to obtain a problem for V in a right-angled sector. Then we can use a reflection argument to establish the announced regularity of V (cf. [3, 8] for instance).

We assume further, as in [6], that V satisfies

$$|\nabla V(x)| \neq 0\,,\ \forall x \in \overline{\Omega}\,. \tag{1.5}$$

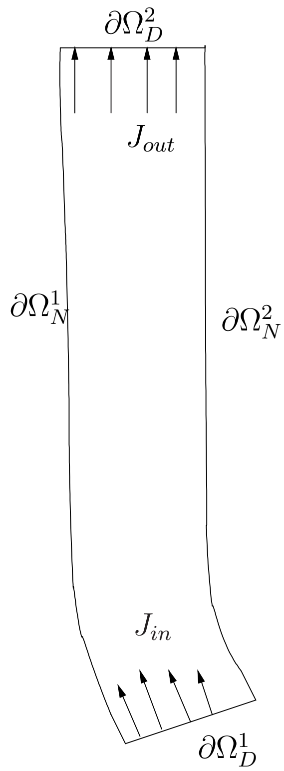

Figure 1: A typical wire-like domain. The arrows denote the direction of the potential gradient (or the current flow: J_{in} for the inlet, and J_{out} for the outlet).

This implies that

$$C_1 \neq C_2 \,.$$

We can indeed follow one component of $\partial\Omega_N$ between two corners and observes that the tangential derivative of V never vanishes (cf. [3]).

The mathematical analysis of Equation (1.4) has a very long record in the literature. We refer to [20], where explicitly known solutions, for many simple domains including the square, are listed. Figure 1 presents a typical sample with properties (R1) and (R2), where the current flows into the sample from one connected component of $\partial\Omega_D$, and exits from another part, disconnected from the first one. Most wires would fall into the above class of domains.
Note that, V being constant on each connected component of $\partial\Omega_D$, we have

$$|\nabla V| = |\partial V/\partial\nu| \ \text{ on } \partial\Omega_D \,.$$

We distinguish in the sequel between two types of potentials satisfying (1.4).

V1 Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}} |\partial V/\partial\nu|$ is attained, lie in $\partial\Omega_D$.

V2 Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}} |\partial V/\partial\nu|$ is attained are corners.

In appendix A we present examples corresponding to both cases. While other cases could be treated by the same techniques, we limit ourselves to these two cases in the interest of simplicity.

The spectral analysis of a Schrödinger operator with a purely imaginary potential has several applications in mathematical physics, among them are the Orr-Sommerfeld equations in fluid dynamics [21], the Ginzburg-Landau equation in the presence of electric current (when magnetic field effects are neglected) [3, 19], the null controllability of Kolmogorov type equations [11], and the diffusion nuclear magnetic resonance [23, 24, 13]. In the present contribution we focus on the Ginzburg-Landau model, in the absence of magnetic field, and choose an electric potential satisfying (1.4). Such a potential was considered in [3] where the asymptotics of a lower bound of $\inf \operatorname{Re} \sigma(\mathcal{A}_h)$ have been obtained as $h \to 0$. Assuming a smooth domain, a similar result has been established in [18], using a more constructive technique, which is employed in the present contribution as well, providing resolvent estimates in addition to the above lower bound.

In [6], improving previous results from [9], we obtained in collaboration with D. Grebenkov, the asymptotic behaviour of an upper bound for $\inf \operatorname{Re} \sigma(-h^2\Delta + iV)$ on smooth bounded domains in $\mathbb{R}^d$. To characterize the potentials addressed in [6] we first define (for $d = 2$, which is the case considered in this work)

$$\partial\Omega_\perp = \{x \in \partial\Omega \,|\, \det(\nabla V(x), \vec{\nu}(x)) = 0\}\,,$$

where $\vec{\nu}(x)$ denotes the outward normal at x. Then it is required in [6] that

$$\inf_{x\in\partial\Omega_\perp} \left|\det\, D^2 V_\partial(x)\right| > 0\,,$$

where V_∂ denotes the restriction of V to $\partial\Omega$, and D^2V_∂ denotes its Hessian matrix. Note that $\partial\Omega_\perp$ must be a discrete set in that case. Clearly, such potentials do not belong to the class of potentials considered in this contribution, as is evident from (1.4). It will become clear in Sections 3 and 4 that the techniques employed in [6] are not applicable for potentials satisfying (1.4). The reason is that the ensuing approximate operators near the boundaries are not separable. Thus, while electric potentials satisfying (1.4) appear very naturally in applications, their spectral analysis poses a significant challenge beyond the potentials addressed in [6].

1.2 Main results

We seek an approximation for $\inf \operatorname{Re} \sigma(\mathcal{A}_h)$ in the limit $h \to 0$. Let

$$J_m = \min_{x\in\partial\Omega_D} |\nabla V(x)|\,. \tag{1.6}$$

Denote by $\mathcal{S}$ the set

$$\mathcal{S} = \{x \in \partial\Omega_D \,:\, |\nabla V(x)| = J_m\}\,. \tag{1.7}$$

1.2.1 Type V1 potentials

In this case, any $x \in \mathcal{S}$ is a minimum point of $|\partial V/\partial\nu|$ on $\partial\Omega_D$. Thus,

$$\partial_\parallel \partial_\nu V(x) = 0\,, \quad \forall x \in \mathcal{S}\,, \tag{1.8}$$

where $\partial_\parallel$ represents the derivative with respect to the arclength along the boundary in the positive trigonometric direction. We next introduce

$$\alpha(x) = \partial_\parallel^2 \partial_\nu V(x) \,, \quad \forall x \in \mathcal{S} \,. \tag{1.9}$$

Let

$$\alpha_m = \min_{x\in\mathcal{S}} |\alpha(x)| \,. \tag{1.10}$$

We then define a new set

$$\mathcal{S}^m = \{ x \in \mathcal{S} \,|\, |\alpha(x)| = \alpha_m \} \,, \tag{1.11}$$

We assume in the following that

$$\alpha_m > 0 \,. \tag{1.12}$$

Consequently any $x \in \mathcal{S}$ is a non-degenerate minimum point of $|\partial V/\partial\nu|$. Furthermore we may conclude from (1.12) that $\mathcal{S}$ is discrete.

Our main result in this case is the following theorem.

Theorem 1.1. *Let $\mathcal{A}_h$ be given by* (1.1)*, in which $\partial\Omega$ satisfies (1.2) and (1.3) for $n = 4$, and let V, the solution of* (1.4)*, be of type V1 and satisfy* (1.5)*. Suppose further that* (1.12) *is satisfied. Then*

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\big\{ \mathrm{Re}\ \sigma(\mathcal{A}_h) \big\} = J_m^{2/3} \frac{|\nu_1|}{2} \,, \tag{1.13}$$

where $\nu_1 < 0$ is the rightmost zero of Airy's function.

Remark 1.2. *As will become evident in the sequel, some of the conditions set above are unnecessary in order to obtain the lower bound on the left hand side of* (1.13).

1.2.2 Type V2 potentials

In this case we similarly define

$$\hat{\alpha}(x) = \partial_\parallel \partial_\nu V(x) \,, \tag{1.14}$$

where

$$\hat{\alpha}_m = \min_{x\in\mathcal{S}} |\hat{\alpha}(x)| \,. \tag{1.15}$$

We then define in $\mathcal{S}$ a new subset

$$\hat{\mathcal{S}}^m = \{ x \in \mathcal{S} \,|\, |\hat{\alpha}(x)| = \hat{\alpha}_m \} \,. \tag{1.16}$$

Theorem 1.3. *Let $\mathcal{A}_h$ be given by* (1.1)*, in which $\partial\Omega$ satisfies* (1.2) *and* (1.3) *for $n = 4$, let V, the solution of* (1.4)*, be of type V2 and satisfy* (1.5)*. Suppose further that $\hat{\alpha}_m > 0$. Then*

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\big\{ \mathrm{Re}\ \sigma(\mathcal{A}_h) \big\} = J_m^{2/3} \frac{|\nu_1|}{2} \,. \tag{1.17}$$

The rest of the contribution is arranged as follows. In the next section we obtain the leading order asymptotic behaviour of a lower bound of $\inf \operatorname{Re} \sigma(\mathcal{A}_h)$ in the limit $h \to 0$. In Sections 3 and 4 we obtain a quasimode for $\mathcal{A}_h$ for potentials of type $V1$ and $V2$ respectively. In Section 5 we obtain some auxiliary resolvent estimates in one dimension, that are employed in Section 6. In Section 6 we obtain resolvent estimates for the approximate operator appearing in Section 3 for type $V1$ potentials. A similar task is carried in Section 7 for type $V2$ potentials. In the last section we complete the proof of Theorems 1.1 and 1.3. Finally, in the appendix we bring examples of potentials of both types.

2 Lower bound

2.1 Main statement

We now state and prove

Proposition 2.1. *Let Ω satisfy* (1.2) *and* (1.3) *with $n = 3$ and V satisfy* (1.5). *Then, we have*

$$\liminf_{h\to 0} \frac{1}{h^{2/3}} \inf\big\{\operatorname{Re}\, \sigma(\mathcal{A}_h)\big\} \geq J_m^{2/3} \frac{|\nu_1|}{2} . \tag{2.1}$$

The proof differs from the proof of the lower bound in [6] only by the need to estimate the resolvent in the vicinity of the Dirichlet-Neumann corners. We thus begin by recalling various lemmas from [6], and then continue by treating the corner case.

Note that (2.1) has already been proved in [3]. Nevertheless, the proof brought in this section, is more constructive and provides resolvent estimates for $\mathcal{A}_h$ in addition for the lower bound on the spectrum.

2.2 Preliminary lemmas

The following lemmas all involve an affine approximation of V.

2.2.1 Complex Airy operator in $\mathbb{R}^2$

Lemma 2.2. *Let*

$$\mathcal{A}_0 = -\Delta + ix_1 ,$$

be defined on

$$D(\mathcal{A}_0) = \{u \in H^2(\mathbb{R}^2) \,|\, x_1 u \in L^2(\mathbb{R}^2)\} .$$

Then, for any $\omega > 0$, there exists C_ω such that

$$\sup_{\operatorname{Re} z \leq \omega} \|(\mathcal{A}_0 - z)^{-1}\| \leq C_\omega .$$

Remark 2.3. *By dilation, we obtain the same result for $-\Delta + i\,\mathfrak{j}\, x_1$ for any $\mathfrak{j} \in \mathbb{R}\backslash\{0\}$. Hence, we can obtain a uniform bound, with respect to $\mathfrak{j}$, of $\sup_{\operatorname{Re} z\leq\omega} \|(-\Delta + i\,\mathfrak{j}\, x_1 - z)^{-1}\|$ on any compact interval excluding 0.*

2.2.2 Complex Airy operator in $\mathbb{R}^2_+$

The next lemma considers the Neumann problem in $\mathbb{R}^2_+ = \mathbb{R}\times\mathbb{R}_+$ which arises while localizing $\mathcal{A}_h$ near $\partial\Omega_N$. It follows immediately from [6, Proposition 4.9].

Lemma 2.4. *Let*

$$\mathcal{A}_N = -\Delta + ix_1 \,,$$

be defined on

$$D(\mathcal{A}_N) = \{u \in H^2(\mathbb{R}^2_+) \,|\, x_1 u \in L^2(\mathbb{R}^2_+) \,,\ \partial u/\partial x_2|_{\partial\mathbb{R}^2_+} = 0\} \,.$$

Then, for any $\omega > 0$, there exists C_ω such that

$$\sup_{\operatorname{Re} z \le \omega} \|(\mathcal{A}_N - z)^{-1}\| \le C_\omega .$$

Remark 2.5. *By the same argument of Remark 2.3 the resolvent of the Neumann realization of $-\Delta + i\,\mathfrak{j}\,x_1$ is uniformly bounded with respect to $\mathfrak{j}$ on any compact interval excluding* 0*.*

We also restate another conclusion of [6, Proposition 4.9] and [6, Proposition 4.5], which is related to the localization of $\mathcal{A}_h$ near $\partial\Omega_D$.

Lemma 2.6. *Let, for $\mathfrak{j} \neq 0$,*

$$\mathcal{A}_D = -\Delta + i\,\mathfrak{j}\,x_2 \,,$$

be defined on

$$D(\mathcal{A}_D) = \{u \in H^2(\mathbb{R}^2_+) \,|\, x_2 u \in L^2(\mathbb{R}^2_+) \,,\ u|_{\partial\mathbb{R}^2_+} = 0\} \,.$$

Then, there exists $C > 0$ such that, for all $0 < \epsilon \le 1$,

$$\sup_{\operatorname{Re} z \le |\mathfrak{j}|^{\frac{2}{3}}|\nu_1|/2-\epsilon} \|(\mathcal{A}_D - z)^{-1}\| + \|\nabla(\mathcal{A}_D - z)^{-1}\| + \|\Delta(\mathcal{A}_D - z)^{-1}\| \le \frac{C}{\epsilon} \,.$$

Moreover, C may be chosen independently of $\mathfrak{j}$ if we confine $\mathfrak{j}$ to a closed bounded interval excluding 0*.*

We continue with the following estimate (cf. [6, Lemma 4.12]) which will become useful in Section 8.

Lemma 2.7. *With the notation of Lemma 2.6, for any compact interval $I = [\mu_1, \mu_2]$, there exists a positive $C(I)$ such that, for any $z = \mu + i\nu$ with $|\nu| > \mu_2 + 4$ and $\mu \in I$,*

$$\|(\mathcal{A}^D - z)^{-1}\| \le C(I) \,. \tag{2.2}$$

2.2.3 Complex Airy operator $\mathbb{R}_+ \times \mathbb{R}_+$

We now present a new result which is useful while using blow-up analysis to obtain the contribution of the corners to the resolvent of $\mathcal{A}_h$:

Lemma 2.8. *Let $\mathcal{A}_c$ denote the operator*

$$\mathcal{A}_c = -\Delta + i\,\mathfrak{j}\,x_1 \tag{2.3a}$$

defined on

$$D(\mathcal{A}_c) = \{u \in H^2(Q)\,|\, u_{\partial Q_\parallel} = 0\,;\, \partial_\nu u_{\partial Q_\perp} = 0\,;\, x_1\, u \in L^2(Q)\}\,, \tag{2.3b}$$

where $\mathfrak{j} \neq 0$ and

$$Q = \mathbb{R}_+ \times \mathbb{R}_+ \quad ; \quad \partial Q_\perp = \mathbb{R}_+ \times \partial\mathbb{R}_+ \quad ; \quad \partial Q_\parallel = \partial\mathbb{R}_+ \times \mathbb{R}_+\,.$$

Then, there exists $C > 0$ such that, for any $\epsilon > 0$,

$$\sup_{\mathrm{Re}\,\lambda \leq |\mathfrak{j}|^{\frac{2}{3}}\,|\nu_1|/2-\epsilon} \|(\mathcal{A}_c - \lambda)^{-1}\| \leq \frac{C}{\epsilon}\,. \tag{2.4}$$

Moreover, C may be chosen independently of $\mathfrak{j}$ if we confine its value to a closed bounded interval excluding 0.

Proof. It can be easily verified that $\mathcal{A}_c : D(\mathcal{A}_c) \to L^2(Q)$ is surjective, injective, and maximally accretive. This can be done either by the separation of variable technique as in [6] or by using generalized Lax-Milgram lemma from [7]. Note that by the arguments presented in [8, Proposition A.3] there exists $C > 0$ such that for every $u \in D(\mathcal{A}_c)$ and $0 < r_1 < r_2$,

$$\|u\|_{H^2(D_{r_1})} \leq C\,\left(\|\Delta u\|_{L^2(D_{r_2})} + \|u\|_{L^2(D_{r_2})}\right)\,,$$

where $D_r = B(0,r) \cap \mathcal{Q}$. Hence, the presence of a corner does not pose a significant obstacle on the way to obtain global regularity estimates.

To prove (2.4) we write

$$\mathcal{A}_c = \mathcal{L}_+ - \partial^2_{x_2}\,,$$

where $\mathcal{L}_+$ is the Dirichlet realization in $\mathbb{R}_+$ of

$$\mathcal{L}_+ = -\frac{d^2}{dx_1^2} + i\,\mathfrak{j}\,x_1\,,$$

and $-\partial^2_{x_2}$ denotes by abuse of notation the Neumann realization of $-\frac{d^2}{dx_2^2}$ in $\mathbb{R}_+$. Since $\mathcal{L}_+$ and $-\partial^2_{x_2}$ commute, we have

$$e^{-t\mathcal{A}_c} = e^{t\partial^2_{x_2}} \otimes e^{-t\mathcal{L}_+}\,,$$

and hence

$$\|e^{-t\mathcal{A}_c}\| \leq \|e^{-t\mathcal{L}_+}\|\,. \tag{2.5}$$

From [6, Proposition 2.4] we learn that

$$\|e^{-t\mathcal{L}_+}\| \leq C\, e^{-t\,|\mathrm{j}|^{\frac{2}{3}}\,|\nu_1|/2}\,,$$

and hence by (2.5) we have

$$\|e^{-t\mathcal{A}_c}\| \leq C\, e^{-t\,|\mathrm{j}|^{\frac{2}{3}}\,|\nu_1|/2}\,.$$

Hence, whenever $\mathrm{Re}\,\lambda < |\mathrm{j}|^{\frac{2}{3}}\,|\nu_1|/2$ we have

$$\|(\mathcal{A}_c-\lambda)^{-1}\| \leq \int_0^\infty \|e^{-t(\mathcal{A}_c-\mathrm{Re}\,\lambda)}\|\,dt \leq \frac{C}{|\mathrm{j}|^{\frac{2}{3}}\,|\nu_1|/2-\mathrm{Re}\,\lambda}\,.$$

∎

Finally, we shall need, in the last section, the following lemma, which is analogous to Lemma 2.7,

Lemma 2.9. *For any compact interval $I=[\mu_1,\mu_2]$, there exists a positive $C(I)$ such that, for any $z=\mu+i\nu$ with $|\nu|>\mu_2+4$ and $\mu\in I$,*

$$\|(\mathcal{A}_c-z)^{-1}\| \leq C(I)\,. \tag{2.6}$$

Proof. To obtain a resolvent estimate, we use an even extension in s, i.e., we define the operator $\mathcal{A}_c^e$ which is associated with the same differential operator as $\mathcal{A}_c$ but whose domain is defined by

$$\mathcal{D}(\mathcal{A}_c^e)=\{u\in H^2(\mathbb{R}^2_+)\,|\,u_{\partial\mathbb{R}^2_+}=0\,;\,\mathcal{R}u=u\,;\,x_1u\in L^2(\mathbb{R}^2_+)\}\,,$$

where $\mathcal{R}$ denotes the reflection $x_2\to -x_2$. The lemma then follows immediately from Lemma 2.7. ∎

2.3 Proof of Proposition 2.1

The proof is similar to the derivation of the lower bound in [6, Section 6] or [18, Section 4]. We thus, recall the main steps only, focusing primarily on the resolvent estimates near the corners (that are absent from [6, 18]).

2.3.1 Partition of unity

For some $1/3<\varrho<2/3$, $h_0>0$, and for every $h\in(0,h_0]$, we choose two sets of indices $\mathcal{J}_i(h)$, $\mathcal{J}_\partial(h)$, and a set of points

$$\big\{a_j(h)\in\Omega : j\in\mathcal{J}_i(h)\big\}\cup\big\{b_k(h)\in\partial\Omega : k\in\mathcal{J}_\partial(h)\big\}\,, \tag{2.7a}$$

such that $B(a_j(h),h^\varrho)\subset\Omega$,

$$\overline{\Omega}\subset\bigcup_{j\in\mathcal{J}_i(h)}B(a_j(h),h^\varrho)\;\cup\bigcup_{k\in\mathcal{J}_\partial(h)}B(b_k(h),h^\varrho)\,, \tag{2.7b}$$

and such that the closed balls $\bar{B}(a_j(h), h^\varrho/2)$, $\bar{B}(b_k(h), h^\varrho/2)$ are all disjoint.
We further split $\mathcal{J}_\partial(h)$ into three disjoint subsets

$$\mathcal{J}_\partial(h) = \mathcal{J}_\partial^D \cup \mathcal{J}_\partial^N \cup \mathcal{J}_\partial^c\,, \tag{2.8}$$

such that $b_k(h) \in \partial\Omega_D$ whenever $k \in \mathcal{J}_\partial^D$, $b_k(h) \in \partial\Omega_N$ whenever $k \in \mathcal{J}_\partial^N$, and for every $k \in \mathcal{J}_\partial^c$ b_k denotes a corner.
We note that $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ is a finite set consisting of the four corners, and that by the above construction

$$\bigcup_{k\in\mathcal{J}_\partial^c} \{b_k\} = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}\,. \tag{2.9}$$

We now construct in $\mathbb{R}^2$ two families of C^∞ functions

$$(\chi_{j,h})_{j\in\mathcal{J}_i(h)} \text{ and } (\zeta_{k,h})_{k\in\mathcal{J}_\partial(h)}\,, \tag{2.10a}$$

such that, for every $x \in \overline{\Omega}$,

$$\sum_{j\in\mathcal{J}_i(h)} \chi_{j,h}(x)^2 + \sum_{k\in\mathcal{J}_\partial(h)} \zeta_{k,h}(x)^2 = 1\,, \tag{2.10b}$$

and such that $\operatorname{supp}\chi_{\rm j,h} \subset {\rm B}({\rm a_j(h), h^\varrho})$ for $j \in \mathcal{J}_i(h)$, $\operatorname{supp}\zeta_{\rm k,h} \subset {\rm B}({\rm b_k(h), h^\varrho})$ for $k \in \mathcal{J}_\partial(h)$, and $\chi_{j,h} \equiv 1$ (respectively $\zeta_{k,h} \equiv 1$) on $\bar{B}(a_j(h), h^\varrho/2)$ (respectively $\bar{B}(b_k(h), h^\varrho/2)$) .

In addition, we also assume that, for all $\alpha \in \mathbb{N}^2$, there exist positive h_0 and C_α, such that, $\forall h \in (0, h_0]$, $\forall x \in \overline{\Omega}$,

$$\sum_j |\partial^\alpha \chi_{j,h}(x)|^2 \leq C_\alpha\, h^{-2|\alpha|\varrho} \quad \text{and} \quad \sum_k |\partial^\alpha \zeta_{k,h}(x)|^2 \leq C_\alpha\, h^{-2|\alpha|\varrho}\,. \tag{2.10c}$$

To satisfy the Neumann boundary condition on $\partial\Omega_N$, and for later reference, we introduce an additional condition

$$\frac{\partial\check{\xi}_h}{\partial\nu}\Big|_{\partial\Omega} = 0\,. \tag{2.11}$$

We further set $\eta_{k,h} = 1_\Omega\, \zeta_{k,h}$.

2.3.2 Definition of the approximate resolvent

Following [6, 18] we construct, for any $\epsilon > 0$, an approximate resolvent, which should be close in operator norm to $(\mathcal{A}_h - \lambda)^{-1}$ as $h \to 0$ for

$$\operatorname{Re}\lambda \leq \left(|J_m|^{2/3}\frac{|\nu_1|}{2} - \epsilon\right) h^{\frac{2}{3}}\,. \tag{2.12}$$

The construction is based on localized resolvents defined on the disks $B(a_j(h), h^\varrho)$ or $B(b_k(h), h^\varrho)$.
For $j \in \mathcal{J}_i$ we set

$$\begin{cases} \mathcal{A}_{j,h} = -h^2\Delta + i\left(V(a_j(h)) + \nabla V(a_j(h))\cdot(x - a_j(h))\right), \\ \mathcal{D}(\mathcal{A}_{j,h}) = H^2(\mathbb{R}^2) \cap L^2(\mathbb{R}^2; |\nabla V(a_j(h))\cdot x|^2 dx)\,. \end{cases} \tag{2.13}$$

By Remark 2.3 (see also [18, Lemma 2.1])

$$\sup_{\operatorname{Re}\lambda\leq\omega\, h^{2/3}} \|(\mathcal{A}_{j,h}-\lambda)^{-1}\| \leq \frac{C_\omega}{h^{2/3}}\,. \tag{2.14}$$

Define in a vicinity of b_k a curvilinear coordinate system (s,ρ) such that $\rho=d(x,\partial\Omega)$ and $s(x)$ denotes the signed arclength along $\partial\Omega$ connecting b_k and the projection of x on $\partial\Omega$. The boundary transformation is denoted by $\mathcal{F}_{b_k}$ and its associated operator by $T_{\mathcal{F}_{b_k}}$.
For $k\in\mathcal{J}_\partial^N$ we have $b_k\in\partial\Omega_N$. Hence, we may use the approximate operator

$$\begin{cases}\tilde{\mathcal{A}}_{k,h}:=-h^2\Delta_{s,\rho}+i\big(V(b_k)\pm \mathfrak{j}_k s\big)\\ \mathcal{D}(\tilde{\mathcal{A}}_{k,h})=\{u\in H^2(\mathbb{R}^2_+)\,|\,\partial_\nu u_{\partial\mathbb{R}^2_+}=0\,;\ s\,u\in L^2(\mathbb{R}^2_+)\}\,,\end{cases} \tag{2.15}$$

where

$$\mathfrak{j}_k=|\nabla V(b_k)|=|\partial_\nu V(b_k)| \tag{2.16a}$$

and the $\pm$ sign is determined by the condition

$$\pm\,\partial_\nu V(b_k)<0\,. \tag{2.16b}$$

Since ∇V is parallel to the boundary, it follows from Remark 2.5 that

$$\sup_{\operatorname{Re}\lambda\leq\omega\, h^{2/3}} \|(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\| \leq \frac{C_\omega}{h^{2/3}}\,. \tag{2.17}$$

For $k\in\mathcal{J}_\partial^D$, we use the approximate operator

$$\begin{cases}\tilde{\mathcal{A}}_{k,h}=-h^2\Delta_{s,\rho}+i\big(V(b_k)\pm \mathfrak{j}_k \rho\big)\\ \mathcal{D}(\tilde{\mathcal{A}}_{kh})=\{u\in H^2(\mathbb{R}^2_+)\cap H^1_0(\mathbb{R}^2_+)\,|\,\rho u\in L^2(\mathbb{R}^2_+)\}\,.\end{cases} \tag{2.18}$$

By Lemma 2.6 we have, for any $\epsilon>0$, the existence of C_ϵ and $h_\epsilon>0$ such that

$$\sup_{\operatorname{Re}\lambda\leq(\mathfrak{j}_k^{\frac{2}{3}}\,|\nu_1|/2-\epsilon)\ h^{2/3}} \|(\mathcal{A}_{k,h}-\lambda)^{-1}\| \leq \frac{C_\epsilon}{h^{2/3}}\,,\ \forall h\in(0,h_\epsilon]\,. \tag{2.19}$$

Hence, by (2.17) and (2.14) all localized resolvents satisfy (2.19).
As in [6] we construct the following approximate resolvent

$$\mathcal{R}(h,\lambda)=\sum_{j\in\mathcal{J}_i(h)}\chi_{j,h}(\mathcal{A}_{j,h}-\lambda)^{-1}\chi_{j,h}+\sum_{k\in\mathcal{J}_\partial(h)}\eta_{k,h}R_{k,h}(\lambda)\eta_{k,h}\,, \tag{2.20}$$

where

$$R_{k,h}(\lambda)=T^{-1}_{\mathcal{F}_{b_k}}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}T_{\mathcal{F}_{b_k}}\,. \tag{2.21}$$

Hence it remains to define $\mathcal{F}_{b_k}$ and $\tilde{\mathcal{A}}_{k,h}$ when b_k is a corner and to establish the corresponding localized resolvent estimates. This is the object of the next few paragraphs.

2.3.3 Conjugate harmonic maps

Let U denote the conjugate harmonic map of V in Ω, which exists under the assumption that Ω is simply connected. Using the Cauchy-Riemann equation, we immediately get that $U \in H^2(\Omega)$ and has the same regularity of V. As a matter of fact, U is a solution of

$$\Delta U = 0\,,\; U = D_\ell \text{ on } \partial\Omega_N^\ell \text{ for } \ell = 1, 2 \text{ and } \partial_\nu U = 0 \text{ on } \partial\Omega_D\,,$$

where D_1 and D_2 are constants.
Due to Assumption (1.3), U and V are also in $C^{n,+}$ at the corners.

2.3.4 Curvilinear coordinates in the neighborhood of a corner

Let $\mathfrak{c}$ be a corner (after a translation we can set $\mathfrak{c} = (0, 0)$) and since $\nabla V(\mathfrak{c}) \neq 0$ and $|\nabla U| = |\nabla V|$ in $\overline{\Omega}$, we define a new local coordinate system (s, ρ) by

$$(s, \rho) = \mathcal{G}_\mathfrak{c}(x) = \Big(\pm \frac{U - D_{\ell(\mathfrak{c})}}{|\nabla V(\mathfrak{c})|}, \pm \frac{V - C_{i(\mathfrak{c})}}{|\nabla V(\mathfrak{c})|} \Big)\,, \tag{2.22a}$$

where $D_{\ell(\mathfrak{c})} = U(\mathfrak{c})$, $C_{i(\mathfrak{c})} = V(\mathfrak{c})$, and the signs are chosen so that both s and ρ are positive. The Jacobian $g_\mathfrak{c}$ of $\mathcal{G}_\mathfrak{c}(x)$ equals 1 at $\mathfrak{c}$ and $\mathcal{G}_\mathfrak{c}$ admits locally an inverse $\mathcal{F}_\mathfrak{c}$ of class $C^{n,+}$. Hence we may write

$$x = \mathcal{F}_\mathfrak{c}(s, \rho)\,. \tag{2.22b}$$

Note that (s, ρ) is an orthogonal coordinate system, and it can be easily verified that

$$\Delta = g_\mathfrak{c}\, \Delta_{s,\rho}\,, \tag{2.23a}$$

where

$$g_\mathfrak{c}(x) = \frac{|\nabla V(x)|^2}{|\nabla V(\mathfrak{c})|^2} = \tilde{g}_\mathfrak{c}(s, \rho)\,. \tag{2.23b}$$

A simple computation yields

$$\frac{\partial \tilde{g}_\mathfrak{c}}{\partial s} = \pm \frac{1}{g_\mathfrak{c} |\nabla V(\mathfrak{c})|} \Big[- \frac{\partial g_\mathfrak{c}}{\partial x_1} \frac{\partial V}{\partial x_2} + \frac{\partial g_\mathfrak{c}}{\partial x_2} \frac{\partial V}{\partial x_1} \Big]\,, \tag{2.24a}$$

and

$$\frac{\partial \tilde{g}_\mathfrak{c}}{\partial \rho} = \pm \frac{1}{g_\mathfrak{c} |\nabla V(\mathfrak{c})|} \Big[- \frac{\partial g_\mathfrak{c}}{\partial x_1} \frac{\partial V}{\partial x_1} - \frac{\partial g_\mathfrak{c}}{\partial x_2} \frac{\partial V}{\partial x_2} \Big]\,. \tag{2.24b}$$

For $n \geq 3$, we may write

$$\tilde{g}_\mathfrak{c}(s, \rho) = 1 + \tilde{\alpha}_\mathfrak{c} s + \tilde{\beta}_\mathfrak{c} \rho + \mathcal{O}(s^2 + \rho^2)\,. \tag{2.25}$$

One can obtain $\tilde{\alpha}_\mathfrak{c}$ and $\tilde{\beta}_\mathfrak{c}$ by setting $(s, \rho) = (0, 0)$ in (2.24).

2.3.5 Estimates of the localized resolvent near the corners

Let

$$T_{\mathcal{F}}^x : L^2(\Omega \cap B(x,\delta)) \longrightarrow L^2(\mathcal{U}) \quad \text{s.t.} \quad T_{\mathcal{F}}^x(u) = u \circ \mathcal{F}_x\,.$$

Let b_k denote a corner point and $\tilde{\eta}_{k,h} = T_{\mathcal{F}_{b_k}}(\eta_{k,h})$. We also introduce

$$\widehat{\mathcal{A}}_{k,h} = T_{\mathcal{F}_{b_k}} \mathcal{A}_h T_{\mathcal{F}_{b_k}}^{-1}\,. \tag{2.26}$$

Let further, with the notation of (2.3),

$$\begin{cases} \tilde{\mathcal{A}}_{k,h} = -h^2 \Delta_{s,\rho} + i\big(V(b_k) \pm \mathfrak{j}_k \rho\big)\,, \\ \mathcal{D}(\tilde{\mathcal{A}}_{k,h}) = \{u \in H^2(Q) \,|\, u|_{\partial Q_\parallel} = 0\,;\, \partial u/\partial\nu|_{\partial Q_\perp} = 0\,;\, \rho\, u \in L^2(Q)\}\,. \end{cases} \tag{2.27}$$

Let $\varepsilon > 0$ and $\lambda \in \mathbb{C}$ satisfy $\operatorname{Re}\lambda \leq (|\nu_1||\mathfrak{j}_k|^{\frac{2}{3}}/2 - \varepsilon)h^{2/3}$. By (2.23) and (2.25) we have

$$\begin{aligned} \|\tilde{\eta}_{k,h}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| &\leq h^2 \|\tilde{\eta}_{k,h}(\hat{g}_{b_k} - 1)\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \\ &\leq C\, h^{2+\varrho}\, \|\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|\,. \end{aligned} \tag{2.28}$$

By (2.4) there exists $C_\varepsilon > 0$ such that

$$\|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq \frac{C_\varepsilon}{h^{2/3}}\,. \tag{2.29}$$

Furthermore, an integration by parts readily yields

$$\|\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|^2 \leq \frac{1}{h^2}\|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| + \frac{\operatorname{Re}\lambda}{h^2}\|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|^2\,,$$

from which, with the aid of (2.29), it follows that

$$\|\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq \frac{C_\varepsilon}{h^{4/3}}\,. \tag{2.30}$$

Finally, as in [18, Eq. (4.26)] we obtain

$$\|\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|^2 \leq \frac{C}{h^2}\, \|\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|\,,$$

which, with the aid of (2.30) and (2.29), yields

$$\|\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq \frac{C}{h^2}\,.$$

Substituting the above together with (2.29) and (2.30) into (2.28) yields

$$\|\tilde{\eta}_{k,h}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq C\, h^\varrho\,. \tag{2.31}$$

We also need the estimate

$$\|1_\Omega[\mathcal{A}_h, \eta_{k,h}]\, R_{k,h}(\lambda)\| \leq C(h^{2-\varrho}\|\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| + h^{2-2\varrho}\|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\|)\,,$$

which follows from (2.10c) and the fact that

$$[\mathcal{A}_h, \eta_{k,h}] = -h^2(\Delta\eta_{k,h}) - 2h^2\nabla\eta_{k,h} \cdot \nabla\,. \tag{2.32}$$

By (2.29) and (2.30) we then have

$$\|1_\Omega[\mathcal{A}_h, \eta_{k,h}]\, R_{k,h}(\lambda)\| \leq C h^{2/3-\varrho}\,.$$

Combining the above with (2.31) yields

$$\|\tilde{\eta}_{k,h}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} + 1_\Omega[\mathcal{A}_h, \eta_{k,h}]\, R_{k,h}(\lambda)\| \leq C\,(h^\varrho + h^{2/3-\varrho})\,. \tag{2.33}$$

2.3.6 Global error estimate

We may now continue as in [6, Section 6]. We recall that

$$(\mathcal{A}_h - \lambda)\,\mathcal{R}(h,\lambda) = I + \mathcal{E}(h,\lambda)\,, \tag{2.34}$$

where

$$\mathcal{E}(h,\lambda) = \sum_{j\in\mathcal{J}_i(h)} \mathcal{B}_j(h,\lambda)\,\chi_{j,h} + \sum_{k\in\mathcal{J}_\partial(h)} \mathcal{B}_k(h,\lambda)\,\eta_{k,h}\,. \tag{2.35}$$

In the above, for $j\in\mathcal{J}_i(h)$,

$$\mathcal{B}_j := \mathcal{B}_j(h,\lambda) = \chi_{j,h}(\mathcal{A}_h-\mathcal{A}_{j,h})(\mathcal{A}_{j,h}-\lambda)^{-1}\widehat{\chi}_{j,h}+[\mathcal{A}_h,\chi_{j,h}](\mathcal{A}_{j,h}-\lambda)^{-1}\widehat{\chi}_{j,h}\,, \tag{2.36a}$$

and, for $k\in\mathcal{J}_\partial(h)$,

$$\mathcal{B}_k := \mathcal{B}_k(h,\lambda) = \eta_{k,h}T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h}-\tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}T_{\mathcal{F}_{b_k}}\widehat{\eta}_{k,h} + 1_\Omega[\mathcal{A}_h,\eta_{k,h}]\,R_{k,h}\,\widehat{\eta}_{k,h}\,, \tag{2.36b}$$

where $\widehat{\chi}_{j,h}$ and $\widehat{\eta}_{k,h}$ are such that

- Supp $\widehat{\chi}_{j,h}\subset B(a_j(h),2h^\varrho)$ for $j\in\mathcal{J}_i(h)$,
- Supp $\widehat{\eta}_{k,h}\subset B(b_k(h),2h^\varrho)$ for $k\in\mathcal{J}_\partial$,
- $\widehat{\chi}_{j,h}\chi_{j,h}=\chi_{j,h}$ and $\widehat{\eta}_{k,h}\eta_{k,h}=\eta_{k,h}$,

and

$$\widehat{\mathcal{A}}_{k,h} = T_{\mathcal{F}_{b_k}}\mathcal{A}_h T^{-1}_{\mathcal{F}_{b_k}}\,.$$

By (2.14), (2.17), (2.19), and (2.29), it follows, as in [6], that $\mathcal{R}(h,\lambda)$ is well defined, for λ satisfying (2.12). Furthermore, we have

$$\|\mathcal{R}(h,\lambda)\| \le C_\epsilon h^{-\frac{2}{3}}\,,\ \forall h\in(0,h_\epsilon]\,. \tag{2.37}$$

We now estimate the remainder $\mathcal{E}(h,\lambda)$. It has been established in [6, Section 6] that there exists h_0 and $C>0$ such that, for $h\in(0,h_0]$, $j\in\mathcal{J}_i(h)$, $k\in\mathcal{J}^D_\partial(h)\cup\mathcal{J}^N_\partial(h)$ and λ satisfying (2.12),

$$\|\mathcal{B}_j(h,\lambda)\| + \|\mathcal{B}_k(h,\lambda)\| \le C\,h^{\min(\varrho,\frac{2}{3}-\varrho)}\,. \tag{2.38}$$

By (2.31) we have also, for $k\in\mathcal{J}^c_\partial$,

$$\|\mathcal{B}_k(h,\lambda)\,\| \le C\,h^{\min(\varrho,\frac{2}{3}-\varrho)}\,.$$

We now observe, using the finite covering property of the partition, (2.10b) and (2.36), that

$$\begin{aligned}\|\mathcal{E}(h,\lambda)f\|_2^2 &\le C_0\Big(\textstyle\sum_{j\in\mathcal{J}_i(h)}\|\mathcal{B}_j(h,\lambda)\|^2\|\chi_{j,h}f\|_2^2 + \sum_{k\in\mathcal{J}_\partial(h)}\|\mathcal{B}_k(h,\lambda)\|^2\|\eta_{k,h}f\|_2^2\Big)\\ &\le Ch^{2\min(\varrho,\frac{2}{3}-\varrho)}\,\|f\|^2\,.\end{aligned} \tag{2.39}$$

Consequently,

$$\sup_{\{\operatorname{Re}\lambda\leq(|J_m|^{\frac{2}{3}}|\nu_1|/2-\epsilon)\;h^{2/3}\}} \|\mathcal{E}(h,\lambda)\| \xrightarrow[h\to 0]{} 0\,, \tag{2.40}$$

and hence, for sufficiently small h, $I+\mathcal{E}(h,\lambda)$ is invertible. With the aid of (2.37), we then obtain that for each λ satisfying (2.12) we must have $\lambda\in\rho(\mathcal{A}_h)$, and by (2.34) and (2.37)

$$\|(\mathcal{A}_h-\lambda)^{-1}\| \leq \|\mathcal{R}(h,\lambda)\|\,\|(I+\mathcal{E}(h,\lambda))^{-1}\| \leq C_\epsilon h^{-\frac{2}{3}}\,.$$

We may now conclude that for each $\epsilon>0$, there exists $h_0(\epsilon)$ such that whenever $0<h\leq h_0(\epsilon)$ and

$$\{\lambda\in\mathbb{C}\,|\,\operatorname{Re}\lambda\leq(|J_m|^{\frac{2}{3}}|\nu_1|/2-\epsilon)\;h^{2/3}\}\subset\rho(\mathcal{A}_h)\,.$$

Proposition 2.1 is proved.

3 Quasimode construction - Type V1

In this section we construct a three terms expansion of a quasimode. Had $\mathcal{A}_h$ been self-adjoint, we could have used from here the spectral theorem to obtain the existence of an eigenvalue. Alternatively, we can use in the self-adjoint case the Min-max Theorem to obtain an upper bound for the left margin of the spectrum. This is, of course, not possible in the non selfadjoint case which is considered in this work.

