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Abstract. We consider the problem of defining an action of Hecke operators on the coherent cohomology of certain integral models of Shimura varieties. We formulate a general conjecture describing which Hecke operators should act integrally and solve the conjecture in certain cases. As a consequence, we obtain p-adic estimates of Satake parameters of certain non-regular self dual automorphic representations of GL_n.

Contents

1. Introduction

This paper studies the problem of defining an action of Hecke operators on certain natural integral coherent cohomology of Shimura varieties.

Let us start by describing the situation for modular curves. Let $N \geq 4$ be an integer, and let $X \to \mathrm{Spec}\ \mathbb{Z}[1/N]$ be the compactified modular curve of level $\Gamma_1(N)$. Let $k \in \mathbb{Z}$ and let ω^k be the sheaf of weight k modular forms. Let p be a prime number, $(p, N) = 1$. When $k \geq 1$, there is a familiar Hecke operator T_p acting on the $\mathbb{C}$-vector space of weight k modular forms. On q-expansions, the operator T_p is given (on forms of nebentypus a character $\chi : \mathbb{Z}/N\mathbb{Z}^\times \to \overline{\mathbb{Z}}^\times$) by the formula $T_p(\sum_{n\geq 0} a_n q^n) = \sum_{n \geq 0} a_{np} q^n + p^{k-1}\chi(p) a_n q^{np}$. This formula is integral and the q-expansion principle implies that the action of T_p actually arises from an action on $\mathrm{H}^0(X, \omega^k)$. We now give a more geometric construction of T_p. Assume first that we work over $\mathbb{Z}[1/Np]$. Then, viewing a modular form f as a rule on triples (E, α_N, ω) where E is an elliptic curve, α_N is a point of order N, and ω is a nowhere vanishing differential form (and satisfying a growth condition near the cusps), there is a geometric formula defining T_p [Kat73, Formula 1.11.0.2]:

$$T_p(f(E, \alpha_N, \omega)) = \frac{1}{p} \sum_{H \subset E[p]} f(E/H, \alpha'_N, \omega')$$

where $H \subset E[p]$ runs over all subgroups of $E[p]$ of order p, and if we let $\pi_H : E \to E/H$ be the isogeny then $\pi_H(\alpha_N) = \alpha'_N$ and $\pi_H^\star \omega' = \omega$.

The above formula does not really make sense over $\mathbb{Z}[1/N]$, but a slight modification of it will be meaningful. We explain this modification.

There is a correspondence $X_0(p)$ over X corresponding to the level $\Gamma_1(N) \cap \Gamma_0(p)$ (so $X_0(p)$ is a compactification of the moduli of triples (E, α_N, H) where $H \subset E[p]$ is a subgroup of order p), and there are two projections : $p_1, p_2 : X_0(p) \to X$ (induced by $(E, H) \mapsto E$ and $(E, H) \mapsto E/H$). Then T_p originates from a cohomological correspondence $T_p : p_2^\star \omega^k \to p_1^! \omega^k$, where $p_1^!$ is the right adjoint functor to $(p_1)_\star$.

The cohomological correspondence T_p is obtained using the differential of the universal isogeny $\pi_H^\star : p_2^\star \omega \to p_1^\star \omega$, the trace of p_1, and a suitable normalization by a power of p that makes everything integral (this corresponds to the coefficient $\frac{1}{p}$ in the displayed formula). More precisely, we first define a (rational if $k \leq 0$) map over $X_0(p)$,

$$T_p^{naive} : p_2^\star \omega^k \xrightarrow{(\pi_H^\star)^{\otimes k}} p_1^\star \omega^k \xrightarrow{\mathrm{tr}_{\mathrm{p}_1}} p_1^! \omega^k$$

and then set $T_p = p^{-1} T_p^{naive}$ when $k \geq 1$ and $T_p = p^{-k} T_p^{naive}$ if $k \leq 1$.