3.1 Local coordinates and approximate operator

Let $x_0\in\mathcal{S}^m$. Recall that for type V1 potentials, x_0 lies in the interior of $\partial\Omega_D$. In the curvilinear coordinate system (s,ρ) centered at x_0 we have

$$\Delta=\Big(\frac{1}{g}\frac{\partial}{\partial s}\Big)^2+\frac{1}{g}\frac{\partial}{\partial\rho}\Big(g\frac{\partial}{\partial\rho}\Big)=\frac{1}{\tilde g^2}\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial\rho^2}-\frac{\rho\kappa'(s)}{\tilde g^3}\frac{\partial}{\partial s}-\frac{\kappa(s)}{\tilde g}\frac{\partial}{\partial\rho}\,, \tag{3.1}$$

where

$$g(x):=\widetilde{g}(s,\rho)=1-\rho\,\kappa(s)\,, \tag{3.2}$$

and $\kappa(s)$ is the curvature at s on $\partial\Omega$.
Note for later reference that (3.2) implies

$$|\widetilde{g}(s,\rho)-1|\leq C\,\rho\ \text{ for } (s,\rho)\in(-s_0,s_0)\times[0,\rho_0)\,. \tag{3.3}$$

We next expand V in the curvilinear coordinates (s,ρ),

$$V(x)-V(x_0)=\widetilde{V}(s,\rho)-V(x_0)=c\,\rho+\frac{1}{2}\hat\beta\rho^2+\frac{1}{2}\alpha s^2\rho+\delta\widetilde{V}(s,\rho)\,, \tag{3.4}$$

where

$$c=\widetilde{V}_\rho(0)\quad,\quad\alpha=\widetilde{V}_{ss\rho}(0)\quad,\quad\hat\beta=\widetilde{V}_{\rho\rho}(0)\,, \tag{3.5}$$

and

$$|\delta\widetilde{V}(s,\rho)|\leq C\,(|s|\rho^2+|\rho|^3)\,,\ \text{ for } (s,\rho)\in(-s_0,s_0)\times[0,\rho_0)\,. \tag{3.6}$$

Remark 3.1. *Using the notation of the introduction, we observe that*

$$|c| = |\nabla V(x_0)| = J_m$$

and note that the sign of c is determined by the values of C_1 and C_2 in (1.4)*. Thus, for $x_0 \in \Omega_D^1$ and $C_2 > C_1$ or $x_0 \in \Omega_D^2$ and $C_2 < C_1$, we have $c > 0$ whereas for $x_0 \in \Omega_D^2$ and $C_2 > C_1$ or $x_0 \in \Omega_D^1$ and $C_2 < C_1$, we have $c < 0$.*

Note that we may deduce from (1.5), (1.11) and (1.12) that:

$$\alpha c > 0\,. \tag{3.7}$$

In the rest of this section we assume $c > 0$, without any loss of generality, since otherwise we consider $\bar{\mathcal{A}}_h$ instead of $\mathcal{A}_h$ and use the relation

$$\sigma(\bar{\mathcal{A}}_h) = \overline{\sigma(\mathcal{A}_h)}\,.$$

Blowup

Applying the transformation

$$\tau = \Big(\frac{J_m}{h^2}\Big)^{1/3}\rho\,,\ \sigma = \Big(\frac{\alpha_m^3}{8J_m h^4}\Big)^{1/12} s\,, \tag{3.8}$$

to (3.1) with

$$u(x) = \widetilde{u}(s,\rho) = \check{u}(\sigma,\tau)\,,$$

yields the identity

$$h^2\Delta u = (hJ_m)^{2/3}\big(\check{u}_{\tau\tau} + \mathfrak{e}(h)\,\check{u}_{\sigma\sigma} - \mathfrak{e}(h)\check{\kappa}(\sigma)[2J_m/\alpha_m]^{1/2}\,\check{u}_\tau + \delta\,u\big)\,, \tag{3.9}$$

where

$$\mathfrak{e}(h) = \frac{\alpha_m^{1/2}h^{2/3}}{2^{\frac{1}{2}}J_m^{5/6}}\,. \tag{3.10}$$

Here $\check{\kappa}(\sigma) = \kappa(s(\sigma))$ and δ is the operator $u \mapsto \delta u$ given by

$$\delta u = \mathfrak{e}(h)\Big(\frac{1}{\tilde{g}^2} - 1\Big)\,\check{u}_{\sigma\sigma} + \mathfrak{e}(h)^{5/2}\frac{2J_m}{\alpha_m}\frac{\tau\kappa'(s(\sigma))}{\tilde{g}^3}\,\check{u}_\sigma - \mathfrak{e}(h)\check{\kappa}(\sigma)[2J_m/\alpha_m]^{1/2}\Big(\frac{1}{\tilde{g}} - 1\Big)\,\check{u}_\tau\,. \tag{3.11}$$

It can be easily verified, using (3.3), that, there exists C, $h_0 > 0$ and $\rho_0 > 0$ such that, for $h \in (0, h_0]$ and u s.t. $\operatorname{supp}\widetilde{\mathrm{u}} \subset (-\mathrm{s}_0, \mathrm{s}_0) \times [0, \rho_0)$,

$$\|\delta u\|_2 \leq C\mathfrak{e}(h)^2\|\check{u}\|_{B^3(\mathbb{R}^2_+)}\,, \tag{3.12}$$

where for $\ell \in \mathbb{N}$,

$$B^\ell(\mathbb{R}^2_+) = \{\check{u} \in L^2(\mathbb{R}^2_+)\,,\ \sigma^p\tau^q\partial_\sigma^m\partial_\tau^n\check{u} \in L^2\,,\ \forall\, p,q,m,n \geq 0 \text{ s.t. } p+q+m+n \leq \ell\}\,.$$

Converting (3.4) to the coordinates (σ,τ) via (3.8) yields

$$\check{V}(\sigma,\tau) - V(x_0) = (hJ_m)^{2/3}\Big(\tau + \mathfrak{e}(h)\Big[\sigma^2\tau + \frac{\hat{\beta}}{2^{1/2}[\alpha_m J_m]^{1/2}}\tau^2\Big] + \delta\check{V}\Big)\,.$$

Using (3.6), we may conclude that there exists C, $h_0 > 0$ and $\rho_0 > 0$ such that, for $h \in (0, h_0]$ and u s.t. $\operatorname{supp} \widetilde{\mathrm{u}} \subset (-\mathrm{s}_0, \mathrm{s}_0) \times [0, \rho_0)$,

$$\|\delta V u\| \leq C\, \mathfrak{e}(h)^{\frac{3}{2}}\, \|\check{u}\|_{B^3(\mathbb{R}^2_+)} . \tag{3.13}$$

We thus obtain the approximate problem (for $c > 0$)

$$\begin{cases} -\check{u}_{\tau\tau} + i\tau\check{u} + \mathfrak{e}(-\check{u}_{\sigma\sigma} + i\sigma^2\tau\check{u} + i\beta\tau^2\check{u} + 2\omega\check{u}_\tau) + \mathcal{O}(\mathfrak{e}^{3/2}) = \lambda\check{u} & \text{in } \mathbb{R}^2_+ \\ \check{u}(0,\sigma) = 0 \text{ for } \sigma \in \mathbb{R} & , \end{cases} \tag{3.14}$$

where the $\mathcal{O}(\mathfrak{e}^{\frac{3}{2}}) = \mathcal{O}(h)$ term is bounded by the right hand side of (3.13), for $\operatorname{supp} \widetilde{\mathrm{u}} \subset (-\mathrm{s}_0, \mathrm{s}_0) \times [0, \rho_0)$, and

$$\omega = \kappa(0) \Big[\frac{J_m}{2\alpha_m}\Big]^{1/2} \quad ; \quad \beta = \frac{\hat{\beta}}{[2\alpha_m J_m]^{1/2}} . \tag{3.15}$$

We recall that for $c < 0$ we obtain (3.14) once again by taking the complex conjugate of the approximate equation, together with the change of parameters $(\beta, \omega) \to (-\beta, -\omega)$.

Remark 3.2. *Although not needed in this section for the formal construction of the quasimode, it will become necessary in Section 8, to define the curvilinear coordinates* (s, ρ) *and their corresponding blowup* (3.8) *centered at a point* y *in* $\mathcal{S}$ *(instead previously at the point* x_0*). All the quantities appearing above* $\kappa, c, \alpha, \hat{\beta}, \beta, \mathfrak{e}$ *are then computed at the point* y *chosen as the origin (for clarity we denote them in Section 8 by* $\kappa(y), c(y), \alpha(y), \ldots, \mathfrak{e}(h, y)$*).*

3.2 The formal construction

We look, in the (σ, τ) variables, for an approximate spectral pair in the form (modulo a multiplication by a cut-off function)

$$u = u_0 + \mathfrak{e} u_1 \, , \; \lambda = \lambda_0 + \mathfrak{e} \lambda_1 \, ,$$

with u_0, u_1 in $\mathcal{S}(\overline{\mathbb{R}^2_+})$.

The leading order balance reads

$$(\mathcal{L}^+ - \lambda_0) u_0 = 0 \, , \tag{3.16}$$

where

$$\mathcal{L}^+ = -\frac{\partial^2}{\partial\tau^2} + i\tau \, , \tag{3.17a}$$

As an unbounded operator on $L^2(\mathbb{R}_+)$ its domain is

$$D(\mathcal{L}^+) = \{u \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \,|\, \tau u \in L^2(\mathbb{R}_+)\} \, , \tag{3.17b}$$

We use the same notation for its natural extension to $L^2(\mathbb{R}^2_+)$ by a tensor product. For u_0 in the form

$$u_0(\sigma, \tau) = v(\tau)\, w_0(\sigma) \, , \tag{3.18}$$

(3.16) leads to

$$(\mathcal{L}^+ - \lambda_0)\, v = 0 \tag{3.19}$$

in $L^2(\mathbb{R}_+)$ and hence

$$v(\tau) = v_1(\tau), \quad \lambda_0 = -e^{-i2\pi/3}\nu_1\,.$$

Here ν_k denotes, for $k \geq 1$ the kth zero of Airy's function, and

$$v_k(\tau) = C_k\, A_i(e^{i\pi/6}\tau + \nu_k)\,, \tag{3.20a}$$

where

$$C_k = \Big[\int_0^\infty |A_i(\tau+\nu_k)|^2\, d\tau\Big]^{-1/2}\,, \tag{3.20b}$$

which follows from the normalization

$$\langle \bar{v}_k, v_k\rangle = 1\,.$$

We thus conclude that u_0 must have the form

$$u_0(\sigma,\tau) = v_1(\tau)\, w_0(\sigma)\,, \tag{3.21}$$

where $w_0 \in \mathcal{S}(\mathbb{R})$. It follows that $u_0 \in \mathcal{S}(\overline{\mathbb{R}^2_+})$ and satisfies the Dirichlet boundary condition at $\tau = 0$. We will determine w_0 from the next order balance.

The next order balance assumes the form

$$(\mathcal{L}^+ - \lambda_0)u_1 = -\Big(-\frac{\partial^2}{\partial\sigma^2} + 2\omega\frac{\partial}{\partial\tau} + i(\sigma^2\tau + \beta\tau^2) - \lambda_1\Big)u_0 \quad ; \quad u_1(0,\sigma) = 0\,. \tag{3.22}$$

Taking the inner product of (3.22) with $\bar{v}_0$ in $L^2(\mathbb{R}_+,\mathbb{C})$ we obtain that the pair (λ_1, w_0) should satisfy

$$(\mathcal{P} - \lambda_1)w_0 = 0\,,$$

where $\mathcal{P}$ is defined on

$$D(\mathcal{P}) = \{u \in H^2(\mathbb{R}) \,|\, \sigma^2 u \in L^2(\mathbb{R})\}$$

by

$$\mathcal{P} := -\frac{\partial^2}{\partial\sigma^2} + e^{i\pi/6}\tau_m\sigma^2 + \beta\tau_{m,2}\,, \tag{3.23}$$

with

$$\tau_m = e^{i\pi/3}\langle \bar{v}_1, \tau v_1\rangle\,, \quad \tau_{m,2} = i\langle \bar{v}_1, \tau^2 v_1\rangle\,. \tag{3.24}$$

Note that by Cauchy's Theorem and deformation of contour, we obtain that τ_m and $\tau_{m,2}$ are real and satisfy

$$\tau_m = \int_{\mathbb{R}_+} \tau \mathrm{Ai}^2(\tau+\nu_1)\, d\tau > 0 \text{ and } \tau_{m,2} = \int_{\mathbb{R}_+} \tau^2 \mathrm{Ai}^2(\tau+\nu_1)\, d\tau > 0\,. \tag{3.25}$$

We now choose λ_1 as the eigenvalue with smallest real part of $\mathcal{P}$ which is a complex harmonic operator and take w_0 as the corresponding eigenfunction

$$w_0(\sigma) = C_0 \exp\Big\{-\Big[\frac{\tau_m}{2}\Big]^{1/2} e^{i\frac{\pi}{12}}\sigma^2\Big\} \quad , \quad \lambda_1 = \sqrt{2\tau_m}e^{i\tau_m\frac{\pi}{12}} + \beta\tau_{m,2}\,, \tag{3.26}$$

where C_0 is chosen so that

$$\int_{\mathbb{R}} w_0(\sigma)^2 \, d\sigma = 1 \, .$$

With this choice of λ_1, the function $u_1 \in \mathcal{S}(\overline{\mathbb{R}^2_+})$ must satisfy

$$(\mathcal{L}^+ - \lambda_0) u_1 = -i[\sigma^2(\tau - e^{-i\pi/3}\tau_m) + \beta(\tau^2 - i\tau_{m,2}) + 2i\omega\partial_\tau]u_0 \quad ; \quad u_1(0,\sigma) = 0 \, . \tag{3.27}$$

Let Π_k denote the spectral projection of $L^2(\mathbb{R}_+, \mathbb{C})$ on $\operatorname{span} v_k$, defined by:

$$\Pi_k u = \langle u, \bar{v}_k \rangle_\tau \, v_k \, , \tag{3.28}$$

where $\langle \cdot, \cdot \rangle_\tau$ denotes the inner product in $L^2(\mathbb{R}_+, \mathbb{C})$ with respect to the τ variable. We use the same notation (instead of $Id \,\widehat{\otimes}\, \Pi_k$) for its natural extension to $L^2(\mathbb{R}) \widehat{\otimes} L^2(\mathbb{R}_+) = L^2(\mathbb{R}^2_+)$.
Consequently we may write

$$u_1(\sigma, \tau) = w_1(\sigma) v_1(\tau) + \hat{u}_1(\sigma, \tau) \, ,$$

where $\hat{u}_1 \in (I - \Pi_1) L^2(\mathbb{R}^2_+)$ and $w_1 \in \mathcal{S}(\mathbb{R})$ is left arbitrary (and should be obtained from higher order balances). We set $w_1 = 0$ in the sequel, as a two-term expansion satisfies our needs in the next sections.
With Fredholm alternative in mind, we look for $u_1(\sigma, \tau)$ in the form

$$u_1(\sigma, \tau) = -i\sigma^2 w_0(\sigma) u_{11}(\tau) + w_0(\sigma)\big(\beta u_{12}(\tau) + \omega u_{13}(\tau)\big) \, ,$$

where $u_{11}(\tau)$ is the unique solution in $\operatorname{Im}(I - \Pi_1)$ of

$$(\mathcal{L}^+ - \lambda_0) u_{11}(\tau) = (\tau - e^{-i\pi/3}\tau_m) v_1(\tau) \, , \; u_{11}(0) = 0 \, ,$$

$u_{12}(\tau)$ is the unique solution in $\operatorname{Im}(I - \Pi_1)$ of

$$(\mathcal{L}^+ - \lambda_0) u_{12}(\tau) = (\tau^2 - i\tau_{m,2}) v_1(\tau) \, , \; u_{12}(0) = 0 \, .$$

and $u_{13}(\tau)$ is the unique solution in $\operatorname{Im}(I - \Pi_1)$ of

$$(\mathcal{L}^+ - \lambda_0) u_{13}(\tau) = v_1'(\tau) \, , \; u_{13}(0) = 0 \, .$$

The above equations are uniquely solvable, since their right hand sides are all orthogonal to $\bar{v}_1$, (and hence both lie in $\operatorname{Im}(I - \Pi_1)$), and since $(\mathcal{L}^+ - \lambda_0)_{/\operatorname{Im}(I - \Pi_1)}$ is invertible. It is not difficult to show that $u_{1,j}$ belongs to $\mathcal{S}(\overline{\mathbb{R}_+})$ for $j = 1, 2, 3$.
We have thus determined λ_1 and $u_1 \in \mathcal{S}(\overline{\mathbb{R}^2_+})$, providing sufficient accuracy for the derivation of the upper bound in the last section.

Remark 3.3. *The above expansion is similar to the one given in [14]. Following the same steps detailed there, one can formally construct an approximate solution, of arbitrary algebraic accuracy (i.e. of $\mathcal{O}(\mathfrak{e}^p)$ for any $p > 0$).*

3.3 Quasimode and remainder

We can now set the approximate eigenpair (U^1, Λ^1) to be given by

$$\Lambda^1(\mathfrak{e}) = \lambda_0 + \mathfrak{e}\lambda_1 \quad ; \quad \check{U}^1(\sigma,\tau) = \eta_{\mathfrak{e}}(\sigma,\tau)\left(\check{u}_0(\sigma,\tau) + \mathfrak{e}\,\check{u}_1(\sigma,\tau)\right), \tag{3.29}$$

where the accent $\check{\cdot}$ is used to denote functions of (σ,τ), and $\eta_{\mathfrak{e}} \in C^\infty(\mathbb{R}^2,[0,1])$ is a cut-off function supported in a neighborhood of x_0

$$\eta_{\mathfrak{e}}(\sigma,\tau) = \begin{cases} 1 & \text{for } |\sigma^2+\tau^2| \le \mathfrak{e}^{-1/4} \\ 0 & \text{for } |\sigma^2+\tau^2| > 2\mathfrak{e}^{-1/4}\,. \end{cases}$$

For latter reference we define

$$\Lambda^1_\gamma(\mathfrak{e}) = \lambda_0 + \gamma\,\mathfrak{e}\,\lambda_1\,, \tag{3.30}$$

for some $0 \le \gamma \le 1$.
We finally state

Proposition 3.4. *Let $x_0 \in \mathcal{S}^m$ and $(\check{U}^1, \Lambda^1)$ be given by* (3.29)*. Let for $c(x_0) > 0$*

$$U^1_h(x) = \widetilde{U}^1_h(s,\rho) = \check{U}^1\Big(\Big[\frac{\alpha_m^3}{8J_m h^4}\Big]^{1/12} s, \Big[\frac{J_m}{h^2}\Big]^{1/3}\rho\Big)\,.$$

and

$$\hat{\Lambda}^1(h,x_0) = iV(x_0) + (J_m h)^{\frac{2}{3}}\Lambda^1(\mathfrak{e}(h))\,,\ \mathfrak{e}(h) = \frac{\alpha_m^{1/2}}{2^{1/2}[J_m]^{5/6}} h^{2/3}\,. \tag{3.31}$$

For $c(x_0) < 0$ set $U^1_h = \overline{\widetilde{U}^1_h}$ and

$$\hat{\Lambda}^1(h,x_0) = iV(x_0) + (J_m h)^{\frac{2}{3}}\overline{\Lambda^1(\mathfrak{e}(h))}\,.$$

Then, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0,h_0)$,

$$\|(\mathcal{A}_h - \hat{\Lambda}^1(h,x_0))U^1_h\|_2 \le C\,h^{5/3}\|U^1_h\|_2\,. \tag{3.32}$$

The proof follows from the preceding asymptotic expansion, the fact that supp $\widetilde{\mathrm{U}}^1_{\mathrm{h}}$ belongs to $(-s_0,s_0)\times[0,\rho_0)$, and the exponential decay of u_0 and u_1 in $\mathbb{R}^2_+$ which implies that $\|(1-\eta_{\mathfrak{e}})u_j\|_{B^3(\mathbb{R}^2_+)} = \mathcal{O}(\mathfrak{e}^{+\infty}) = \mathcal{O}(h^{+\infty})$.

4 Quasimode construction - Type V2

In this section we present a similar construction to the previous section for type V2 potentials.
Let $x_0 \in \hat{\mathcal{S}}^m$ which for type V2 potentials is a corner point. We use the curvilinear system of coordinates (s,ρ) given by (2.22). The corner is set to be the origin, and (s,ρ) varies in a neighborhood of $(0,0)$ in $Q = [0,+\infty)\times[0,+\infty)$. We then use the

diffeomorphism $\mathcal{F}_{x_0}(h)$ given by (2.22b). The potential V is given in the vicinity of x_0 by

$$V(x) = \widetilde{V}(s,\rho) = V(x_0) + c\,\rho\,, \tag{4.1}$$

where $c = \pm|\nabla V(x_0)|$.

As in the previous section we assume, without any loss of the generality of the proof, that $c > 0$, otherwise we move to consider, as before, $\bar{\mathcal{A}}_h$ instead of $\mathcal{A}_h$. The Laplacian operator is given according to (2.23) by

$$\Delta = \tilde{g}_c\,(\partial_\rho^2 + \partial_s^2)\,,$$

where

$$g_c(x) = |\nabla V|^2/c^2 = \widetilde{g}_c(s,\rho)\,.$$

By the smoothness of V, $\widetilde{g}$ admits, in the vicinity of $(0,0)$ the expansion

$$\widetilde{g}_c(s,\rho) = 1 + \tilde{\alpha}_{x_0}s + \tilde{\beta}_{x_0}\rho + \mathcal{O}(s^2+\rho^2)\,, \tag{4.2}$$

where $\tilde{\alpha}_{x_0} > 0$ since x_0 is a minimum of $|\partial V/\partial\nu|$ on $\overline{\partial\Omega_D}$. Note that by (2.24), at every corner we have

$$\tilde{\alpha}_{x_0} = 2\hat{\alpha}(x_0)/c > 0\,, \tag{4.3}$$

where $\hat{\alpha}$ is given by (1.14). Since $|c| = J_m$ on $\mathcal{S}$ it follows

$$\tilde{\alpha} := \tilde{\alpha}_{x_0} = 2\hat{\alpha}_m/J_m\,,\ \text{ for } x_0 \in \hat{\mathcal{S}}^m\,. \tag{4.4}$$

We now apply the transformation

$$\tau = \Big[\frac{J_m}{h^2}\Big]^{1/3}\rho \quad ; \quad \sigma = \Big[\frac{2^3\hat{\alpha}_m^3}{J_m h^4}\Big]^{1/9} s\,, \tag{4.5}$$

to obtain from (2.23) that

$$h^{4/3}J_m^{-2/3}\Delta = (1 + \varepsilon\sigma + \mathcal{O}(\varepsilon^{\frac{3}{2}}))\Big(\frac{\partial^2}{\partial\tau^2} + \varepsilon\frac{\partial^2}{\partial\sigma^2}\Big)\,.$$

In the above,

$$\varepsilon(h) = \Big[2^6\hat{\alpha}_m^6 J_m^{-8}\Big]^{1/9} h^{\frac{4}{9}}\,. \tag{4.6}$$

Consequently, we may write

$$(hJ_m)^{-2/3}\mathcal{A}_h u = -(1+\varepsilon\sigma)\frac{\partial^2\check{u}}{\partial\tau^2} + i\,\mathrm{sign}\,c\,\tau\check{u} - \varepsilon\frac{\partial^2\check{u}}{\partial\sigma^2} + \delta u\,, \tag{4.7}$$

where, for u such that $\mathrm{supp}\,\widetilde{\mathrm{u}} \subset (-\mathrm{s}_0,\mathrm{s}_0)\times[0,\rho_0)$,

$$\|\delta u\|_2 \leq C\,\varepsilon^{\frac{3}{2}}\,\|\check{u}\|_{B^4(Q)}\,.$$

We now continue as in the previous section. The eigenvalue problem can be formulated, for $c > 0$, as finding an approximate pair $(\check{u},\lambda)$ such that

$$\begin{cases} -(1+\varepsilon\sigma)\check{u}_{\tau\tau} + i\tau\check{u} - \varepsilon\check{u}_{\sigma\sigma} + \mathcal{O}(\varepsilon^{\frac{3}{2}}) = \lambda\check{u} & \text{in } Q\,, \\ \check{u}(\sigma,0) = 0 & \text{for } \sigma \in \mathbb{R}_+\,, \\ \frac{\partial\check{u}}{\partial\sigma}(0,\tau) = 0 & \text{for } \tau \in \mathbb{R}_+\,. \end{cases} \tag{4.8}$$

Note that for $c < 0$ we obtain (4.8) once again by taking the complex conjugate of the approximate problem.

Omitting the accent $\check{\cdot}$, we first assume

$$u = u_0 + \varepsilon\, u_1 \quad ; \quad \lambda = \lambda_0 + \varepsilon\, \check{\lambda}_1 \,,$$

with u_0 and u_1 in $\mathcal{S}(\overline{Q})$.
The leading order balance is precisely (3.29), and hence, as before,

$$u_0 = v_1(\tau)\, w_0^+(\sigma) \quad ; \quad \lambda_0 = -e^{-i2\pi/3}\nu_1 \,, \tag{4.9}$$

with w_0 arbitrary in $\mathcal{S}(\overline{\mathbb{R}}_+)$, as long as it satisfies, the Neumann condition at $\sigma = 0$.

The next order balance assumes the form

$$(\mathcal{L}^+ - \lambda_0)u_1 = \Big(\frac{\partial^2}{\partial\sigma^2} + \sigma\frac{\partial^2}{\partial\tau^2} + \check{\lambda}_1\Big)u_0 \quad ; \quad u_1(0,\sigma) = 0 \,,$$

where $\mathcal{L}^+$ is defined by (3.17).
As

$$\frac{\partial^2 u_0}{\partial\tau^2} = (i\tau - \lambda_0)u_0 \,,$$

we obtain that

$$(\mathcal{L}^+ - \lambda_0)u_1 = -\Big(\mathcal{P}_+(\tau) - \check{\lambda}_1\Big)u_0 \quad ; \quad u_1(0,\sigma) = 0 \,, \tag{4.10}$$

where

$$\mathcal{P}_+(\tau) = -\frac{\partial^2}{\partial\sigma^2} - (i\sigma\tau - \lambda_0\sigma) \,. \tag{4.11}$$

Taking the inner product of (3.22) with $\bar{v}_1$ in $L^2(\mathbb{R}_+, \mathbb{C})$ we obtain that

$$-\frac{\partial^2 w_0^+}{\partial\sigma^2} + (\theta_0\sigma - \check{\lambda}_1)w_0^+ = 0 \,, \quad (w_0^+)'(0) = 0 \,,$$

where

$$\theta_0 = \lambda_0 - e^{i\pi/6}\tau_m \,, \tag{4.12}$$

in which τ_m is given by (3.24). As

$$\theta_0 = -\int_{\mathbb{R}_+} (i\tau - \lambda_0)v_1^2(\tau)\, d\tau = \int_{\mathbb{R}_+} (v_1'(\tau))^2\, d\tau = e^{i\pi/3}\int_{\mathbb{R}_+} (\mathrm{Ai}'(e^{i\pi/6}\tau + \nu_1'))^2\, d\tau \,,$$

it easily follows that $\arg\theta_0 = \pi/6$.

As a Neumann realization of a complex Airy operator on $\mathbb{R}_+$, the spectrum of the operator $-\partial^2/\partial\sigma^2 + \theta_0\sigma$ is discrete and λ_1 can explicitly be found as function of the zeros of the derivative of the Airy function. Thus,

$$w_0^+(\sigma) = C_0 A_i(\theta_0^{1/3}\sigma + \nu_1') \quad ; \quad \check{\lambda}_1 = -\theta_0^{2/3}\nu_1' \,, \tag{4.13}$$

where C_0 is chosen so that

$$\int_{\mathbb{R}_+} w_0^+(\sigma)^2\, d\sigma = 1 \,,$$

and ν_1' is the first zero of A_i'.

The problem for u_1 then assumes the form

$$(\mathcal{L}^+ - \lambda_0)u_1 = -i\sigma(\tau - e^{-i\pi/3}\tau_m)u_0 \quad ; \quad u_1(\sigma,0) = 0\,.$$

As in the previous section it follows that there exists a unique solution to the above problem in $(I-\Pi_1)L^2(\mathbb{R}_+,\mathbb{C})$, which in addition is in $\mathcal{S}(\overline{Q})$ and satisfies the Dirichlet-Neumann condition. We can now set

$$\Lambda^2(\varepsilon) = \lambda_0 + \varepsilon\check{\lambda}_1 \quad ; \quad \check{U}^2 = (u_0 + \varepsilon u_1)\,\eta_\varepsilon\,, \tag{4.14}$$

where

$$\eta_\varepsilon(\sigma,\tau) = \begin{cases} 1 & \text{for } |\sigma^2+\tau^2| \leq \varepsilon^{-1/4}\,, \\ 0 & \text{for } |\sigma^2+\tau^2| > 2\varepsilon^{-1/4}\,, \end{cases}$$

to obtain the approximate eigenpair $(\check{U}^2,\Lambda^2)$. In a similar manner to the previous section we define, for later reference

$$\Lambda^2_\gamma(\varepsilon) = \lambda_0 + \gamma\,\varepsilon\,\check{\lambda}_1 \tag{4.15}$$

with λ_0 and $\check{\lambda}_1$ defined in (4.9) and (4.13).

The following proposition is an immediate consequence of the foregoing discussion.

Proposition 4.1. *Let $x_0 \in \hat{\mathcal{S}}^m$. Let $(\check{U}^2,\Lambda^2)$ be given by (4.14). Let for $c(x_0) > 0$*

$$U_h^2(x) = \widetilde{U}_h^2(s,\rho) = \check{U}^2\Big(\Big[\frac{8\hat{\alpha}_m^3}{J_m h^4}\Big]^{1/9} s,\, \Big[\frac{J_m}{h^2}\Big]^{1/3}\rho\Big)\,,$$

and

$$\hat{\Lambda}^2(h,x_0) = iV(x_0) + (J_m h)^{\frac{2}{3}}\Lambda^2(\varepsilon(h))\,,\ \varepsilon(h) = \Big[2^6\hat{\alpha}_m^6 J_m^{-8}\Big]^{1/9} h^{\frac{4}{9}}\,. \tag{4.16}$$

For $c(x_0) < 0$, set $U_h^2 = \overline{\widetilde{U}_h^2}$ and

$$\hat{\Lambda}^2(h,x_0) = iV(x_0) + (J_m h)^{\frac{2}{3}}\overline{\Lambda^2(\varepsilon(h))}\,.$$

Then, we have

$$\|(\mathcal{A}_h - \hat{\Lambda}^2(h,x_0))U_h^2\|_2 \leq C\,h^{4/3}\,\|U_h^2\|_2\,. \tag{4.17}$$

Remark 3.1 still holds for $x_0 \in \hat{\mathcal{S}}^m$.

5 V1 potentials: 1D operators

5.1 Motivation

To prove the existence of an eigenvalue of $\mathcal{A}_h$ in the vicinity of the approximated value $\hat{\Lambda}^1(h)$, one needs an estimate of $\|(\mathcal{A}_h-\lambda)^{-1}\|$ for λ in an annulus whose interior circle encloses $\hat{\Lambda}^1(h)$. The relevant eigenmode is expected to decay exponentially fast away from a point x_0 on $\mathcal{S}^m$. We thus replace the type V1 potential by the leading orders in its Taylor expansion around x_0 as in (3.4) and renormalize the operator by considering an approximation of $\check{\mathcal{A}}_h = (J_m h)^{-\frac{2}{3}}(\mathcal{A}_h - iV(x_0))$. The spectral parameter λ is thus replaced by $\check{\lambda} = (J_m h)^{-\frac{2}{3}}(\lambda - iV(x_0))$ and the parameter $\mathfrak{e} \sim h^{\frac{2}{3}}$ given by (3.10) is introduced. In the next two sections we estimate the resolvent of the ensuing approximate operator, after the dilation (3.8) centered at x_0 is applied, and the blowup coordinates (σ, τ) are being introduced. The estimation of the error generated through the use of an approximate potential, instead of V, is left to the last section.

A necessary first step towards the above mentioned resolvent estimate is to consider a one-dimensional simplification of it. We recall from (3.14), where we state the eigenvalue problem for the approximate operator after dilation, that the approximate potential includes the term $\mathfrak{e}\beta\tau^2$. Compared to the leading order term, τ, this term is much smaller for all $\tau \ll \mathfrak{e}^{-1}$. However, any attempt to drop this term completely from the expansion and to account for the error afterwards would fail, as by (3.26) it has an $\mathcal{O}(\mathfrak{e})$ effect on $\Lambda^1(\mathfrak{e})$. Since we seek an estimate of $\|(\check{\mathcal{A}}_h - \check{\lambda})^{-1}\|$ on a circle centered at $\Lambda_1(\mathfrak{e})$ of radius much smaller than $\mathfrak{e}$, a complete neglect of this term seems impossible. However, since we consider $\tau \in \mathbb{R}_+$, it makes sense to avoid problems resulting from the fact that for $\beta < 0$, $\tau + \mathfrak{e}\beta\tau^2$ changes sign for sufficiently large $|\tau|$. We thus multiply $\mathfrak{e}\beta\tau^2$ by an appropriate cutoff function, so that the error generated by it need not be accounted for in the last section, considering the fact that the resolvent is multiplied there by a cutoff function as in Section 2.

5.2 Realization on the entire real line

Let, for $\mathfrak{e} > 0$, $\mathcal{L}_2(\mathfrak{e})$ be given by

$$\mathcal{L}_2(\mathfrak{e}) = -\frac{d^2}{d\tau^2} + i(\tau + \mathfrak{e}\beta\chi(\mathfrak{e}^b\tau)\tau^2)\,, \tag{5.1}$$

where $\beta \in \mathbb{R}$, $1/2 < b < 3/4$, and $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ is chosen such that

$$\chi(x) = \begin{cases} 1 & |x| < 1\,, \\ 0 & |x| > 2\,, \end{cases} \tag{5.2}$$

and so that

$$\check{\chi} = \sqrt{1 - \chi^2} \text{ in } \mathbb{R}\,,$$

is in $C^\infty(\mathbb{R}, [0, 1])$. We shall frequently drop the reference to $\mathfrak{e}$ in $\mathcal{L}_2(\mathfrak{e})$ and write instead $\mathcal{L}_2$ when no ambiguity is expected.

Clearly, $\mathcal{L}_2$ is a closed operator whose domain is given by

$$D(\mathcal{L}_2) = \{u \in H^2(\mathbb{R}) \,|\, \tau u \in L^2(\mathbb{R})\}\,.$$

We now need to establish that $\mathcal{L}$ and $\mathcal{L}_2$ share some properties in common. In particular, from [3, 18] we know that $\mathcal{L}$ has a compact resolvent, empty spectrum and that, for all $\mu_0 \in \mathbb{R}$, the resolvent norm is uniformly bounded in the half space $\operatorname{Re}\lambda \leq \mu_0$:

$$\sup_{\operatorname{Re}\lambda\leq\mu_0} \|(\mathcal{L}-\lambda)^{-1}\| < +\infty\,. \tag{5.3}$$

Moreover, we will make use of the following regularity property for $\mathcal{L}$ (cf. [6, Proposition 5.4]),

$$\|\tau u\| \leq C\,(\|\mathcal{L}u\| + \|u\|), \forall u \in D(\mathcal{L})\,, \tag{5.4}$$

which implies together with (5.3),

$$\|\tau u\| \leq C_{\mu_0}\,\|(\mathcal{L}-\mu)u\|, \forall u \in D(\mathcal{L})\,,\ \forall \mu \in [-1,\mu_0]\,. \tag{5.5}$$

It can also be easily verified, by integrating by parts $\operatorname{Re}\langle(\mathcal{L}-\mu)u\,,\,u\rangle$, that

$$|\mu|\,\|u\|_2 \leq C\|(\mathcal{L}-\mu)u\|, \forall u \in D(\mathcal{L})\,,\ \forall \mu < -1\,,$$

which implies (5.5) for $\mu < -1$ using again (5.4).

Similarly, we get the following properties for $\mathcal{L}_2$:

Proposition 5.1. *For any $\mathfrak{e} > 0$, $\mathcal{L}_2 = \mathcal{L}_2(\mathfrak{e})$ has a compact resolvent. Moreover, for all $\mu_0 \in \mathbb{R}$, there exists $\mathfrak{e}_0 > 0$ and $C_{\mu_0} > 0$ such that for all $0 < \mathfrak{e} \leq \mathfrak{e}_0$, the spectrum of $\mathcal{L}_2(\mathfrak{e})$ lies outside $\{\operatorname{Re}\lambda \leq \mu_0\}$ and*

$$\sup_{\operatorname{Re}\lambda\leq\mu_0} \|(\mathcal{L}_2(\mathfrak{e})-\lambda)^{-1}\| \leq C_{\mu_0}\,. \tag{5.6}$$

Proof.
The first statement is an immediate consequence of the boundedness of $\mathcal{L}-\mathcal{L}_2$.
To prove (5.6) we first show that

$$\sup_{\{\operatorname{Re}\lambda\leq\mu_0\}\cap\rho(\mathcal{L}_2(\mathfrak{e}))} \|(\mathcal{L}_2(\mathfrak{e})-\lambda)^{-1}\| \leq C_{\mu_0}\,, \tag{5.7}$$

where $\rho(\mathcal{L}_2)$ denotes the resolvent set of $\mathcal{L}_2$.
The spectrum of $\mathcal{L}_2$ being discrete, this uniform bound implies that $\sigma(\mathcal{L}_2)$ lies outside $\{\operatorname{Re}\lambda < \mu_0\}$, and hence, it also implies (5.6).