The first basic result for elliptic modular forms is the following (see also [Con07, §4.5] and [ERX17, Proposition 3.11]):

Proposition 1.1. *For any $k \in \mathbb{Z}$, we have a cohomological correspondence $T_p : p_2^\star \omega^k \to p_1^! \omega^k$ over $X_0(p)$, inducing a Hecke operator $T_p \in \mathrm{End}(\mathrm{R}\Gamma(X, \omega^k))$*

Given the cohomological correspondence, the operator on the cohomology is simply obtained using pullback and pushforward as follows:

$$T_p : \mathrm{H}^\star(X, \omega^k) \to \mathrm{H}^\star(X_0(p), p_2^\star \omega^k) \xrightarrow{T_p} \mathrm{H}^\star(X_0(p), p_1^! \omega^k) \to \mathrm{H}^\star(X, \omega^k).$$

The action of T_p on formal q-expansions (of level $\Gamma_1(N)$ and Nebentypus χ) is given by :

- $T_p(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} a_{np} q^n + p^{k-1} \chi(p) a_n q^{np}$ if $k \geq 1$,
- $T_p(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} p^{1-k} a_{np} q^n + \chi(p) a_n q^{np}$ if $k \leq 1$,

therefore T_p appears to be optimally integral.

One would like to generalize this to other Shimura varieties. To do this, we introduce some notations and recall the fundamental results concerning the cohomology of automorphic vector bundles on Shimura varieties. A standard reference for this material is [Har90a].

Let (G, X) be a Shimura datum, let $K \subset G(\mathbb{A}_f)$ be a neat compact open subgroup, and let Sh_K be the associated Shimura variety. This is a smooth scheme defined over the reflex field E. Over Sh_K there is a large supply of automorphic vector bundles $\mathcal{V}_{\kappa,K}$, naturally parametrized by weights κ of a maximal torus in G, dominant for the compact roots. For a choice of polyhedral cone decomposition Σ, one can construct a toroidal compactification $Sh^{tor}_{K,\Sigma}$ of Sh_K, such that $D_{K,\Sigma} = Sh^{tor}_{K,\Sigma} \setminus Sh_{K,\Sigma}$ is a Cartier divisor. Moreover, there is a canonical extension $\mathcal{V}_{\kappa,K,\Sigma}$ of the vector bundle $\mathcal{V}_{\kappa,K}$, as well as a sub-canonical extension $\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma})$.

The coherent cohomology complexes we are interested in (which generalize the classical notion of modular forms) are:

$$\mathrm{R}\Gamma(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}) \text{ and } \mathrm{R}\Gamma(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})).$$

These cohomology complexes are independant of Σ, they have good functorial properties in the level K and they carry an action of the Hecke algebra $\mathcal{H}_K = \mathcal{C}_c^\infty(K \backslash G(\mathbb{A}_f)/K, \mathbb{Z})$.

Over $\mathbb{C}$ this coherent cohomology can be computed via the $(\mathfrak{p}, K)$-cohomology of the space of automorphic forms on the group G [Jun18]. Automorphic representations contributing to these cohomology groups therefore possess a rational structure. We now fix a prime p. In order to study p-adic properties of automorphic forms contributing to the coherent cohomology, we introduce integral structures on $\mathrm{R}\Gamma(Sh^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(Sh^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$.

Let λ be a place of E above p. If the Shimura datum (G,X) is of abelian type, G is unramified at p, and $K=K^pK_p$, where $K^p\subset G(\mathbb{A}_f^p)$ and $K_p\subset G(\mathbb{Q}_p)$ is hyperspecial, we have natural integral models $\mathfrak{Sh}_K$ for Sh_K, as well as an integral model of $\mathcal{V}_{\kappa,K}$, over $\mathrm{Spec}\ \mathcal{O}_{E,\lambda}$ [MFK94], [Kot92], [Kis10], [KMP16]. Moreover, there is a theory of toroidal compactifications (at least in the Hodge case) $\mathfrak{Sh}^{tor}_{K,\Sigma}$ of $\mathfrak{Sh}_K$, with integral canonical extension $\mathcal{V}_{\kappa,K,\Sigma}$, and integral sub-canonical extension $\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})$ [FC90], [Lan13], [Lan12], [MP19].

We are therefore led to consider the cohomology complexes $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma})$ as well as $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$. Our work investigates the question of extending the action of the Hecke algebra $\mathcal{H}_K$ to these cohomology groups. The action of the prime to p Hecke algebra extends without much difficulty, so our main task is to investigate the action of (a suitable sub-algebra of) the local Hecke algebra $\mathcal{C}_c^\infty(K_p\backslash G(\mathbb{Q}_p)/K_p,\mathbb{Q})$ on $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma})$.

We formulate a conjecture (see Conjecture 4.16, as well as Conjectures 4.7 and 4.15) precisely describing a sub $\mathbb{Z}$-algebra $\mathcal{H}^{int}_{p,\kappa,\iota}$ (where ι is an isomorphism of $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$) of $\mathcal{C}_c^\infty(K_p\backslash G(\mathbb{Q}_p)/K_p,\mathbb{Q})$ which should act on the cohomology complexes $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma}))$.

The definition of $\mathcal{H}^{int}_{p,\kappa,\iota}$ is given in Definition 4.8, with the help of the Satake basis $[V_\lambda]$ of $\mathcal{H}_p$ (where λ runs through the set of dominant and Galois invariant chararacters of the dual group $\hat{G}$ of G) normalized by a certain power of p determined by λ and the weight κ.

In the modular curve case, the description of this sub-algebra precisely reflects the normalizing factors $p^{-\inf\{1,k\}}$ in the definition of the T_p-operator on the cohomology in weight k.

This conjecture is a translation for the coherent cohomology of Shimura varieties of results obtained by V. Lafforgue on the Betti cohomology of locally symmetric spaces in [Laf11]. It is inspired by Katz–Mazur inequality: For X a proper smooth scheme defined over $\mathbb{Z}_p$, Katz and Mazur gave p-adic estimates for the eigenvalues of the geometric Frobenius acting on the cohomology $\mathrm{H}^i(X_{\overline{\mathbb{Q}}_p},\mathbb{Q}_\ell)$ $(\ell\neq p)$. The estimates say that p-adic Newton polygon of the characteristic polynomial of Frobenius lies above the Hodge polygon (determined by the filtration on the de Rham cohomology of $X_{\mathbb{Q}_p}$).

One believes that similar estimates hold for algebraic automorphic representations because of the conjectural correspondence between motives and automorphic representations [Clo90], [BG14]. Roughly speaking, the weight κ determines the Hodge polygon, while the characters of $\mathcal{H}_p$ determine the characteristic polynomials of Frobenii. We postulate the Hodge–Newton inequality to determine the precise shape of the maximal sub-algebra $\mathcal{H}^{int}_{p,\kappa,\iota}\subset\mathcal{H}_p$ which should act integrally on the cohomology.

We then try to prove our conjecture in certain special cases. To the charateristic functions of all double cosets K_pgK_p in the local Hecke algebra, one can associate Hecke correspondences $p_1,p_2: Sh_{K\cap gKg^{-1}}\to Sh_K$ over Sh_K. These correspondences rarely admit integral models whose geometry is understood, except when g is associated with a minuscule coweight. In this case, $K_p\cap gK_pg^{-1}$ is a parahoric subgroup and there is a good theory of integral models whose local geometry is described by the local model theory. We are thus led to work with Hecke operators associated to minuscule coweights (this is a serious restriction).

At this point a second obstacle arises: the projections p_i almost never extend integrally to finite flat morphisms. This means that defining the necessary trace maps in cohomology is in principle complicated. We develop, using Grothendieck–Serre duality [Har66], a formalism of cohomological correspondences in coherent cohomology which solves the problem under some assumptions on the correspondences. In particular, we assume that our correspondences are given by Cohen–Macaulay schemes. This suffices for our pusposes since using the theory of local models (see, e.g., [PRS13]), one can often prove that integral models of Shimura varieties with parahoric level structure are Cohen–Macaulay.