Consider first the case $|\operatorname{Im}\lambda| \leq \mathfrak{e}^{-(1+2b)/3}$.
Let $\lambda = \mu + i\nu \in \rho(\mathcal{L}_2)$, $w \in D(\mathcal{L}_2)$ and $g = (\mathcal{L}_2 - \lambda)w$. It follows that

$$|\nu| \leq \mathfrak{e}^{-(1+2b)/3}\,. \tag{5.8}$$

Let further

$$\nu_\mathfrak{e} := \nu + \mathfrak{e}\beta\chi(\mathfrak{e}^b\nu)\nu_1(\mathfrak{e})\,,$$

and

$$\nu_1(\mathfrak{e}) = \frac{\nu^2}{1-2\beta\mathfrak{e}\nu\chi(\mathfrak{e}^b\nu)}\,,$$

which is by (5.8) well defined when $4\,\mathfrak{e}^{1-b}|\beta| < 1$.
We assume in the sequel that

$$4\,\mathfrak{e}_0^{1-b}|\beta| \leq \frac{1}{2}\,,\ \text{and}\ 0 < \mathfrak{e}_0 \leq 1\,. \tag{5.9}$$

Note that under these assumptions, we have

$$\nu_{\mathfrak{e}} = \nu\,(1 + \mathcal{O}(\mathfrak{e}^{1-b}))\,. \tag{5.10}$$

Applying the transformation

$$\tau' = \tau - \nu_{\mathfrak{e}}\,,$$

yields (dropping the superscript $'$)

$$(\mathcal{L} - \mu)w = g - i\mathfrak{e}\beta\Big((\tau + \nu_{\mathfrak{e}})^2\varphi_{\mathfrak{e},\nu}(\tau) - \nu_1(\mathfrak{e})\chi(\mathfrak{e}^b\nu)\Big)w\,,$$

where we have introduced

$$\varphi_{\mathfrak{e},\nu}(\tau) = \chi(\mathfrak{e}^b(\tau + \nu_{\mathfrak{e}}))\,.$$

By (5.3), (5.5) and (5.10), there exists, for any $\mu_0 \in \mathbb{R}$, a constant C_{μ_0} such that, for $\mu \leq \mu_0$ and $\mathfrak{e} \in (0, \mathfrak{e}_0]$,

$$\|w\|_2+\|\tau w\|_2 \leq C_{\mu_0}\Big(\|g\|_2+\mathfrak{e}\|\tau^2\varphi_{\mathfrak{e},\nu}w\|_2+\mathfrak{e}|\nu|\|\tau\varphi_{\mathfrak{e},\nu}w\|_2+\mathfrak{e}\,\|\,\big(\nu_{\mathfrak{e}}^2\varphi_{\mathfrak{e},\nu} - \nu_1(\mathfrak{e})\chi(\mathfrak{e}^b\nu)\big)\,w\|\Big)\,. \tag{5.11}$$

To estimate the second term of the right hand side, we first observe that, for some constant $C_0 > 0$,

$$|\tau\varphi_{\mathfrak{e},\nu}(\tau)| \leq |\nu_{\mathfrak{e}}| + 2\mathfrak{e}^{-b} \leq C_0(|\nu| + \mathfrak{e}^{-b})\,,$$

and hence, using the assumptions on ν, b and $\mathfrak{e}$

$$\mathfrak{e}\|\tau^2\varphi_{\mathfrak{e},\nu}w\|_2 \leq C_0\mathfrak{e}(|\nu| + \mathfrak{e}^{-b})\|\tau w\|_2 \leq 2C_0\mathfrak{e}^{2(1-b)/3}\|\tau w\|_2\,. \tag{5.12}$$

For the third term, we simply observe that

$$\mathfrak{e}|\nu|\|\tau\varphi_{\mathfrak{e},\nu}w\|_2 \leq \mathfrak{e}^{2(1-b)/3}\|\tau w\|_2\,. \tag{5.13}$$

It remains to obtain a bound for the last term on the right-hand-side of (5.11)

$$r_{\mathfrak{e}} := \mathfrak{e}\|\,\big(\nu_{\mathfrak{e}}^2\varphi_{\mathfrak{e},\nu} - \nu_1(\mathfrak{e})\chi(\epsilon^b\nu)\big)\,w\|\,.$$

To this end we first observe that

$$r_{\mathfrak{e}} \leq \mathfrak{e}\nu_{\mathfrak{e}}^2\|(\varphi_{\mathfrak{e},\nu} - \chi(\epsilon^b\nu))w\| + \mathfrak{e}|\nu_{\mathfrak{e}}^2 - \nu_1(\mathfrak{e})|\chi(\mathfrak{e}^b\nu)\|w\|\,. \tag{5.14}$$

Using the fact that

$$|\varphi_{\mathfrak{e},\nu}(\tau) - \chi(\mathfrak{e}^b\nu)| = |\chi(\mathfrak{e}^b\tau + \mathfrak{e}^b\nu_{\mathfrak{e}}) - \chi(\mathfrak{e}^b\nu)| \leq \epsilon^b\,(\sup|\chi'|)\,(|\tau| + |\nu_\epsilon - \nu|)\,,$$

Equation (5.10) and the assumptions on $\mathfrak{e}, \nu, b$, yield for the first term on the right-hand-side of (5.14)

$$\mathfrak{e}\nu_{\mathfrak{e}}^2\|(\varphi_{\mathfrak{e},\nu} - \chi(\epsilon^b\nu))w\| \leq C\,\mathfrak{e}^{(1-b)/3}\left(\|\tau w\|_2 + \mathfrak{e}^{(1-b)/3}\|w\|_2\right)\,. \tag{5.15}$$

For the second term on the right-hand-side of (5.14), we get from the identity

$$\nu^2 + 2\beta\mathfrak{e}\frac{\nu^3\chi(\mathfrak{e}^b\nu)}{1-2\beta\mathfrak{e}\nu\chi(\mathfrak{e}^b\nu)} = \frac{\nu^2}{1-2\beta\mathfrak{e}\nu\chi(\mathfrak{e}^b\nu)} = \nu_1(\mathfrak{e})\,,$$

the estimate

$$\mathfrak{e}|\nu_{\mathfrak{e}}^2 - \nu_1(\mathfrak{e})|\,|\chi(\mathfrak{e}^b\nu)|\,\|w\| \leq C\,\mathfrak{e}^3\nu^4|\chi(\mathfrak{e}^b\nu)|\|w\| \leq C\,\mathfrak{e}^{3-4b}\|w\|\,. \tag{5.16}$$

Using (5.11)-(5.16), we obtain

$$\|w\|_2 + \|\tau w\|_2 \leq \hat{C}_{\mu_0}\big(\|g\|_2 + \mathfrak{e}^{(1-b)/3}\|\tau w\|_2 + (\mathfrak{e}^{3-4b} + \mathfrak{e}^{\frac{2(1-b)}{3}})\|w\|_2\big)\,,$$

and choosing sufficiently small $\mathfrak{e}_0$ (which could depend on μ_0) we finally obtain, for $b < \frac{3}{4}$, the existence of C_{μ_0} such that, for any $\mathfrak{e} \in (0, \mathfrak{e}_0]$, any λ s.t. $\operatorname{Re}\lambda \leq \mu_0$, and any $w \in \mathcal{D}(\mathcal{L})$,

$$\|w\|_2 \leq \hat{C}_{\mu_0}\|g\|_2\,. \tag{5.17}$$

Consequently,

$$\sup_{\substack{\lambda\in\rho(\mathcal{L}_2(\mathfrak{e}))\\ \mu\leq\mu_0\\ |\nu|\leq\mathfrak{e}^{-(1+2b)/3}}} \|(\mathcal{L}_2(\mathfrak{e}) - \lambda)^{-1}\| \leq C_{\mu_0}\,. \tag{5.18}$$

Consider now λ such that $\operatorname{Re}\lambda \leq \mu_0$ and $|\operatorname{Im}\lambda| > \mathfrak{e}^{-(1+2b)/3}$.

As before let $w \in D(\mathcal{L}_2)$ and $g = (\mathcal{L}_2 - \lambda)w$. Let $\chi_2(\tau) = \chi(\mathfrak{e}^b\tau/2)$ and $\check{\chi}_2(\tau) = \check{\chi}(\mathfrak{e}^b\tau/2)$. Hence we have $\check{\chi}_2^2 + \chi_2^2 = 1$. Clearly,

$$\operatorname{Im}\langle\chi_2^2 w, (\mathcal{L}_2 - \lambda)w\rangle = -\nu\|\chi_2 w\|_2^2 + \langle\tau(1 + \mathfrak{e}\beta\tau\chi)\chi_2^2 w, w\rangle + 2\operatorname{Im}\langle\chi_2' w, (\chi_2 w)'\rangle\,.$$

Consequently,

$$\mathfrak{e}^{-(1+2b)/3}\|\chi_2 w\|_2^2 \leq C(\mathfrak{e}^{-b}\|\chi_2 w\|_2^2 + \mathfrak{e}^b\|w\|_2^2 + \mathfrak{e}^b\|(\chi_2 w)'\|_2^2 + \mathfrak{e}^b\|g\|_2^2 + \mathfrak{e}^{1-2b}\|\chi_2 w\|_2^2)\,. \tag{5.19}$$

Furthermore, as

$$\operatorname{Re}\langle\chi_2^2 w, (\mathcal{L}_2 - \lambda)w\rangle = \|(\chi_2 w)'\|_2^2 - \mu\|\chi_2 w\|_2^2 - \|\chi_2' w\|_2^2\,,$$

we obtain that

$$\|(\chi_2 w)'\|_2^2 \leq C_{\mu_0}(\|\chi_2 w\|_2^2 + \mathfrak{e}^{2b}\|w\|_2^2 + \|g\|_2^2)\,.$$

Substituting the above into (5.19) yields for a new constant C_{μ_0}

$$\|\chi_2 w\|_2 \leq C_{\mu_0}\,\mathfrak{e}^{(1+5b)/6}(\|g\|_2 + \|w\|_2)\,. \tag{5.20}$$

We now write, observing that $\check{\chi}_2(\tau)\,\chi(e^b\tau) = 0$ on $\mathbb{R}$,

$$(\mathcal{L}_2 - \lambda)(\check{\chi}_2 w) = (\mathcal{L} - \lambda)(\check{\chi}_2 w) = \check{\chi}_2 g + 2\check{\chi}_2' w' + \check{\chi}_2'' w\,.$$

Hence, using (5.3),

$$\|\check{\chi}_2 w\|_2 \leq C_{\mu_0}(\|g\|_2 + \mathfrak{e}^b\|w'\|_2 + \mathfrak{e}^{2b}\|w\|_2)\,. \tag{5.21}$$

As

$$\operatorname{Re}\langle w, (\mathcal{L}_2 - \lambda)w\rangle = \|w'\|_2^2 - \mu\|w\|_2^2\,,$$

we easily obtain that

$$\|w'\|_2 \leq C_{\mu_0}(\|g\|_2 + \|w\|_2)\,.$$

Substituting the above into (5.21) yields

$$\|\check{\chi}_2 w\|_2 \leq C_{\mu_0}(\|g\|_2 + \mathfrak{e}^b\|w\|_2)\,,$$

which combined with (5.20) yields

$$\sup_{\substack{\lambda\in\rho(\mathcal{L}_2)\\ \mu\leq\mu_0\\ |\nu|>\mathfrak{e}^{-(1+2b)/3}}} \|(\mathcal{L}_2 - \lambda)^{-1}\| \leq C_{\mu_0}\,. \tag{5.22}$$

The above together with (5.18) yields (5.7). ∎

5.3 Dirichlet realization in the half-line

Denote the Dirichlet realization of (5.1) in $\mathbb{R}_+$ by $\mathcal{L}_2^+(\mathfrak{e})$ (or $\mathcal{L}_2^+$ for simplicity). Its domain is given by $D(\mathcal{L}^+)$ (see (3.17)). We recall that $\mathcal{L}^+$ has compact resolvent and that its spectrum consists of eigenvalues with multiplicity 1. In the sequel we denote these eigenvalues (ordered by non decreasing real part) by $\{\vartheta_n\}_{n=1}^\infty$ and their associated eigenfunctions by $\{v_n\}_{n=1}^\infty$ (recall that $\vartheta_n = |\nu_n|e^{i\pi/3}$).

Proposition 5.2. *Let $\mu_0 < \operatorname{Re}\vartheta_2$, $\delta_0 > 0$ and*

$$\Lambda(\mathfrak{e},\delta,\mu_0)) = \{\lambda\in\mathbb{C}\,,\ -1\leq \operatorname{Re}\lambda\leq\mu_0 \text{ and } |\lambda-\vartheta_1-\mathfrak{e}\beta\tau_{m,2}|\geq\delta\}\,. \tag{5.23}$$

There exist positive $\mathfrak{e}_0$ and C such that $(\mathcal{L}_2^+(\mathfrak{e})-\lambda)$ is invertible whenever $\lambda\in\Lambda(\mathfrak{e},\delta,\mu_0)$, for all $\mathfrak{e}\in(0,\mathfrak{e}_0]$ and $\mathfrak{e}^{2-b}\leq\delta\leq\delta_0$. Moreover

$$\sup_{\lambda\in\Lambda(\mathfrak{e},\delta,\mu_0)} \|(\mathcal{L}_2^+(\mathfrak{e})-\lambda)^{-1}\| \leq \frac{C}{\delta}\,. \tag{5.24}$$

Proof. Let $\lambda\in\rho(\mathcal{L}_2^+)\cap\Lambda(\mathfrak{e},\delta,\mu_0)$. Let $w\in D(\mathcal{L}_2^+)$, $g=(\mathcal{L}_2^+-\lambda)w$ and $\tau_{m,2}$ be given by (3.24). Write

$$(\mathcal{L}^+-\lambda-\mathfrak{e}\beta\tau_{m,2})w = g-\mathfrak{e}\beta(i\chi_{\mathfrak{e}}\tau^2-\tau_{m,2})w\,, \tag{5.25}$$

with

$$\chi_{\mathfrak{e}}(\tau)=\chi(\mathfrak{e}^b\tau)\,.$$

Recall the definition of Π_k from (3.28). Applying Π_1 to (5.25), we obtain

$$(\vartheta_1-\lambda-\mathfrak{e}\beta\tau_{m,2})\Pi_1 w = \Pi_1 g - \mathfrak{e}\beta\Pi_1\big((\chi_{\mathfrak{e}}i\tau^2-\tau_{m,2})w\big)\,. \tag{5.26}$$

From the definition of $\tau_{m,2}$ in (3.24) we have

$$\Pi_1\big((i\tau^2-\tau_{m,2})w\big) = \Pi_1\big((i\tau^2-\tau_{m,2})(I-\Pi_1)w\big)\,. \tag{5.27}$$

Furthermore, from the definition of Π_1 we have

$$\left\|\Pi_1\big((\chi_{\mathfrak{e}} i\tau^2-\tau_{m,2})w\big)-\Pi_1\big((i\tau^2-\tau_{m,2})w\big)\right\|_2\leq C\exp\Big\{-\frac{\sqrt{2}}{3}|\mathfrak{e}|^{-\frac{3b}{2}}\Big\}\|w\|_2\,. \quad (5.28)$$

Here we have used the decay properties of the Airy function v_1 as $\tau\to+\infty$. See for example [1].
Consequently, we may write

$$\left\|\Pi_1\big((\chi_{\mathfrak{e}} i\tau^2-\tau_{m,2})w\big)\right\|_2\leq C\Big(\|(I-\Pi_1)w\|_2+\exp\Big\{-\frac{\sqrt{2}}{3}|\mathfrak{e}|^{-\frac{3b}{2}}\Big\}\|w\|_2\Big).$$

By the above and (5.26) we then have

$$\|\Pi_1 w\|_2\leq\frac{C}{|\lambda-\vartheta_1-\mathfrak{e}\beta\tau_{m,2}|}\Big(\|\Pi_1 g\|_2+\mathfrak{e}\|(I-\Pi_1)w\|_2+\exp\Big\{-\frac{\sqrt{2}}{3}|\mathfrak{e}|^{-\frac{3b}{2}}\Big\}\|w\|_2\Big). \quad (5.29)$$

Since $\lambda\in\Lambda(\mathfrak{e},\delta,\mu_0))$, we obtain

$$\|\Pi_1 w\|_2\leq\frac{C}{|\lambda-\vartheta_1-\mathfrak{e}\beta\tau_{m,2}|}(\|g\|_2+\mathfrak{e}\|(I-\Pi_1)w\|_2). \quad (5.30)$$

Next, we apply $(I-\Pi_1)$ to both sides of (5.25) to obtain

$$(\mathcal{L}^+-\lambda+\mathfrak{e}\beta\tau_{m,2})(I-\Pi_1)w=(I-\Pi_1)g-\mathfrak{e}\beta(I-\Pi_1)\big((\chi_{\mathfrak{e}} i\tau^2-\tau_{m,2})w\big).$$

It has been established in [6] that

$$\|(\mathcal{L}^+-\hat\lambda)^{-1}(I-\Pi_1)\|\leq C\text{ for }\operatorname{Re}\hat\lambda\leq\mu_0\,. \quad (5.31)$$

Hence,

$$\|(I-\Pi_1)w\|_2\leq C(\|(I-\Pi_1)g\|_2+\mathfrak{e}\|\chi_{\mathfrak{e}}\tau^2 w\|_2+\mathfrak{e}\|w\|_2)\,.$$

Having in mind the support of $\chi_{\mathfrak{e}}$ we obtain

$$\mathfrak{e}\|\chi_{\mathfrak{e}}\tau^2 w\|_2\leq 2\mathfrak{e}^{1-b}\|\tau w\|_2\,, \quad (5.32)$$

and hence the existence of $C>0$ such that, for all $w\in D(\mathcal{L}^+)$,

$$\|(I-\Pi_1)w\|_2\leq C\,(\|g\|_2+\mathfrak{e}^{1-b}\|\tau w\|_2+\mathfrak{e}\|w\|_2)\,. \quad (5.33)$$

Since $w\in D(\mathcal{L}^+)$, we can apply [6, Proposition 5.8] and (5.4) in the case of $\mathcal{L}$ to (5.25) to obtain

$$\|\tau w\|_2\leq C(\|g\|_2+|\lambda|\|w\|_2+\mathfrak{e}\|\chi_{\mathfrak{e}}\tau^2 w\|_2)\,.$$

From (5.32) we then get for $\mathfrak{e}_0$ small enough and $\mathfrak{e}\in(0,\mathfrak{e}_0)$

$$\|\tau w\|_2\leq C(\|g\|_2+|\lambda|\|w\|_2)\,. \quad (5.34)$$

Combining (5.34) with (5.30) and (5.33) yields

$$\begin{aligned}\|w\|_2 &\leq\|\Pi_1 w\|_2+\|(I-\Pi_1)w\|_2\\ &\leq C\Big(\tfrac{1}{|\lambda-\vartheta_1-\mathfrak{e}\beta\tau_{m,2}|}+1\Big)\left(\|g\|_2+\mathfrak{e}^{2-b}[|\lambda|+1]\|w\|_2\right),\end{aligned}$$

which implies the existence of $\mathfrak{e}_0$ and $\hat{C} > 0$ such that, for $\mathfrak{e} \in (0, \mathfrak{e}_0]$ and $|\mathrm{Im}\,\lambda| \leq \frac{1}{\hat{C}}\mathfrak{e}^{b-2}$,

$$\|w\|_2 \leq \hat{C}\Big(\frac{1}{|\lambda - \vartheta_1 - \mathfrak{e}\beta\tau_{m,2}|} + 1\Big)\|g\|_2 \,. \tag{5.35}$$

To complete the proof of (5.24) we need to consider the case $|\mathrm{Im}\,\lambda| > \hat{C}^{-1}\mathfrak{e}^{b-2}$. To this end we need only observe that the proof of (5.22), where a bound on $\|(\mathcal{L}_2-\lambda)^{-1}\|$ is obtained, can be adapted without any changes for the Dirichlet realization $\mathcal{L}_2^+$, whenever $|\mathrm{Im}\,\lambda| \geq \mathfrak{e}^{-(1+2b)/3}$. As $\mathfrak{e}^{b-2} > \mathfrak{e}^{-(1+2b)/3}$ we may conclude that there exist $C > 0$ and $\mathfrak{e}_0$ such that for $(\delta, \mathfrak{e})$ satisfying the assumption of the proposition we have

$$\sup_{\substack{\lambda\in\rho(\mathcal{L}_2^+(\mathfrak{e}))\\ \lambda\in\Lambda(\mathfrak{e},\delta,\mu_0)}} \|(\mathcal{L}_2^+(\mathfrak{e}) - \lambda)^{-1}\| \leq \frac{C}{\delta}\,. \tag{5.36}$$

By the discreteness of $\sigma(\mathcal{L}_2^+(\mathfrak{e}))$ it now follows that $\sigma(\mathcal{L}_2^+(\mathfrak{e})) \cap \Lambda(\mathfrak{e}, \delta, \mu_0) = \emptyset$. This completes the proof of the proposition. ∎

Using the quantitative version of the Gearhart-Prüss Theorem (see [22] or [16]), we immediately obtain

Corollary 5.3. *Let* $b \in (\frac{1}{2}, \frac{3}{4})$. *Let* $e^{-t\mathcal{L}_2^+(\mathfrak{e})}$ *denote the semigroup associated with* $-\mathcal{L}_2^+(\mathfrak{e})$. *There exist positive* $\mathfrak{e}_0$ *and* $C > 0$ *such that, for* $\mathfrak{e} \in (0, \mathfrak{e}_0]$ *and* $\mathfrak{e}^{2-b} \leq \delta \leq \mathrm{Re}\,\vartheta_1 + 1$, *we have*

$$\|e^{-t\mathcal{L}_2^+(\mathfrak{e})}\| \leq \frac{C}{\delta} e^{-t(\mathrm{Re}\,\vartheta_1 - \delta)}\,. \tag{5.37}$$

Remark 5.4. *Note that* C *and* $\mathfrak{e}_0$, *in both* (5.37) *and* (5.24) *a priori depend on* β. *Nevertheless, as the proof of* (5.24) *simply assumes that* β *is bounded, we may drop this dependence by confining* β *to a bounded interval.*

By [7, Example 4.1.2] $\mathcal{L}_2^+(\mathfrak{e})$ possesses a complete system of generalized eigenfunctions in $L^2(\mathbb{R}_+)$. Denote by $\{\tilde{\vartheta}_k\}_{k=1}^{+\infty}$ the sequence of distinct eigenvalues ordered by non decreasing real part and the corresponding projection operators by $\tilde{\Pi}_k$. Whenever an emphasis of the dependence on $\mathfrak{e}$ is necessary, we use the notation

$$\tilde{\vartheta}_k = \vartheta_{k,\mathfrak{e}} \text{ and } \tilde{\Pi}_k = \Pi_{k,\mathfrak{e}}\,.$$

We now attempt to obtain a bound for the variation of the eigenvalues and eigenfunctions of $\mathcal{L}_2^+(\mathfrak{e})$ as function of $\mathfrak{e}$.

Proposition 5.5. *For any* $\beta_0 > 0$, $b \in (\frac{1}{2}, \frac{3}{4})$ *and* $\mu_0 < \mathrm{Re}\,\vartheta_2$, *there exists a positive* $\mathfrak{e}_1$ *such that, for all* $\beta \in [-\beta_0, \beta_0]$ *and* $\mathfrak{e} \in (0, \mathfrak{e}_1]$, $\mathcal{L}_2(\mathfrak{e})$ *has in the half plane* $\{\mathrm{Re}\,\lambda \leq \mu_0\}$ *at most a single eigenvalue of multiplicity* 1 *denoted by* $\vartheta_{1,\mathfrak{e}}$ *which satisfies*

$$|\vartheta_{1,\mathfrak{e}} - \vartheta_1 - \mathfrak{e}\beta\tau_{m,2}| \leq \mathfrak{e}^{2-b}\,. \tag{5.38}$$

Proof. By (5.24) there exists $\mathfrak{e}_1 > 0$ such that for any $\mathfrak{e} \in (0, \mathfrak{e}_1]$, the set $\sigma(\mathcal{L}_2^+(\mathfrak{e})) \cap \{\mathrm{Re}\,\lambda \leq \mu_0\}$ is either empty or

$$\sigma(\mathcal{L}_2^+(\mathfrak{e})) \cap \{\mathrm{Re}\,\lambda \leq \mu_0\} \subset B(\vartheta_1 + \mathfrak{e}\beta\tau_{m,2}, \mathfrak{e}^{2-b})\,. \tag{5.39}$$

Throughout the proof we use the notation $\mathcal{L}_2(\mathfrak{e},\beta)$ instead of $\mathcal{L}_2(\mathfrak{e})$ in order to emphasize the dependence on β of $\mathcal{L}_2$. Let $\Pi_1(\mathfrak{e},\beta)$ denote the projector associated with the spectrum of $\mathcal{L}_2^+(\mathfrak{e},\beta)$ in the disk $D(\vartheta_1+\mathfrak{e}\tau_{m,2},\mathfrak{e}^{2-b})$. For fixed $\mathfrak{e}\in(0,\mathfrak{e}_1]$, the operator-valued function $\beta\mapsto(\widehat{\mathcal{L}_2^+}(\mathfrak{e},\beta)-\lambda)^{-1}\in\mathcal{L}(L^2(\mathbb{R}_+))$, in view of Remark 5.4, is uniformly continuous for λ on $\partial D(\vartheta_1+\mathfrak{e}\tau_{m,2},\mathfrak{e}^{2-b})$ and $\beta\in[-\beta_0,\beta_0]$. We indeed observe that

$$\begin{aligned}&(\mathcal{L}_2^+(\mathfrak{e},\beta)-\lambda)^{-1}-(\mathcal{L}_2^+(\mathfrak{e},\beta')-\lambda)^{-1}\\&\quad=i\mathfrak{e}(\beta-\beta')(\mathcal{L}_2^+(\mathfrak{e},\beta)-\lambda)^{-1}\circ(\chi(\mathfrak{e}^b\tau)i\tau^2-\tau_{m,2})\circ(\mathcal{L}_2^+(\mathfrak{e},\beta')-\lambda)^{-1}\,,\end{aligned}\;.$$

The projector $\Pi_1(\mathfrak{e},\beta)$ (which can be expressed by a Cauchy integral of the resolvent along $\partial D(\vartheta_1+\mathfrak{e}\tau_{m,2},\mathfrak{e}^{2-b})$) is Lipschitz continuous in β in $[-\beta_0,\beta_0]$, and its rank is therefore a continuous integer valued function of β. This rank is consequently constant and, noting that for $\beta=0$ we have $\Pi_1(\mathfrak{e},0)=\Pi_1$, must be equal to one. Hence $\sigma(\mathcal{L}_2^+(\mathfrak{e},\beta))\cap D(\vartheta_1+\mathfrak{e}\beta\tau_{m,2},\mathfrak{e}^{2-b})$ contains precisely one eigenvalue of multiplicity one for all $\beta\in[-\beta_0,\beta_0]$. Moreover $\Pi_1(\mathfrak{e},\beta)=\widetilde{\Pi}_1=\Pi_{1,\mathfrak{e}}$. ∎

Remark 5.6. *Note that, expressing $\Pi_{1,\mathfrak{e}}$ by a Cauchy integral along a fixed circle centered at ϑ_1 and contained in the half space $\{\operatorname{Re}\lambda<\operatorname{Re}\vartheta_2\}$, we obtain by (5.24) that there exists a constant $C>0$ such that, $\forall\mathfrak{e}\in(0,\mathfrak{e}_1]$,*

$$1\leq\|\Pi_{1,\mathfrak{e}}\|\leq C\,. \tag{5.40}$$

We can now obtain the following complement to Proposition 5.3:

Proposition 5.7. *For every $\mu_0<\operatorname{Re}\vartheta_2$ there exists $M_{\mu_0}>0$ and $\mathfrak{e}_{\mu_0}>0$ such that, $\forall\mathfrak{e}\in(0,\mathfrak{e}_{\mu_0}]$,*

$$\|e^{-t\mathcal{L}_2^+(\mathfrak{e})}(I-\Pi_{1,\mathfrak{e}})\|\leq M_{\mu_0}\,e^{-t\mu_0}\,. \tag{5.41}$$

Proof. Let $\operatorname{Re}\vartheta_1<\mu_0<\operatorname{Re}\vartheta_2$. Recalling (5.24), we observe that

$$r(\mu_0):=\sup_{\operatorname{Re}\lambda=\mu_0}\|(\mathcal{L}_2^+(\mathfrak{e})-\lambda)^{-1}\|<+\infty\,.$$

By [17, Theorem 1.6] we have

$$\|e^{-t\mathcal{L}_2^+(\mathfrak{e})}(I-\Pi_{1,\mathfrak{e}})\|\leq\frac{e^{-\mu_0 t}}{r(\mu_0)\int_0^{t/2}\|e^{-s\mathcal{L}_2^+(\mathfrak{e})}\|^{-2}e^{-2\mu_0 s}\,ds}\,\|\,I-\Pi_{1,\mathfrak{e}}\|\,.$$

We may now use (5.24), (5.40) and (5.37) to prove that (5.41) holds uniformly for $t\in\mathbb{R}_+$. ∎

As

$$(\mathcal{L}_2^+(\mathfrak{e})-\lambda)^{-1}(I-\Pi_{1,\mathfrak{e}})=\int_0^\infty e^{-t(\mathcal{L}_2^+(\mathfrak{e})-\lambda)}(I-\Pi_{1,\mathfrak{e}})\,dt\,,$$

we obtain from (5.41) the following corollary:

Corollary 5.8. *For every $\mu_0 < \operatorname{Re}\vartheta_2$, there exists $M_{\mu_0} > 0$ and $\mathfrak{e}_{\mu_0} > 0$ such that, $\forall \mathfrak{e} \in (0, \mathfrak{e}_{\mu_0}]$,*

$$\sup_{\operatorname{Re}\lambda \leq \mu_0} \|(I - \Pi_{1,\mathfrak{e}})(\mathcal{L}_2^+(\mathfrak{e}) - \lambda)^{-1}\| \leq M_{\mu_0}\,. \tag{5.42}$$

We conclude by obtaining the dependence on $\mathfrak{e}$ and the decay as $\tau \to +\infty$ of the eigenfunction $\tilde{v}_1 := v_{1,\mathfrak{e}}$.

Proposition 5.9. *Under the assumptions of Proposition 5.5, the corresponding eigenfunction $v_{1,\mathfrak{e}}$ can be normalized such that*

$$\int_0^{+\infty} v_{1,\mathfrak{e}}^2(\tau)\, d\tau = 1\,, \tag{5.43}$$

and

$$\|v_{1,\mathfrak{e}} - v_1\|_2 \leq C_1 \mathfrak{e}\,. \tag{5.44}$$

Moreover, for any $\Upsilon < \sqrt{2}/3$, there exists $C_\Upsilon > 0$ and $\mathfrak{e}_1 > 0$ such that, for all $0 < \mathfrak{e} \leq \mathfrak{e}_1$,

$$\|e^{\Upsilon \tau^{3/2}}\, v_{1,\mathfrak{e}}\|_2 \leq C_\alpha\,. \tag{5.45}$$

Proof. We first observe, by [10], that we can normalize the eigenfunction $\tilde{v}_1 = v_{1,\mathfrak{e}}$ so that (5.43) holds. Once this normalization is applied, we may write

$$\tilde{\Pi}_1 = \Pi_{1,\mathfrak{e}} = \langle\, \cdot\,,\, \bar{v}_{1,\mathfrak{e}} \rangle\, v_{1,\mathfrak{e}}\,. \tag{5.46}$$

Note that the above normalization determines $\tilde{v}_1$ up to a multiplication by ± 1. Since $\Pi_{1,\mathfrak{e}}$ is a rank one projection, it follows by [10]

$$\|\Pi_{1,\mathfrak{e}}\| = \|v_{1,\mathfrak{e}}\|^2\,,$$

which implies together with (5.40) that

$$1 \leq \|v_{1,\mathfrak{e}}\|_2 \leq C\,. \tag{5.47}$$

To prove (5.44) we first observe that

$$(\mathcal{L}_2^+(\mathfrak{e}) - \vartheta_1)v_1 = -i\mathfrak{e}\beta\tau^2\chi v_1\,.$$

By (5.42) we then have

$$\|(I - \Pi_{1,\mathfrak{e}})v_1\|_2 \leq M_1\, \mathfrak{e}\,. \tag{5.48}$$

Hence, for some $M_1 > 0$ and for any $\mathfrak{e} \in (0, \mathfrak{e}_1]$, there exist $\alpha_{1,\mathfrak{e}}$ and a function $\psi_\mathfrak{e} \in (I - \Pi_{1,\mathfrak{e}})L^2(\mathbb{R}_+)$ satisfying

$$\|\psi_\mathfrak{e}\|_2 \leq M_1 \mathfrak{e}\,,\ |\alpha_{1,\mathfrak{e}}| \leq M_1\,, \tag{5.49}$$

such that

$$v_1 = \alpha_{1,\mathfrak{e}}\, v_{1,\mathfrak{e}} + \psi_\mathfrak{e}\,. \tag{5.50}$$

Taking the inner product with $\bar{v}_1$ and having in mind the normalization of v_1 and $\tilde{v}_1$ yields

$$1 = \alpha_{1,\mathfrak{e}} \int_0^{+\infty} v_{1,\mathfrak{e}}(\tau) v_1(\tau)\, d\tau + \int_0^{+\infty} \psi_\mathfrak{e}(\tau) v_1(\tau)\, d\tau\,.$$

Taking the inner product with $\bar{v}_{1,\mathfrak{e}}$ yields

$$\int_0^{+\infty} v_{1,\mathfrak{e}}(\tau)v_1(\tau)\, d\tau = \alpha_{1,\mathfrak{e}}\, .$$

Consequently,

$$1 = \alpha_{1,\mathfrak{e}}^2 + \int_0^{+\infty} \psi_{\mathfrak{e}}(\tau)v_1(\tau)\, d\tau\, .$$

By (5.49) we must therefore have

$$|\alpha_{1,\mathfrak{e}}^2 - 1| \leq C\, \mathfrak{e}\, .$$

Possibly changing $v_{1,\mathfrak{e}}$ into $-v_{1,\mathfrak{e}}$ we get (5.44).

To obtain the decay of $v_{1,\mathfrak{e}}$ we observe, following Agmon [2] that

$$\begin{aligned} 0 &= \operatorname{Re}\langle e^{2\Upsilon\tau^{3/2}}\tilde{v}_1, (\mathcal{L}_2^+ - \tilde{\vartheta}_1)\tilde{v}_1\rangle \\ &= \|(e^{\Upsilon\tau^{3/2}}\tilde{v}_1)'\|_2^2 - \tfrac{9\Upsilon^2}{4}\|\tau^{1/2}e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2 - \operatorname{Re}\tilde{\vartheta}_1\|e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2\, , \end{aligned}$$

and

$$\begin{aligned} 0 &= \operatorname{Im}\langle e^{2\Upsilon\tau^{3/2}}\tilde{v}_1, (\mathcal{L}_2^+ - \tilde{\vartheta}_1)\tilde{v}_1\rangle \\ &= \|\tau^{1/2}e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2 - \operatorname{Im}\tilde{\vartheta}_1\|e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2 \\ &\quad +3\Upsilon \operatorname{Im}\langle \tau^{1/2}e^{\Upsilon\tau^{3/2}}\tilde{v}_1, (e^{\Upsilon\tau^{3/2}}\tilde{v}_1)'\rangle + \beta\mathfrak{e}\|\chi^{1/2}\tau e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2\, . \end{aligned} \tag{5.51}$$

Combining the above identities yields

$$\Big[1 - \frac{9\Upsilon^2}{2} - C\mathfrak{e}^b\Big]\|\tau^{1/2}e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2 \leq C\|e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2\, .$$

Consequently, for $0 < \Upsilon < \sqrt{2}/3$, there exists $\hat{C}_\Upsilon > 0$ and $\mathfrak{e}_1 > 0$ such that, for $\mathfrak{e} \in (0, \mathfrak{e}_1]$,

$$\|e^{\Upsilon\tau^{3/2}}\tilde{v}_1\|_2^2 \leq \hat{C}_\Upsilon\, \|\mathbf{1}_{\{\tau\leq\hat{C}_\Upsilon\}}\, \tilde{v}_1\|_2^2 \leq \hat{C}_\Upsilon\, \|\tilde{v}_1\|\, .$$

We can then conclude by using (5.40). ∎

Remark 5.10. *More generally, one can prove that, for every $k \in \mathbb{N}$ there exist positive C_k and $\mathfrak{e}_k$ such that, for all $\mathfrak{e} \in (0, \mathfrak{e}_k]$, $\vartheta_{k,\mathfrak{e}}$ is simple and satisfies*

$$|\vartheta_{k,\mathfrak{e}} - \vartheta_k| \leq C_k\, \mathfrak{e}\, .$$

We can normalize the corresponding eigenfunction $v_{k,\mathfrak{e}}$ by

$$\int_0^{+\infty} v_{k,\mathfrak{e}}^2(\tau)\, d\tau = 1\, ,$$

and with this normalization

$$\|v_{k,\mathfrak{e}} - v_k\|_2 \leq C_k\, \mathfrak{e}\, .$$

Moreover for any $\Upsilon < \sqrt{2}/3$ and any $k \in \mathbb{N}$, there exist $C_k^\Upsilon > 0$ and $\mathfrak{e}_k > 0$ such that, for all $\mathfrak{e} \in (0, \mathfrak{e}_k]$,

$$\|e^{\Upsilon\tau^{3/2}}\, v_{k,\mathfrak{e}}\|_2 \leq C_\Upsilon^k\,.$$

The proof for $k \geq 2$ can indeed be similarly obtained by considering

$$(\mathcal{L}_2^+(\mathfrak{e}) - \lambda)^{-1}\Big(I - \sum_{n=1}^{k-1} \tilde{\Pi}_n\Big)$$

instead of $(\mathcal{L}_2^+(\mathfrak{e}) - \lambda)^{-1}(I - \tilde{\Pi}_1)$.

5.4 Application to 2D separable operators

The above-derived one-dimensional estimates can now be used to derive similar estimates for some operators that can be represented as a sum of $\mathcal{L}_2^+$ and an operator that depends on σ only (see the definition of (σ, τ) in (3.8)). We begin the estimation with the following auxiliary lemma which will be used in the next section.

Lemma 5.11. *Let $g \in L^2(\mathbb{R}_+)$, and τ_m be given by* (3.24)*. Then,*

$$\big\|\tilde{\Pi}_1\big((\tau - e^{-i\pi/3}\tau_m)g\big) - \tilde{\Pi}_1\big((\tau - e^{-i\pi/3}\tau_m)(I - \tilde{\Pi}_1)g\big)\big\|_2 \leq C\mathfrak{e}\, \|g\|_2\,. \tag{5.52}$$

Proof. By (5.44) we have

$$\|\Pi_1 - \tilde{\Pi}_1\| \leq C\,\mathfrak{e}\,. \tag{5.53}$$

From the definition of τ_m we have

$$\Pi_1\big((\tau - e^{-i\pi/3}\tau_m)\Pi_1 g\big) = 0\,.$$

We may thus write

$$\tilde{\Pi}_1\big((\tau - e^{-i\pi/3}\tau_m)\tilde{\Pi}_1 g\big) = (\tilde{\Pi}_1 - \Pi_1)\big((\tau - e^{-i\pi/3}\tau_m)\tilde{\Pi}_1 g\big) + \Pi_1\big((\tau - e^{-i\pi/3}\tau_m)(\tilde{\Pi}_1 - \Pi_1)g\big)\,.$$

By (5.53) and (5.45) we have

$$\|(\tilde{\Pi}_1 - \Pi_1)\big((\tau - e^{-i\pi/3}\tau_m)\tilde{\Pi}_1 g\big)\|_2 \leq C\mathfrak{e}\, \|(\tau - e^{-i\pi/3}\tau_m)\tilde{\Pi}_1 g\|_2 \leq C\mathfrak{e}\, \|g\|_2\,,$$

and since

$$\Pi_1\big((\tau - e^{-i\pi/3}\tau_m)(\tilde{\Pi}_1 - \Pi_1)g\big) = \langle \bar{v}_1, (\tau - e^{-i\pi/3}\tau_m)(\tilde{\Pi}_1 - \Pi_1)g\rangle v_1\,,$$

we obtain from (5.53) and the decay properties of the Airy function that

$$\|\Pi_1\big((\tau - e^{-i\pi/3}\tau_m)(\tilde{\Pi}_1 - \Pi_1)g\big)\|_2 \leq C\,\mathfrak{e}\, \|g\|_2\,.$$

∎

The next proposition will also be needed in the next section.

Proposition 5.12. *Let, for $\mathfrak{e} > 0$, $\beta \in \mathbb{R}$, $1/2 < b < 3/4$, and I an open interval in $\mathbb{R}$ (we may set $I = \mathbb{R}$ as well),*

$$\mathcal{M}(\mathfrak{e}, I, \beta, \chi) = \mathcal{L}_2^+(\mathfrak{e}, \beta) - \mathfrak{e}\partial_\sigma^2 \,, \tag{5.54a}$$

be defined on

$$D(\mathcal{M}(\mathfrak{e}, I, \beta, \chi)) = \{u \in H^2(S_I) \cap H_0^1(S_I) \,|\, \tau u \in L^2(S_I)\} \,, \tag{5.54b}$$

where $S_I = I \times \mathbb{R}_+$.
Then, there exist $\mathfrak{e}_0 > 0$ and $C > 0$ such that, for any triple $(\mathfrak{e}, \delta, I)$ satisfying $\delta \in [\mathfrak{e}^{2-b}, \operatorname{Re}\tilde{\vartheta}_1 + 1)$ and $\mathfrak{e} \in (0, \mathfrak{e}_0]$, the spectrum $\mathcal{M}(\mathfrak{e}, I, \beta)$ lies outside $\{\operatorname{Re}\lambda \leq \operatorname{Re}\tilde{\vartheta}_1 - \delta\}$ and

$$\sup_{\operatorname{Re}\lambda \leq \operatorname{Re}\tilde{\vartheta}_1 - \delta} \|(\mathcal{M}(\mathfrak{e}, I, \beta, \chi) - \lambda)^{-1}\| \leq \frac{C}{\delta} \,. \tag{5.55}$$

The proof can easily be obtained from (5.24) by using Fourier series in σ, or by using a Fourier transform in the case $I = \mathbb{R}$ (see also in [6, Lemma 4.12]).
Finally we will also make use, in the next section, of the following proposition:

Proposition 5.13. *Let τ_m and $\mathcal{P}$ be defined by* (3.23) *and* (3.24). *Let further*

$$\mathcal{M}_\mathfrak{e}^2 = \mathcal{L}_2^+(\mathfrak{e}) + \mathfrak{e}\,\mathcal{P} \,, \tag{5.56}$$

be the closed operator on $L^2(\mathbb{R}_+^2)$ with domain

$$D(\mathcal{M}_\mathfrak{e}^2) = \{u \in H^2(\mathbb{R}_+^2) \cap H_0^1(\mathbb{R}_+^2) \,|\, (\sigma^2 + \tau)u \in L^2(\mathbb{R}_+^2)\,\} \,.$$

Then, there exist $\mathfrak{e}_0$ and $C > 0$, such that, for all pairs $(\mathfrak{e}, \delta)$ for which $\mathfrak{e}^{2-b} \leq \delta \leq \operatorname{Re}\vartheta_1 + 1$ and $\mathfrak{e} \in (0, \mathfrak{e}_0]$, we have

$$\|e^{-t\mathcal{M}_\mathfrak{e}^2}\| \leq \frac{C}{\delta} e^{-t(\operatorname{Re}\vartheta_1 - \delta + \mathfrak{e}\mu_1^\tau)} \,, \tag{5.57a}$$

where $\mu_1^\tau = \inf \operatorname{Re}\sigma(\mathcal{P})$.
Furthermore, for each $\varpi < \operatorname{Re}\vartheta_2$, there exists $M_\varpi > 0$ such that

$$\|e^{-t\mathcal{M}_\mathfrak{e}^2}(I - \tilde{\Pi}_1)\| \leq M_\varpi e^{-t\varpi} \,. \tag{5.57b}$$

Proof. Note first that the potential

$$\hat{V}_\mathfrak{e}(\tau, \sigma) = \tau + \beta\mathfrak{e}\tau^2\chi(\mathfrak{e}^b\tau) + \mathfrak{e}\, e^{-i\pi/3}\sigma^2\tau_m$$

has positive real and imaginary parts for $\mathfrak{e} \in (0, \mathfrak{e}_0]$ (with $\mathfrak{e}_0$ small enough) and satisfies for some $C(\mathfrak{e}, \beta, b) > 0$

$$|\nabla\hat{V}_\mathfrak{e}| \leq C\sqrt{1 + \hat{V}_\mathfrak{e}^2} \quad \text{in } \mathbb{R}_+^2 \,.$$

Hence we may use the technique in [7] to obtain that $\mathcal{M}_\mathfrak{e}^2 : D(\mathcal{M}_\mathfrak{e}^2) \to L^2(\mathbb{R}_+^2)$ has a bounded inverse.