In order to prove Conjecture 4.16, our strategy is to switch to the local model, where we can make all the computations, and then transfer back the information to the Shimura variety.

We completely solve our conjecture for Hilbert modular varieties ($G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$), when p is unramified in the totally real field F. To formulate the result precisely in this case, let E' be the finite extension of $E = \mathbb{Q}$ equal to the Galois closure of F. Weights for Hilbert modular forms are tuples of integers $\kappa = ((k_\sigma)_{\sigma\in\mathrm{Hom}(F,E')}; k)$ where k_σ and k all have the same parity. We let $\iota : E' \to \overline{\mathbb{Q}}_p$ be an embedding. Let $p = \prod_i \mathfrak{p}_i$ be the decomposition of p in $\mathcal{O}_F$ has a product of prime ideals.

Theorem 1.2 (Theorem 5.9 and §5.6.4)**.** *The Hecke algebra $\mathcal{H}^{int}_{p,\kappa,\iota} = \otimes_i \mathbb{Z}[T_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}^{-1}]$ acts on $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$.*

Under the assumption that the weight κ belongs to a certain cone (conjecturally the ample cone), the theorem was first proved in [ERX17, Proposition 3.11] by different methods.

We can be explicit about normalization factors. Let $I_i = \{\sigma \in \mathrm{Hom}(F, E'),\ \iota\circ\sigma(\mathfrak{p}_i) \subset \mathfrak{m}_{\overline{\mathbb{Z}}_p}\}$ and let

$$T^{naive}_{\mathfrak{p}_i} = \mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i^{-1}, 1)\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})$$

and

$$S^{naive}_{\mathfrak{p}_i} = \mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i^{-1}, \mathfrak{p}_i^{-1})\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})$$

be the naive, "unnormalized" version of the Hecke operators attached to the familiar double classes (the reason we have to use these double classes and not their inverse is that we set up the theory in such a way that the Hecke algebra acts naturally on the left and not on the right on the cohomology). Our normalized operators are:

$$T_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i}\sup\{\frac{k_\sigma+k}{2}-1, \frac{k-k_\sigma}{2}\}} T^{naive}_{\mathfrak{p}_i}$$

and

$$S_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i} k} S^{naive}_{\mathfrak{p}_i}.$$

For more general Shimura varieties of symplectic type we only have partial results because there is essentially only one minuscule coweight (see Theorem 5.9). However, the situation is better for unitary Shimura varieties. Let F be a totally real field, and let L be a totally imaginary quadratic extension of F. We let $G \subset \mathrm{Res}_{L/\mathbb{Q}}\mathrm{GL}_n$ be a unitary group of signature $(p_\tau, q_\tau)_{\tau:F\hookrightarrow\mathbb{C}}$. We assume that p is unramified in L and we let $\mathfrak{Sh}^{tor}_{K,\Sigma}$ be a toroidal compactification of the (smooth) integral model of the unitary Shimura variety attached to G.

Theorem 1.3 (Theorem 7.5)**.** *Assume that all finite places $v \mid p$ in F split in L. Let $\mathcal{H}^{int}_{p,\kappa,\iota} = \otimes_{0\le i\le m}\mathbb{Z}[T_{\mathfrak{p}_i,j},\ 0 \le j \le n, T^{-1}_{\mathfrak{p}_i,0}, T^{-1}_{\mathfrak{p}_i,n}]$ be the normalized integral Hecke algebra in weight κ (where the $T_{\mathfrak{p}_i,j}$ are standard generators of this Hecke algebra). All the operators $T_{\mathfrak{p}_i,j}$ act on $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$. In particular,*

$\mathrm{Im}\big(\mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}) \to \mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\big)$ *is a lattice which is stable under the action of* $\mathcal{H}^{int}_{p,\kappa,\iota}$.

Remark 1.4. The limitation of this last result is that we have not been able to prove that the operators $T_{\mathfrak{p}_i,j}$ commute with each other in general.