Since $\mathcal{M}_{\mathfrak{e}}^2$ is separable, we have (cf. [6])

$$e^{-t\mathcal{M}_{\mathfrak{e}}^2} = e^{-t\mathcal{L}_2^+(\mathfrak{e})} \otimes e^{-t\mathcal{P}}\,.$$

The proof of (5.57a) follows from (5.37) and the fact, proven in [12, Corollary 14.5.2],

$$\|e^{-t\mathcal{P}}\| \leq Ce^{-t\,\mu_r^1}\,. \tag{5.58}$$

∎

Remark 5.14. *The validity of* (5.57) *remains intact, if we define an operator* $\mathcal{M}_{\mathfrak{e}}^2(L)$ *as the Dirichlet realization of* $\mathcal{M}_{\mathfrak{e}}^2$ *in the semi-infinite strip* $S_L = (-L,L)\times\mathbb{R}_+$.

6 V1 potentials: 2D simplification

In this section we estimate the resolvent of the operator appearing on the left hand side of (3.14), which is obtained from the Taylor expansion of V near some $x_0 \in \mathcal{S}^m$. To remove the large τ effect of the term $i\mathfrak{e}\beta\tau^2$, we attach to it a cutoff function $\chi(\mathfrak{e}^b\tau)$ as in the previous section. Since the term $i\mathfrak{e}\sigma^2\tau$ has a significant effect for large values of $|\sigma|$ we separately estimate the resolvent in the region $|\sigma|\gg 1$. We address the effect of the term $2\mathfrak{e}\omega\partial_\tau$ in a later stage. Other error terms appearing in (3.9)-(3.11) will be treated in Subsection 6.4.

6.1 The operator $\mathcal{B}_{\mathfrak{e}}$

Let

$$\mathcal{B}_{\mathfrak{e}} = \mathcal{L}_2^+(\mathfrak{e},\beta) + \mathfrak{e}\,\mathcal{K} \tag{6.1}$$

where $\mathcal{L}_2^+(\mathfrak{e},\beta)$ is defined by (5.1), with domain given by (3.17b), and

$$\mathcal{K} = -\partial_\sigma^2 + i\sigma^2\tau\,. \tag{6.2}$$

We first give a characterization of the domain of the Dirichlet realization of $\mathcal{B}_{\mathfrak{e}}$ in $\mathbb{R}^2_+$. We may assume that $\mathfrak{e}_0 > 0$ is small enough so that

$$V_{\mathfrak{e}}(\tau,\sigma) := \tau(1+\mathfrak{e}\sigma^2+\mathfrak{e}\beta\tau\chi(\mathfrak{e}^b\tau))$$

is non negative for $\mathfrak{e}\in(0,\mathfrak{e}_0]$.
The operator $\mathcal{B}_{\mathfrak{e}}$ has the form $-\Delta + iV_{\mathfrak{e}}$. It can be verified that $|\nabla V_{\mathfrak{e}}|^2 + V_{\mathfrak{e}}^2$ tends to $+\infty$ as $\sigma^2+\tau^2$ tends to $+\infty$ and that there exist $C := C(\mathfrak{e},\beta,b) > 0$ such that

$$|D^2 V_{\mathfrak{e}}| \leq C\sqrt{1+|\nabla V_{\mathfrak{e}}|^2+V_{\mathfrak{e}}^2} \quad \text{in } \mathbb{R}^2_+\,.$$

Hence

$$D(\mathcal{B}_{\mathfrak{e}}) = \{u\in H^2(\mathbb{R}^2_+)\cap H^1_0(\mathbb{R}^2_+)\,|\,\tau(1+\sigma^2)u\in L^2(\mathbb{R}^2_+)\}\,, \tag{6.3}$$

and the resolvent of $\mathcal{B}_{\mathfrak{e}}$ is compact by [6, Corollary 5.10]. Note that while the corollary in [6] refers separable operators, we may still apply it, given the Dirichlet boundary conditions in (6.3). As a matter of fact, we can use all the estimates of [6, Section 5.2], obtained in the absence of boundaries.

6.2 Large $|\sigma|$ simplification

In the following we estimate the resolvent of (6.1) for large $|\sigma|$, or more precisely, in $(\mathfrak{e}^{-\mathfrak{a}}, +\infty) \times \mathbb{R}_+$. It is convenient to shift $\mathcal{B}_\mathfrak{e}$ to a fixed domain $Q = \mathbb{R}_+ \times \mathbb{R}_+$ by using the transformation $\sigma \to \sigma + \mathfrak{e}^{-\mathfrak{a}}$.

Proposition 6.1. *Let $\mathfrak{a} > 0$ such that $1/6 < \mathfrak{a} < 1/4$, and let*

$$\mathcal{C}_\mathfrak{e} = \mathcal{L}_2^+(\mathfrak{e}) - \mathfrak{e}\partial_\sigma^2 + i\mathfrak{e}((\sigma + \mathfrak{e}^{-\mathfrak{a}})^2\tau)\,, \tag{6.4a}$$

be defined on

$$D(\mathcal{C}_\mathfrak{e}) = \{u \in H^2(Q) \cap H_0^1(Q) \,|\, \tau(1+\sigma^2)u \in L^2(Q)\}\,, \tag{6.4b}$$

where $Q = \mathbb{R}_+ \times \mathbb{R}_+$. Then, for all $\gamma_0 > 0$ there exist positive $C(\gamma_0)$ and $\mathfrak{e}_0$ such that for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$, we have $B(\vartheta_1, \gamma_0\mathfrak{e}) \subset \rho(\mathcal{C}_\mathfrak{e})$,

$$\sup_{\lambda \in B(\vartheta_1, \gamma_0\mathfrak{e})} \|(\mathcal{C}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,, \tag{6.5a}$$

and

$$\sup_{\lambda \in B(\vartheta_1, \gamma_0\mathfrak{e})} \|\partial_\sigma(\mathcal{C}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{\mathfrak{e}^{3/2-2\mathfrak{a}}}\,. \tag{6.5b}$$

Proof.

Partition of unity. We begin the proof by introducing an appropriate partition of unity. Let $\{\phi_k\}_{k=0}^{+\infty}$ denote a sequence of cutoff function in $C^\infty(\mathbb{R}, [0,1])$ satisfying

$$\phi_k(x) = \begin{cases} 1 & |x-k)| < 1/4 \\ 0 & |x-k| > 3/4\,, \end{cases} \text{ and } |\phi_k'| + |\phi_k''| \leq C\,, \tag{6.6a}$$

and

$$\sum_{k=0}^{+\infty} \phi_k^2 = 1 \text{ in } \mathbb{R}_+\,. \tag{6.6b}$$

For $\mathfrak{b}$ satisfying $1/6 < \mathfrak{b} < \mathfrak{a}$, we introduce

$$\phi_k^\mathfrak{e}(\sigma) = \phi_k(\mathfrak{e}^\mathfrak{b}\sigma)\,.$$

Let

$$S_k = ((k-1)\mathfrak{e}^{-\mathfrak{b}}, (k+1)\mathfrak{e}^{-\mathfrak{b}}) \times \mathbb{R}_+ \tag{6.7}$$

and $\mathcal{C}_{\mathfrak{e},k}$ denote the Dirichlet realization in S_k of the differential operator given by (6.4a). Its domain, for $k \geq 1$, is given by

$$D(\mathcal{C}_{\mathfrak{e},k}) = \{u \in H^2(S_k) \cap H_0^1(S_k) \,|\, \tau u \in L^2(S_k)\}\,. \tag{6.8a}$$

For $k = 0$ the domain is given by

$$D(\mathcal{C}_0) = \{u \in H^2(S_0^+) \cap H_0^1(S_0^+) \,|\, \tau^2 u \in L^2(S_0^+)\}\,, \tag{6.8b}$$

where $S_0^+ = S_0 \cap Q$.
We attempt to estimate $(\mathcal{C}_\mathfrak{e} - \lambda)^{-1}$ by the following approximate resolvent

$$\mathcal{R}_\mathcal{C}^{app} = \sum_{k=0}^{+\infty} \phi_k^\mathfrak{e} (\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1} \phi_k^\mathfrak{e} \,. \tag{6.9}$$

Clearly,

$$(\mathcal{C}_\mathfrak{e} - \lambda)\mathcal{R}_\mathcal{C}^{app} = I + \mathcal{E}_\mathcal{C} \,, \tag{6.10a}$$

where

$$\mathcal{E}_\mathcal{C} = - \sum_{k=0}^{+\infty} \mathfrak{e}[\partial_\sigma^2, \phi_k^\mathfrak{e}](\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1} \phi_k^\mathfrak{e} \,. \tag{6.10b}$$

Note that since $\mathcal{C}_\mathfrak{e}$ has a compact resolvent, boundedness of the right inverse of $(\mathcal{C}_\mathfrak{e} - \lambda)$ immediately implies its surjectivity and injectivity, and hence an identity between its right and the left inverses. To bound $\|(\mathcal{C}_\mathfrak{e} - \lambda)^{-1}\|$ we have to establish that $\|\mathcal{E}_\mathcal{C}\| \to 0$ as $\mathfrak{e} \to 0$. To this end we need to show the existence of $\mathfrak{e}_0$ such that, for any k and any $\mathfrak{e} \in (0, \mathfrak{e}_0]$, the disk $B(\vartheta_1, \gamma_0 \mathfrak{e})$ belongs to $\rho(\mathcal{C}_{\mathfrak{e},k})$ and to obtain an estimate of $\|(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}\|$ in this disc.

Control of $(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}$.

Let $w \in D(\mathcal{C}_{\mathfrak{e},k})$ and $g \in L^2(S_k)$ (or $L^2(S_0^+)$ when $k = 0$) such that

$$(\mathcal{C}_{\mathfrak{e},k} - \lambda) w = g \,. \tag{6.11}$$

We rewrite (6.11) in the form

$$\begin{aligned}&\big(- \partial_\tau^2 + i(\mathfrak{d}(\mathfrak{e},k)^3 + \mathfrak{e}\beta\chi(\mathfrak{e}^b\tau)\tau)\tau - \mathfrak{e}\partial_\sigma^2 - \lambda\big) w \\ &\qquad = g - i\big\{\mathfrak{e}[\sigma - k\mathfrak{e}^{-\mathfrak{b}}]^2 + 2[\mathfrak{e}^{1-\mathfrak{a}} + k\mathfrak{e}^{1-\mathfrak{b}}](\sigma - k\mathfrak{e}^{-\mathfrak{b}})\big\}\tau w \,,\end{aligned}$$

where

$$\mathfrak{d}(\mathfrak{e},k) := (1 + \mathfrak{e}[\mathfrak{e}^{-\mathfrak{a}} + k\mathfrak{e}^{-\mathfrak{b}}]^2)^{1/3} \,.$$

Using the dilation

$$(\sigma, \tau) \to \mathfrak{d}(\mathfrak{e},k)(\sigma,\tau) \tag{6.12}$$

yields, in $I(\mathfrak{e}, k, \mathfrak{d}(\mathfrak{e},k)) \times \mathbb{R}_+$, with

$$\begin{aligned}&I(\mathfrak{e},k,\mathfrak{d}) = \mathfrak{d}((k-1)\mathfrak{e}^{-\mathfrak{b}}, (k+1)\mathfrak{e}^{-\mathfrak{b}}) \,, \\ &\beta(\mathfrak{e},k) = \beta\mathfrak{d}(\mathfrak{e},k)^{-4} \,, \; \chi^\mathfrak{d}(\tau) = \chi(\mathfrak{e}^b \mathfrak{d}(\mathfrak{e},k)^{-1}\tau) \,,\end{aligned}$$

and

$$\Big(\mathcal{M}(\mathfrak{e}, I(\mathfrak{e},k,\mathfrak{d}(\mathfrak{e},k)), \beta(\mathfrak{e},k), \chi^{\mathfrak{d}(\mathfrak{e},k)}) - \frac{\lambda}{\mathfrak{d}(\mathfrak{e},k)^2}\Big)\widehat{w} = \mathfrak{d}(\mathfrak{e},k)^{-2}\widehat{h} \,. \tag{6.13}$$

Here

$$\begin{aligned}&h = g - i\big\{\mathfrak{e}[\sigma - k\mathfrak{e}^{-\mathfrak{b}}]^2 + 2[\mathfrak{e}^{1-\mathfrak{a}} + k\mathfrak{e}^{1-\mathfrak{b}}](\sigma - k\mathfrak{e}^{-\mathfrak{b}})\big\}\tau w \,, \\ &\widehat{w} = w \circ \mathfrak{d}(\mathfrak{e},k)^{-1} \,, \; \widehat{h} = h \circ \mathfrak{d}(\mathfrak{e},k)^{-1} \,,\end{aligned} \tag{6.14}$$

and $\mathcal{M}(\mathfrak{e}, I, \beta, \chi)$ is the Dirichlet realization in the interval $I \times \mathbb{R}_+$ of

$$\mathcal{M}(\mathfrak{e}, I, \beta, \chi) := \mathcal{L}_2^+(\mathfrak{e}, \beta, \chi) - \mathfrak{e}\partial_\sigma^2 .$$

Since $\mathfrak{d}(\mathfrak{e}, k) \geq 1$, the new parameter $\beta(\mathfrak{e}, k) = \beta\mathfrak{d}(\mathfrak{e}, k)^{-4}$ is bounded. More caution should be used below while assessing the effect of (6.12) on $\chi^{\mathfrak{d}(\mathfrak{e},k)}$. Nevertheless, it is safe to apply (5.55) as long as $\mathfrak{d}(\mathfrak{e}, k)$ remains in a bounded interval $[1, \mathfrak{d}_0]$.

We now assume that $\lambda \in B(\vartheta_1, \gamma_0\mathfrak{e})$. Observe that

$$\mathfrak{d}(\mathfrak{e}, k)^3 \geq 1 + \mathfrak{e}^{1-2\mathfrak{a}} .$$

We first assume $k \leq \mathfrak{e}^{-(1/2-\mathfrak{b})}$, so that

$$1 \leq \mathfrak{d}(\mathfrak{e}, k) \leq \mathfrak{d}_0 := (3 + 2\mathfrak{e}_0^{1-2\mathfrak{a}})^{\frac{1}{3}} .$$

We now attempt to apply (5.55), with $\delta = \operatorname{Re}\vartheta_1(1 - \mathfrak{d}^{-2})/2$, $\beta = \beta(\mathfrak{e}, k)$, and $I = I(\mathfrak{e}, k)$. Here we note that the constant C in (5.55) is independent of I and that, for $\mathfrak{e}$ small enough, we have

$$\delta \geq \operatorname{Re}\vartheta_1\mathfrak{d}_0^{-1}(\mathfrak{d}_0^2 + \mathfrak{d}_0 + 1)^{-1}(\mathfrak{d}^3 - 1) \geq \operatorname{Re}\vartheta_1\mathfrak{d}_0^{-1}(\mathfrak{d}_0^2 + \mathfrak{d}_0 + 1)^{-1}\mathfrak{e}^{1-2\mathfrak{a}} .$$

Obviously, $1 - 2\mathfrak{a} < 1 < 2 - b$, and hence we have, for $\mathfrak{e}$ small enough,

$$\mathfrak{e}^{2-b} \leq \delta .$$

In addition, for sufficiently small $\mathfrak{e}$,

$$\mathfrak{d}(\mathfrak{e}, k)^{-2}\operatorname{Re}\lambda \leq (\operatorname{Re}\vartheta_1 + \gamma_0\mathfrak{e})\mathfrak{d}(\mathfrak{e}, k)^{-2} \leq \operatorname{Re}\vartheta_1 - \delta , \forall k .$$

Hence all the conditions, needed for the sake of applying (5.55), are met, and with the aid of the identity

$$\mathfrak{e}^{1-\mathfrak{a}} + k\mathfrak{e}^{1-\mathfrak{b}} = [\mathfrak{e}(\mathfrak{d}^3 - 1)]^{1/2} .$$

we obtain that

$$\|w\|_2 \leq \frac{C}{\mathfrak{d}^2 - 1}\left(\|g\|_2 + \mathfrak{e}^{1/2-\mathfrak{b}}(\mathfrak{d}^3 - 1)^{1/2}\|\tau w\|_2\right) .$$

Clearly,

$$\frac{[(\mathfrak{d}^3 - 1)]^{1/2}}{\mathfrak{d}^2 - 1} = \frac{1}{[\mathfrak{d} - 1]^{1/2}}\frac{[\mathfrak{d}^2 + \mathfrak{d} + 1]^{1/2}}{\mathfrak{d} + 1} < \frac{1}{[\mathfrak{d} - 1]^{1/2}} .$$

Substituting the above into (6.2) and taking account of the fact that, for $k \leq \mathfrak{e}^{-(1/2-\mathfrak{b})}$,

$$\mathfrak{d}(\mathfrak{e}, k)^2 - 1 \geq \mathfrak{e}^{1-2\mathfrak{a}}/(2(\mathfrak{d}_0 + 1)) ,$$

we obtain

$$\|w\|_2 \leq \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\left(\|g\|_2 + \mathfrak{e}^{1-\mathfrak{b}-\mathfrak{a}}\|\tau w\|_2\right) = C\mathfrak{e}^{1-2\mathfrak{a}}\|g\|_2 + C\,\mathfrak{e}^{\mathfrak{a}-\mathfrak{b}}\|\tau w\|_2 . \tag{6.15}$$

We now consider the case $k > \mathfrak{e}^{-(1/2-\mathfrak{b})}$. We begin by observing that in this case

$$\mathfrak{d}^3 \geq 2 \quad \text{and} \quad \frac{\operatorname{Re}\lambda}{\mathfrak{d}(\mathfrak{e}, k)^2} \leq (2^{-\frac{2}{3}} + \gamma_0\mathfrak{e})\operatorname{Re}\vartheta_1 .$$

Hence there exists $\gamma_1 < 1$ and $\mathfrak{e}_0$ such that for $\mathfrak{e} \in (0, \mathfrak{e}_0]$

$$\frac{\operatorname{Re}\lambda}{\mathfrak{d}(\mathfrak{e},k)^2} \le \gamma_1 \operatorname{Re}\vartheta_1 \,.$$

We then use resolvent estimates for $\mathcal{M}(\mathfrak{e}, I(\mathfrak{e}, k, \mathfrak{d}(\mathfrak{e}, k)), 0)$. Thus, writing

$$\Big(\mathcal{M}(\mathfrak{e}, I(\mathfrak{e}, k, \mathfrak{d}(\mathfrak{e}, k)), 0, 0) - \frac{\lambda}{\mathfrak{d}(\mathfrak{e},k)^2}\Big)\widehat{w} = (\mathfrak{d}(\mathfrak{e},k))^{-2}\Big(\hat{h} - i\mathfrak{e}(\mathfrak{d}(\mathfrak{e},k))^{-2}\beta\chi^{\mathfrak{d}}(\tau)\tau^2\,\widehat{w}\Big)\,,$$

we may use (5.55) (with $\beta = 0$) and the fact that

$$(\mathfrak{d}(\mathfrak{e},k))^{-4}\|\chi^{\mathfrak{d}}(\tau)\tau^2\hat{w}\|_2 \le \mathfrak{e}^{-b}(\mathfrak{d}(\mathfrak{e},k))^{-3}\|\tau\hat{w}\|_2$$

to obtain

$$\|w\|_2 \le C\left(\|g\|_2 + \mathfrak{e}^{1-b}\|\tau w\|_2\right)\,. \tag{6.16}$$

Since

$$\begin{aligned}&\operatorname{Im}\langle w, (\mathcal{C}_{\mathfrak{e},k} - \lambda)w\rangle \\ &\quad = \|\tau^{1/2}w\|_2^2 + \mathfrak{e}\beta\|\chi^{1/2}(\mathfrak{e}^b\tau)\tau w\|_2^2 + \mathfrak{e}\big\|[|\sigma + \mathfrak{e}^{-\mathfrak{a}}|^2\tau]^{1/2}w\big\|_2^2 - \operatorname{Im}\lambda\|w\|_2^2\,,\end{aligned}$$

we obtain, with the aid of the inequality

$$\|\chi^{1/2}(\mathfrak{e}^b\tau)\tau w\|_2^2 \le \sqrt{2}\,\mathfrak{e}^{-b}\,\|\tau^{\frac{1}{2}}w\|_2^2\,,$$

and the fact that $\operatorname{Im}\lambda \le \gamma_0\mathfrak{e}_0$, the estimate

$$\|\tau^{1/2}w\|_2 \le C(\|g\|_2 + \|w\|_2)\,. \tag{6.17}$$

Furthermore, as

$$\operatorname{Re}\langle w, (\mathcal{C}_{\mathfrak{e},k} - \lambda)w\rangle = \|\partial_\tau w\|_2^2 + \mathfrak{e}\|\partial_\sigma w\|_2^2 - \operatorname{Re}\lambda\|w\|_2^2\,,$$

we readily deduce that

$$\mathfrak{e}^{1/2}\|\partial_\sigma w\|_2 + \|\partial_\tau w\|_2 \le C(\|g\|_2 + \|w\|_2)\,. \tag{6.18}$$

Finally, as

$$\begin{aligned}\operatorname{Im}\langle \tau w, (\mathcal{C}_{\mathfrak{e},k} - \lambda)w\rangle \;&= \|\tau w\|_2^2 + \mathfrak{e}\beta\|\chi^{1/2}(\mathfrak{e}^b\tau)\tau^{3/2}w\|_2^2 \\ &\quad + \mathfrak{e}\big\|[|\sigma + \mathfrak{e}^{-\mathfrak{a}}|\tau]w\big\|_2^2 \\ &\quad - \operatorname{Im}\lambda\|\tau^{1/2}w\|_2^2 + 2\operatorname{Im}\langle w, w_\tau\rangle\,,\end{aligned}$$

we may use (6.17) and (6.18) to establish that

$$\|\tau w\|_2 \le C(\|g\|_2 + \|w\|_2)\,. \tag{6.19}$$

Substituting the above into either (6.15) or (6.16) we get the existence of $\mathfrak{e}_0 > 0$, such that for any k, any $\mathfrak{e} \in (0, \mathfrak{e}_0]$, and $\lambda \in B(\vartheta_1, \gamma_0\mathfrak{e}) \cap \rho(\mathcal{C}_{\mathfrak{e},k})$,

$$\|(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}\| \le \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \tag{6.20}$$

Using the discreteness of the spectrum in $B(\vartheta_1, \gamma_0\mathfrak{e})$ we indeed get from the above uniform estimate that $\sigma(\mathcal{C}_{\mathfrak{e},k}) \cap B(\vartheta_1, \gamma_0\mathfrak{e}) = \emptyset$. By the above and (6.18) we then have

$$\|\partial_\sigma(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}\| \le \frac{C}{\mathfrak{e}^{3/2-2\mathfrak{a}}}\,. \tag{6.21}$$

Hence we have established

$$\|(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}\| + \mathfrak{e}^{1/2}\|\partial_\sigma(\mathcal{C}_{\mathfrak{e},k} - \lambda)^{-1}\| \le \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \tag{6.22}$$

Estimation of $\|(\mathcal{C}_\mathfrak{e}-\lambda)^{-1}\|$
From (6.22) it follows that there exists $\mathfrak{e}_0>0$, such that for any k and any $\mathfrak{e}\in(0,\mathfrak{e}_0]$

$$\|\mathfrak{e}(\partial_\sigma^2\phi_k^\mathfrak{e})(\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\phi_k^\mathfrak{e}g\|_2\leq C\,\mathfrak{e}^{2\mathfrak{b}+2\mathfrak{a}}\,\|\mathbf{1}_{S_k}g\|_2\,,$$

whereas from (6.21) it follows that (for $k=0$ we write S_0^+ instead of S_k)

$$\|\mathfrak{e}(\partial_\sigma\phi_k^\mathfrak{e})\partial_\sigma(\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\phi_k^\mathfrak{e}g\|_2\leq C\,\mathfrak{e}^{\mathfrak{b}+2\mathfrak{a}-1/2}\,\|\mathbf{1}_{S_k}g\|_2\,.$$

Since

$$\langle\mathfrak{e}[\partial_\sigma^2,\phi_k^\mathfrak{e}](\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\phi_k^\mathfrak{e}g,\mathfrak{e}[\partial_\sigma^2,\phi_m^\mathfrak{e}](\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\phi_m^\mathfrak{e}g\rangle=0$$

whenever $|k-m|\geq 2$, we may conclude that

$$\begin{aligned}\|\textstyle\sum_{k=0}^\infty\mathfrak{e}[\partial_\sigma^2,\phi_k^\mathfrak{e}](\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\phi_k^\mathfrak{e}g\|_2^2&\leq 4C\,\mathfrak{e}^{2(\mathfrak{b}+2\mathfrak{a}-1/2)}\Big(\|\mathbf{1}_{S_0^+}g\|_2^2+\textstyle\sum_{k=1}^\infty\|\mathbf{1}_{S_k}g\|_2^2\Big)\\&\leq 4C\,\mathfrak{e}^{2(\mathfrak{b}+2\mathfrak{a}-1/2)}\,\|g\|_2^2\,.\end{aligned}$$

Consequently, by (6.10c) we obtain that

$$\lim_{\mathfrak{e}\to0}\mathcal{E}_\mathcal{C}=0\,. \tag{6.23}$$

To complete the proof we use the fact that by (6.23) the operator $I+\mathcal{E}_\mathcal{C}$ is invertible for sufficiently small $\mathfrak{e}$ to obtain

$$(\mathcal{C}_\mathfrak{e}-\lambda)^{-1}=\mathcal{R}_\mathcal{C}^{app}(I+\mathcal{E}_\mathcal{C})^{-1}\,.$$

Hence, we can conclude from (6.22) the existence of $\mathfrak{e}_0>0$ and $C>0$ such that, for $\mathfrak{e}\in(0,\mathfrak{e}_0]$,

$$\|(\mathcal{C}_\mathfrak{e}-\lambda)^{-1}\|\leq 2\,\|\mathcal{R}_\mathcal{C}^{app}\|\leq 8\sup_{k\geq0}\|(\mathcal{C}_{\mathfrak{e},k}-\lambda)^{-1}\|\leq\frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,.$$

This completes the proof of (6.5a). The proof of (6.5b) easily follows from the fact that

$$\mathrm{Re}\,\langle w,(\mathcal{C}_\mathfrak{e}-\lambda)w\rangle=\|w_\tau\|_2^2+\mathfrak{e}\|w_\sigma\|_2^2-\mathrm{Re}\,\lambda\|w\|_2^2\,,$$

for all $w\in D(\mathcal{C}_\mathfrak{e})$. ∎

6.3 Resolvent estimates for $\mathcal{B}_\mathfrak{e}$

Let

$$\Lambda_\gamma^1(\mathfrak{e})=\vartheta_1+\gamma\mathfrak{e}\lambda_1\text{ for some }\gamma\in[0,1]\,,$$

where λ_1 is given by (3.26). Let further b, in the definition of $\mathcal{B}_\mathfrak{e}$ (see (6.1)) satisfy

$$\frac{1}{2}<b<\frac{3}{4}\,, \tag{6.24}$$

and $r(\mathfrak{e})$ satisfy, for some $q<1/6$,

$$\lim_{\mathfrak{e}\to0}r(\mathfrak{e})=0\ \text{ and }\ \lim_{\mathfrak{e}\to0}\mathfrak{e}^{-q}r(\mathfrak{e})=+\infty\,. \tag{6.25}$$

In the following we prove the inclusion of $\partial B(\Lambda_\gamma^1(\mathfrak{e}),r(\mathfrak{e})\mathfrak{e})$ in the resolvent set of $\mathcal{B}_\mathfrak{e}$ and obtain a bound on the resolvent norm there.

Proposition 6.2. *Under the previous conditions, there exist positive C and $\mathfrak{e}_0$ such that $\partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e}) \subset \rho(\mathcal{B}_\mathfrak{e})$ for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$ and $\gamma \in [0,1]$. Furthermore, the inequality*

$$\|(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e}) + 1 - \gamma)\mathfrak{e}}\,, \tag{6.26}$$

holds true.

Proof.

Construction of the right approximate resolvent.
We introduce a C^∞ partition of unity (ζ_-, η, ζ_+) of $\mathbb{R}$ such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| > 2\,, \end{cases} \tag{6.27}$$

and

$$\zeta_+(x) = 0 \text{ if } x < 1\,,\ \zeta_-(x) = \zeta_+(-x)\,. \tag{6.28}$$

Let further

$$\eta_\mathfrak{e}(\sigma) = \eta(\mathfrak{e}^\mathfrak{a}\sigma) \text{ and } \zeta^\pm_\mathfrak{e}(\sigma) = \zeta_\pm(\mathfrak{e}^\mathfrak{a}\sigma) \tag{6.29}$$

for some $\mathfrak{a}$ satisfying

$$1/6 < \mathfrak{a} < (1-q)/4\,. \tag{6.30}$$

Next, let $S_N = (-2\mathfrak{e}^{-\mathfrak{a}}, 2\mathfrak{e}^{-\mathfrak{a}}) \times \mathbb{R}_+$ and $\mathcal{C}_N$ denote the Dirichlet realization in S_N associated with the differential operator given by (6.1).
Let further $S^+_D = (\mathfrak{e}^{-\mathfrak{a}}, +\infty) \times \mathbb{R}_+$, $S^-_D = (-\infty, -\mathfrak{e}^{-\mathfrak{a}}) \times \mathbb{R}_+$, and $\mathcal{C}^\pm_D$ denote the Dirichlet realizations in $S^\pm_D$ associated with the differential operator given by (6.1). The corresponding domains are

$$D(\mathcal{C}^\pm_D) = \{u \in H^2(S^\pm_D) \cap H^1_0(S^\pm_D)\,|\,\sigma^2(1+\tau)u \in L^2(S^\pm_D)\}\,.$$

We can now formally introduce the approximate resolvent in the form

$$\mathcal{R}^{app}_\mathcal{B} = \eta_\mathfrak{e}(\mathcal{C}_N - \lambda)^{-1}\eta_\mathfrak{e} + \zeta^+_\mathfrak{e}(\mathcal{C}^+_D - \lambda)^{-1}\zeta^+_\mathfrak{e} + \zeta^-_\mathfrak{e}(\mathcal{C}^-_D - \lambda)^{-1}\zeta^-_\mathfrak{e}\,. \tag{6.31}$$

Estimation of $\|(\mathcal{C}^\pm_D - \lambda)^{-1}\|$.
By (6.5), observing that $B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e}) \subset B(\vartheta_1, \gamma_0\mathfrak{e})$ for some $\gamma_0 > \gamma$ and $\mathfrak{e}_0$ small enough, we have, for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$,

$$\|(\mathcal{C}^\pm_D - \lambda)^{-1}\| + \mathfrak{e}^{1/2}\|\partial_\sigma(\mathcal{C}^\pm_D - \lambda)^{-1}\| \leq \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \tag{6.32}$$

Note that the estimates for $\mathcal{C}^-_D$ are deduced from the estimates for $\mathcal{C}^+_D$ by using the intertwining relation $\mathcal{C}^+_D = \mathcal{R}^{-1}\mathcal{C}^-_D\mathcal{R}$, where $\mathcal{R}$ represents the reflection $\sigma \to -\sigma$.

Estimation of $\|(\mathcal{C}_N-\lambda)^{-1}\|$.
It remains to obtain an estimate for $\|(\mathcal{C}_N-\lambda)^{-1}\|$. Let $w\in D(\mathcal{C}_N)$ and $g\in L^2(S_N)$ satisfy

$$(\mathcal{C}_N-\lambda)w=g\,. \tag{6.33}$$

Let further

$$\tilde{\mathcal{C}}_N=\mathcal{L}_2^+(\mathfrak{e})+\mathfrak{e}(\mathcal{P}-\beta\tau_{m,2})=\mathcal{L}_2^+(\mathfrak{e})-\mathfrak{e}\partial_\sigma^2+e^{i\pi/6}\mathfrak{e}\sigma^2\tau_m\,,$$

where τ_m is given by (3.24) and $\mathcal{P}$ is given by (3.23).
We can now write (6.33) in the following form

$$(\tilde{\mathcal{C}}_N-\lambda)w=g-i\mathfrak{e}\sigma^2(\tau-e^{-i\pi/3}\tau_m)w\,.$$

Applying the projection $\tilde{\Pi}_1$, given by (5.46) (which stands for $\mathrm{Id}\,\widehat{\otimes}\,\tilde{\Pi}_1$ as in Section 3) , to the above balance yields

$$(\tilde{\mathcal{C}}_N-\lambda)\tilde{\Pi}_1w=\tilde{\Pi}_1g-i\mathfrak{e}\sigma^2\tilde{\Pi}_1\big((\tau-e^{-i\pi/3}\tau_m)w\big)\,. \tag{6.34}$$

From (5.52) it follows that

$$\|\tilde{\Pi}_1\big((\tau-e^{-i\pi/3}\tau_m)\phi\big)-\tilde{\Pi}_1\big((\tau-e^{-i\pi/3}\tau_m)(I-\tilde{\Pi}_1)\phi\big)\|_2\leq C\mathfrak{e}\|\phi\|_2\,,\ \forall\phi\in L^2(S_N)\,.$$

Let $\phi(\sigma,\tau)=\sigma^2w(\sigma,\tau)$. We can now conclude, as $|\sigma|\leq 2\mathfrak{e}^{-\mathfrak{a}}$ in S_N, that

$$\begin{aligned}\big\|\sigma^2\tilde{\Pi}_1\big((\tau-e^{-i\pi/3}\tau_m)w\big)\big\| &\leq C(\|\tilde{\Pi}_1(\tau-e^{-i\pi/3}\tau_m)\sigma^2(I-\tilde{\Pi}_1)w\|_2+\mathfrak{e}\|\sigma^2w\|_2)\\ &\leq C\,\mathfrak{e}^{-2\mathfrak{a}}\,(\|(I-\tilde{\Pi}_1)w\|_2+\mathfrak{e}\|w\|_2)\,.\end{aligned} \tag{6.35}$$

Since

$$(\tilde{\mathcal{C}}_N-\lambda)\tilde{\Pi}_1=\Big(-\mathfrak{e}\partial_\sigma^2+e^{i\pi/6}\mathfrak{e}\sigma^2\tau_m+(\tilde{\vartheta}_1-\lambda)\Big)\,\tilde{\Pi}_1\,, \tag{6.36}$$

we obtain from (5.38), (6.34), (6.35), and the Riesz-Schauder theory, that for $\lambda\in\partial B(\Lambda_\gamma^1,r(\mathfrak{e})\mathfrak{e})$,

$$\|(\tilde{\mathcal{C}}_N-\lambda)^{-1}\widetilde{\Pi}_1\|\leq\frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,. \tag{6.37}$$

By (5.57b) and Remark 5.14 we have,

$$\|(\tilde{\mathcal{C}}_N-\lambda)^{-1}(I-\widetilde{\Pi}_1)\|\leq C\,. \tag{6.38}$$

Applying (6.37) to (6.34) yields, with the aid of (6.35),

$$\|\tilde{\Pi}_1w\|_2\leq\frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}(\|g\|_2+\mathfrak{e}^{1-2\mathfrak{a}}\|(I-\tilde{\Pi}_1)w\|_2+\mathfrak{e}^{2-2\mathfrak{a}}\|w\|_2)\,. \tag{6.39}$$

We now apply $I-\tilde{\Pi}_1$ to (6.34) to obtain

$$(\tilde{\mathcal{C}}_N-\lambda)(I-\tilde{\Pi}_1)w=(I-\tilde{\Pi}_1)g-i\mathfrak{e}\sigma^2(I-\tilde{\Pi}_1)\big((\tau-e^{-i\pi/3}\tau_m)w\big)\,. \tag{6.40}$$

Since the norm of $I-\tilde{\Pi}_1$ is uniformly bounded (see (5.40)), we have

$$\big\|\sigma^2(I-\widetilde{\Pi}_1)\big((\tau-e^{-i\pi/3}\tau_m)w\big)\big\|_2\leq C\,\|\sigma^2(\tau-e^{-i\pi/3}\tau_m)w\|_2\leq C\mathfrak{e}^{-2\mathfrak{a}}\|(\tau-e^{-i\pi/3}\tau_m)w\|_2\,.$$

In the same manner we have obtained (6.19) we can now obtain

$$\|\tau w\|_2 \leq C\left(\|w\|_2 + \|g\|_2\right). \tag{6.41}$$

Consequently,

$$\big\|\sigma^2(I-\tilde{\Pi}_1)\big((\tau - e^{-i\pi/3}\tau_m)w\big)\big\|_2 \leq C\mathfrak{e}^{-2\mathfrak{a}}(\|w\|_2 + \|g\|_2). \tag{6.42}$$

We now apply (6.38) to (6.40) to obtain, with the aid of (6.35) and the above inequality,

$$\|(I-\tilde{\Pi}_1)w\|_2 \leq C(\|g\|_2 + \mathfrak{e}^{1-2\mathfrak{a}}\|w\|_2). \tag{6.43}$$

Substituting the above into (6.39) yields

$$\|\tilde{\Pi}_1 w\|_2 \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,\|g\|_2 + \frac{C\mathfrak{e}^{1-4\mathfrak{a}}}{r(\mathfrak{e})+1-\gamma}\,\|w\|_2\,.$$

The above together with (6.43), (6.25), and (6.30) yield the existence of $\mathfrak{e}_0 > 0$ and C such that, for all $\mathfrak{e} \in (0,\mathfrak{e}_0]$, $\partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})$ belongs to $\rho(\mathcal{C}_N)$ and

$$\|(\mathcal{C}_N-\lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,,\ \forall\lambda\in\partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})\,. \tag{6.44}$$

A bound on $\partial_\sigma(\mathcal{C}_N-\lambda)^{-1}$

As in the proof of Proposition 6.1 we need an estimate for $\partial_\sigma(\mathcal{C}_N-\lambda)^{-1}$. While (6.18) still holds, it is unsatisfactory in the present context. Let (w,g) satisfy (6.33). To achieve a better estimate of $\|w_\sigma\|_2$, we separately estimate $\|\tilde{\Pi}_1 w_\sigma\|_2$ and $\|(I-\tilde{\Pi}_1)w_\sigma\|_2$.