Let L be a CM or a totally real number field. One can realize regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of GL_n/L in the Betti cohomology of Shimura varieties. The interest of coherent cohomology is that it captures more automorphic representations. Namely, one can weaken the regularity condition to a condition that we call weakly regular odd (the oddness property is automatically satisfied in the regular case). Weakly regular, algebraic, odd, essentially (conjugate) self dual, cuspidal automorphic representation on GL_n/L admit a compatible system of Galois representations, but at the moment these compatible systems are not known to be de Rham in general (and local-global compatibility is not known). We can nevertheless prove the following result which is to be viewed as the Katz–Mazur inequality.

Theorem 1.5 (Theorem 9.11)**.** *Let* π *be a weakly regular, algebraic, odd, essentially (conjugate) self dual, cuspidal automorphic representation of* GL_n/L *with infinitesimal character* $\lambda = (\lambda_{i,\tau}, 1 \le i \le n, \tau \in \mathrm{Hom}(L, \overline{\mathbb{Q}}))$ *and* $\lambda_{1,\tau} \ge \cdots \ge \lambda_{n,\tau}$*. Let* p *be a prime unramified in* L *and* w *be a finite place of* L *dividing* p*. Assume also that* π_w *is spherical, and corresponds to a semi-simple conjugacy class* $\mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ *by the Satake isomorphism. We let* $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ *be an embedding and* v *the associated* p*-adic valuation normalized by* $v(p) = 1$*. After permuting we assume that* $v(a_1) \le \cdots \le v(a_n)$*. Let* $I_w \subset \mathrm{Hom}(L, \overline{\mathbb{Q}})$ *be the set of embeddings* τ *such that* $\iota \circ \tau$ *induces the* w*-adic valuation on* L*. Then we have*

$$\sum_{i=1}^{k} v(a_i) \ge \sum_{\tau \in I_w} \sum_{\ell=1}^{k} -\lambda_{\ell,\tau},$$

for $1 \le k \le n$*, with equality if* $k = n$*.*

1.1. **Organisation of the paper.** In Section 2 we develop a formalism of cohomological correspondences in coherent cohomology. In Section 3 we give a number of classical results concerning the structure of the local spherical Hecke algebra of an unramified group. In Section 4, we introduce Shimura varieties and their coherent cohomology and formulate Conjecture 4.16 on the action of the integral Hecke algebra on the integral coherent cohomology. In Section 5 and 6 we consider the case of Shimura varieties of symplectic type and their local models. In Section 7 and 8 we consider the case of Shimura varieties of unitary type and their local models. Finally, the last section deals with applications to automorphic representations and Galois representations.

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2. Correspondences and coherent cohomology

2.1. **Preliminaries on residues and duality.** We start by recalling some results of Grothendieck duality theory for coherent cohomology. The original reference for this, which we use below, is [Har66]; although this and [Con00], which is based on it, suffice for our purposes, the more abstract approaches of [Ver69], [Nee96] and [Lip09] give more general results with arguably more efficient proofs. In particular, the latter two references show that the noetherian and finite Krull dimension hypotheses of [Har66] can be eliminated and that most results extend to the unbounded derived category.

For a scheme X we let $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ be the subcategory of the derived category $\mathbf{D}(\mathscr{O}_X)$ of $\mathscr{O}_X$-modules whose objects have quasi-coherent cohomology sheaves. We let $\mathbf{D}^+_{qcoh}(\mathscr{O}_X)$ (resp. $\mathbf{D}^-_{qcoh}(\mathscr{O}_X)$) be the full subcategory of $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ whose objects have 0 cohomology sheaves in sufficiently negative (resp. positive) degree. We let $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)$ be the full subcategory of $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ whose objects have 0 cohomology sheaves for all but finitely many degrees. We remark that if X is locally notherian $\mathbf{D}^+_{qcoh}(\mathscr{O}_X)$ is also the derived category of the category of bounded below complexes of quasi-coherent sheaves on X [Har66, I, Corollary 7.19]. We let $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)_{fTd}$ be the full subcategory of $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)$ whose objects are quasi-isomorphic to bounded complexes of flat sheaves of $\mathscr{O}_X$-modules [Har66, II, Definition 4.13]. We fix for the rest of this section a noetherian affine scheme S.