To facilitate the estimation of $\|\tilde{\Pi}_1 w_\sigma\|_2$, we rewrite (6.34)-(6.36) in the following manner

$$\mathfrak{e}\Big(-\partial_\sigma^2 + e^{i\pi/6}\tau_m\sigma^2 - \frac{\lambda-\tilde{\vartheta}_1}{\mathfrak{e}}\Big)\tilde{\Pi}_1 w = \tilde{\Pi}_1 g - i\mathfrak{e}\sigma^2\tilde{\Pi}_1\big((\tau - e^{-i\pi/3}\tau_m)w\big)\,. \tag{6.45}$$

Taking the inner product of (6.45) with $\tilde{\Pi}_1 w$ we obtain from the real part and (6.35)

$$\mathfrak{e}\|\tilde{\Pi}_1 w_\sigma\|_2^2 \leq C\mathfrak{e}\|\tilde{\Pi}_1 w\|_2^2 + \|\tilde{\Pi}_1 w\|_2\left(\|\tilde{\Pi}_1 g\|_2 + \mathfrak{e}^{1-2\mathfrak{a}}\|(I-\tilde{\Pi}_1)w\|_2 + \mathfrak{e}^{2-2\mathfrak{a}}\|w\|_2\right).$$

Hence,

$$\|\tilde{\Pi}_1 w_\sigma\|_2 \leq \hat{C}\left(\|\tilde{\Pi}_1 w\|_2 + \mathfrak{e}^{-1}\|\tilde{\Pi}_1 g\|_2 + \mathfrak{e}^{-2\mathfrak{a}}\|(I-\tilde{\Pi}_1)w\|_2 + \mathfrak{e}^{1-2\mathfrak{a}}\|w\|_2\right).$$

Using (6.43) and (6.30) we then obtain

$$\begin{aligned}\|\tilde{\Pi}_1 w_\sigma\|_2 &\leq C\left(\|\tilde{\Pi}_1 w\|_2 + \mathfrak{e}^{1-4\mathfrak{a}}\|w\|_2 + \mathfrak{e}^{-1}\|g\|_2\right)\\ &\leq \tilde{C}\left(\|w\|_2 + \mathfrak{e}^{-1}\|g\|_2\right)\right),\end{aligned}$$

from which we deduce, with the aid of (6.33) and (6.44),

$$\|\tilde{\Pi}_1 w_\sigma\|_2 \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\|g\|_2\,. \tag{6.46}$$

To estimate $\|(I-\tilde{\Pi}_1)w_\sigma\|_2$, we now take the inner product of (6.40) with $(I-\tilde{\Pi}_1)w$ to obtain, with the aid of (6.41),

$$\mathfrak{e}\|(I-\tilde{\Pi}_1)w_\sigma\|_2^2 \leq C\|(I-\tilde{\Pi}_1)w\|_2^2 + C\|(I-\tilde{\Pi}_1)w\|_2(\|g\|_2+\mathfrak{e}^{1-2\mathfrak{a}}\|w\|_2)\,.$$

Making use of (6.43) then yields

$$\|(I-\tilde{\Pi}_1)w_\sigma\|_2 \leq C(\mathfrak{e}^{-1/2}\|g\|_2+\mathfrak{e}^{1/2-2\mathfrak{a}}\|w\|_2) \leq \check{C}(\mathfrak{e}^{-1}\|g\|_2+\|w\|_2)\,,$$

which as above leads to

$$\|(I-\tilde{\Pi}_1)w_\sigma\|_2 \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\|g\|_2\,. \tag{6.47}$$

Then (6.46) and (6.47) give the existence of C and $\mathfrak{e}_0$ such that, for all $\mathfrak{e}\in(0,\mathfrak{e}_0]$,

$$\|\partial_\sigma(\mathcal{C}_N-\lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,. \tag{6.48}$$

The approximate resolvent

The preceding paragraphs prove that the approximate resolvent $\mathcal{R}_{\mathcal{B}}^{app}$, introduced in (6.31) is well defined. We now prove that it serves as a good approximation for the resolvent. We note that

$$(\mathcal{B}_\mathfrak{e}-\lambda)\mathcal{R}_{\mathcal{B}}^{app} = I+\mathcal{E}_{\mathcal{B}}\,, \tag{6.49a}$$

where

$$\mathcal{E}_{\mathcal{B}} = -\mathfrak{e}[\partial_\sigma^2,\eta_\mathfrak{e}](\mathcal{C}_N-\lambda)^{-1}\eta_\mathfrak{e}-\mathfrak{e}[\partial_\sigma^2,\zeta_\mathfrak{e}^+](\mathcal{C}_D^+-\lambda)^{-1}\zeta_\mathfrak{e}^+-\mathfrak{e}[\partial_\sigma^2,\zeta_\mathfrak{e}^-](\mathcal{C}_D^--\lambda)^{-1}\zeta_\mathfrak{e}^-\,. \tag{6.49b}$$

As

$$\mathfrak{e}[\partial_\sigma^2,\zeta_\mathfrak{e}^\pm](\mathcal{C}_D^\pm-\lambda)^{-1}\zeta_\mathfrak{e}^\pm = \left(\mathfrak{e}^{1+2\mathfrak{a}}(\zeta^{\pm,\prime\prime})_\mathfrak{e}+2\mathfrak{e}^{1+\mathfrak{a}}(\zeta^{\pm,\prime})_\mathfrak{e}\partial_\sigma\right)(\mathcal{C}_D^\pm-\lambda)^{-1}\zeta_\mathfrak{e}^\pm\,,$$

we obtain by (6.32) that

$$\|\mathfrak{e}[\partial_\sigma^2,\zeta_\mathfrak{e}^\pm](\mathcal{C}_D^\pm-\lambda)^{-1}\zeta_\mathfrak{e}\| \leq C\mathfrak{e}^{3\mathfrak{a}-1/2}\,. \tag{6.50}$$

Furthermore, since

$$\mathfrak{e}[\partial_\sigma^2,\eta_\mathfrak{e}](\mathcal{C}_N-\lambda)^{-1}\eta_\mathfrak{e} = (\mathfrak{e}^{1+2\mathfrak{a}}(\eta'')_\mathfrak{e}+2\mathfrak{e}^{1+\mathfrak{a}}(\eta')_\mathfrak{e}\,\partial_\sigma)(\mathcal{C}_N-\lambda)^{-1}\eta_\mathfrak{e}\,,$$

we obtain from (6.44) and (6.48)

$$\|\mathfrak{e}[\partial_\sigma^2,\eta_\mathfrak{e}](\mathcal{C}_N-\lambda)^{-1}\eta_\mathfrak{e}\| \leq C\frac{\mathfrak{e}^\mathfrak{a}}{r(\mathfrak{e})+1-\gamma}\,.$$

The above, together with (6.50) and (6.49b), yields

$$\|\mathcal{E}_{\mathcal{B}}\| \leq C\Big(\frac{\mathfrak{e}^{\mathfrak{a}}}{r(\mathfrak{e})+1-\gamma} + \mathfrak{e}^{3\mathfrak{a}-1/2}\Big)\,.$$

Hence, (6.25), and (6.30) imply that $\|\mathcal{E}_{\mathcal{B}}\|$ tends to 0 as $\mathfrak{e}\to 0$. Consequently, for sufficiently small $\mathfrak{e}$, $I+\mathcal{E}_{\mathcal{B}}$ is invertible and we may use (6.49a) to obtain the right inverse to $(\mathcal{B}_{\mathfrak{e}}-\lambda)$.
For $\lambda\in\rho(\mathcal{B}_{\mathfrak{e}})\cap\partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})$, this right inverse is identical with the left inverse and we get

$$\|(\mathcal{B}_{\mathfrak{e}}-\lambda)^{-1}\| \leq C\,\mathcal{R}^{app}_{\mathcal{B}} \leq \frac{\widehat{C}}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,.$$

The spectrum of $\mathcal{B}_{\mathfrak{e}}$ being discrete, we may conclude from the above estimate that $\sigma(\mathcal{B}_{\mathfrak{e}})\cap\partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})=\emptyset$, which completes the proof of the proposition. ∎

For later reference we separately estimate the σ-derivatives of $(\mathcal{B}_{\mathfrak{e}}-\lambda)^{-1}$

Proposition 6.3. *Under the conditions of Proposition 6.2, for any $\mathfrak{a}\in(1/6,(1-q)/4)$, there exists $\mathfrak{e}_0$ and $C_{\mathfrak{a}}$ such that, for all $\lambda\in\partial B(\Lambda^1_\gamma,\mathfrak{e}r(\mathfrak{e}))$, we have*

$$\|\partial_\sigma(\mathcal{B}_{\mathfrak{e}}-\lambda)^{-1}\| \leq C_{\mathfrak{a}}\Big(\frac{1}{\mathfrak{e}^{3/2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\Big)\,, \tag{6.51a}$$

and

$$\|\partial^2_{\sigma\sigma}(\mathcal{B}_{\mathfrak{e}}-\lambda)^{-1}\| \leq C_{\mathfrak{a}}\Big(\frac{1}{\mathfrak{e}^{2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}^{3/2}}\Big)\,. \tag{6.51b}$$

Proof.

Estimation of $\partial_\sigma(\mathcal{B}_{\mathfrak{e}}-\lambda)^{-1}$
Let $w\in D(\mathcal{B}_{\mathfrak{e}})$ and $g\in L^2(\mathbb{R}^2_+)$ satisfy $(\mathcal{B}_{\mathfrak{e}}-\lambda)w=g$. Clearly,

$$(\mathcal{B}_{\mathfrak{e}}-\lambda)(\eta_{\mathfrak{e}}w) = \eta_{\mathfrak{e}}g - 2\mathfrak{e}^{1+\mathfrak{a}}(\eta')_{\mathfrak{e}}w_\sigma - \mathfrak{e}^{1+2\mathfrak{a}}(\eta'')_{\mathfrak{e}}\,w\,,$$

where $(\eta')_{\mathfrak{e}}(\sigma)=\eta'(\mathfrak{e}^{\mathfrak{a}}\sigma)$ and $(\eta'')_{\mathfrak{e}}(\sigma)=\eta''(\mathfrak{e}^{\mathfrak{a}}\sigma)$.
By (6.48) we then have

$$\|(\eta_{\mathfrak{e}}w)_\sigma\|_2 \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}[\|\eta_{\mathfrak{e}}g\|_2 + \mathfrak{e}^{1+\mathfrak{a}}\|(\eta')_{\mathfrak{e}}\,w_\sigma\|_2 + \mathfrak{e}^{1+2\mathfrak{a}}\|(\eta'')_{\mathfrak{e}}\,w\|_2]\,. \tag{6.52}$$

Similarly, as

$$(\mathcal{B}_{\mathfrak{e}}-\lambda)(\zeta^\pm_{\mathfrak{e}}w) = \zeta^\pm_{\mathfrak{e}}g - 2\mathfrak{e}^{1+\mathfrak{a}}(\zeta^{\pm,\prime})_{\mathfrak{e}}w_\sigma - \mathfrak{e}^{1+2\mathfrak{a}}(\zeta^{\pm,\prime\prime})_{\mathfrak{e}}w\,,$$

we may use (6.32) to obtain

$$\|(\zeta^\pm_{\mathfrak{e}}w)_\sigma\|_2 \leq \frac{C}{\mathfrak{e}^{3/2-2\mathfrak{a}}}[\|\zeta^\pm_{\mathfrak{e}}g\|_2 + \mathfrak{e}^{1+\mathfrak{a}}\|(\zeta^{\pm,\prime})_{\mathfrak{e}}w_\sigma\|_2 + \mathfrak{e}^{1+2\mathfrak{a}}\|(\zeta^{\pm,\prime\prime})_{\mathfrak{e}}\,w\|_2]\,. \tag{6.53}$$

Combining (6.52) and (6.53) yields (recalling that $\mathfrak{a} > 1/6$ and $q < \frac{1}{6}$)

$$\|w_\sigma\|_2 \leq C\Big(\frac{1}{\mathfrak{e}^{3/2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\Big)[\|g\|_2 + \mathfrak{e}^{1+2\mathfrak{a}}\|w\|_2]\,.$$

With the aid of (6.26) we then obtain, for any pair (w,g) satisfying $(\mathcal{B}_\mathfrak{e} - \lambda)w = g$,

$$\|w_\sigma\|_2 \leq C\Big(\frac{1}{\mathfrak{e}^{3/2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\Big)\|g\|_2\,,$$

from which (6.51a) easily follows.

Estimation of $\partial^2_{\sigma\sigma}(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}$
For the same pair (w,g), an integration by parts yields

$$-\mathrm{Re}\,\langle w_{\sigma\sigma}, (\mathcal{B}_\mathfrak{e} - \lambda)w\rangle = \|w_{\tau\sigma}\|_2^2 + \mathfrak{e}\|w_{\sigma\sigma}\|_2^2 + 2\mathfrak{e}\mathrm{Im}\,\langle w_\sigma, \sigma\tau w\rangle - \mathrm{Re}\,\lambda\|w_\sigma\|_2^2\,.$$

Note here that $< w_{\sigma\sigma}, w_{\tau\tau} >= \|w_{\sigma\tau}\|^2$ for all $w \in H^2(\mathbb{R}^2_+) \cap H^1_0(\mathbb{R}^2_+)$ and hence also for all $w \in D(\mathcal{B}_\mathfrak{e})$.
Hence,

$$\|w_{\sigma\sigma}\|_2 \leq \frac{C}{\mathfrak{e}^{1/2}}(\|w_\sigma\|_2 + \mathfrak{e}\|\sigma\tau w\|_2 + \mathfrak{e}^{-1/2}\|g\|_2)\,. \tag{6.54}$$

As

$$\mathrm{Im}\,\langle \tau w, (\mathcal{B}_\mathfrak{e}-\lambda)w\rangle = \mathrm{Im}\,\langle w, w_\tau\rangle + \|\tau w\|_2^2 + \beta\mathfrak{e}\|\tau^{3/2}\chi(\mathfrak{e}^b\tau)w\|_2^2 + \mathfrak{e}\|\sigma\tau w\|_2^2 - \mathrm{Im}\,\lambda\|\tau^{1/2}w\|_2^2\,,$$

and since both (6.18) and (6.19) hold in this case as well, we easily obtain, in view of the fact that $\tau\chi(\mathfrak{e}^b\tau) \leq 2\mathfrak{e}^{-b}$, that

$$\|\sigma\tau w\|_2 \leq \frac{C}{\mathfrak{e}^{1/2}}(\|g\|_2 + \|w\|_2)\,. \tag{6.55}$$

Substituting (6.51a) and (6.55) into (6.54) then yields

$$\|w_{\sigma\sigma}\|_2 \leq C\Big(\frac{1}{\mathfrak{e}^{2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}^{3/2}}\Big)\|g\|_2\,,$$

for any pair (w,g) which satisfies $(\mathcal{B}_\mathfrak{e} - \lambda)w = g$, which completes the proof of (6.51b). ∎

For later reference we also need the following additional estimate:

Proposition 6.4. *Under the conditions preceding Proposition 6.2, for all $\mathfrak{a}$ in $(1/6, (1-q)/4)$, there exists $C_\mathfrak{a} > 0$ and $\mathfrak{e}_0 > 0$ such that, for any $\mathfrak{e} \in (0, \mathfrak{e}_0]$,*

$$\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}^{1/2}\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C_\mathfrak{a}}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \tag{6.56}$$

Proof. Since for sufficiently small $\mathfrak{e}$ we have

$$(\mathcal{B}_\mathfrak{e} - \lambda)^{-1} = \mathcal{R}_\mathcal{B}^{app}(I + \mathcal{E}_\mathcal{B})^{-1},$$

it follows by (6.31) that

$$\mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}(\mathcal{B}_\mathfrak{e} - \lambda)^{-1} = \mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}(\mathcal{C}_D^+ - \lambda)^{-1}\zeta_\mathfrak{e}^+(I + \mathcal{E}_\mathcal{B})^{-1}.$$

By (6.32) we then have

$$\|\mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C_\mathfrak{a}}{\mathfrak{e}^{1-2\mathfrak{a}}}.$$

In a similar manner we write

$$\mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\mathcal{B}_\mathfrak{e} - \lambda)^{-1} = \mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\mathcal{C}_D^+ - \lambda)^{-1}\zeta_\mathfrak{e}^+(I + \mathcal{E}_\mathcal{B})^{-1}.$$

Once again by (6.32) we obtain

$$\mathfrak{e}^{\frac{1}{2}}\,\|\mathbf{1}_{\sigma \geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\mathcal{B}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C_\mathfrak{a}}{\mathfrak{e}^{1-2\mathfrak{a}}}.$$

∎

6.4 Curvature effects

In the following, we estimate the effect of some of the error terms in (3.9) and (3.11). Since the estimation of these terms is complex, it is preferable to consider them as modifications of $\mathcal{B}_\mathfrak{e}$ and not in the context of the original operator $\mathcal{A}_h$, which is addressed in Section 8.

6.4.1 Effect 1

The first effect is generated by the first error term in (3.11).

Proposition 6.5. *Consider on $D(\mathcal{B}_\mathfrak{e})$ the operator*

$$\hat{\mathcal{B}}_\mathfrak{e} = \mathcal{B}_\mathfrak{e} - \theta\mathfrak{e}^2\tau\chi(\mathfrak{e}^{\tilde{b}}\tau)\partial_\sigma^2\,, \tag{6.57}$$

where $\theta \in \mathbb{R}$, $\tilde{b} \in (0, 1/2 - q)$ and $\chi \in C^\infty(\mathbb{R}_+, [0,1])$ is given by (5.2). Then, there exist positive C and $\mathfrak{e}_0$ such that, for every $\mathfrak{e} \in (0, \mathfrak{e}_0]$, $\partial B(\Lambda_\gamma^1, r(\mathfrak{e})\mathfrak{e}) \cap \sigma(\hat{\mathcal{B}}_\mathfrak{e}) = \emptyset$ and

$$\sup_{\lambda \in \partial B(\Lambda_\gamma^1, r(\mathfrak{e})\mathfrak{e})} \|(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e}) + 1 - \gamma)\mathfrak{e}}\,,\ \forall \lambda \in \partial B(\Lambda_\gamma^1, r(\mathfrak{e})\mathfrak{e})\,. \tag{6.58a}$$

Furthermore, we have, for all $\lambda \in \partial B(\Lambda_\gamma^1, r(\mathfrak{e})\mathfrak{e})$,

$$\|\partial_\tau(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}^{1/2}\|\partial_\sigma(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e}) + 1 - \gamma)\mathfrak{e}}\,, \tag{6.58b}$$

and

$$\|\partial_{\tau\tau}(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}\|\partial_{\sigma\sigma}(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C}{(r(\mathfrak{e}) + 1 - \gamma)\mathfrak{e}}\,. \tag{6.58c}$$

Proof. For sake of brevity we use the notation $\tilde{\chi}(\tau) = \chi(\mathfrak{e}^{\tilde{b}}\tau)$ where χ is given by (5.2). We keep the same notation as in the previous subsection for the cut-off functions given by (6.27)-(6.30) .

Let $u \in D(\mathcal{B}_\mathfrak{e})$ and $g \in L^2(\mathbb{R}^2_+)$ satisfy

$$(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u = g\,.$$

We rewrite the above balance in the following manner

$$(\mathcal{B}_\mathfrak{e} - \lambda)u = g + \theta\mathfrak{e}^2\tau\tilde{\chi}u_{\sigma\sigma}\,. \tag{6.59}$$

Keeping in mind that $|\tau\tilde{\chi}(\tau)| \leq 2\mathfrak{e}^{-\tilde{b}}$ we use (6.51b) with $\mathfrak{a} \in (\frac{1}{4} - \frac{q}{2}, \frac{1}{4} - \frac{q}{4})$ to obtain

$$\|u_{\sigma\sigma}\|_2 \leq C\Big(\mathfrak{e}^{2\mathfrak{a}} + \frac{\mathfrak{e}^{1/2}}{(r(\mathfrak{e}) + 1 - \gamma)}\Big)(\mathfrak{e}^{-2}\|g\|_2 + \mathfrak{e}^{-\tilde{b}}\|u_{\sigma\sigma}\|_2)\,.$$

Since $\tilde{b} \in (0, 1/2 - q)$ we may conclude that

$$\|u_{\sigma\sigma}\|_2 \leq C\Big(\frac{1}{\mathfrak{e}^{2-2\mathfrak{a}}} + \frac{1}{(r(\mathfrak{e}) + 1 - \gamma)\mathfrak{e}^{3/2}}\Big)\|g\|_2\,. \tag{6.60}$$

Applying (6.26) to (6.59) yields

$$\|u\|_2 \leq \frac{C}{(r(\mathfrak{e})\mathfrak{e} + 1 - \gamma)}(\|g\|_2 + \mathfrak{e}^{2-\tilde{b}}\|u_{\sigma\sigma}\|_2)\,.$$

We first establish (6.58a) for $\lambda \in \rho(\hat{\mathcal{B}}_\mathfrak{e}) \cap \partial B(\Lambda^1_\gamma, \mathfrak{e}r(\mathfrak{e}))$, by substituting (6.60) (observing that $\tilde{b} < \frac{1}{2} - q < 2\mathfrak{a}$) into the above inequality. Since the spectrum of $\hat{\mathcal{B}}_\mathfrak{e}$ is discrete, we can deduce, as for $\mathcal{B}_\mathfrak{e}$, that $\sigma(\hat{\mathcal{B}}_\mathfrak{e}) \cap \partial B(\Lambda^1_\gamma, \mathfrak{e}r(\mathfrak{e})) = \emptyset$, and hence, that (6.58a) is satisfied without restriction.

The proof of (6.58b) follows immediately from (6.58a) and the identity

$$\mathrm{Re}\,\langle u, (\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u\rangle = \|u_\tau\|_2^2 + \mathfrak{e}\|u_\sigma\|_2^2 + \theta\mathfrak{e}^2\|[\tilde{\chi}\tau]^{1/2}u_\sigma\|_2^2 - \mathrm{Re}\,\lambda\|u\|_2^2\,,$$

which holds for every $u \in D(\hat{\mathcal{B}}_\mathfrak{e})$. To prove (6.58c) we use (6.60) and the following identity, that holds for every $u \in D(\hat{\mathcal{B}}_\mathfrak{e})$,

$$\begin{aligned}&-\mathrm{Re}\,\langle u_{\tau\tau}, (\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u\rangle \\ &\qquad = \|u_{\tau\tau}\|_2^2 + \mathfrak{e}\|u_{\tau\sigma}\|_2^2 + \theta\mathfrak{e}^2\big(\|[\tilde{\chi}\tau]^{1/2}u_{\sigma\tau}\|_2^2 - \mathrm{Re}\,\langle[\tilde{\chi}\tau]'u_\sigma, u_{\tau\sigma}\rangle\big) - \mathrm{Re}\,\lambda\|u\|_2^2\,,\end{aligned}$$

together with (6.58a,b) and the fact that $\|[\tilde{\chi}\tau]'\|_\infty \leq 1 + 2\|\chi'\|_\infty$. ∎

We shall also need in the sequel the following estimate

Proposition 6.6. *Under the conditions of Proposition 6.5, for any $\mathfrak{a}$ in the interval $(1/6, (1-q)/4)$ there exists $C_\mathfrak{a} > 0$ and $\mathfrak{e}_0 > 0$ such that for any $\mathfrak{e} \in (0, \mathfrak{e}_0]$, and $\lambda \in \partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})$,*

$$\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}^{1/2}\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq \frac{C_\mathfrak{a}}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \tag{6.61}$$

Proof. Let, as in the previous proof, $u \in D(\hat{\mathcal{B}}_\mathfrak{e})$, $\lambda \in \mathbb{C}$ and $g \in L^2(\mathbb{R}^2_+)$ such that $g = (\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u$. Since

$$(\mathcal{B}_\mathfrak{e} - \lambda)u = g - \theta\mathfrak{e}^2\tau\tilde{\chi}u_{\sigma\sigma}\,,$$

we obtain from (6.56) that

$$\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}u\|_2 \leq \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}(\|g\|_2 + \mathfrak{e}^{2-\tilde{b}}\|u_{\sigma\sigma}\|)\,.$$

By (6.60) we then have, using the fact that $\tilde{b} < 1/2 - q$,

$$\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}u\|_2 \leq C\Big(\frac{1}{\mathfrak{e}^{1-2\mathfrak{a}}} + \frac{1}{r(\mathfrak{e})\mathfrak{e}^{1/2+\tilde{b}-2\mathfrak{a}}}\Big)\|g\|_2 \leq \frac{\tilde{C}}{\mathfrak{e}^{1-2\mathfrak{a}}}\,\|g\|_2\,.$$

In a similar manner we show that

$$\mathfrak{e}^{1/2}\|\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}u_\sigma\|_2 \leq \frac{C_\mathfrak{a}}{\mathfrak{e}^{1-2\mathfrak{a}}}\,\|g\|_2\,.$$

∎

We finally establish an asymptotic estimate, which is needed in Section 8. It is valid in a region where τ is large, but the the cutoff function χ is still 1 (i.e. $1 \ll \tau \leq \mathfrak{e}^{-b}$).

Proposition 6.7. *Let $0 < a < b$. Then, there exist positive C_a and $\mathfrak{e}_0$ such that for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$ and $\lambda \in \partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})$,*

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}^{a/2}\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\tau(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}^{(a+1)/2}\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\sigma(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq C_a\mathfrak{e}^a\,, \quad (6.62a)$$

and

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\tau^2(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| + \mathfrak{e}\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\sigma^2(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)^{-1}\| \leq C_a\,. \quad (6.62b)$$

Proof. Let ζ_+ be given by (6.28) and for some $\xi > 0$, $\zeta_\xi(\tau) = \zeta_+(2\tau/\xi)$. Let $u \in D(\hat{\mathcal{B}}_\mathfrak{e})$, $\lambda \in \partial B(\Lambda^1_\gamma, r(\mathfrak{e})\mathfrak{e})$, and $g \in L^2(\mathbb{R}^2_+)$ satisfy

$$(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u = g\,.$$

Proof of (6.62a) As the identity

$$\begin{aligned}\mathrm{Re}\,\langle(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u\,,\,\zeta^2u\rangle + \mathrm{Im}\,\langle(\hat{\mathcal{B}}_\mathfrak{e} - \lambda)u\rangle\,,\,\zeta^2u\rangle \\ = \|\partial_\tau(\zeta u)\|_2^2 - (\mathrm{Im}\,\lambda + \mathrm{Re}\,\lambda)\|\zeta u\|_2^2 + \mathfrak{e}\|\zeta\partial_\sigma u\|_2^2 + \theta\mathfrak{e}^2\|\tau^{1/2}\zeta\tilde{\chi}^{1/2}\partial_\sigma u\|_2^2 \\ -\|\zeta' u\|_2^2 + 2\mathrm{Im}\,\langle\zeta' u, \partial_\tau(\zeta u)\rangle + \|\tau^{1/2}\zeta u\|_2^2 + \beta\mathfrak{e}\|\tau\zeta(\chi(\mathfrak{e}^b\tau))^{\frac{1}{2}}u\|_2^2 + \mathfrak{e}\|\tau^{1/2}\sigma\zeta u\|_2^2\,,\end{aligned} \quad (6.63)$$

holds for any C^∞ function ζ with support in $\mathbb{R}_+$, we get, for $\zeta = \zeta_\xi$,

$$\|\tau^{1/2}\zeta_\xi u\|_2^2 - (\mathrm{Im}\,\lambda + \mathrm{Re}\,\lambda)\|\zeta_\xi u\|_2^2 \leq 4\|\zeta_\xi' u\|_2^2 + 4\|\zeta_\xi u\|_2\|\zeta_\xi g\|_2\,.$$

Observing that $|\zeta_\xi'| \leq \frac{C_0}{\xi}$, we deduce

$$\left(\frac{\xi}{2} - (\operatorname{Im}\lambda + \operatorname{Re}\lambda)\right) \|\zeta_\xi u\|_2^2 \leq \frac{4C_0}{\xi^2}\|\mathbf{1}_{\tau\geq\xi/2}u\|_2^2 + 4\|\zeta_\xi u\|_2\|g\|_2 \,.$$

Hence, for $\xi \geq 4(\operatorname{Im}\lambda + \operatorname{Re}\lambda) \geq \operatorname{Re}\vartheta_1 > 0$,

$$\|\mathbf{1}_{\tau\geq\xi}u\|_2 \leq \frac{C}{\xi}(\|g\|_2 + \xi^{-\frac{1}{2}}\|\mathbf{1}_{\tau\geq\xi/2}u\|_2) \leq \frac{\hat{C}}{\xi}(\|g\|_2 + \|\mathbf{1}_{\tau\geq\xi/2}u\|_2) \,.$$

Applying the above inequality k times recursively yields, for any ξ satisfying $\xi \geq 4^k(\operatorname{Im}\lambda + \operatorname{Re}\lambda)$,

$$\|\mathbf{1}_{\tau\geq\xi}u\|_2 \leq \frac{C_k}{\xi}(\|g\|_2 + \xi^{-(k-1)}\|\mathbf{1}_{\tau\geq\xi/2^k}u\|_2) \,. \tag{6.64}$$

Choosing $\xi = \mathfrak{e}^{-a}$, we obtain, for $\mathfrak{e}^a\, 4^k(\operatorname{Im}\lambda + \operatorname{Re}\lambda) \leq 1$,

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}u\|_2 \leq C_k\mathfrak{e}^a\,(\|g\|_2 + \mathfrak{e}^{a(k-1)}\|u\|_2) \,.$$

Choosing $k \geq \frac{3}{a}$ yields,

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}u\|_2 \leq C_a(\mathfrak{e}^a\|g\|_2 \,. + \mathfrak{e}^3\|u\|_2) \,,$$

With the aid of (6.58), we may now conclude the existence, of $\mathfrak{e}_0$ and C such that, for any $\mathfrak{e} \in (0, \mathfrak{e}_0]$,

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}u\|_2 \leq C\,\mathfrak{e}^a\|g\|_2 \,. \tag{6.65}$$

An additional conclusion that can be drawn from (6.63) is

$$\|\partial_\tau(\zeta_\xi u)\|_2^2 + \mathfrak{e}\|\zeta_\xi\partial_\sigma u\|_2^2 \leq 8\|\zeta_\xi' u\|_2^2 + \frac{C}{\xi}\|\zeta_\xi g\|_2^2 \leq \frac{\hat{C}}{\xi}(\|g\|_2^2 + \xi^{-1}\|\mathbf{1}_{\tau\geq\xi/2}u\|_2^2) \,,$$

which leads to

$$\|\mathbf{1}_{\tau\geq\xi}\partial_\tau u\|_2 + \mathfrak{e}^{1/2}\|\mathbf{1}_{\tau\geq\xi}\partial_\sigma u\|_2 \leq \frac{C}{\xi^{1/2}}(\|g\|_2 + \xi^{-\frac{1}{2}}\|\mathbf{1}_{\tau\geq\xi/2}u\|_2) \,.$$

Using (6.64) with ξ replaced by $\xi/2$, we obtain for any k the existence of positive constants C_k and ξ_k such that for all $\xi \geq \xi_k$

$$\|\mathbf{1}_{\tau\geq\xi}\partial_\tau u\|_2 + \mathfrak{e}^{1/2}\|\mathbf{1}_{\tau\geq\xi}\partial_\sigma u\|_2 \leq \frac{C_k}{\xi^{1/2}}(\|g\|_2 + \xi^{-\frac{1}{2}-k}\|u\|_2) \,. \tag{6.66}$$

Setting once again $\xi = \mathfrak{e}^{-a}$, and k to be sufficiently large completes the proof of (6.62a).

Proof of (6.62b)
We first prove that

$$\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\tau^2(\hat{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\|\leq C_a\,. \tag{6.67}$$

Let, for some C^∞ function ζ supported in $\mathbb{R}^+$,

$$\mathcal{G}:=-\langle(\hat{\mathcal{B}}_\mathfrak{e}-\lambda)u,\zeta^2u_{\tau\tau}\rangle=-\langle(\mathcal{L}_2^+-\lambda)u,\zeta^2u_{\tau\tau}\rangle+\mathfrak{e}\langle(1+\mathfrak{e}\theta\tilde{\chi}\tau)u_{\sigma\sigma}-i\sigma^2\tau u,\zeta^2u_{\tau\tau}\rangle\,. \tag{6.68}$$

We estimate each term separately, repeatedly applying the following integration by parts formula

$$\langle\tau v,\hat{\zeta}^2v_{\tau\tau}\rangle=-\|\tau^{\frac{1}{2}}\hat{\zeta}v_\tau\|^2-\langle v,\partial_\tau(\tau\hat{\zeta}^2)v_\tau\rangle\,,$$

for various choices of v and $\hat{\zeta}$. We thus have

$$-\langle\mathcal{L}_2^+(\mathfrak{e})u,\zeta^2u_{\tau\tau}\rangle=\|\zeta u_{\tau\tau}\|^2-i\langle\tau(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau))u,\zeta^2u_{\tau\tau}\rangle\,,$$

$$\begin{aligned}&-i\langle\tau(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau))u,\zeta^2u_{\tau\tau}\rangle\\&\qquad=i\left(\|\tau^{\frac{1}{2}}(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau)^{\frac{1}{2}}\zeta u_\tau\|^2+\langle u\,,\,\partial_\tau(\tau(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau))\zeta^2)\,u_\tau\rangle\right)\,,\\&\langle\lambda u,\zeta^2u_{\tau\tau}\rangle=\lambda\left(-\|\partial_\tau(\zeta u)\|^2+\|\zeta'u\|^2+\langle\partial_\tau(\zeta u),\zeta'u\rangle-\langle\zeta'u,\partial_\tau(\zeta u)\rangle\right)\,,\\&\qquad-i\mathfrak{e}\langle\sigma^2\tau u,\zeta^2u_{\tau\tau}\rangle=i\mathfrak{e}\left(\|\tau^{\frac{1}{2}}\sigma\zeta u_\tau\|^2+\langle\sigma^2u,\partial_\tau(\tau\zeta^2)u_\tau\rangle\right)\,,\end{aligned}$$

and

$$\begin{aligned}\langle(-(1+\mathfrak{e}\theta\tilde{\chi}\tau)u_{\sigma\sigma},\zeta^2u_{\tau\tau}\rangle&=-\langle((1+\mathfrak{e}\theta\tilde{\chi}\tau)u_\sigma,\zeta^2(u_\sigma)_{\tau\tau}\rangle\\&=\|(1+\mathfrak{e}\theta\tilde{\chi}\tau)^{\frac{1}{2}}\zeta u_{\sigma\tau}\|^2+\langle u_\sigma\,,\,\partial_\tau((1+\mathfrak{e}\theta\tilde{\chi}\tau)\zeta^2)u_{\sigma\tau}\rangle\,.\end{aligned}$$

We now decompose $\mathcal{G}$ in the following manner

$$\mathcal{G}=\mathcal{G}_1+i\mathcal{G}_2+\mathcal{G}_3\,,$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are positive terms defined by

$$\mathcal{G}_1=\|\zeta u_{\tau\tau}\|_2^2+\operatorname{Re}\lambda\|\zeta'u\|^2+\mathfrak{e}\|(1+\mathfrak{e}\theta\tilde{\chi}\tau)^{\frac{1}{2}}\zeta u_{\sigma\tau}\|^2\,, \tag{6.69a}$$

$$\mathcal{G}_2=\|\tau^{\frac{1}{2}}(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau)^{\frac{1}{2}}\zeta u_\tau\|^2+\operatorname{Im}\lambda\|\zeta'u\|^2+\mathfrak{e}\|\tau^{\frac{1}{2}}\sigma\zeta u_\tau\|^2\,, \tag{6.69b}$$

and $\mathcal{G}_3$ is given by

$$\begin{aligned}\mathcal{G}_3=&\,i\langle u\,,\,\partial_\tau(\tau(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau)\zeta^2))\,u_\tau\rangle\\&+\lambda\left(-\|\partial_\tau(\zeta u)\|^2+<\langle\partial_\tau(\zeta u),\zeta'u\rangle-<\langle\zeta'u,\partial_\tau(\zeta u)\rangle\right)\\&+i\mathfrak{e}\langle\sigma^2u,\partial_\tau(\tau\zeta^2)\,u_\tau\rangle\\&+\mathfrak{e}\langle u_\sigma\,,\,\partial_\tau((1+\mathfrak{e}\theta\tilde{\chi}\tau)\zeta^2)\,u_{\sigma\tau}\rangle\,.\end{aligned} \tag{6.69c}$$

For the first term on the right-hand side of (6.69c), we have

$$|\langle u\,,\,\partial_\tau(\tau(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau))\zeta^2)\,u_\tau\rangle|\leq C\left(\|\zeta u\|_2^2+\|\zeta u_\tau\|_2^2+\|\tau\zeta'u_\tau\|_2^2\right)\,.$$

It is readily verified that the second line is bounded by $C(\|\zeta' u\|_2^2+\|\partial_\tau(\zeta u)\|_2^2)$. Let $\epsilon>0$. For the third line of (6.69c) we have

$$|\langle\sigma\tau^{\frac{1}{2}}\zeta u_\tau,\tau^{-\frac{1}{2}}\zeta\,\sigma u\rangle+2\langle\zeta\sigma\tau^{\frac{1}{2}}u_\tau,\zeta'\tau^{\frac{1}{2}}\,\sigma u\rangle|\;\leq\epsilon\|\sigma\tau^{1/2}\zeta u_\tau\|_2^2+\frac{C}{\epsilon}\left(\|\zeta'\tau^{\frac{1}{2}}\;\sigma u\|_2^2+\|\tau^{-\frac{1}{2}}\zeta\sigma u\|_2^2\right).$$

Finally for the forth term, we have for some $C_\epsilon>0$

$$\mathfrak{e}|\langle u_\sigma\,,\,\partial_\tau((1+\mathfrak{e}\theta\tilde{\chi}\tau)\zeta^2)u_{\sigma\tau}\rangle|\leq\mathfrak{e}\epsilon\|(1+\mathfrak{e}\theta\tilde{\chi}\tau)^{\frac{1}{2}}\zeta u_{\sigma\tau}\|_2^2+\mathfrak{e}C_\epsilon\|\partial_\tau(\zeta[1+\mathfrak{e}\theta\tau\tilde{\chi}]^{1/2})\,u_\sigma\|_2^2\,.$$

Combining the above yields that for every $\epsilon>0$ there exists $C_\epsilon>0$ such that

$$|\mathcal{G}_3|\leq\epsilon\,(\mathcal{G}_1+\mathcal{G}_2)+C_\epsilon\,\mathcal{G}_4 \tag{6.70}$$

where

$$\mathcal{G}_4=\|\mathbf{1}_{\tau\geq\xi/2}u\|_2^2+\mathfrak{e}\|\mathbf{1}_{\tau\geq\xi/2}u_\sigma\|_2^2+\|\zeta u_\tau\|_2^2+\mathfrak{e}\|\mathbf{1}_{\tau\geq\xi/2}\tau^{1/2}\sigma u\|_2^2\,. \tag{6.71}$$

To obtain (6.70) we have used the pointwise inequality

$$|\zeta_\xi|+|\zeta_\xi'|+\mathfrak{e}|\zeta(\tau\tilde{\chi})'|\leq C\mathbf{1}_{\tau\geq\xi/2}\,.$$

We now observe that, by (6.68),

$$\|\zeta u_{\tau\tau}\|_2^2\leq\mathcal{G}_1+\mathcal{G}_2\leq\sqrt{2}|\mathcal{G}_1+i\mathcal{G}_2|\leq\sqrt{2}(|\mathcal{G}|+|\mathcal{G}_3|)\leq\epsilon(\mathcal{G}_1+\mathcal{G}_2+\|\zeta u_{\tau\tau}\|_2^2)+C_\epsilon(\mathcal{G}_4+\|g\|_2^2)\,,$$

which implies

$$\|\zeta u_{\tau\tau}\|_2^2\leq C(\|g\|_2^2+\mathcal{G}_4)\,. \tag{6.72}$$

A proper bound of $\mathcal{G}_4$ would thus complete the proof of (6.67). To this end, we now show that there exists C and $\mathfrak{e}_0$ such that we have, with $\zeta=\zeta_\xi$, $\mathfrak{e}\in(0,\mathfrak{e}_0]$ and $\xi=\mathfrak{e}^{-\mathfrak{a}}$,

$$\mathcal{G}_4\leq C\,\|g\|_2^2\,. \tag{6.73}$$

The first term appearing on the right-hand-side of (6.71), may be estimated by using (6.65) (which remains valid if $\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}$ is replaced by $\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}/2}$). To estimate the second term and the third term on the right-hand-side of (6.71), we use (6.66) (with ξ replaced by $\xi/2$). Finally, to estimate the last term on the right-hand-side of (6.71), we may use (6.63) to obtain

$$\mathfrak{e}\|\tau^{\frac{1}{2}}\sigma\mathbf{1}_{\tau\geq\xi}u\|^2\leq\mathfrak{e}\|\tau^{\frac{1}{2}}\sigma\zeta u\|^2\leq C\,(\|g\|^2+\|\zeta u\|^2+\|\zeta' u\|^2)\leq\hat{C}\,\|g\|^2\,.$$

Consequently, by (6.71) we have (6.73) which when substituted into (6.72) yields (6.67).

Note, for future use, that the proof provides us, for sufficiently small $\mathfrak{e}_0$,

$$\|\tau^{\frac{1}{2}}\zeta u\|^2\leq 2\mathcal{G}_2\leq C\,\|g\|^2\,. \tag{6.74}$$

Estimation of $\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\sigma^2(\hat{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\|$
To complete the proof of (6.62b), it remains necessary to show that

$$\mathfrak{e}\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}\partial_\sigma^2(\hat{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\|\leq C_a\,. \tag{6.75}$$

To this end we write

$$\hat{\mathcal{G}}:=-\langle(\hat{\mathcal{B}}_\mathfrak{e}-\lambda)u,\zeta^2u_{\sigma\sigma}\rangle=\langle(\mathcal{L}_2^+(\mathfrak{e})-\lambda)u_\sigma,\zeta^2u_\sigma\rangle-\mathfrak{e}\langle(-(1+\mathfrak{e}\theta\tilde{\chi}\tau)u_{\sigma\sigma}+i\sigma^2\tau u),\zeta^2u_{\sigma\sigma}\rangle\,.$$

We have

$$\begin{aligned}\hat{\mathcal{G}}\;&=\Big(\mathfrak{e}\|(1+\mathfrak{e}\theta\tilde{\chi}\tau)^{\frac12}\zeta u_{\sigma\sigma}\|_2^2+\|\zeta u_{\tau\sigma}\|_2^2\Big)\\&\quad+i\Big(\mathfrak{e}\|\sigma\tau^{\frac12}\zeta u_\sigma\|_2^2+\|\tau^{\frac12}(1+\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau))^{\frac12}\zeta u_\sigma\|_2^2\Big)\\&\quad-\lambda\|\zeta u_\sigma\|_2^2+2i\mathfrak{e}\langle\tau^{\frac12}\sigma\zeta u,\zeta\tau^{\frac12}u_\sigma\rangle+2\langle\zeta u_{\tau\sigma}\,,\,\zeta'u_\sigma\rangle\\&:=\hat{\mathcal{G}}_1+i\,\hat{\mathcal{G}}_2+\hat{\mathcal{G}}_3\,,\end{aligned}$$

from which we obtain as in the proof of (6.67) (recall that $\|\mathfrak{e}\theta\tau\tilde{\chi}\|_\infty\leq C\mathfrak{e}^{1-\tilde{b}}$ and $|\beta\mathfrak{e}\tau\chi(\mathfrak{e}^b\tau)|\leq C\mathfrak{e}^{1-b}$)

$$\mathfrak{e}\|\zeta u_{\sigma\sigma}\|_2^2\leq C\left(\mathfrak{e}^{-1}\|g\|_2^2+\|\zeta'u_\sigma\|_2^2+\|\zeta u_\sigma\|_2^2+\mathfrak{e}^2\|\tau^{1/2}\zeta u\|_2^2\right)\,. \tag{6.76}$$

To bound the last term on the right-hand-side we use (6.74), whereas for first two terms we obtain from (6.66) (with ξ replaced by $\xi/2$)

$$\|\zeta'u_\sigma\|_2^2+\|\zeta u_\sigma\|_2^2\leq C\|\mathbf{1}_{\tau\geq\xi/2}u_\sigma\|_2^2\leq\hat{C}\mathfrak{e}^{-1}\|g\|_2^2\,.$$

Hence,

$$\mathfrak{e}\|\mathbf{1}_{\tau\geq\mathfrak{e}^{-a}}u_{\sigma\sigma}\|_2\leq C_a\|g\|_2\,,$$

which completes the proof of (6.75). ∎

6.4.2 Effect 2

We now address an additional modification of $\mathcal{B}_\mathfrak{e}$, resulting from the third term on the right-hand-side of (3.9).