2.1.1. *Embeddable morphisms.* Let X, Y be two S-schemes and $f : X \to Y$ be a morphism of S-schemes. The morphism f is embeddable if there exists a smooth S-scheme P and a finite map $i : X \to P \times_S Y$ such that f is the composition of i and the second projection [Har66, p. 189]. The morphism f is projectively embeddable if it is embeddable and P can be taken to be a projective space over S [Har66, p. 206].

2.1.2. *The functor $f^!$.* For $f : X \to Y$ a morphism of S-schemes, there is a functor $Rf_\star : \mathbf{D}_{qcoh}(\mathscr{O}_X) \to \mathbf{D}_{qcoh}(\mathscr{O}_Y)$. By [Har66, III, Theorem 8.7] if f is embeddable, we can define a functor $f^! : \mathbf{D}^+_{qcoh}(\mathscr{O}_Y) \to \mathbf{D}^+_{qcoh}(\mathscr{O}_X)$. If f is projectively embeddable, by [Har66, III, Theorem 10.5] there is a natural transformation (trace map) $Rf_\star f^! \Rightarrow Id$ of endofunctors of $\mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$. Moreover, by [Har66, III, Theorem 11.1] this natural transformation induces a duality isomorphism:

$$\mathrm{Hom}_{\mathbf{D}_{qcoh}(\mathscr{O}_X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{qcoh}(\mathscr{O}_Y)}(Rf_\star\mathscr{F}, \mathscr{G}) \tag{2.1.A}$$

for $\mathscr{F} \in \mathbf{D}^-_{qcoh}(\mathscr{O}_X)$ and $\mathscr{G} \in \mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$.

The functor $f^!$ for embeddable morphism enjoys many good properties. Let us record one that will be crucially used.

Proposition 2.1. [Har66, III, Proposition 8.8] *If $\mathscr{F} \in \mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$ and $\mathscr{G} \in \mathbf{D}^b_{qcoh}(\mathscr{O}_Y)_{fTd}$, there is a functorial isomorphism $f^!\mathscr{F} \otimes^L Lf^\star\mathscr{G} = f^!(\mathscr{F} \otimes^L \mathscr{G})$.*

2.2. **Cohen–Macaulay, Gorenstein and lci morphisms.**

2.2.1. *Cohen–Macaulay morphisms.* Recall that a morphism $h : X \to S$ is called Cohen–Macaulay (abbreviated to *CM* for the rest of this section) if h is flat, locally of finite type and has CM fibres.

Lemma 2.2. *For a CM morphism of pure relative dimension n, $h^!\mathscr{O}_S = \omega_{X/S}[n]$, where $\omega_{X/S}$ is a coherent sheaf which is flat over S. Assume moreover that S is CM. Then $\omega_{X/S}$ is a CM sheaf.*

Proof. For the first point, see [Con00, Theorem 3.5.1]. For the second point, first observe that if S is Gorenstein (e.g., S is the spectrum of a field), so $\mathscr{O}_S[0]$ is a dualizing complex for S, then $\omega_{X/S}[n] = h^!\mathscr{O}_S$ is a dualizing complex for X [Har66, V, §§2,10]. Using local duality [BH98, Theorem 1.2.8], it follows from [BH98, Corollary 3.5.11] that $\omega_{X/S}$ is a CM sheaf. In general, for a CM map h as above, the formation of $\omega_{X/S}$ is compatible with base change [Con00, Theorem 3.6.1], so it follows from [Gro65, Proposition 6.3.1] that $\omega_{X/S}$ is a CM sheaf.

□

2.2.2. *Gorenstein morphisms.* A local ring A is Gorenstein if it satisfies the equivalent properties of [Har66, V, Theorem 9.1] (for instance, if A is a dualizing complex for itself). Note that local complete intersection rings, in particular regular rings, are Gorenstein. Furthermore, Gorenstein rings are CM. A locally noetherian scheme is Gorenstein if all its local rings are Gorenstein. Finally, a morphism $h : X \to S$ is Gorenstein if it is flat and all its fibres are Gorenstein.

Lemma 2.3. *For a proper Gorenstein morphism $h : X \to S$ of pure relative dimension n, $h^!\mathscr{O}_S = \omega_{X/S}[n]$, where $\omega_{X/S}$ is an invertible sheaf on X.*

Proof. A Gorenstein morphism is CM so by Lemma 2.2, $h^!\mathscr{O}_S = \omega_{X/S}[n]$, where $\omega_{X/S}$ is a coherent sheaf on X. Since for a CM map h the formation of $\omega_{X/S}$ is compatible with base change [Con00, Theorem 3.6.1], for any point $s \in S$ $\omega_{X/S}|_{X_s} = \omega_{X_s/s}$. By [Har66, V, Proposition 2.4, Theorem 8.3] (and the remark after Theorem 8.3), $\omega_{X_s/s}$ is a dualizing sheaf for X_s. By [Har66, V, Theorem 3.1], the Gorenstein hypothesis on X_s implies that $\omega_{X_s/s}$ is an invertible sheaf. Since $\omega_{X/S}$ is flat over S by Lemma 2.2, it is flat over X by the fibrewise flatness criterion [Gro66, Théorème 11.3.10], so it is an invertible sheaf over X. □

2.2.3. *Local complete intersection morphisms.* A morphism $h : X \to S$ is called a local complete intersection (henceforth *lci*) morphism if locally on X we have a factorization $h : X \xrightarrow{i} Z \to S$ where i is a regular immersion [Gro66, Définition 16.9.2], and Z is a smooth S-scheme. If h is lci, then the cotangent complex of h, denoted by $\mathbb{L}_{X/S}$, is a perfect complex concentrated in degree -1 and 0 [Ill71, Proposition 3.2.9]. Its determinant in the sense of [KM76] is denoted by $\omega_{X/S}$.

Lemma 2.4. *If $h : X \to S$ is an embeddable morphism and a local complete intersection of pure relative dimension n, then $h^!\mathscr{O}_X = \omega_{X/S}[n]$ where $\omega_{X/S}$ is the determinant of the cotangent complex $\mathbb{L}_{X/S}$.*

Proof. This follows from the very definition of $h^!$ given in [Har66, III, Theorem 8.7]. □

2.3. **An application: construction of a trace map.**

Proposition 2.5. *Let X, Y be two embeddable S-schemes and let $f : X \to Y$ be an embeddable morphism. Assume that $X \to S$ is CM and that $Y \to S$ is Gorenstein. Assume that X and Y have the same pure relative dimension over S. Then $f^!\mathscr{O}_Y := \omega_{X/Y}$ is a coherent sheaf. If moreover S is assumed to be CM, then $\omega_{X/S}$ is a CM sheaf. If X and Y are smooth over S, $\omega_{X/S} = \det \Omega^1_{X/S} \otimes_{f^{-1}\mathscr{O}_Y} (f^{-1} \det \Omega^1_{Y/S})^{-1}$.*

Proof. We have $h^!\mathscr{O}_S = \omega_{X/S}[n]$. On the other hand,

$$\begin{aligned} h^!\mathscr{O}_S &= f^!(g^!\mathscr{O}_S) \\ &= f^!(\omega_{Y/S}[n]) \\ &= f^!(\mathscr{O}_Y \otimes \omega_{Y/S}[n]) \\ &= f^!(\mathscr{O}_Y) \otimes f^\star\omega_{Y/S}[n]. \end{aligned}$$

□

We observe that by adjunction we have a universal trace map $Rf_\star\omega_{X/Y} \to \mathscr{O}_Y$. In particular for any section $s \in \mathrm{H}^0(X, \omega_{X/Y})$ (or equivalently morphism $\mathscr{O}_X \to \omega_{X/Y}$), we get a corresponding trace $\mathrm{Tr}_s : Rf_\star\mathscr{O}_X \to \mathscr{O}_Y$.