Proposition 6.8. *Let, for $\omega\in\mathbb{R}$,*

$$\tilde{\mathcal{B}}_\mathfrak{e}=\hat{\mathcal{B}}_\mathfrak{e}-2\omega\mathfrak{e}\partial_\tau\,, \tag{6.77}$$

be defined on $e^{-\mathfrak{e}\omega\tau}D(\mathcal{B}_\mathfrak{e})$. Let further, for some $0<a<a'<1$, $I_\mathfrak{e}=(\mathfrak{e}^{-a},\mathfrak{e}^{-a'})$.

Then, there exist positive C and $\mathfrak{e}_0$ such that, for every $\mathfrak{e}\in(0,\mathfrak{e}_0]$, the circle $\partial B(\Lambda_\gamma^1,\mathfrak{e}r(\mathfrak{e}))$ is included in $\rho(\tilde{\mathcal{B}}_\mathfrak{e})$, and, for $\lambda\in\partial B(\Lambda_\gamma^1,\mathfrak{e}r(\mathfrak{e}))$,

$$\begin{aligned}&\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|+\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\partial_\tau(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|\\&\quad+\mathfrak{e}^{1/2}\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\partial_\sigma(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|+\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\partial_\tau^2(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|\\&\quad+\mathfrak{e}\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\partial_\sigma^2(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|\leq\frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,,\end{aligned} \tag{6.78a}$$

$$\mathfrak{e}^{-a}\|\mathbf{1}_{\tau\in I_\mathfrak{e}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| + \mathfrak{e}^{-a/2}\|\mathbf{1}_{\tau\in I_\mathfrak{e}}\partial_\tau(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|$$
$$+ \mathfrak{e}^{1/2}\|\mathbf{1}_{\tau\in I_\mathfrak{e}}\partial_\sigma(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| + \|\mathbf{1}_{\tau\in I_\mathfrak{e}}\partial_\tau^2(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|$$
$$+ \mathfrak{e}\|\mathbf{1}_{\tau\in I_\mathfrak{e}}\partial_\sigma^2(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| \leq C\,, \quad (6.78b)$$

and for every $1/6<\mathfrak{a}<1/4$

$$\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\|$$
$$+ \mathfrak{e}^{1/2}\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\mathbf{1}_{|\sigma|\geq 2\mathfrak{e}^{-\mathfrak{a}}}\partial_\sigma(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| \leq \frac{C}{\mathfrak{e}^{1-2\mathfrak{a}}}\,. \quad (6.78c)$$

Proof. It can be easily verified that

$$\tilde{\mathcal{B}}_\mathfrak{e} = e^{-\mathfrak{e}\omega\tau}(\hat{\mathcal{B}}_\mathfrak{e}-\mathfrak{e}^2\omega^2)e^{\mathfrak{e}\omega\tau}\,.$$

For the first statement in (6.78a), we have

$$\begin{aligned}\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| &= \|e^{-\mathfrak{e}\omega\tau}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}(\hat{\mathcal{B}}_\mathfrak{e}-\lambda-\mathfrak{e}^2\omega^2)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}e^{\mathfrak{e}\omega\tau}\| \\ &\leq e^{2|\omega|\mathfrak{e}^{1-a'}}\|(\hat{\mathcal{B}}_\mathfrak{e}-\lambda-\mathfrak{e}^2\omega^2)^{-1}\|\,,\end{aligned}$$

and we can use (6.58) (note that (6.58) is valid in the ring

$$A_\mathfrak{e} = B(\Lambda_\gamma^1, 2\mathfrak{e}r(\mathfrak{e}))\setminus B(\Lambda_\gamma^1, \frac{1}{2}\mathfrak{e}r(\mathfrak{e}))$$

and that $\partial B(\Lambda_\gamma^1+\mathfrak{e}^2\omega^2, \mathfrak{e}r(\mathfrak{e}))\subset A_\mathfrak{e}$ for $\mathfrak{e}_0>0$ small enough) to obtain

$$\|\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-a'}}\| \leq \frac{C}{(r(\mathfrak{e})+1-\gamma)\mathfrak{e}}\,.$$

The rest of the inequalities embedded in (6.78a) are similarly obtained by using (6.58b) and (6.58c).

To bound the first term on the right-hand-side of (6.78b), we use (6.62) to obtain

$$\|\mathbf{1}_{\tau\in I_\mathfrak{e}}(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-(1-a)}}\| \leq e^{2|\omega|\mathfrak{e}^{1-a'}}\|\mathbf{1}_{\mathfrak{e}^{-a}\leq\tau}(\hat{\mathcal{B}}_\mathfrak{e}-\lambda-\mathfrak{e}^2\omega^2)^{-1}\| \leq C_a\,.$$

As

$$\partial_\tau(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1} = \partial_\tau e^{\mathfrak{e}\omega\tau}(\hat{\mathcal{B}}_\mathfrak{e}-\lambda-\mathfrak{e}^2\omega^2)^{-1}e^{-\mathfrak{e}\omega\tau} = e^{\mathfrak{e}\omega\tau}[\partial_\tau+\mathfrak{e}\omega](\hat{\mathcal{B}}_\mathfrak{e}-\lambda-\mathfrak{e}^2\omega^2)^{-1}e^{\mathfrak{e}\omega\tau}\,,$$

we may conclude, once again with the aid of (6.62), that

$$\|\mathbf{1}_{\tau\in I_\mathfrak{e}}\partial_\tau(\tilde{\mathcal{B}}_\mathfrak{e}-\lambda)^{-1}\mathbf{1}_{\tau\leq\mathfrak{e}^{-(1-a)}}\| \leq C_a\,\mathfrak{e}^{\frac{a}{2}}\,.$$

The rest of the inequalities in (6.78) can be proved in a similar manner. ∎

6.5 A linear potential estimate

We conclude this section by the following estimate, which is somewhat similar to [6, Lemma 7.5]. Let

$$\check{\mathcal{B}}_\delta = -\partial_\tau^2 + i(1+\delta)\tau - \partial_\sigma^2 \,,$$

be defined on

$$D(\check{\mathcal{B}}_\delta) = \{u \in H^2(\mathbb{R}_+^2) \cap H_0^1(\mathbb{R}_+^2) \,|\, \tau u \in L^2(\mathbb{R}_+^2)\,\}.$$

Proposition 6.9. *For any $a > 0$, there exist $\delta_0 > 0$ and $C > 0$ such that for all $\delta \in (0, \delta_0]$, $p < \frac{2}{3\,\mathrm{Re}\,\vartheta_1}$, and $\mathrm{Re}\,\lambda \leq \mathrm{Re}\,\vartheta_1 + p\delta$ we have*

$$\|\mathbf{1}_{\tau \geq \delta^{-a}}(\check{\mathcal{B}}_\delta - \lambda)^{-1}\| + \|\mathbf{1}_{\tau \geq \delta^{-a}}\partial_\tau(\check{\mathcal{B}}_\delta - \lambda)^{-1}\| \leq C \,, \tag{6.79a}$$

and

$$\|\partial_\sigma(\check{\mathcal{B}}_\delta - \lambda)^{-1}\| \leq \frac{C}{\delta^{1/2}} \,. \tag{6.79b}$$

Proof. Since the proof is similar to the proof of [6, Lemma 7.5] we bring only its outlines. We first apply the transformation $(t, s) = (1+\delta)^{1/3}(\tau, \sigma)$ and argue for $\lambda' = (1+\delta)^{-2/3}\lambda$. The operator then assumes the form $\check{\mathcal{B}}_0 = -\partial_t^2 + it - \partial_s^2$, and λ' satisfies $\mathrm{Re}\,\lambda' \leq (\mathrm{Re}\,\vartheta_1 + p\delta)(1+\delta)^{-\frac{2}{3}}$.

We next observe that

$$\|(\check{\mathcal{B}}_0 - \lambda')^{-1}(I - \Pi_1)\| + \|\nabla_{t,s}(\check{\mathcal{B}}_0 - \lambda')^{-1}(I - \Pi_1)\| \leq C \,. \tag{6.80}$$

Consequently, the proof of (6.79a) follows immediately from the decay of the Airy function v_1 and its derivative. To prove (6.79b) we begin by writing

$$(\check{\mathcal{B}}_0 - \lambda')\Pi_1 = (-\partial_s^2 + \vartheta_1 - \lambda')\Pi_1 \,.$$

Integration by parts then yields, using the fact that $\mathrm{Re}\,\lambda' < \mathrm{Re}\,\vartheta_1$ for sufficiently small δ_0,

$$\|\partial_s(\check{\mathcal{B}}_0 - \lambda')^{-1}\Pi_1\|_2^2 \leq \|(\check{\mathcal{B}}_0 - \lambda')^{-1}\Pi_1\|\,\|\Pi_1\| \,,$$

which together with (6.80) yields (6.79b) . ∎

7 Simplified operators: V2 potentials

In this section we estimate the resolvent norm of the operator whose eigenvalues were formally found in (4.8). For convenience, we use an even extension to $(\sigma, \tau) \in \mathbb{R}\times\mathbb{R}_+$, instead of considering it on $\mathbb{R}_+ \times \mathbb{R}_+$ with a Neumann boundary condition for $\sigma = 0$, as in Section 4.

7.1 Definition and preliminary estimates

We begin by defining for $\varepsilon > 0$

$$\mathcal{U}_\varepsilon = -[1+\varepsilon|\sigma|]\partial_\tau^2 - \varepsilon\partial_\sigma^2 + i\tau\,. \tag{7.1}$$

To define $\mathcal{U}_\varepsilon$ and characterize its domain, we look at the typical case $\varepsilon = 1$. We start from the bilinear form given by

$$a(u,v) = \langle u_\tau, (1+|\sigma|)v_\tau\rangle + \langle u_\sigma, v_\sigma\rangle + i\langle u, \tau v\rangle\,,$$

defined on $\mathcal{V}\times\mathcal{V}$ where

$$\mathcal{V} = \{u \in H_0^1(\mathbb{R}_+^2)\,|\,|\tau^{1/2}u \in L^2(\mathbb{R}_+^2)\,;\ |\sigma|^{1/2}u_\tau \in L^2(\mathbb{R}_+^2)\}\,,$$

is equipped with the norm

$$\|u\|_{\mathcal{V}}^2 = \|u_\sigma\|_2^2 + \|(1+|\sigma|)^{1/2}u_\tau\|_2^2 + \|(1+\tau^{1/2})u\|_2^2\,.$$

It can be easily verified that

$$|a(u,v)| \leq \|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}\,,$$

and that there exists $c>0$ such that

$$|a(u,u)| \geq c\|u\|_{\mathcal{V}}^2\,,\ \forall u \in \mathcal{V}\,.$$

It follows from the Lax-Milgram Theorem (cf. [16, 7]) that we can define $\mathcal{U}_1$ as a closed semibounded operator on $L^2(\mathbb{R}_+^2)$, whose domain is given by

$$\begin{aligned} D(\mathcal{U}_1) \ &= \{u\in\mathcal{V}, s.t\, \mathcal{V}\ni v\mapsto a(u,v) \text{ can be extended} \\ &\qquad \text{as a continuous antilinear map on } L^2(\mathbb{R}_+^2)\}\,. \end{aligned}$$

Since we consider a Dirichlet problem, we then have

$$D(\mathcal{U}_1) = \{u\in\mathcal{V},\ s.t.\,\mathcal{U}_1 u \in L^2(\mathbb{R}_+^2)\}\,. \tag{7.2}$$

Moreover (see the proof of Lemma 3.3 in [7]) the subspace $\tilde{\mathcal{V}}$ of the functions in $C^\infty(\overline{\mathbb{R}_+^2})\cap H_0^1(\mathbb{R}_+^2)$ compactly supported in $\overline{\mathbb{R}_+^2}$ is dense in $\mathcal{V}$ and in $D(\mathcal{U}_1)$ for the graph norm.

Lemma 7.1. *$\mathcal{U}_1$ has compact resolvent.*

Proof. The operator being semi-bounded it is enough to prove the compact injection of $\mathcal{V}$ into $L^2(\mathbb{R}_+^2)$. We observe that, for all $u\in\tilde{\mathcal{V}}$,

$$\begin{aligned} \|u\|_{\mathcal{V}}^2 \ &\geq \int_{\mathbb{R}_+^2}(|\sigma|\,|u_\tau(\sigma,\tau)|^2 + \tau|u(\sigma,\tau)|^2)\,d\sigma d\tau \\ &\geq \tfrac{1}{2}\int_{\mathbb{R}_+}\big(\int_{\mathbb{R}}(|\sigma|\,|u_\tau(\sigma,\tau)|^2+\tau|u(\sigma,\tau)|^2)\,d\sigma\big)\,d\tau + \tfrac{1}{2}\int_{\mathbb{R}_+^2}\tau|u(\sigma,\tau)|^2\,d\sigma d\tau \\ &\geq \tfrac{|\nu_1|}{2}\int_{\mathbb{R}_+^2}|\sigma|^{1/3}|u(\sigma,\tau)|^2 d\sigma\,d\tau + \tfrac{1}{2}\int_{\mathbb{R}_+^2}\tau|u(\sigma,\tau)|^2\,d\sigma d\tau\,, \end{aligned}$$

where we recall that $|\nu_1|$ is the first eigenvalue of the Dirichlet realization of the Airy operator $D_\tau^2+\tau$ in $\mathbb{R}^+$.
By density the inequality is true for $u\in\mathcal{V}$ and proves the continuous injection of $\mathcal{V}$ into an L^2 weighted space whose weight $(|\tau|+|\sigma|^{\frac{1}{3}})$ tends to $+\infty$ as $(|\sigma|+|\tau|)$ tends to $+\infty$. This injection combined with the fact that $\mathcal{V}\subset H^{1,loc}(\overline{\mathbb{R}_+^2})$ completes the proof of the lemma. ∎

Lemma 7.2.

$$D(\mathcal{U}_1) = \{u \in \mathcal{V}\,,\ (1+|\sigma|)u_{\tau\tau} \in L^2 \ \textit{and}\ u_{\sigma\sigma} \in L^2\}\,. \tag{7.3}$$

Proof. It is enough to establish an inequality for $u \in \tilde{\mathcal{V}}$. We begin with the identity

$$\|\mathcal{U}_1 u\|_2^2 = \|(1+|\sigma|)u_{\tau\tau} + u_{\sigma\sigma}\|_2^2 + \|\tau u\|_2^2 - 2\mathrm{Im}\,\langle u_\tau, (1+|\sigma|)u\rangle\,. \tag{7.4}$$

Then, we deduce from

$$\begin{aligned}\langle(1+|\sigma|)^{1/3}u, \mathcal{U}_1 u\rangle \ &= \|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|^{1/6})u_\sigma\|_2^2 \\ &\quad + i\|(1+|\sigma|)^{1/6}\tau^{1/2}u\|_2^2 \\ &\quad + \tfrac{1}{3}\langle \mathrm{sign}\,\sigma(1+|\sigma|)^{-2/3}u, u_\sigma\rangle\,,\end{aligned}$$

that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned}&\|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|^{1/6})u_\sigma\|_2^2 + \|(1+|\sigma|)^{1/6}\tau^{1/2}u\|_2^2 \\ &\quad \leq |\tfrac{1}{3}\langle \mathrm{sign}\,\sigma(1+|\sigma|)^{-2/3}u, u_\sigma\rangle\,| + C_\epsilon\|\mathcal{U}_1 u\|^2 + \epsilon\|(1+|\sigma|)^{1/3}u\|^2\,,\end{aligned}$$

from which we conclude

$$\begin{aligned}&\|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|^{1/6})u_\sigma\|_2^2 + \|(1+|\sigma|)^{1/6}\tau^{1/2}u\|_2^2 \\ &\quad \leq C_\epsilon(\|\mathcal{U}_1 u\|^2 + \|u\|^2) + \epsilon\|(1+|\sigma|)^{1/3}u\|^2\,.\end{aligned}$$

On the other hand, we have (see the proof of the previous lemma)

$$\|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|)^{1/6}\tau^{1/2}u\|_2^2 \geq |\nu_1|\,\|(1+|\sigma|)^{1/3}u\|_2^2\,.$$

For sufficiently small $\epsilon > 0$ the above two inequalities imply

$$\|(1+|\sigma|)^{1/3}u\|_2^2 + \|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|^{1/6})u_\sigma\|_2^2 \leq C(\|\mathcal{U}_1 u\|^2 + \|u\|_2^2)\,. \tag{7.5}$$

Returning to (7.4), we estimate the third term of the right hand side in the following manner

$$2|\langle u_\tau, (1+|\sigma|)u\rangle| \leq \|(1+|\sigma|)^{\frac{2}{3}}u_\tau\|_2^2 + \|(1+|\sigma|)^{1/3}u\|_2^2 \leq C(\|\mathcal{U}_1 u\|_2^2 + \|u\|_2^2)\,,$$

to obtain

$$\|(1+|\sigma|)u_{\tau\tau} + u_{\sigma\sigma}\|_2^2 + \|\tau u\|_2^2 \leq C(\|\mathcal{U}_1 u\|_2^2 + \|u\|_2^2)\,. \tag{7.6}$$

Finally, since

$$\mathrm{Re}\,\langle(1+|\sigma|)u_{\tau\tau}, u_{\sigma\sigma}\rangle = \|[1+|\sigma|]^{1/2}u_{\tau\sigma}\|_2^2 + \mathrm{Re}\,\langle u_{\tau\sigma}, \mathrm{sign}\,\sigma\, u_\tau\rangle\,,$$

(where we have used the fact that u_σ and $u_{\sigma\sigma}$ vanish on $\partial\mathbb{R}^2_+$ when $u \in \tilde{\mathcal{V}}$) we may conclude that

$$\|(1+|\sigma|)u_{\tau\tau} + u_{\sigma\sigma}\|_2^2 \geq \|(1+|\sigma|)u_{\tau\tau}\|_2^2 + \|u_{\sigma\sigma}\|_2^2 - \|u_\tau\|_2^2\,.$$

By (7.5) and (7.6) we then obtain

$$\|(1+|\sigma|)u_{\tau\tau}\|_2^2 + \|u_{\sigma\sigma}\|_2^2 \leq C(\|\mathcal{U}_1 u\|_2^2 + \|u\|_2^2)\,,\ \forall u \in \tilde{\mathcal{V}}\,. \tag{7.7}$$

By density (7.7) is extended to $u \in D(\mathcal{U}_1)$ establishing, thereby, (7.3). ∎

7.2 Large $|\sigma|$ simplification

In the similar fashion to (6.4a) we define in $Q := \mathbb{R}_+ \times \mathbb{R}_+$ the operator

$$\mathcal{T}_\varepsilon = -[1 + \varepsilon(\sigma + \varepsilon^{-\mathfrak{a}})]\partial_\tau^2 - \varepsilon\partial_\sigma^2 + i\tau\,, \tag{7.8}$$

associated with the bilinear form given by

$$a^+(u, v) = \langle u_\tau, (1 + (\sigma + \varepsilon^{-\mathfrak{a}}))v_\tau\rangle + \varepsilon\langle u_\sigma, v_\sigma\rangle + i\langle u, \tau v\rangle\,,$$

defined on $\mathcal{V}^+ \times \mathcal{V}^+$ where

$$\mathcal{V}^+ = \{u \in H_0^1(Q)\,|\,\tau^{1/2}u \in L^2(Q)\,;\ \sigma^{1/2}u_\tau \in L^2(Q)\}\,,$$

is equipped with its natural Hilbertian norm. In the same manner we have established (7.5), we can prove that the domain of $\mathcal{T}_\varepsilon$ is

$$D(\mathcal{T}_\varepsilon) = \{u \in H^2(Q) \cap H_0^1(Q)\,|\,\tau u \in L^2(Q)\,; \sigma u_{\tau\tau} \in L^2(Q)\}\,. \tag{7.9}$$

We can also show as for $\mathcal{U}_\varepsilon$ that $\mathcal{T}_\varepsilon$ has compact resolvent.
Let

$$\Lambda_\gamma^2(\varepsilon) = \lambda_0 + \gamma\varepsilon\check{\lambda}_1$$

be given by (4.15), (4.9) and (4.13), and let $r(\varepsilon)$ satisfy (6.25). We can now state and prove

Proposition 7.3. *Let $1/4 < \mathfrak{a} < (1-q)/2$. Then, there exist positive C and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0]$, $\partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon) \subset \rho(\mathcal{T}_\varepsilon)$ and such that for $\lambda \in \partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon)$*

$$\|(\mathcal{T}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}}\,, \tag{7.10a}$$

$$\|\partial_\tau\,(\mathcal{T}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}}\,, \tag{7.10b}$$

$$\|\partial_\tau^2\,(\mathcal{T}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}}\,, \tag{7.10c}$$

and

$$\|\partial_\sigma(\mathcal{T}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{3/2-\mathfrak{a}}}\,. \tag{7.10d}$$

Proof. The proof is similar to the proof of Proposition 6.1, and we therefore bring only its outlines. We begin by defining the partition of unity (6.6), S_k as in (6.7) and $\mathcal{T}_k$ as in (6.8) with $\mathfrak{e}$ replaced by ε, where we recall that $\frac{1}{6} < \mathfrak{b} < \mathfrak{a}$. Then, we set

$$\mathcal{R}_{\mathcal{T}}^{app} = \sum_{k=0}^{\infty} \phi_k^\varepsilon(\mathcal{T}_k - \lambda)^{-1}\phi_k^\varepsilon\,, \tag{7.11}$$

yielding

$$(\mathcal{T}_\varepsilon - \lambda)\mathcal{R}_{\mathcal{T}}^{app} = I + \mathcal{E}_{\mathcal{T}}\,, \tag{7.12a}$$

where

$$\mathcal{E}_{\mathcal{T}} = -\sum_{k=0}^{\infty} \varepsilon[\partial_\sigma^2, \phi_k^\varepsilon](\mathcal{T}_k - \lambda)^{-1}\phi_k^\varepsilon\,. \tag{7.12b}$$

We now prove that $\mathcal{E}_{\mathcal{T}} \to 0$ as $\varepsilon \to 0$. To this end we set

$$(\mathcal{T}_k - \lambda)w = g$$

for $g \in L^2(S_k)$ and $w \in D(\mathcal{T}_k)$. As in the derivation of (6.15) and (6.16) we write

$$(-[1+\varepsilon(k\varepsilon^{-\mathfrak{b}}+\varepsilon^{-\mathfrak{a}})]\partial_\tau^2 - \varepsilon\partial_\sigma^2 + i\tau - \lambda)w = g + \varepsilon(\sigma - k\varepsilon^{-\mathfrak{b}} + \varepsilon^{-\mathfrak{a}})w_{\tau\tau}\,.$$

This allows us to conclude by extending the proof of (5.55), using a dilation in the τ variable with $\mathfrak{e}$ replaced by ε and $\beta = 0$, to get, under our assumption that λ is $\mathcal{O}(\varepsilon)$-close to λ_0,

$$\|w\|_2 \leq \frac{C}{\varepsilon^{1-\mathfrak{a}} + k\varepsilon^{1-\mathfrak{b}}}(\|g\|_2 + \varepsilon^{1-\mathfrak{b}}\|w_{\tau\tau}\|_2)\,. \tag{7.13}$$

As

$$\mathrm{Re}\,\langle(\mathcal{T}_k-\lambda)w, w\rangle = \|[1+\varepsilon(\sigma+\varepsilon^{-\mathfrak{a}})]^{\frac{1}{2}}w_\tau\|_2^2 + \varepsilon\|w_\sigma\|_2^2 - \mathrm{Re}\,\lambda\|w\|_2^2\,,$$

we immediately obtain

$$\|[1+\varepsilon(\sigma+\varepsilon^{-\mathfrak{a}})]^{\frac{1}{2}}w_\tau\|_2 \leq C(\|w\|_2 + \|g\|_2)\,. \tag{7.14}$$

Furthermore, as $w_\sigma|_{\tau=0} \equiv 0$, we have

$$\begin{aligned} -\mathrm{Re}\,\langle(\mathcal{T}_k-\lambda)w, w_{\tau\tau}\rangle \;&= \|[1+\varepsilon(|\sigma|+\varepsilon^{-\mathfrak{a}})]^{\frac{1}{2}}w_{\tau\tau}\|_2^2 + \varepsilon\|w_{\tau\sigma}\|_2^2 \\ &\quad -\mathrm{Im}\,\langle w_\tau, (1+\varepsilon(\sigma+\varepsilon^{-\mathfrak{a}}))w\rangle - \mathrm{Re}\,\lambda\|w_\tau\|_2^2\,. \end{aligned} \tag{7.15}$$

Consequently, by (7.14) we obtain

$$\|w_{\tau\tau}\|_2 \leq C\big([1+\varepsilon^{1-\mathfrak{a}}+k\varepsilon^{1-\mathfrak{b}}]^{1/2}\|w\|_2 + \|g\|_2\big)\,,$$

which when substituted into (7.13) yields

$$\|(\mathcal{T}_k-\lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}+k\varepsilon^{1-\mathfrak{b}}}\,. \tag{7.16}$$

Note, for future use, that we may conclude in addition

$$\|w_{\tau\tau}\|_2 \leq C\big(\|w\|_2 + \|g\|_2\big)\,,$$

and hence also that

$$\|\partial_\tau^2(\mathcal{T}_k-\lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}+k\varepsilon^{1-\mathfrak{b}}}\,. \tag{7.17}$$

We now proceed as in the proof of Proposition 6.1 to show that $\mathcal{E}_{\mathcal{T}} \to 0$ as $\varepsilon \to 0$. Then, as

$$\|(\mathcal{T}_\varepsilon-\lambda)^{-1}\| \leq C\|\mathcal{R}_{\mathcal{T}}^{app}\| \leq C\sup_{k\geq 0}\|(\mathcal{T}_k-\lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-2\mathfrak{a}}}\,,$$

we have established (7.10a). The proofs of (7.10b) and (7.10c) are now respectively deduced from (7.14) and (7.17), and the proof of (7.10d) follows from the identity

$$\mathrm{Re}\,\langle(\mathcal{T}_\varepsilon-\lambda)w, w\rangle = \|(1+\varepsilon|\sigma|)^{1/2}w_\tau\|_2^2 + \varepsilon\|w_\sigma\|_2^2 - \mathrm{Re}\,\lambda\|w\|_2^2\,,$$

which holds for every $w \in D(\mathcal{T}_\varepsilon)$. ∎

7.3 Second simplified operator

Let Λ_γ^2 be given by (4.15), and let $r(\varepsilon)$ satisfy (6.25).

Proposition 7.4. *There exist positive C and $\varepsilon_0 > 0$ such that for all ε in $(0, \varepsilon_0]$, $\partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon)$ is included in $\rho(\mathcal{U}_\varepsilon)$ and*

$$\|(\mathcal{U}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{(r(\varepsilon) + 1 - \gamma)\varepsilon}, \, \forall \lambda \in \partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon) . \tag{7.18}$$

Proof. Let η and $\zeta_\pm$ be defined by (6.27) and (6.28) respectively. Let $1/4 < \mathfrak{a} < (1-q)/2$. Next, let $S_N = (-2\varepsilon^{-\mathfrak{a}}, 2\varepsilon^{-\mathfrak{a}}) \times \mathbb{R}_+$ and $\mathcal{T}_N$ denote the operator associated with the differential operator given by (7.1) with domain

$$D(\mathcal{T}_N) = \{u \in H^2(S_N) \cap H_0^1(S_N) \,|\, \tau u \in L^2(S_N)\} .$$

Let further $S_D^+ = (\varepsilon^{-\mathfrak{a}}, +\infty) \times \mathbb{R}_+$, $S_D^- = (-\infty, -\varepsilon^{-\mathfrak{a}}) \times \mathbb{R}_+$, and $\mathcal{T}_D^\pm$ denote the operator associated with the differential operator given by (6.1), whose domain can be characterized as

$$D(\mathcal{T}_D^\pm) = \{u \in H^2(S_D^\pm) \cap H_0^1(S_D^\pm) \,|\, \tau u \text{ and } \sigma u_{\tau\tau} \in L^2(S_D^\pm)\} .$$

We can now define the approximate resolvent

$$\mathcal{R}_\mathcal{T}^{app} = \eta_\varepsilon(\mathcal{T}_N - \lambda)^{-1}\eta_\varepsilon + \zeta_\varepsilon^-(\mathcal{T}_D^+ - \lambda)^{-1}\zeta_\varepsilon^- + \zeta_\varepsilon^+(\mathcal{T}_D^+ - \lambda)^{-1}\zeta_\varepsilon^+ . \tag{7.19}$$

By (6.5) we have

$$\|(\mathcal{T}_D^\pm - \lambda)^{-1}\| + \varepsilon^{1/2}\|\partial_\sigma(\mathcal{T}_D^\pm - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{1-\mathfrak{a}}} . \tag{7.20}$$

We seek an estimate for $\|(\mathcal{T}_N - \lambda)^{-1}\|$. Let $w \in D(\mathcal{T}_N)$ and $g \in L^2(S_N)$ satisfy

$$(\mathcal{T}_N - \lambda)w = g . \tag{7.21}$$

Applying the projection Π_1, given by (3.28), to the above balance yields

$$\begin{aligned}(\mathcal{L}^+ - \varepsilon\partial_\sigma^2 - \lambda)\Pi_1 w &= \Pi_1 g + \varepsilon\,|\sigma|\,\Pi_1\partial_{\tau\tau}^2 w \\ &= \Pi_1 g + \varepsilon|\sigma|(i\Pi_1(\tau w) - \lambda_0\Pi_1 w)\end{aligned} . \tag{7.22}$$

Let

$$\tilde{\mathcal{T}}_N = \mathcal{L} + \varepsilon(-\partial_\sigma^2 + \theta_0|\sigma|) ,$$

where θ_0 is given by (4.12). We now rewrite (7.22) in the form

$$(\tilde{\mathcal{T}}_N - \lambda)\Pi_1 w = \Pi_1 g - i\varepsilon|\sigma|\Pi_1((\tau - e^{-i\pi/3}\tau_m)w) . \tag{7.23}$$

From the definition of τ_m in (3.24) it follows that

$$\Pi_1\big((\tau - e^{-i\pi/3}\tau_m)w\big) = \Pi_1\big((\tau - e^{-i\pi/3}\tau_m)(I - \Pi_1)w\big) .$$

We can thus conclude that

$$\big\|\sigma\Pi_1\big((\tau - e^{-i\pi/3}\tau_m)w\big)\big\| \leq C\varepsilon^{-\mathfrak{a}}\|(I - \Pi_1)w\|_2 . \tag{7.24}$$

In the same manner as it is established in [6, Lemma 7.1] or in Proposition 5.13 we have

$$\|(\tilde{\mathcal{T}}_N-\lambda)^{-1}\Pi_1\|\leq\frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\quad;\quad\|(\tilde{\mathcal{T}}_N-\lambda)^{-1}(I-\Pi_1)\|\leq C\,. \tag{7.25a,b}$$

Applying (7.25a) to (7.23) yields, with the aid of (7.24),

$$\|\Pi_1w\|_2\leq\frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}(\|g\|_2+\varepsilon^{1-\mathfrak{a}}\|(I-\Pi_1)w\|_2)\,. \tag{7.26}$$

We now apply $I-\Pi_1$ to (7.21) to obtain

$$(\mathcal{L}^+-\varepsilon\partial_\sigma^2-\lambda)(I-\Pi_1)w=(I-\Pi_1)g+\varepsilon|\sigma|(I-\Pi_1)w_{\tau\tau}\,. \tag{7.27}$$

Since $I-\Pi_1$ is bounded, we have

$$\|\sigma(I-\Pi_1)w_{\tau\tau}\|_2\leq C\varepsilon^{-\mathfrak{a}}\|w_{\tau\tau}\|_2\,.$$

We can now obtain, using (7.15), (7.17), and the fact that $|\sigma|\leq 2\varepsilon^{-2\mathfrak{a}}$ in S_N, that

$$\|w_{\tau\tau}\|_2\leq C(\|w\|_2+\|g\|_2)\,. \tag{7.28}$$

Consequently,

$$\|\sigma(I-\Pi_1)w_{\tau\tau}\|_2\leq C\varepsilon^{-\mathfrak{a}}(\|w\|_2+\|g\|_2)\,. \tag{7.29}$$

We now apply [6, Eq. (7.16)] to (7.27) to obtain, with the aid of the above inequality,

$$\|(I-\Pi_1)w\|_2\leq C(\|g\|_2+\varepsilon^{1-\mathfrak{a}}\|w\|_2)\,. \tag{7.30}$$

Substituting the above into (7.26) yields

$$\|\Pi_1w\|_2\leq\frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\|g\|_2+\frac{C\varepsilon^{1-2\mathfrak{a}}}{r(\varepsilon)+1-\gamma}\|w\|_2\,.$$

Together with (6.43) (recall that $\mathfrak{a}<(1-q)/2$, and r satisfies (6.25)), this yields

$$\|(\mathcal{T}_N-\lambda)^{-1}\|\leq\frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\,. \tag{7.31}$$

Note, for future use, that together with (7.28) the above inequality implies

$$\|\partial_\tau^2\,(\mathcal{T}_N-\lambda)^{-1}\|\leq\frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\,. \tag{7.32}$$

We also need an estimate for $\partial_\sigma(\mathcal{T}_N-\lambda)^{-1}$. To this end we rewrite (7.23) in the following manner

$$\varepsilon\Big(-\partial_\sigma^2+i\theta_0|\sigma|-\frac{\lambda-\lambda_0}{\varepsilon}\Big)\Pi_1w=\Pi_1g-i\varepsilon|\sigma|\Pi_1\big((\tau-e^{-i\pi/3}\tau_m)w\big)\,.$$

Taking the inner product with Π_1w then yields for the real part

$$\varepsilon\|\Pi_1w_\sigma\|_2^2\leq C\left(\varepsilon\|\Pi_1w\|_2^2+\|\Pi_1w\|_2(\|\Pi_1g\|_2+\varepsilon^{1-\mathfrak{a}}\|(I-\Pi_1)w\|_2\right)\,.$$

Using (7.30) we then obtain

$$\|\Pi_1 w_\sigma\|_2 \leq C(\|\Pi_1 w\|_2 + \varepsilon^{\frac{3}{2}-2\mathfrak{a}}\|w\|_2 + \varepsilon^{-1}\|g\|_2)\,,$$

from which we deduce, using (7.31) and the fact that $\mathfrak{a} < (1-q)/2$,

$$\|\Pi_1 w_\sigma\|_2 \leq \frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\|g\|_2\,. \tag{7.33}$$

We now take the real part of the inner product of (7.27) with $(I-\Pi_1)w$, to obtain with the aid of (7.29) and (7.30),

$$\begin{aligned}
&\|((I-\Pi_1)w)_\tau\|_2^2 + \varepsilon\|(I-\Pi_1)w_\sigma\|_2^2 \\
&\qquad \leq C\left(\varepsilon\|(I-\Pi_1)w\|_2^2 + \|(I-\Pi_1)w\|_2\left(\|g\|_2 + \varepsilon^{1-\mathfrak{a}}\|w\|_2\right)\right) \\
&\qquad \leq \hat{C}\left(\|g\|_2 + \varepsilon^{1-\mathfrak{a}}\|w\|_2\right)^2\,.
\end{aligned}$$

Hence we have obtained

$$\|(I-\Pi_1)w_\sigma\|_2 \leq C(\varepsilon^{-1/2}\|g\|_2 + \varepsilon^{1/2-\mathfrak{a}}\|w\|_2)\,,$$

which combined with (7.31) and (7.33) yields

$$\|\partial_\sigma(\mathcal{T}_N-\lambda)^{-1}\| \leq \frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\,. \tag{7.34}$$

We may now proceed to obtain (7.18) in precisely the same manner as in the proof of Proposition 6.2. ∎

We now return to the problem appearing in Section 4. The operator is defined on the quarter plane $Q = \mathbb{R}_+ \times \mathbb{R}_+$ with a Dirichlet-Neumann condition. Hence, we consider $\mathcal{Q}_\varepsilon$ to be defined by (7.1) (via a Lax-Milgram theorem) and whose domain can be characterized as

$$D(\mathcal{Q}_\varepsilon) = \{u \in H^2(Q)\,|\,u(0,\sigma)=0\,;\,\partial_\sigma u(0,\tau)=0\,;\,\tau u \text{ and } \sigma u_{\tau\tau} \in L^2(Q)\}\,. \tag{7.35}$$

We can now make the following statement

Proposition 7.5. *There exist positive C and ε_0 such that for every $\varepsilon \in (0,\varepsilon_0]$, $\partial B(\Lambda_\gamma^2, r\varepsilon)$ belongs to $\rho(\mathcal{Q}_\varepsilon)$ and*

$$\|(\mathcal{Q}_\varepsilon-\lambda)^{-1}\| \leq \frac{C}{(r(\varepsilon)+1-\gamma)\varepsilon}\,,\ \forall\lambda\in\partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon)\,. \tag{7.36}$$

The proof follows immediately from Proposition 7.4 and the fact that $\mathcal{U}_\varepsilon$ is an even extension of $\mathcal{Q}_\varepsilon$.
We shall continue to obtain results for $\mathcal{U}_\varepsilon$. All of them, by the same token, are valid for $\mathcal{Q}_\varepsilon$ as well.
Another consequence Proposition 7.4 is the following:

Proposition 7.6. *Under the conditions of Proposition 7.4, for every* $1/4 < \mathfrak{a} < (1-q)/2$ *there exists* $C_\mathfrak{a} > 0$ *and* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$,

$$\|\mathbf{1}_{|\sigma|\geq 2\varepsilon^{-\mathfrak{a}}}(\mathcal{U}_\varepsilon - \lambda)^{-1}\| + \varepsilon^{1/2}\|\mathbf{1}_{|\sigma|\geq 2\varepsilon^{-\mathfrak{a}}}\partial_\sigma(\mathcal{U}_\varepsilon - \lambda)^{-1}\| \leq \frac{C_\mathfrak{a}}{\varepsilon^{1-\mathfrak{a}}}\,. \tag{7.37}$$

Proof. The proof is identical with the proof of Proposition 6.4 and is therefore omitted. ∎

We continue by proving another estimate:

Proposition 7.7. *Under the conditions of Proposition 7.4 we have*

$$\|\partial_\tau^2(\mathcal{U}_\varepsilon - \lambda)^{-1}\| + \|\partial_\tau(\mathcal{U}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{r(\varepsilon)\varepsilon}\,, \tag{7.38a}$$

and

$$\|\partial_\sigma^2(\mathcal{U}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{\varepsilon^{2-\mathfrak{a}}}\,. \tag{7.38b}$$

Proof. The proof of (7.38a) follows from (7.10b), (7.10c), and (7.32). To obtain (7.38b) we use the identity

$$-\mathrm{Re}\,\langle w_{\sigma\sigma}, (\mathcal{U}_\varepsilon - \lambda)w\rangle = \|w_{\tau\sigma}\|_2^2 + \varepsilon\,\|w_{\sigma\sigma}\|_2^2 + \varepsilon\,\mathrm{Im}\,\langle w_\sigma, w_{\tau\tau}\mathrm{sign}\,\sigma\rangle - \mathrm{Re}\,\lambda\|w_\sigma\|_2^2\,.$$

Hence,

$$\varepsilon\,\|w_{\sigma\sigma}\|_2^2 \leq C\left(\|w_\sigma\|_2^2 + \varepsilon^2\|w_{\tau\tau}\|_2^2 + \varepsilon^{-1}\|g\|_2^2\right)\,,$$

which together with (7.38a) and (7.10d) leads to

$$\|w_{\sigma\sigma}\|_2 \leq \frac{C}{\varepsilon^{1/2}}(\varepsilon^{\mathfrak{a}-\frac{3}{2}} + \frac{1}{r(\varepsilon)} + \varepsilon^{-1/2})\,\|g\|_2)\,.$$

∎

As in the previous section we also need the following estimate:

Proposition 7.8. *Let* $0 < a$. *Then, there exist positive* C_a *and* ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0]$, $\partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon)$ *belongs to* $\rho(\mathcal{U}_\varepsilon)$ *and for all* $\lambda \in \partial B(\Lambda_\gamma^2, r(\varepsilon)\varepsilon)$,

$$\|\mathbf{1}_{\tau\geq\varepsilon^{-a}}(\mathcal{U}_\varepsilon-\lambda)^{-1}\|+\varepsilon^{a/2}\|\mathbf{1}_{\tau\geq\varepsilon^{-a}}\partial_\tau(\mathcal{U}_\varepsilon-\lambda)^{-1}\|+\varepsilon^{a/2+1/2}\|\mathbf{1}_{\tau\geq\varepsilon^{-a}}\partial_\sigma(\mathcal{U}_\varepsilon-\lambda)^{-1}\| \leq C_a\varepsilon^a\,, \tag{7.39a}$$

and

$$\|\mathbf{1}_{\tau\geq\varepsilon^{-a}}\partial_\tau^2(\mathcal{U}_\varepsilon - \lambda)^{-1}\| + \varepsilon\|\mathbf{1}_{\tau\geq\varepsilon^{-a}}\partial_\sigma^2(\mathcal{U}_\varepsilon - \lambda)^{-1}\| \leq C_a\,. \tag{7.39b}$$

Proof. As in the proof of Proposition 6.7, let ζ_+ be given by (6.27). Let further $\zeta_\xi(\tau) = \zeta_+(2\tau/\xi)$. Let $w \in D(\mathcal{U}_\varepsilon)$ and $g \in L^2(\mathbb{R}_+^2)$ satisfy

$$(\mathcal{U}_\varepsilon - \lambda)w = g\,.$$

As, with $\zeta = \zeta_\xi$,

$$
\begin{array}{l}
\operatorname{Re}\langle(\mathcal{U}_\varepsilon - \lambda)w, \zeta^2 w\rangle + \operatorname{Im}\langle(\mathcal{U}_\varepsilon - \lambda)w, \zeta^2 w\rangle \\
\quad = \|[1+\varepsilon|\sigma|]^{1/2}\partial_\tau(\zeta w)\|^2 \\
\qquad -(\operatorname{Im}\lambda + \operatorname{Re}\lambda)\|\zeta w\|^2 + \varepsilon\|\zeta\partial_\sigma w\|^2 - \|w\zeta'\|^2 \\
\qquad +2\operatorname{Im}\langle w\zeta', \partial_\tau(\zeta w)\rangle + \|\tau^{1/2}\zeta w\|^2\,,
\end{array}
\tag{7.40}
$$

we get first

$$
\|\tau^{1/2}\zeta_\xi w\|_2^2 - (\operatorname{Im}\lambda + \operatorname{Re}\lambda)\|\zeta_\xi w\|_2^2 \le 2\|\zeta_\xi' w\|_2^2 + 2\|\zeta_\xi w\|_2\|\zeta_\xi g\|_2\,.
$$

The proof then continues along the same lines of the proof of Proposition 6.7 and is therefore omitted. ∎

8 Upper bound

8.1 Goals and notation

Let $x_0 \in \mathcal{S}^m$ (resp. $x_0 \in \hat{\mathcal{S}}^m$), where $\mathcal{S}^m$ (resp. $\hat{\mathcal{S}}^m$) is defined by (1.10) –(1.12) (resp.(1.15)–(1.16)), for type V1 (resp. V2) potentials. We have proven in Proposition 3.4 (resp. Proposition 4.1) the existence of an approximate eigenvalue $\hat{\Lambda}^1(h,x_0)$ (resp. $\hat{\Lambda}^2(h,x_0)$) with a corresponding approximate eigenstate localized (as $h \to 0$) near x_0. In this section, we prove the existence of an eigenvalue inside the disk $B(\hat{\Lambda}^i(h,x_0), \hat{r}_i(h)h^{k_i})$ where $i = 1$ (resp. $i = 2$) corresponds to potentials of type V1 (resp. V2), $k_1 = 4/3$ and $k_2 = 10/9$, $\hat{\Lambda}_1(h,x_0)$ (resp. $\hat{\Lambda}^2(h,x_0)$) is defined, for $c(x_0) > 0$, in (3.31) (resp. in (4.16)) by:

$$
\begin{aligned}
\hat{\Lambda}^1(h,x_0) &= iV(x_0) + (c(x_0)h)^{2/3}(\lambda_0 + \mathfrak{e}(h)\lambda_1)\,,\\
\hat{\Lambda}^2(h,x_0) &= iV(x_0) + (c(x_0)h)^{2/3}(\lambda_0 + \varepsilon(h)\check{\lambda}_1)\,,
\end{aligned}
$$

and for $c(x_0) < 0$ by

$$
\begin{aligned}
\hat{\Lambda}^1(h,x_0) &= iV(x_0) + (-c(x_0)h)^{2/3}\overline{(\lambda_0 + \mathfrak{e}(h)\lambda_1)}\,,\\
\hat{\Lambda}^2(h,x_0) &= iV(x_0) + (-c(x_0)h)^{2/3}\overline{(\lambda_0 + \varepsilon(h)\check{\lambda}_1)}\,.
\end{aligned}
$$

We also recall from (3.10) and (4.6) that

$$
\mathfrak{e}(h) = (2^{-1/2}\,\alpha_m^{1/2} J_m^{-\frac{5}{6}})\,h^{2/3} \ \text{ and } \ \varepsilon(h) = [2^6\hat{\alpha}_m^6 J_m^{-8}]^{1/9}\,h^{4/9}\,,
$$

and keep in mind Remark 3.1 and that $|c(x_0)| = J_m$.
The functions $\hat{r}_i(h)$ ($i = 1,2$) are determined from $r_1(\mathfrak{e})$ and $r_2(\varepsilon)$, respectively appearing in Sections 6 and 7, via the relations

$$
J_m^{-\frac{2}{3}} h^{k_1 - 2/3}\hat{r}_1(h) = \mathfrak{e}(h) r_1(\mathfrak{e}(h)) \text{ and } J_m^{-\frac{2}{3}} h^{k_2 - 2/3}\hat{r}_2(h) = \varepsilon(h) r_2(\varepsilon(h))\,. \tag{8.1}
$$

We now choose

$$
\hat{r}_1(h) = h^{\hat{q}_1} \ \text{ and } \hat{r}_2(h) = h^{\hat{q}_2}\,, \tag{8.2}
$$

where,
$$\hat{q}_1 = \frac{2}{3}q \quad \text{and} \quad \hat{q}_2 = \frac{4}{9}q\,. \tag{8.3}$$
With this choice and from (3.10) and (4.6), we get
$$r_1(\mathfrak{e}) = \left(2^{\frac{1+q}{2}} J_m^{\frac{(1+5q)}{6}}\, \alpha_m^{-\frac{1+q}{2}}\right) \mathfrak{e}^q\,,\; r_2(\varepsilon) = \left(J_m^{\frac{2+8q}{9}}\, 2^{-\frac{2(1+q)}{3}}\, \hat{\alpha}_m^{-\frac{2(1+q)}{3}}\right) \varepsilon^q\,. \tag{8.4}$$
Recall that $q < 1/6$ (in both Sections 6 and 7) and hence
$$\hat{q}_1 < 1/9 \text{ and } \hat{q}_2 < 2/27\,. \tag{8.5a,b}$$

Using the preliminary estimates established in the previous sections, we obtain in this section a bound on the resolvent norm of $\mathcal{A}_h$ given by (1.1) on a suitable circle centered at $\hat{\Lambda}^i(h)$. The method is similar to the one used in [18, 6], i.e., we obtain localized approximations of the resolvent $(\mathcal{A}_h - \lambda)^{-1}$, that facilitate the application of the various estimates obtained in Sections 6 and 7. The combination of these estimates with the construction of quasi-modes leads to the proof of existence of an eigenvalue.

8.2 Refined partition of unity

We start from the partition of unity of size h^ϱ constructed in paragraph 2.3.1 (which we denote in this section by $(\bar{\chi}_{j,h}, \bar{\zeta}_{k,h})$ to avoid the confusion with a future notation), with $j \in \mathcal{J}_i$, $k \in \mathcal{J}_\partial$. Note that $\bar{\chi}_{j,h}$ and $\bar{\zeta}_{k,h}$ are respectively supported in $B(a_j, h^\varrho)$ or $B(b_k, h^\varrho)$. Recall also the decomposition of $\mathcal{J}_\partial$ in Section 2 into three disjoint subsets $\mathcal{J}_\partial^c$, $\mathcal{J}_\partial^D$ and $\mathcal{J}_\partial^N$, so that all corners are included in $\mathcal{J}_\partial^c$. To simplify our resolvent construction, we impose in addition the condition
$$\mathcal{S} \subset \bigcup_{k\in\mathcal{J}_\partial} \{b_k\}\,. \tag{8.6}$$
When the potential is of type V1, we split further $\mathcal{J}_\partial^D$ by setting
$$\mathcal{J}_\partial^s = \{k \in \mathcal{J}_\partial^D \,|\, b_k \in \mathcal{S}\} \text{ and } \mathcal{J}_\partial^r = \mathcal{J}_\partial^D \setminus \mathcal{J}_\partial^s\,.$$
When the potential is of type V2, we set
$$\mathcal{J}_\partial^s = \{k \in \mathcal{J}_\partial^c \,|\, b_k \in \mathcal{S}\} \text{ and } \mathcal{J}_\partial^r = \mathcal{J}_\partial^D\,.$$
We further set
$$\mathcal{J}_\partial^{s,0} = \{k \in \mathcal{J}_\partial^s \,|\, V(b_k) \neq V(x_0)\}\,.$$
As in [6] (Subsection 7.3) we need to use two different scales (or disk sizes), i.e. as before h^ϱ for $k \in (\mathcal{J}_\partial \setminus \mathcal{J}_\partial^s) \cup \mathcal{J}_\partial^{s,0}$ or $j \in \mathcal{J}_i$ but now $h^{\varrho_\perp}$ for $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$ where
$$\frac{2}{3} > \varrho > \frac{1}{3} > \varrho_\perp > 0\,. \tag{8.7}$$

We now proceed in two steps.
We first construct a finite (independent of h) partition of unity of size $h^{\varrho_\perp}$, $\check\xi_h, \check\zeta_{k,h}$ with $k \in \mathcal{J}_\partial^s$ such that

$$\check\xi_h^2 + \sum_{k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}} \check\zeta_{k,h}^2 = 1 \text{ in } \Omega\,, \tag{8.8a}$$

with

$$\begin{aligned}\check\zeta_{k,h} \equiv 1 \text{ in } B(b_k, h^{\varrho_\perp}/2)\,,\, \check\zeta_{k,h} \equiv 0 \text{ in } \Omega\setminus B(b_k, h^{\varrho_\perp})\,,\\ |\nabla\check\zeta_{k,h}| + h^{\varrho_\perp}|D^2\check\zeta_{k,h}| \leq C\, h^{-\varrho_\perp}\,,\ \forall k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}\,.\end{aligned} \tag{8.8b}$$

and

$$|\nabla\check\xi_h| + h^{\varrho_\perp}|D^2\check\xi_h| \leq C\, h^{-\varrho_\perp}\,. \tag{8.8c}$$

Then, we set for $k\in\mathcal{J}_\partial$

$$\tilde\zeta_{k,h} = \bar\zeta_{k,h}\,\check\xi_h \quad \tilde\chi_{j,h} = \bar\chi_{j,h}\,\check\xi_h\,.$$

To satisfy the Neumann boundary condition on $\partial\Omega_N$, and for later reference, we introduce an additional condition

$$\frac{\partial\check\xi_h}{\partial\nu}\Big|_{\partial\Omega} = 0 \quad ; \quad \frac{\partial\bar\zeta_{k,h}}{\partial\nu}\Big|_{\partial\Omega} = 0\,. \tag{8.9}$$

As $\varrho>\varrho_\perp$, we have for sufficiently small $h_0>0$, $\tilde\zeta_{k,h}\equiv 0$ for $k\in\mathcal{J}^s$ and $h\in(0,h_0]$. Note however that we may have for sufficiently small h, $j\in\mathcal{J}_i$, and $k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}$ the inclusion $B(a_j,h^\varrho)\subset B(b_k,h^{\varrho_\perp}/2)$ and hence $\tilde\chi_{j,h}\equiv 0$. A similar observation can be made for $k\in(\mathcal{J}_\partial\setminus\mathcal{J}_\partial^s)\cup\mathcal{J}_\partial^{s,0}$. We thus define

$$\begin{aligned}&\tilde{\mathcal{J}}_i = \{j\in\mathcal{J}_i\,|\,\tilde\chi_{j,h}\not\equiv 0\}\,, \qquad &&\tilde{\mathcal{J}}_\partial^N = \{k\in\mathcal{J}_\partial^N\,|\,\tilde\zeta_{k,h}\not\equiv 0\}\,,\\ &\tilde{\mathcal{J}}_\partial^D = \{k\in\mathcal{J}_\partial^D\,|\tilde\zeta_{k,h}\not\equiv 0\}\,, &&\tilde{\mathcal{J}}_\partial^r = \{k\in\mathcal{J}_\partial^r\,|\,\tilde\zeta_{k,h}\not\equiv 0\}\,.\end{aligned}$$

Clearly, we have

$$\sum_{k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}}\check\zeta_{k,h}^2 + \sum_{k\in(\tilde{\mathcal{J}}_\partial^c\setminus\mathcal{J}_\partial^s)\cup\mathcal{J}_\partial^{s,0}}\tilde\zeta_{k,h}^2 + \sum_{k\in\tilde{\mathcal{J}}_\partial^N\cup\tilde{\mathcal{J}}_\partial^r}\tilde\zeta_{k,h}^2 + \sum_{j\in\tilde{\mathcal{J}}_i}\tilde\chi_{j,h}^2 = 1 \text{ in } \Omega\,.$$

Note that for type V1 potentials, $\mathcal{J}_\partial^c\setminus\mathcal{J}_\partial^s=\mathcal{J}_\partial^c$, whereas for type V2, $\mathcal{J}_\partial^D\setminus\mathcal{J}_\partial^s=\mathcal{J}_\partial^D$. For simplicity of notation we drop the tilde and check accents in the sequel and use $(\chi_{j,h},\zeta_{k,h})$ instead of $(\tilde\chi_{j,h},\tilde\zeta_{k,h})$ and $(\mathcal{J}_i,\mathcal{J}_\partial^D,\mathcal{J}_\partial^N,\mathcal{J}_\partial^r)$ instead of $(\tilde{\mathcal{J}}_i,\tilde{\mathcal{J}}_\partial^D,\tilde{\mathcal{J}}_\partial^N,\tilde{\mathcal{J}}_\partial^r)$. Note further that by the previous construction we have that

$$\begin{cases}|\nabla\chi_{j,h}| + h^{\varrho_\perp}|D^2\chi_{j,h}| \leq C\,h^{-\varrho_\perp} & \text{in } B(a_j,h^\varrho/2)\\ |\nabla\chi_{j,h}| + h^{\varrho}|D^2\chi_{j,h}| \leq C\,h^{-\varrho} & \text{in } B(a_j,h^\varrho)\end{cases}\,,\ \forall j\in\mathcal{J}_i\,. \tag{8.10a}$$

At the boundary, we have

$$\begin{cases}|\nabla\zeta_{k,h}| + h^{\varrho_\perp}|D^2\zeta_{k,h}| \leq Ch^{-\varrho_\perp} & \text{in } B(b_k,h^\varrho/2)\\ |\nabla\zeta_{k,h}| + h^{\varrho}|D^2\zeta_{k,h}| \leq Ch^{-\varrho} & \text{in } B(b_k,h^\varrho)\end{cases}\,. \tag{8.10b}$$

As in Section 2, each point of Ω belongs to at most N_0 disks with N_0 independent of h, and

$$\sum_j |\partial^{\hat\gamma}\chi_{j,h}(x)|^2 + \sum_k |\partial^{\hat\gamma}\zeta_{k,h}(x)|^2 \leq C_{\hat\gamma}\, h^{-2|\hat\gamma|\varrho}\,,\ \forall \hat\gamma \in \mathbb{N}^2 \text{ s.t. } |\hat\gamma| \leq 2\,. \tag{8.10c}$$

As in Section 2 once again, we introduce $\eta_{k,h} = 1_\Omega\, \zeta_{k,h}$.
Note that, as a result of (8.9), we have

$$\frac{\partial \zeta_{k,h}}{\partial \nu}\big|_{\partial\Omega} = 0\,,\ \forall k \in \mathcal{J}_\partial^N\,. \tag{8.11}$$

To be compatible with future constraints we impose from now on the following restrictions for potentials of type V1

$$\frac{13}{63} < \varrho_\perp < \frac{2}{9} \text{ and } \frac{\varrho_\perp}{2} + \frac{1}{3} < \varrho < \frac{2}{3} - \varrho_\perp\,, \tag{8.12a}$$

and for potentials of type V2

$$\frac{7}{27} < \varrho_\perp < \frac{1}{3} \text{ and } \frac{\varrho_\perp}{2} + \frac{1}{3} < \varrho < \frac{2}{3} - \frac{\varrho_\perp}{2}\,. \tag{8.12b}$$

These new conditions are, clearly, more restrictive than the ones previously given in (8.7).
As in Section 2 (see (2.20)) the approximate resolvent has the form

$$\mathcal{R}(h,\lambda) = \sum_{j\in\mathcal{J}_i(h)} \chi_{j,h}(\mathcal{A}_{j,h}-\lambda)^{-1}\chi_{j,h} + \sum_{k\in\mathcal{J}_\partial(h)} \eta_{k,h}R_{k,h}(\lambda)\eta_{k,h}\,, \tag{8.13}$$

but this time we need to estimate the localized resolvents (some of them account now for higher order terms in the Taylor expansion of V near b_k) $R_{k,h}(\lambda)$, and the remainder

$$\mathcal{E}(h,\lambda) = (\mathcal{A}_h - \lambda)\mathcal{R}(h,\lambda) - I\,, \tag{8.14}$$

for $\lambda \in \partial B(\hat\Lambda^i(h), h^{k_i}\hat r_i(h))$.

8.3 Localized resolvent estimates

8.3.1 Approximation associated with $j \in \mathcal{J}_i$ and $k \in \mathcal{J}_\partial^N$

In this case we can directly apply the estimates (2.14) and (2.17) given in Section 2, as $\partial B(\hat\Lambda^i(h),\, h^{k_i}\hat r_i(h)))$ is included in $\{\mathrm{Re}\,\lambda \leq \omega\, h^{\frac{2}{3}}\}$ for any $\omega > J_m^{\frac{2}{3}} \frac{|\nu_1|}{2}$.

8.3.2 Decomposition of $\mathcal{J}_\partial^s$

Having in mind the definition of $\mathcal{J}_\partial^{s,0}$ we further split $\mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$ in the following manner:

1. $\mathcal{J}_\partial^{s,1} = \{k \in \mathcal{J}_\partial^s \,|\, V(b_k) = V(x_0)\,;\ |\alpha(b_k)| > \alpha_m\,\}$,
2. $\mathcal{J}_\partial^{s,2} = \{k \in \mathcal{J}_\partial^s \,|\, V(b_k) = V(x_0)\,;\ |\alpha(b_k)| = \alpha_m\,\}$,

where $\alpha(x)$ is given by (1.9) for potentials of type V1 (resp. by (1.14) for potentials of type V2, where in this case α is replaced by $\hat\alpha$ and α_m by $\hat\alpha_m$).
Note that by Remark 3.1, when $V(b_k) = V(x_0)$, we have $c(b_k)c(x_0) > 0$ as x_0 and b_k belong to the same connected component of $\overline{\partial\Omega^D}$.

8.3.3 Localization associated with $k \in \mathcal{J}_\partial^{s,0}$

For potentials of type $V1$, we have for any $k \in \mathcal{J}_\partial^{s,0}$, $b_k \in \partial\Omega_D$ and we can therefore use the approximate operator $\tilde{\mathcal{A}}_{k,h}$ introduced in (2.18). Upon dilation we then use Lemma 2.7 (with $\nu = [(V(b_k) - \operatorname{Im}\lambda)]\,(J_m h)^{-\frac{2}{3}}$ and $\mu = \operatorname{Re}\lambda\,(J_m h)^{-\frac{2}{3}}$) to obtain that $\partial B(\hat{\Lambda}^1(h), h^{k_1}\hat{r}_1(h)) \subset \rho(\tilde{\mathcal{A}}_{k,h})$ and

$$\max_{k\in\mathcal{J}_\partial^{s,0}} \sup_{\lambda\in\partial B(\hat{\Lambda}^1(h),\, h^{k_1}\hat{r}_1(h))} \|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq \frac{C}{h^{2/3}}\,. \tag{8.15}$$

For potentials of type $V2$, we have for any $k \in \mathcal{J}_\partial^{s,0}$, $b_k \in \mathcal{J}_\partial^c$ and we may therefore use the approximate operator $\tilde{\mathcal{A}}_{k,h}$ introduced in (2.27). After dilation we may apply Lemma 2.9 to obtain that $\partial B(\hat{\Lambda}^2(h), h^{k_2}\hat{r}_2(h)) \subset \rho(\tilde{\mathcal{A}}_{k,h})$ and

$$\max_{k\in\mathcal{J}_\partial^{s,0}} \sup_{\lambda\in\partial B(\hat{\Lambda}^2(h),\, \hat{r}_2(h)h^{k_2})} \|(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\| \leq \frac{C}{h^{2/3}}\,. \tag{8.16}$$

8.3.4 $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$

For $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$, we need the approximation of $\mathcal{A}_h$, to be more refined than (2.18) or (2.27). We thus introduce, for potentials of type V1, (see (3.1))

$$\begin{cases} \tilde{\mathcal{A}}_{k,h} = -h^2\Delta_{s,\rho} + h^2\kappa(b_k)\partial_\rho - 2h^2\kappa(b_k)\rho\chi(h^{-2(1-\tilde{b})/3}\rho)\partial_s^2 + iV_{b_k}^{(2)}\,, \\ \mathcal{D}(\tilde{\mathcal{A}}_{k,h}) = \{u \in H^2(\mathbb{R}^2_+) \cap H^1_0(\mathbb{R}^2_+) \,|\, \rho(1+s^2)u \in L^2(\mathbb{R}^2_+)\}\,, \end{cases} \tag{8.17}$$

where

$$V_{b_k}^{(2)} = V(x_0) \pm J_m\rho + \frac{1}{2}\alpha(b_k)s^2\rho + \frac{1}{2}\hat{\beta}_k\chi(h^{-2(1-b)/3}\rho)\rho^2\,. \tag{8.18}$$

The curvilinear coordinates (s,ρ), defined in Section 3.1, are centered at b_k (see Remark 3.2). The curvature κ is approximated in (8.17) by its value at b_k and $\hat{\beta}_k = \hat{\beta}(b_k)$ is given by (3.5).
The cutoff function χ is the restriction to $\mathbb{R}_+$ of (5.2), the positive parameters b and $\tilde{b}$ satisfy the limitations set in Sections 5 and 6, i.e.,

$$\frac{1}{2} < b < \frac{3}{4} \quad ; \quad 0 < \tilde{b} < \frac{1}{2} - q\,. \tag{8.19}$$

Further restrictions for b and $\tilde{b}$ will be imposed at a later stage.

For potentials of type V2 we use the coordinates introduced in Paragraph 2.3.4, centered at the corner b_k, and consider the approximate operator

$$\begin{cases} \tilde{\mathcal{A}}_{k,h} = -h^2[(1+\tilde{\alpha}_{b_k}s)\partial_\rho^2 + \partial_s^2] + i\big(V(b_k) \pm J_m\rho\big)\,, \\ \mathcal{D}(\tilde{\mathcal{A}}_{k,h}) = \{u \in H^2(Q) \,|\, u_{\partial Q_\parallel} = 0\,;\, \partial_\nu u_{\partial Q_\perp} = 0\,;\, \rho\, u \text{ and } s\,\partial_\rho^2 u \in L^2(Q)\}\,, \end{cases} \tag{8.20}$$

where the coordinates (s,ρ) are given by (2.22), and $\hat{\alpha}_{b_k}$ is the same as in (2.25) with $\mathfrak{c} = b_k$.

For potentials of type $V1$ (with Remark 3.2 in mind once again) we apply to (8.17) the dilation

$$s=\Big[\frac{J_m h^4}{8|\alpha(b_k)|^3}\Big]^{1/12}\sigma \quad ; \quad \rho=\Big[\frac{h^2}{J_m}\Big]^{1/3}\tau\,, \tag{8.21}$$

to obtain the unitary equivalent operator (for $c(x_0)>0$ which is equivalent to $c(b_k)>0$)

$$iV(x_0)+[hJ_m]^{\frac{2}{3}}\,\tilde{\mathcal{B}}_{\mathfrak{e}_k}\,. \tag{8.22}$$

In the above, $\tilde{\mathcal{B}}_{\mathfrak{e}}$ is defined in (6.1)-(6.57)-(6.77), and (see also (3.15))

$$\beta_k=\frac{\hat\beta_k}{[8|\alpha(b_k)|J_m]^{1/2}}\,,\quad \omega=\kappa(b_k)\Big[2\frac{J_m}{\alpha(b_k)}\Big]^{1/2}\,,$$

$$\theta=2^{\frac{3}{2}}\,\frac{J_m^{1/2}\kappa(b_k)}{|\alpha(b_k)|^{1/2}}\,,\qquad \mathfrak{e}_k(h)=\,J_m^{-5/6}2^{-\frac{1}{2}}\,|\alpha(b_k)|^{1/2}\,h^{2/3}\,.$$

When $c(x_0)<0$ (and hence $c(b_k)<0$) we obtain $\overline{\tilde{\mathcal{B}}_{\mathfrak{e}}}$ instead of $\tilde{\mathcal{B}}_{\mathfrak{e}}$ in (8.22). Note that since $\mathcal{S}$ is finite, there exists $C>1$ such that, $\forall k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}$,

$$1\leq \mathfrak{e}_k(h)/\mathfrak{e}(h)=\alpha_m^{-\frac{1}{2}}|\alpha(b_k)|^{\frac{1}{2}}\leq C\,. \tag{8.23}$$

Note further, that since $\lambda\in\partial B(\hat\Lambda^1(h),\hat r_1(h)h^{k_1})$ we obtain, in view of (8.22),

$$\check\lambda:=[hJ_m]^{-\frac{2}{3}}(\lambda-iV(x_0))\in\partial B(\Lambda^1_{\gamma_k}(\mathfrak{e}_k(h)),r_1(\mathfrak{e}_k(h);b_k)\mathfrak{e}_k(h))$$

where, for $k\in\mathcal{J}_\partial^{s,2}$,

$$r_1(\mathfrak{e};b_k)=r_1(\mathfrak{e};x_0)=r_1(\mathfrak{e})\,,\ \gamma_k=1\,,\ \mathfrak{e}_k(h)=\mathfrak{e}(h)\,,$$

and for $k\in\mathcal{J}_\partial^{s,1}$,

$$r_1(\mathfrak{e};b_k)=\Big(2^{\frac{1}{2}(1+q)}J_m^{\frac{1+5q}{6}}|\alpha(b_k)|^{-\frac{1+q}{2}}\Big)\,\mathfrak{e}^q\,,\ \gamma_k=[\alpha_m/|\alpha(b_k)|]^{1/2}\,.$$

Note that by (8.4) there exists $c>0$ such that, $\forall k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}$,

$$c\leq r_1(\mathfrak{e};b_k)/r_1(\mathfrak{e})=\gamma_k^{1+q}\leq 1\,.$$

By (6.78a) we then have that $\partial B(\hat\Lambda^1(h),h^{k_1}\hat r_1(h))\subset\rho(\tilde{\mathcal{A}}_{k,h})$ and

$$\sup_{k\in\mathcal{J}_\partial^{s,1}\cup\mathcal{J}_\partial^{s,2}}\sup_{\lambda\in\partial B(\hat\Lambda^1(h),\hat r_1(h)h^{k_1})}\|\mathbf{1}_{B(b_k,h^{\varrho_\perp})}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\tilde\eta_{k,h}\|\ \leq\frac{C}{h^{k_1+\hat q_1}}\,, \tag{8.24}$$

where the cutoff function $\tilde\eta_{k,h}$ is given by the boundary operator defined in paragraph 2.3.5 (see also [6, 18])

$$\tilde\eta_{k,h}=T_{\mathcal{F}_{b_k}}(\eta_{k,h})\,.$$

To show that (6.78a) can be applied we first observe that, for all $x=(s,\rho)\in B(b_k,h^{\varrho_\perp})$ we have, by (8.21) and (3.10), that

$$\tau\leq J_m^{\frac{1}{3}}h^{\varrho^\perp-\frac{2}{3}}=J_m^{1/3}(J_m^{5/6}2^{\frac{1}{2}}|\alpha(b_k)|^{-\frac{1}{2}})^{3\varrho_\perp/2-1}\,\mathfrak{e}_k(h)^{3\varrho_\perp/2-1}\leq C\,\mathfrak{e}_k(h)^{3\varrho_\perp/2-1}\,,$$

where C is independent of k. Hence, for $1-3\varrho_\perp/2 < a' < 1$, there exists $h_0 > 0$, such that for all $h \in (0, h_0]$ and $k \in \mathcal{J}_\partial^s \backslash \mathcal{J}_\partial^{s,0}$ we have $\tau \leq \mathfrak{e}_k(h)^{-a'}$, which is precisely what we need to apply (6.78a).

Similarly, for potentials of type V2 we apply (with Remark 3.2 and (4.5) in mind) the transformation

$$\rho = \Big[\frac{h^2}{J_m}\Big]^{1/3}\tau \quad ; \quad s = \Big[\frac{J_m h^4}{2|\hat{\alpha}(b_k)|^3}\Big]^{1/9}\sigma\,, \tag{8.25}$$

and set (see (4.6)),

$$\gamma_k = [\hat{\alpha}_m/|\hat{\alpha}(b_k)|]^{2/3} \quad \text{and} \quad \varepsilon_k(h) = \gamma_k^{-1}\ \varepsilon(h)\,.$$

As in the case of potentials of type V1, we have

$$r_2(\varepsilon; b_k) = (2^{-\frac{2(1+q)}{3}}\,|\hat{\alpha}(b_k)|^{-\frac{2(1+q)}{3}}\,J_m^{\frac{2(q+4)}{9}})\,\varepsilon^q = \gamma_k^{1+q}\,r_2(\varepsilon)\,.$$

We then obtain via (7.18) that there exists $h_0 > 0$ such that, for $h \in (0, h_0]$, $\partial B(\hat{\Lambda}^2(h), \hat{r}_2(h)h^{k_2}) \subset \rho(\tilde{\mathcal{A}}_{k,h})$ and

$$\sup_{k\in\mathcal{J}_\partial^s\backslash\mathcal{J}_\partial^{s,0}}\ \sup_{\lambda\in\partial B(\hat{\Lambda}^2(h),\hat{r}_2(h)h^{k_2})} \|(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\tilde{\eta}_{k,h}\| \leq \frac{C}{h^{k_2+\hat{q}_2}}\,. \tag{8.26}$$

8.3.5 $k \in \mathcal{J}_\partial^c \setminus \mathcal{J}_\partial^s$.

In this case we use as our approximate operator

$$\begin{cases} \tilde{\mathcal{A}}_{k,h} = -h^2\Delta_{s,\rho} + i\big(V(b_k) \pm \mathfrak{j}_k\rho\big) \\ \mathcal{D}(\tilde{\mathcal{A}}_{k,h}) = \{u \in H^2(Q)\,|\,u|_{\partial Q_\perp} = 0\,,\ \partial_\nu u_{\partial Q_\parallel} = 0\,;\ \rho u \in L^2(Q)\}\,. \end{cases} \tag{8.27}$$

Applying the dilation

$$(\rho, s) = h^{2/3}(x_1, x_2)\,, \tag{8.28}$$

$\tilde{\mathcal{A}}_{k,h}$ is transformed into (2.3). By (2.4), inverse dilation, and the fact that $|\mathfrak{j}_k| > J_m$ we have

$$\sup_{\lambda\in\partial B(\hat{\Lambda}^i(h),\,h^{k_i}\hat{r}_i(h))} \|(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\| \leq \frac{C}{h^{2/3}}\,, \tag{8.29}$$

where $i = 1$ for potentials of type V1 and $i = 2$ for potentials of type V2.

8.3.6 $k \in \mathcal{J}_\partial^r$.

In this case, we use (2.18) as the approximate resolvent. Let

$$\delta_k(h) = \mathfrak{j}_k - J_m = |\nabla V(b_k(h))| - J_m\,.$$

Upon the dilation (8.28) we use Lemma 2.6 to obtain that, for $i = 1, 2$, $\partial B(\hat{\Lambda}^i(h), \hat{r}_i(h)h^{k_i}) \subset \rho(\tilde{\mathcal{A}}_{k,h})$, and that for some $C > 0$

$$\sup_{\lambda\in\partial B(\hat{\Lambda}^i,\,h^{k_i}\hat{r}_i(h)} \|(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\| \leq \frac{C}{\delta_k(h)\,h^{2/3}}\,. \tag{8.30}$$

Note that, in contrast with [6], card $\mathcal{J}_\partial^r(h)$ is not bounded as $h \to 0$. Consequently, $\delta_k(h)$ depends on h through the distance between $b_k(h)$ and $\mathcal{S}$. Depending on the potential type there exist positive constants h_0, c, and C such that

$$\delta_k(h) \geq C\, d(b_k(h), \mathcal{S})^{p_i} \geq c\, h^{p_i \varrho_\perp}\,, \; \forall h \in (0, h_0], \forall k \in \mathcal{J}_\partial^r(h)\,, \tag{8.31}$$

where $p_1 = 2$ for potentials of type $V1$ and $p_2 = 1$ for potentials of type $V2$.

8.3.7 Approximate resolvent norm

We have shown that there exists $h_0 > 0$ such that the approximate resolvent (8.13) is well defined when $\lambda \in \partial B(\hat{\Lambda}_i(h), h^{k_i}\hat{r}_i(h))$ for all $h \in (0, h_0]$. To summarize, we state the following

Lemma 8.1. *For $i = 1, 2$, there exists C and h_0 such that, for any $h \in (0, h_0]$ and any $\lambda \in \partial B(\hat{\Lambda}_i(h), h^{k_i + \hat{q}_i})$, the approximate resolvent satisfies*

$$\|\mathcal{R}(h, \lambda)\| \leq C\, h^{-k_i - \hat{q}_i}\,. \tag{8.32}$$

The proof follows immediately from (2.14), (2.17), (8.15), (8.16), (8.24), (8.26), (8.29), and (8.30).

8.4 Approximate resolvent error

In this subsection, we show that $\mathcal{R}(h, \lambda)$ is a good approximation of $(\mathcal{A}_h - \lambda)^{-1}$. We proceed as in Paragraph 2.3.6, albeit with the refined partition of unity defined in Section 8.2 . From (2.35)) we recall that

$$\mathcal{E}(h, \lambda) = \sum_{j \in \mathcal{J}_i(h)} \mathcal{B}_j(h, \lambda)\, \chi_{j,h} + \sum_{k \in \mathcal{J}_\partial(h)} \mathcal{B}_k(h, \lambda)\, \eta_{k,h}\,,$$

and keep the same definition for $\mathcal{B}_j(h, \lambda)$ and $\mathcal{B}_k(h, \lambda)$ as in (2.36).

For the present partition of unity, we set, as in Section 2,

- Supp $\widehat{\chi}_{j,h} \subset B(a_j(h), 2h^\varrho)$ for $j \in \mathcal{J}_i(h)$,
- Supp $\widehat{\eta}_{k,h} \subset B(b_k(h), 2h^{\varrho_\perp})$ for $k \in \mathcal{J}_\partial^\perp$,
- Supp $\widehat{\eta}_{k,h} \subset B(b_k(h), 2h^\varrho)$ for $k \in \mathcal{J}_\partial^N$,
- $\widehat{\chi}_{j,h}\chi_{j,h} = \chi_{j,h}$ and $\widehat{\eta}_{k,h}\eta_{k,h} = \eta_{k,h}$,

and

$$\widehat{\mathcal{A}}_{k,h} = T_{\mathcal{F}_{b_k}} \mathcal{A}_h T_{\mathcal{F}_{b_k}}^{-1}\,.$$

In the sequel we prove the following generalization of [6, Lemma 7.6] (see also paragraph 2.3.6).

Proposition 8.2. *Let* $\hat{q}_i$ *be defined by* (8.3) *with* $0 < q < \frac{1}{6}$*, and* $(\varrho_\perp\,,\,\varrho)$ *satisfy* (8.12)*. Let further* $\tilde{b}$ *and* b *respectively satisfy (8.19) and*

$$1 - \frac{3\varrho_\perp}{2} < b < 3/4\,. \tag{8.33}$$

Then, under the assumptions of either Theorem 1.1 (V1 potentials) or Theorem 1.3 (V2 potentials),we have

$$\lim_{h\to 0} \sup_{\lambda\in\partial B(\hat{\Lambda}^i(h),h^{k_i+\hat{q}_i})} \|\mathcal{E}(h,\lambda)\| = 0\,. \tag{8.34}$$

Note that by (8.12), we have $\varrho_\perp > \frac{1}{6}$ which implies that the interval $(1-\frac{3\varrho_\perp}{2}, 3/4)$ is not empty, though (8.33) is certainly more restrictive than (8.19).

Keeping (2.39) in mind, (8.34) follows, under the assumptions of Proposition 8.2, from the following lemma

Lemma 8.3. *Under the assumptions of Proposition 8.2, there exist* $C > 0$*,* $h_0 > 0$ *and* $d_0 > 0$ *such that, for all* $h \in (0, h_0]$ *and* $\lambda \in \partial B(\hat{\Lambda}^i(h), h^{k_i+\hat{q}_i})$*, we have*

$$\sup_{j\in\mathcal{J}_i(h)} \|\mathcal{B}_j(h,\lambda)\| + \sup_{k\in\mathcal{J}_\partial(h)} \|\|\mathcal{B}_k(h,\lambda)\| \le Ch^{d_0}\,. \tag{8.35}$$

Proof.
We split the proof into several steps, estimating the $\|\mathcal{B}_j(h,\lambda)\|$ or $\|\mathcal{B}_k(h,\lambda)\|$ in the various cases listed in Subsection 8.3. An explicit formula of d_0 is provided at the end of the proof. Throughout the proof of the lemma, all the constants C and h_0 appearing at each step can be chosen independently of $h \in (0, h_0]$, $j \in \mathcal{J}_i(h)$, $k \in \mathcal{J}_\partial(h)$ and $\lambda \in \partial B(\hat{\Lambda}_i(h), h^{k_i+\hat{q}_i})$.

Step 1: **Estimate $\|\mathcal{B}_j\|$ for $j \in \mathcal{J}_i$ and $\|\mathcal{B}_k\|$ for $k \in \mathcal{J}_\partial^{s,0} \cup (\mathcal{J}_\partial^c \setminus \mathcal{J}_\partial^s)$.**

We refer the reader to [6, (7.50) -(7.51)] where it is shown that there exist $C > 0$ and $h_0 > 0$, such that, for all $h \in (0, h_0]$, $j \in \mathcal{J}_i(h)$, $k \in \mathcal{J}_\partial^N(h) \cup \mathcal{J}_\partial^{s,0}$, and $\lambda \in \partial B(\hat{\Lambda}^i(h), h^{k_i+\hat{q}_i})$,

$$\|\mathcal{B}_j(h,\lambda)\| + \|\mathcal{B}_k(h,\lambda)\| \le C\,(h^{2/3-\varrho} + h^{2(\varrho-1/3)})\,. \tag{8.36}$$

For $k \in \mathcal{J}_\partial^c \setminus \mathcal{J}_\partial^s$ we use (2.33) (which remains valid under the assumptions set above on λ) to obtain

$$\|\mathcal{B}_k(h,\lambda)\| \le C\,(h^{2/3-\varrho} + h^{\varrho})\,. \tag{8.37}$$

Hence (8.35) is satisfied for $k \in \mathcal{J}_\partial^c \setminus \mathcal{J}_\partial^s$ if d_0 satisfies

$$0 < d_0 \le d_1 := \inf(\varrho\,, 2/3 - \varrho\,, 2(\varrho - 1/3))\,. \tag{8.38}$$

Note that by (8.12) d_1 is positive.

Step 2: **Estimate $\left\|\eta_{k,h} T_{\mathcal{F}_{b_k}}^{-1}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} T_{\mathcal{F}_{b_k}} \hat{\eta}_{k,h}\right\|$ for $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$ and $V1$ potentials.**

Note that the above term is the first one on the right-hand-side of (2.36b), and we must provide an effective bound for it in order to obtain a proper estimate of $\|\mathcal{B}_k(h,\lambda)\|$. We recall that $\widehat{\mathcal{A}}_{k,h}$ has been introduced in (2.26) and that, for type V1 potentials, $\tilde{\mathcal{A}}_{k,h}$ is introduced in (8.17). Let $(a_0,\tilde{b})$ satisfy

$$0 < a_0 < 2\tilde{b}/3\,. \tag{8.39}$$

Decomposing $(\widehat{\mathcal{A}}_{k,h}-\tilde{\mathcal{A}}_{k,h})$, we write (cf. also [6, (7.52)-(7.55)])

$$\begin{aligned}
&\big\|\eta_{k,h}T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h}-\tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}T_{\mathcal{F}_{b_k}}\hat{\eta}_{k,h}\big\| \\
&\quad\le C\Big(\|\tilde{\eta}_{k,h}\delta_1\mathbf{1}_{\rho\le h^{2/3-a_0}}h^2\partial_s^2(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad+\|\tilde{\eta}_{k,h}\delta_1\mathbf{1}_{\rho> h^{2/3-a_0}}h^2\partial_s^2(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad+\|\tilde{\eta}_{k,h}\delta_2h^2\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda(h))^{-1}\breve{\eta}_{k,h}\| \\
&\qquad+\|\tilde{\eta}_{k,h}\delta_3h^2\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda(h))^{-1}\breve{\eta}_{k,h}\| \\
&\qquad+\|\mathbf{1}_{\rho\le h^{2/3-a_0}}(V\circ\mathcal{F}_{b_k}-V^{(2)}_{b_k})\tilde{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad+\|\mathbf{1}_{\rho> h^{2/3-a_0}}(V\circ\mathcal{F}_{b_k}-V^{(2)}_{b_k})\tilde{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\Big)\,,
\end{aligned} \tag{8.40}$$

where

$$\delta_1=\tilde{g}^{-2}-1-2\kappa(b_k)\rho\chi(h^{-2(1-\tilde{b})/3}\rho)\,,\ \delta_2=\kappa(s)\tilde{g}^{-1}-\kappa(b_k)\,,\ \delta_3=\rho\kappa'(s)\tilde{g}^{-3}\,,$$

$$\breve{\eta}_{k,h}=T_{\mathcal{F}_{b_k}}\hat{\eta}_{k,h}\,,\quad \tilde{\eta}_{k,h}=T_{\mathcal{F}_{b_k}}\eta_k\,,$$

and where $V^{(2)}_{b_k}$ is introduced in (8.18),

For the first term on the right-hand-side of (8.40), observing that

$$\delta_1=\mathcal{O}(\rho^2)+\mathcal{O}(\rho s)+\mathcal{O}(\rho)(1-\chi(h^{-2(1-\tilde{b})/3}\rho))\,,$$

we obtain with the aid of (8.39),

$$\|\delta_1\tilde{\eta}_{k,h}\mathbf{1}_{\rho\le h^{2/3-a_0}}\|_\infty\le C\,h^{2/3+\varrho_\perp-a_0}\,.$$

Using the dilation (8.21) together with (6.78a) (with $3a_0/2<a'<1$) we obtain

$$h^2\|\tilde{\eta}_{k,h}\delta_1\mathbf{1}_{\rho\le h^{2/3-a_0}}\partial_s^2(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\le C\,h^{\varrho_\perp-a_0-\hat{q}_1}\,. \tag{8.41}$$

To confirm the effectiveness of the above bound we need to restrict a_0 further by imposing

$$0<a_0<\varrho_\perp-\hat{q}_1\,,$$

which is possible by (8.5) and (8.12a).
As

$$\|\delta_1\tilde{\eta}_{k,h}\|_\infty\le h^{\varrho_\perp}\,,$$

the second term can be estimated using the above dilation together with (6.78b) for $a\le\frac{3}{2}a_0$ and $a'\ge 1-\frac{3}{2}\varrho_\perp$, (thus satisfying $0<a<a'<1$ as is required in Proposition 6.8) to obtain

$$h^2\|\tilde{\eta}_{k,h}\delta_1\mathbf{1}_{\rho> h^{2/3-a_0}}\partial_s^2(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\le Ch^{\varrho_\perp}\,. \tag{8.42}$$

Since

$$\|\tilde{\eta}_{k,h}\delta_2\|_\infty + \|\tilde{\eta}_{k,h}\delta_3\|_\infty \leq Ch^{\varrho_\perp}\,,$$

the third term and the fourth terms on the right-hand-side of (8.40), can be bounded via the same dilation and (6.78a) (with $a' \geq 1 - \frac{3}{2}\varrho_\perp$), yielding

$$h^2[\|\tilde{\eta}_{k,h}\delta_2\partial_\rho(\tilde{\mathcal{A}}_{k,h} - \lambda(h))^{-1}\breve{\eta}_{k,h}\| + \|\tilde{\eta}_{k,h}\delta_3\partial_s(\tilde{\mathcal{A}}_{k,h} - \lambda(h))^{-1}\breve{\eta}_{k,h}\|] \leq Ch^{\varrho_\perp - \hat{q}_1}\,. \tag{8.43}$$

Note that $\varrho_\perp - \hat{q}_1 > 0$ by (8.12a) and (8.3). To control the fifth term on the right-hand-side of (8.40) we use (3.6) (with x_0 replaced by b_k) to obtain

$$\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}(V\circ\mathcal{F}_{b_k} - V^{(2)}_{b_k})\,\tilde{\eta}_{k,h}\|_\infty \leq Ch^{4/3+\varrho_\perp-2a_0}\,,$$

and, with the aid of the same dilation and (6.78a),

$$\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}(V\circ\mathcal{F}_{b_k} - V^{(2)}_{b_k})\tilde{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{\varrho_\perp-\hat{q}_1-2a_0}\,. \tag{8.44}$$

For (8.44) to be an effective bound we must further restrict a_0 so that

$$0 < a_0 < (\varrho_\perp - \hat{q}_1)/2\,.$$

Finally, to bound the last term on the right-hand-side of (8.40) we have, by (6.78b) and the same dilation

$$\|\mathbf{1}_{\rho> h^{2/3-a_0}}(V\circ\mathcal{F}_{b_k} - V^{(2)}_{b_k})\tilde{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{3\varrho_\perp-2/3+a_0}\,, \tag{8.45}$$

which leads to a further restriction on a_0

$$3\varrho_\perp - 2/3 + a_0 > 0\,.$$

Altogether a_0 should satisfy:

$$-3\Big(\varrho_\perp - \frac{2}{9}\Big) < a_0 < \frac{\varrho_\perp - \hat{q}_1}{2}\,. \tag{8.46}$$

Such a_0 exists if and only if

$$-3\Big(\varrho_\perp - \frac{2}{9}\Big) < \frac{\varrho_\perp - \hat{q}_1}{2}\,,$$

or, equivalently, when

$$\frac{4}{3} + \hat{q}_1 < 7\varrho_\perp\,.$$

Note that the above condition is satisfied by the upper bound of $\hat{q}_1$ in (8.5) and the lower bound of $\varrho_\perp$ in (8.12a).

In view of (8.46) we set

$$a_0 = \frac{1}{3} - \frac{5\varrho_\perp}{4} - \frac{\hat{q}_1}{4} > \frac{1}{36}\,, \tag{8.47}$$

(wherein the lower bound follows from (8.12a) and (8.5)).
By (8.39), $\tilde{b} \in (0, \frac{1}{2} - q)$ must satisfy

$$\frac{1}{2} - \frac{15\varrho_\perp}{8} - \frac{3\hat{q}_1}{8} < \tilde{b} < \frac{1}{2} - \frac{3}{2}\hat{q}_1\,.$$

For the above inequality to make any sense we must have $\hat{q}_1 < \frac{5}{3}\varrho_\perp$, which clearly holds by (8.5) and (8.12a).

By (8.41)-(8.47), there exists $C > 0$ such that for sfficiently small h

$$\big\|\eta_{k,h}T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}T_{\mathcal{F}_{b_k}}\hat{\eta}_{k,h}\big\| \leq Ch^{d_0}\,,$$

for any

$$0 < d_0 \leq d_2 := \inf(\varrho_\perp - \hat{q}_1 - 2a_0, 3\varrho_\perp - 2/3 + a_0) = 7\varrho_\perp/4 - 1/3 - \hat{q}_1/4\,. \quad (8.48)$$

Note that the positivity of d_2 has been verified above.

Step 3: **Estimate** $\big\|\eta_{k,h}T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h}-\tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}T_{\mathcal{F}_{b_k}}\hat{\eta}_{k,h}\big\|$ **for** $k \in \mathcal{J}^s_\partial\backslash\mathcal{J}^{s,0}_\partial$ **and** $V2$ **potentials.**

For potentials of type V2, we recall that $\tilde{\mathcal{A}}_{k,h}$ is introduced in (8.20) and write

$$\begin{aligned}\big\|\eta_{k,h}&T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}T_{\mathcal{F}_{b_k}}\hat{\eta}_{k,h}\big\| \\ &\leq Ch^2\big(\|\mathbf{1}_{\rho>h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1)\partial_s^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \\ &\qquad +\|\mathbf{1}_{\rho>h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1 - \alpha s)\partial_\rho^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \\ &\qquad +\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1)\partial_s^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \\ &\qquad +\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1 - \alpha s)\partial_\rho^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\|\big)\,.\end{aligned} \quad (8.49)$$

For the first two terms we use the dilation (8.25) together with (7.39b) (for $a = \frac{9}{4}a_0$) to obtain

$$\begin{aligned}h^2\big(\|&\mathbf{1}_{\rho>h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1)\partial_s^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \\ &+ \|\mathbf{1}_{\rho>h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1 - \alpha s)\partial_\rho^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\|\big) \leq Ch^{\varrho_\perp}\,.\end{aligned} \quad (8.50)$$

For the third term we use the same dilation and (7.38b) to obtain for $\mathfrak{a}$ satisfying $1/4 < \mathfrak{a} = 5/12 < (1-q)/2$ that

$$h^2\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1)\partial_s^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{\varrho_\perp - 7/27}\,. \quad (8.51)$$

Recall that $\varrho_\perp > \frac{7}{27}$ by (8.12b).

Finally, for the last term on the right-hand-side of (8.49) we use the fact that

$$\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1 - \alpha s)\|_\infty \leq C(h^{2\varrho_\perp} + h^{2/3-a_0})\,,$$

to obtain by the same dilation and (7.38a)

$$h^2\|\mathbf{1}_{\rho\leq h^{2/3-a_0}}\tilde{\eta}_{k,h}(\tilde{g}_c - 1 - \alpha s)\partial_\rho^2(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1}\breve{\eta}_{k,h}\| \leq C\left(h^{2/9-a_0-\hat{q}_2} + h^{2\varrho_\perp - 4/9 - \hat{q}_2}\right)\,. \quad (8.52)$$

As $2\varrho_\perp - 4/9 - \hat{q}_2 > 0$ by (8.5) and (8.12b) we need to select $a_0 < \frac{2}{9} - \hat{q}_2$ (yielding $a_0 < \frac{4}{27}$). Setting

$$a_0 = 2(1/3 - \varrho_\perp)\,, \quad (8.53)$$

yields by (8.49)-(8.52) that (8.35) is satisfied in the present context if

$$0 < d_0 \leq d_3 := 2\varrho_\perp - 4/9 - \hat{q}_2\,, \quad (8.54)$$

where the positivity of d_3 follows from (8.5) and (8.12b).

Step 4: **Estimate $\|[\mathcal{A}_h, \eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}\|$ for $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$ and V1 potentials.**

Note that the above term is the second term on the right-hand side of (2.36b), and an effective bound for it in is necessary order to obtain a proper estimate of $\|\mathcal{B}_k(h,\lambda)\|$. . Having in mind (2.32), we decompose $[\mathcal{A}_h, \eta_{k,h}]$ in the form

$$[\mathcal{A}_h, \eta_{k,h}] = -h^2(\Delta\eta_{k,h}) - 2h^2 1_{\rho>h^{2/3-a_0}}\nabla\eta_{k,h}\cdot\nabla - 2h^2 1_{\rho<h^{2/3-a_0}}\nabla\eta_{k,h}\cdot\nabla\,, \quad (8.55)$$

where a_0 is given by (8.47). Note by (8.8b) that $(\Delta\eta_{k,h})$ and $\nabla\eta_{k,h}$ are supported in $B(b_k, h^{\varrho_\perp}) \setminus B(b_k, h^{\varrho_\perp}/2)$. Hence, whenever $\rho < h^{2/3-a_0}$ we have, for sufficiently small h,

$$|s| \geq \frac{1}{2}h^{\varrho_\perp} - h^{2/3-a_0} > \frac{1}{3}h^{\rho_\perp}\,,$$

since by (8.47) we have $\varrho_\perp < \frac{2}{3} - a_0$.
Consequently, we may represent (8.55) in the form

$$[\mathcal{A}_h, \eta_{k,h}] = -h^2(\Delta\eta_{k,h}) - 2h^2 1_{\rho>h^{2/3-a_0}}\nabla\eta_{k,h}\cdot\nabla - 2h^2 1_{\rho<h^{2/3-a_0}} 1_{|s|>\frac{h^{\rho_\perp}}{3}}\nabla\eta_{k,h}\cdot\nabla\,.$$

Recall from (8.8b) that for all $k \in \mathcal{J}_\partial^s \setminus \mathcal{J}_\partial^{s,0}$,

$$|\nabla\eta_{k,h}| + h^{\varrho_\perp}|D^2\eta_{k,h}| \leq C\,h^{-\varrho_\perp}\,.$$

We finally note that by (8.9) and (8.11), we have, for $\rho < h^{2/3-a_0}$, that

$$|\partial\tilde{\eta}_{k,h}/\partial\rho| \leq C\,h^{2/3-a_0-2\varrho_\perp}\,.$$

With these remarks in mind we obtain the existence of C and h_0 such that for $h \in (0, h_0]$ and $\lambda \in \rho(\tilde{A}_{k,h})$

$$\begin{aligned}
&\big\|[\mathcal{A}_h, \eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}\big\| \\
&\quad \leq C\,h^{2(1-\varrho_\perp)}\|\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad + C\,h^{2-\varrho_\perp}\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad + C\,h^{2-\varrho_\perp}\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad + C\,h^{2-\varrho_\perp}\,\|\breve{\eta}_{k,h}\mathbf{1}_{s>\frac{h^{\rho_\perp}}{3}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad + C\,h^{8/3-a_0-2\varrho_\perp}\|\breve{\eta}_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\,,
\end{aligned} \quad (8.56)$$

where $\tilde{\mathcal{A}}_{k,h}$ is given by (8.17).
We now estimate term by term the right hand side of (8.56) using the dilation (8.21). To this end we use (6.78) below with $a = \frac{3}{2}a_0$ and $a' \geq 1 - \frac{3}{2}\varrho_\perp$.
For the first term, we get from (6.78a)

$$h^{2(1-\varrho_\perp)}\|\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{\frac{2}{3}-2\varrho_\perp-\hat{q}_1}\,.$$

We note that by (8.5) and (8.12a) it follows that $\frac{2}{3} - 2\varrho_\perp - \hat{q}_1 > 0$.
For the second term, we use (6.78b) to obtain

$$h^{2-\varrho_\perp}\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\| \leq Ch^{\frac{2}{3}-\varrho_\perp+\frac{a_0}{2}}\,.$$

For the third term, using (6.78b) yields

$$h^{2-\varrho_\perp}\|\breve\eta_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\|\leq Ch^{\frac{2}{3}-\varrho_\perp}\,.$$

For the forth term, we use (6.78c) whose assumptions hold when $a'\geq 1-\frac{3}{2}\varrho_\perp$ and $\mathfrak{a}$ satisfies

$$\frac{1}{6}<\mathfrak{a}<\frac{1}{4}\ \text{and}\ \frac{3}{2}\varrho_\perp-\frac{1}{2}+\mathfrak{a}<0\,. \tag{8.57}$$

Such a choice for $\mathfrak{a}$ is possible since $\varrho_\perp<\frac{2}{9}$ is satisfied by (8.12a). We obtain

$$\|\breve\eta_{k,h}\mathbf{1}_{s>\frac{h^{\rho_\perp}}{3}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|\leq Ch^{\frac{4}{3}\mathfrak{a}-\varrho_\perp}\,.$$

Note by (8.12a) that

$$4\mathfrak{a}/3-\varrho_\perp>2/9-\varrho_\perp>0\,.$$

Finally, for the fifth term, we get from (6.78a)

$$h^{8/3-a_0-2\varrho_\perp}\|\breve\eta_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|\leq C\,h^{\frac{2}{3}-a_0-2\varrho_\perp-\hat q_1}\,.$$

Using (8.47) together with (8.5) and (8.12a) then yields

$$\frac{2}{3}-a_0-2\varrho_\perp-\hat q_1=\frac{1}{3}-\frac{3}{4}(\hat q_1+\varrho_\perp)>0\,.$$

In conclusion, we have established, for potentials of type V1 that $[\mathcal{A}_h,\eta_{k,h}]R_{k,h}\breve\eta_{k,h}$ satisfies (8.35) when d_0 satisfies

$$0<d_0\leq d_4^1:=\min\Big\{\frac{2}{3}-2\varrho_\perp-\hat q_1\,,\frac{4}{3}\mathfrak{a}-\varrho_\perp\,,\frac{1}{3}-\frac{3}{4}(\hat q_1+\varrho_\perp)\Big\}\,, \tag{8.58}$$

for $\mathfrak{a}$ satisfying (8.57).
The positivity of d_4^1 is established by the foregoing discussion.

Step 5: **Estimate $\|[\mathcal{A}_h,\eta_{k,h}]R_{k,h}\breve\eta_{k,h}\|$ for $k\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}$ for V2 potentials.**

We begin the estimate, as in the V1 case, starting from (8.56) but with $\tilde A_{k,h}$ introduced in (8.20), a_0 given by (8.53). We use here the dilation (8.25). For the first term on the right-hand-side of (8.56), we use (7.36) to obtain

$$h^{2(1-\varrho_\perp)}\|\breve\eta_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|\leq Ch^{8/9-2\varrho_\perp-\hat q_2}\,.$$

Note that by (8.12b) and (8.5) we have $8/9-2\varrho_\perp-\hat q_2>4/27$.

Upon dilation we use (7.39a) with $a=9a_0/4$, for the second and third terms on the right-hand-side of (8.56) to obtain

$$h^{2-\varrho_\perp}\big(\|\breve\eta_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|+\|\breve\eta_{k,h}\mathbf{1}_{\rho>h^{2/3-a_0}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|\big)\leq h^{2/3-\varrho_\perp+a_0/2}\,.$$

The fourth term on the right-hand-side of (8.56) can be estimated, upon dilation, with the aid of (7.37), where $1/4<\mathfrak{a}<(1-q)/2$. We get

$$h^{2-\varrho_\perp}\,\|\breve\eta_{k,h}\mathbf{1}_{s>\frac{h^{\rho_\perp}}{3}}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve\eta_{k,h}\|\leq h^{2/9-\varrho_\perp+\frac{4}{9}\mathfrak{a}}\,.$$

To have $s > h^{\rho_\perp}/3 \Rightarrow \sigma > \varepsilon^{-\mathfrak{a}}$ we further require

$$\frac{9}{4}\varrho_\perp - 1 + \mathfrak{a} < 0\,, \tag{8.59}$$

which can be satisfied since $\varrho_\perp < \frac{1}{3}$ by (8.12b) that implies, in addition,

$$2/9 - \varrho_\perp + \frac{4}{9}\mathfrak{a} > 0\,.$$

Finally, by (7.38a), applied upon (8.25), we have for the last term on the right-hand-side of (8.56),

$$h^{8/3-a_0-2\varrho_\perp}\|\breve{\eta}_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq C\, h^{8/9-2\varrho_\perp-a_0-\hat{q}_2}\,,$$

By (8.12b) and (8.53) we have

$$\frac{8}{9} - 2\varrho_\perp - a_0 - \hat{q}_2 > \frac{2}{9} - \frac{2}{27} > 0\,.$$

In conclusion, we have obtained for potentials of type V2 that $[\mathcal{A}_h, \eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}$ satisfies (8.35) for

$$0 < d_0 \leq d_4^2 := \inf\{\frac{8}{9} - 2\varrho_\perp - a_0 - \hat{q}_2, 2/3 - \varrho_\perp + a_0/2, \frac{8}{9} - \varrho_\perp + \frac{4}{9}\mathfrak{a}\}\,, \tag{8.60}$$

with $\mathfrak{a} \in (1/6, (1-q)/2)$ satisfying (8.59).
The positivity of d_4^2 has already been established in the foregoing discussion.

Step 6: **Estimate of** $\|\eta_{k,h}T^{-1}_{\mathcal{F}_{b_k}}(\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|$ **for** $k \in \mathcal{J}_\partial^r$**.**

We recall that $\tilde{\mathcal{A}}_{k,h}$ is introduced in (2.18) and we first observe that

$$\begin{aligned}\|(V\circ\mathcal{F}_{b_k} - V^{(1)}_{b_k})\breve{\eta}_{k,h}\|_\infty &\leq C\, h^{2\varrho}\Big(\Big\|\frac{\partial^2 V}{\partial s^2}\Big\|_{L^\infty(B(b_k,h^\varrho))} \\ &\qquad + \Big\|\frac{\partial^2 V}{\partial s\partial\rho}\Big\|_{L^\infty(B(b_k,h^\varrho))} \\ &\qquad + \Big\|\frac{\partial^2 V}{\partial \rho^2}\Big\|_{L^\infty(B(b_k,h^\varrho))}\Big) \\ &\leq \tilde{C}\, h^{2\varrho}\,,\end{aligned} \tag{8.61}$$

where

$$V^{(1)}_{b_k}(\rho,s) := V(b_k) \pm \mathfrak{j}_k\rho$$

as in (2.18).

V1 Potentials For potentials of type V1 we now observe that

$$\Big\|\frac{\partial^2 V}{\partial s\partial\rho}\Big\|_{L^\infty(B(b_k,h^\varrho))} \leq C\,\left(d(b_k,\mathcal{S}) + h^\varrho\right)\,,$$

and that $b_k \notin \bigcup_{n\in\mathcal{J}_\partial^s\setminus\mathcal{J}_\partial^{s,0}} B(b_n, h^{\varrho_\perp})$ for all $k\in\mathcal{J}_\partial^r$ we have

$$d(b_k,\mathbb{S}) \geq h^{\varrho_\perp}\,. \tag{8.62}$$

Clearly,

$$\Big\|\frac{\partial^2 V}{\partial s\partial\rho}\Big\|_{L^\infty(B(b_k,h^\varrho))} \leq C\, d(b_k,\mathbb{S})\,.$$

Furthermore, since $V_{ss}(0,s)=0$, we have

$$\Big\|\frac{\partial^2 V}{\partial s^2}\Big\|_{L^\infty(B(b_k,h^\varrho))} \leq Ch^\rho\,,$$

and, for any $a_1>0$,

$$\Big\|\mathbf{1}_{\rho\leq h^{2/3-a_1}}\frac{\partial^2 V}{\partial s^2}\Big\|_{L^\infty(B(b_k,h^\varrho))} \leq Ch^{2/3-a_1}\,.$$

Consequently, using a Taylor expansion of order 2, we obtain from the preceding inequalities

$$\|\mathbf{1}_{\rho\leq h^{2/3-a_1}}(V\circ\mathcal{F}_{b_k}-V_{b_k}^{(1)})\breve{\eta}_{k,h}\|_\infty \leq C\left(h^{2\varrho}d(b_k,\mathcal{S})+h^{\frac{2}{3}-a_1+2\varrho}+h^{\frac{4}{3}-2a_1}\right)\,.$$

Set $a_1=1/3-\varrho/2$. As $\varrho>\frac{2}{9}$ and $\varrho+\varrho_\perp<2/3$ by (8.12a) The above inequality together with (8.62), lead to

$$\|\mathbf{1}_{\rho\leq h^{2/3-a_1}}(V\circ\mathcal{F}_{b_k}-V_{b_k}^{(1)})\breve{\eta}_{k,h}\|_\infty \leq C\,h^{2\varrho}d(b_k,\mathcal{S})\,. \tag{8.63}$$

We now write

$$\begin{aligned}\Big\|\eta_{k,h}T_{\mathcal{F}_{b_k}}^{-1}&(\widehat{\mathcal{A}}_{k,h}-\tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\Big\|\\ &\leq C\,h^\varrho\|h^2\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\\ &\quad+C\|h^2\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h}-\lambda(h))^{-1}\breve{\eta}_{k,h}\|\\ &\quad+Ch^{2\varrho}\|\mathbf{1}_{\rho\geq h^{2/3-a_1}}\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\\ &\quad+Ch^{2\varrho}d(b_k,\mathcal{S})\|\mathbf{1}_{\rho\leq h^{2/3-a_1}}\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\,.\end{aligned} \tag{8.64}$$

By (8.30) and Lemma 2.6 we have

$$h^\varrho\|h^2\Delta_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq C\,h^{\varrho-2\varrho_\perp}\,. \tag{8.65}$$

Note that by (8.12a)

$$\varrho-2\varrho_\perp>\frac{1}{3}-\frac{3}{2}\varrho_\perp>0\,.$$

Similarly,

$$\|h^2\nabla_{(s,\rho)}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq C\,h^{2/3-2\varrho_\perp}\,. \tag{8.66}$$

Upon the dilation (8.28) we may use (6.79) with $a=\frac{a_1}{3\varrho_\perp}$ to obtain for the third term

$$h^{2\varrho}\|\mathbf{1}_{\rho\geq h^{2/3-a_1}}\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{2(\varrho-1/3)}\,. \tag{8.67}$$

Note that the cutoff $\rho > h^{2/3-a_1}$ leads after dilation to $\tau > h^{-a_1}$ which should be compared with $\tau \geq \delta^{-a}$ for the application of (6.79). Hence, we must have $h^{-a_1} >> \delta^{-a}$ which leads, in view of (8.62), to $2a\varrho_\perp - a_1 < 0$. For the application of (6.79) we must have in addition $d(b_k, \mathcal{S}) < \delta_1$ for some $\delta_1 > 0$. If $d(b_k, \mathcal{S}) \geq \delta_1$ (8.67) follows immediately from Lemma 2.6.

For the last term, we use (8.30) to obtain

$$h^{2\varrho} d(b_k, \mathcal{S}) \|\mathbf{1}_{\rho \leq h^{2/3-a_1}} \breve{\eta}_{k,h} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \leq C d(b_k, \mathcal{S})^{-1} h^{2\varrho - \frac{2}{3}} \leq \tilde{C}\, h^{2\varrho - \varrho_\perp - 2/3} . \tag{8.68}$$

Note that by (8.12a) $2\varrho - \varrho_\perp - 2/3 > 0$.
Substituting the above into (8.64) yields, that $\eta_{k,h} T^{-1}_{\mathcal{F}_{b_k}} (\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}$ satisfies (8.35) with

$$0 < d_0 \leq d_5^1 := \inf\{2\varrho - \varrho_\perp - 2/3, \varrho - 2\varrho_\perp, 2(\varrho - 1/3), 2/3 - 2\varrho_\perp\} , \tag{8.69}$$

where positivity of d_5^1 has been established above.

Type V2 potentials For potentials of type V_2, we write, using (8.61) directly,

$$\begin{aligned} &\left\|\eta_{k,h} T^{-1}_{\mathcal{F}_{b_k}} (\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\right\| \\ &\quad \leq C \Big(h^\varrho \|h^2 \Delta_{(s,\rho)} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \\ &\qquad\qquad + \|h^2 \nabla_{(s,\rho)} (\tilde{\mathcal{A}}_{k,h} - \lambda(h))^{-1} \breve{\eta}_{k,h}\| \\ &\qquad\qquad\qquad + h^{2\varrho} \|\tilde{\eta}_{k,h} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \Big) . \end{aligned} \tag{8.70}$$

Proceeding as in the V1 case, we obtain, similarly to the proof of (8.65) and (8.66) but with (8.31) in mind,

$$h^\varrho \|h^2 \Delta_{(s,\rho)} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \leq C h^{\varrho - \varrho_\perp}$$

and

$$\|h^2 \nabla_{(s,\rho)} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \leq C h^{\frac{2}{3} - \varrho_\perp} .$$

For the third term, we may now use (8.30) and (8.31), with $p_2 = 1$, to obtain that

$$h^{2\varrho} \|\tilde{\eta}_{k,h} (\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}\| \leq C\, h^{2\varrho - \varrho_\perp - 2/3} .$$

Hence, we obtain that $\eta_{k,h} T^{-1}_{\mathcal{F}_{b_k}} (\widehat{\mathcal{A}}_{k,h} - \tilde{\mathcal{A}}_{k,h})(\tilde{\mathcal{A}}_{k,h} - \lambda)^{-1} \breve{\eta}_{k,h}$ satisfies (8.35) with

$$0 < d_0 \leq d_5^2 := \inf\{2\varrho - \varrho_\perp - 2/3, \varrho - \varrho_\perp\} , \tag{8.71}$$

where positivity of d_6^2 follows from (8.12b).

Step 7: **Estimate of** $[\mathcal{A}_h, \eta_{k,h}] R_{k,h} \breve{\eta}_{k,h}$ **for** $k \in \mathcal{J}_\partial^r$**.**

We now estimate the rest of the contribution of $\mathcal{J}_\partial^r$, as in Steps 4 and 5, by writing

(dropping the cut-off in the s variable)

$$\begin{aligned}
&\Big\|[\mathcal{A}_h,\eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}\Big\| \\
&\quad\leq C\Big(h^{2(1-\varrho)}\|\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad +h^{2-\varrho}\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_1}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad +h^{2-\varrho}\|\breve{\eta}_{k,h}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \\
&\qquad +h^{8/3-a_1-2\varrho}\|\breve{\eta}_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\Big)\,.
\end{aligned} \tag{8.72}$$

V1 potentials For potentials of type V1, we use Lemma 2.6 upon the dilation (8.28) to obtain for the first term

$$h^{2(1-\varrho)}\|\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq Ch^{\frac{4}{3}-2\varrho-2\varrho_\perp}\,, \tag{8.73}$$

and observe that $\frac{4}{3}-2\varrho-2\varrho_\perp$ is positive by (8.12a).
Upon dilation we then use (6.79a), with $a=\frac{a_1}{3\varrho_\perp}$, to obtain for the second term

$$h^{2-\varrho}\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_1}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq Ch^{2/3-\varrho}\,. \tag{8.74}$$

Then we write, using (6.79b) upon dilation, with δ satisfying (8.31),

$$h^{2-\varrho}\|\breve{\eta}_{k,h}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq Ch^{2/3-\varrho-\varrho_\perp}\,. \tag{8.75}$$

Finally, we obtain with the aid of (8.30) and Lemma 2.6, setting $a_1=2/3-\varrho-\varrho_\perp$,

$$h^{8/3-a_1-2\varrho}\|\breve{\eta}_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq Ch^{2/3-\varrho-\varrho_\perp}\,.$$

Hence (8.35) holds for $\breve{\eta}_{k,h}[\mathcal{A}_h,\eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}$ if d_0 satisfies

$$0<d_0\leq d_6^1:=\frac{2}{3}-\varrho-\varrho_\perp\,, \tag{8.76}$$

where the positivity of d_6^1 follows from (8.12a).

V2 potentials In a similar manner to the V1 case, we begin by employing Lemma 2.6 upon dilation to obtain

$$h^{2(1-\varrho)}\|\breve{\eta}_{k,h}(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq C\,h^{\frac{4}{3}-2\varrho-\varrho_\perp}\,.$$

Then by (6.79a), with $a:=\frac{a_1}{2\varrho_\perp}$, we have

$$h^{2-\varrho}\,\|\breve{\eta}_{k,h}\mathbf{1}_{\rho>h^{2/3-a_1}}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq C\,h^{2/3-\varrho}\,,$$

and by (6.79b),

$$h^{2-\varrho}\,\|\breve{\eta}_{k,h}\partial_s(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\|\leq C\,h^{2/3-\varrho-\varrho_\perp/2}\,.$$

Finally, with the aid of (8.30) and Lemma 2.6, we obtain by setting first $a_1 = 2/3 - \varrho - \frac{1}{2}\varrho_\perp$,

$$h^{8/3-a_1-2\varrho}\|\breve{\eta}_{k,h}\partial_\rho(\tilde{\mathcal{A}}_{k,h}-\lambda)^{-1}\breve{\eta}_{k,h}\| \leq Ch^{2/3-\varrho-\varrho_\perp/2}\,.$$

Hence (8.35) holds in the V2 case for the term $[\mathcal{A}_h, \eta_{k,h}]R_{k,h}\breve{\eta}_{k,h}$ if d_0 satisfies

$$0 < d_0 \leq d_6^2 := \frac{2}{3} - \varrho - \frac{\varrho_\perp}{2}\,, \tag{8.77}$$

where the positivity of d_6^2 results of (8.12b).

Conclusion: **Computation of d_0.**

Combining (8.38), (8.48), (8.54), (8.58), (8.60), (8.69), (8.71), (8.76), and (8.77), we have established (8.35) with

$$d_0 = \inf\{d_1, d_2, d_3, d_4^1, d_4^2, d_5^1, d_5^2, d_6^1, d_6^2\}\,.$$

This completes the proof of the lemma. ∎

8.5 Eigenvalue existence

From (8.34) it follows that $(I + \mathcal{E}(h,\lambda))^{-1}$ is uniformly bounded as $h \to 0$. Since by Lemma 8.1 we have $\|\mathcal{R}(h,\lambda)\| \leq C\,h^{-(k_i+\hat{q}_i)}$, we get the existence of positive h_0 and C , such that for $h \in (0, h_0]$ the circle $\partial B(\hat{\Lambda}^i(h), h^{k_i+\hat{q}_i})$ is in $\rho(\mathcal{A}_h)$ and the resolvent, which is given by

$$(\mathcal{A}_h - \lambda)^{-1} = \mathcal{R}(h,\lambda)(I + \mathcal{E}(h,\lambda))^{-1}\,,$$

consequently satisfies there

$$\|(\mathcal{A}_h - \lambda)^{-1}\| \leq C\,h^{-(k_i+\hat{q}_i)}\,.$$

Hence, we obtain the following result:

Proposition 8.4. *Let $q \in (0, \frac{1}{6})$ and for $i = 1, 2$, $\hat{q}_i = (\frac{2}{3})^i q$, $k_i = \frac{2}{3} + (\frac{2}{3})^i$. Under the assumptions of Theorem 1.1, for potentials of type V1 (where $i = 1$) and the assumptions of Theorem 1.3, for potentials of type V2 (where $i = 2$) there exist positive constants C and h_0 such that, for all $h \in (0, h_0]$,*

$$\sup_{\lambda\in\partial B(\hat{\Lambda}^i(h),h^{k_i+\hat{q}_i})} \|(\mathcal{A}_h - \lambda)^{-1}\| \leq C\,h^{-(k_i+\hat{q}_i)}\,. \tag{8.78}$$

We can now prove the upper bound for the spectrum.

Proposition 8.5. *Let $i \in \{1, 2\}$ and suppose that V is of type Vi. There exist $h_0 > 0$ and, for $h \in (0, h_0]$ an eigenvalue $\lambda \in \sigma(\mathcal{A}_h)$ satisfying*

$$\lambda - \hat{\Lambda}^i(h) = o(h^{k_i}) \quad \text{as } h \to 0\,. \tag{8.79}$$

Proof. Let U^1 be given by (3.29) and U^2 by (4.14) and let $f_i = (\mathcal{A}_h - \hat{\Lambda}^i(h))U^i$. Clearly,

$$(\mathcal{A}_h - \lambda)U^i = f_i + (\hat{\Lambda}^i(h) - \lambda)U^i\,.$$

Hence, for $\lambda \in \partial B(\hat{\Lambda}^i(h), h^{k_i+\hat{q}_i})$, we can write

$$\langle U^i, (\mathcal{A}_h - \lambda)^{-1}U^i\rangle = -\frac{1}{\lambda - \hat{\Lambda}^i(h)}[\langle U^i, U^i\rangle - \langle U^i, (\mathcal{A}_h - \lambda)^{-1} f_i\rangle]\,.$$

By (8.78) and either (3.32) for $i = 1$ or (4.17) for $i = 2$, we then obtain

$$\|(\mathcal{A}_h - \lambda)^{-1} f_i\|_2 \leq \frac{C}{h^{k_i+\hat{q}_i}}\,\|f_i\|_2 \leq C\,h^{m_i-\hat{q}_i}\|U^i\|\,,$$

where $m_1 = 1/3$ and $m_2 = 2/9$.
Consequently, observing that $\hat{q}_i < m_i$ $(i = 1, 2)$,

$$\Big|\frac{1}{2\pi i}\oint_{\partial B(\Lambda^i(h),h^{k_i+\hat{q}_i})}\langle U^i, (\mathcal{A}_h - \lambda)^{-1} f_i\rangle\, d\lambda + \|U^i\|^2\Big| \leq C\,h^{m_i-\hat{q}_i}\,\|U^i\|^2\,.$$

Hence there exists $h_0 > 0$ such that, for $h \in (0, h_0]$, $(\mathcal{A}_h - \lambda)^{-1}$ is not holomorphic in $B(\Lambda^i(h), h^{k_i+\hat{q}_i})$ and the proposition follows. ∎

The existence of an eigenvalue satisfying (8.79) provides an effective upper bound for $\inf \operatorname{Re}\sigma(\mathcal{A}_h)$. Together with the lower bound (2.1), this completes the proof of Theorems 1.1 and 1.3.

A Examples of potentials satisfying (1.4)

We now derive some simple examples of potentials satisfying (1.4), demonstrating that both types of potentials may exist. The basic idea is again that the problem introduced in (1.4) exhibits some invariance to conformal mapping (see [20]). Hence starting form a problem defined on the square $\square$, where the solution of (1.4) is a linear function, we can get from family of conformal maps a corresponding family of potentials satisfying (1.4) in various domains, together with (1.5), (1.2), and (1.3).

Let $\square = (0, 1) \times (0, 1) \subset \mathbb{C}$ and $\Omega = f(\square)$ where, for $w = u + iv$, f is the conformal map

$$f(w) = w + \delta\Big(\frac{1}{2}w^2 + \frac{\gamma}{3}w^3\Big)\,,$$

in which $\delta > 0$ and $\gamma \in \mathbb{R}$. Let further $f(w) = z = x + iy \in \Omega$, and set $g = f^{-1}$: $\Omega \to \square$, which clearly exists for sufficiently small δ. We may now set

$$\partial\Omega_D^1 = \{f(u)\,,\, u \in (0, 1)\} \quad ; \quad \partial\Omega_D^2 = \{f(u + i)\,,\, u \in (0, 1)\}\,,$$

and

$$\partial\Omega_N^1 = \{f(iv)\,,\, v \in (0, 1)\} \quad ; \quad \partial\Omega_N^2 = \{f(1 + iv)\,,\, v \in (0, 1)\}\,.$$

Let $g = U + iV$ (or $U = \operatorname{Re} g$ and $V = \operatorname{Im} g$). Clearly, $V \equiv 0$ on $\partial\Omega_D^1$ and $V \equiv 1$ on $\partial\Omega_D^2$. Furthermore, V is harmonic and since f is conformal, we must have

$\partial V/\partial\vartheta = 0$ on $\partial\Omega_N$. It follows that V is a solution of (1.4) with $C_1 = 0$ and $C_1 = 1$. Since V is constant on each connected component of $\partial\Omega_D$ we have there, using the Cauchy-Riemann equations satisfied by g,

$$|\partial V/\partial\vartheta| = |\nabla V| = |g'| \,. \tag{A.1}$$

The same argument shows that for fixed $\gamma \in \mathbb{R}$, and for δ small enough, Assumption 1.5 is also satisfied in $\overline{\Omega}$. Consequently, we need to identify the location of

$$\inf_{z\in\partial\Omega_D} |g'(z)| = \inf_{0<u<1} \min\Big(\frac{1}{|f'(u)|}\,,\,\frac{1}{|f'(u+i)|}\Big)\,.$$

It can be easily verified that whenever $-2 < \gamma < 0$ we have

$$\sup_{0<u<1} |f'(u)| = \Big|f'\Big(-\frac{1}{2\gamma}\Big)\Big| = 1 + \frac{\delta}{4\gamma}\,,$$

and for sufficiently small δ we have

$$\sup_{0<u<1} |f'(u+i)| = \Big|f'\Big(-\frac{1}{2\gamma}+i\Big)\Big| + \mathcal{O}(\delta^2) = 1 - \frac{\delta}{4\gamma} - \delta\gamma + \mathcal{O}(\delta^2)\,.$$

It follows that whenever $\gamma < -1/2$ the minimum of $|\partial V/\partial\vartheta|$ over $\partial\Omega_D$ is obtained, for sufficiently small δ at an interior point (close to $i-\frac{1}{2\gamma}$), whereas for $0 > \gamma > -1/2$ the minimum is attained, for sufficiently small δ at one of the corners.
Note that

$$f^{(3)}(z) = \delta\gamma\,,$$

and hence, for $\gamma < -1/2$, the maximum of $|f'|$ (or the minimum of $|g'|$) is non-degenerate.
Hence, depending on the value of γ, we can either find δ and a pair (V,Ω) for which (V1) is satisfied or find δ and a pair (V,Ω) for which (V2) is satisfied.

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