Proposition 2.6. *Let X, Y be two embeddable S-schemes and let $f : X \to Y$ be an embeddable morphism. Assume that:*

(1) $X \to S$ is CM,
(2) $Y \to S$ is Gorenstein,
(3) S is CM,
(4) X and Y have the same pure relative dimension over S,
(5) there are open sets $V \subset X$, $U \subset Y$ such that $f(V) \subset U$, U and V smooth over S and $X \backslash V$ is of codimension 2 in X.

Then there is a canonical morphism $\Theta : \mathscr{O}_X \to \omega_{X/Y}$ called the fundamental class. Moreover if $W \subset Y$ is an open subscheme and $X \times_Y W \to W$ is finite flat, then the trace Tr_Θ restricted to W is the usual trace map for the finite flat morphism $X \times_Y W \to W$.

Proof. It is enough to specify the fundamental class over V because it will extend to all of X by Lemma 2.2. Then over V, we have a map $\mathrm{d}f : f^\star\Omega^1_{U/S} \to \Omega^1_{V/S}$ and we define the fundamental class as the determinant $\det(\mathrm{d}f) \in \mathrm{H}^0(V, \det\Omega^1_{V/S} \otimes (f^\star \det\Omega^1_{U/S})^{-1})$. To prove the second claim, we can assume that X, Y are smooth over S and the map $X \to Y$ is finite flat. In this situation, $X \to Y$ is lci. We claim that the cotangent complex $\mathbb{L}_{X/Y}$ is represented by the complex in degree -1 and 0 : $[f^\star\Omega^1_{Y/S} \xrightarrow{\mathrm{d}f} \Omega^1_{X/S}]$, and that the determinant $\det(\mathrm{d}f) \in \mathrm{H}^0(X, \omega_{X/Y}) = \mathrm{Hom}(\mathscr{O}_X, f^!\mathscr{O}_Y)$ is the trace map tr_f. We have a closed embedding $i : X \hookrightarrow X \times_S Y$ of X into the smooth Y-scheme $X \times_S Y$. We have an exact sequence:

$$0 \to \mathcal{I}_Y \to \mathscr{O}_{Y\times_S Y} \to \mathscr{O}_Y \to 0$$

which gives after tensoring with $\mathscr{O}_X$ above $\mathscr{O}_Y$

$$0 \to \mathcal{I}_X \to \mathscr{O}_{X\times_S Y} \to \mathscr{O}_X \to 0$$

where $\mathcal{I}_X$ is the ideal sheaf of the immersion i. It follows that $\mathcal{I}_X/\mathcal{I}_X^2 = \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_X = \Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{O}_X$.

On the other hand, $i^\star\Omega^1_{X\times_S Y/Y} = \Omega^1_{X/S}$. The cotangent complex is represented by $[\mathcal{I}_X/\mathcal{I}_X^2 \to i^\star\Omega^1_{X\times_S Y/Y}]$ which is the same as $[f^\star\Omega^1_{Y/S} \to \Omega^1_{X/S}]$.

The morphism $f_\star \det \mathbb{L}_{X/Y} = \underline{\mathrm{Hom}}(f_\star\mathscr{O}_X, \mathscr{O}_Y) \to \mathscr{O}_Y$ is the residue map which associates to $\omega \in f_\star\Omega^1_{X/S}$ and to $(t_1, \cdots, t_n)$ local generators of the ideal $\mathcal{I}_X$ over Y the function $\mathrm{Res}[\omega, t_1, ..., t_n]$. It follows from [Har66, Property (R6), p. 198] that the determinant of $[f^\star\Omega^1_{Y/S} \to \Omega^1_{X/S}]$ maps to the usual trace map. □

2.3.1. *Fundamental class and divisors.* We need a slight variant of the last proposition, where we also have some divisors.

Proposition 2.7. *Let X, Y be two embeddable S-schemes and let $f : X \to Y$ be an embeddable morphism. Let $D_X \hookrightarrow X$ and $D_Y \hookrightarrow Y$ be relative (with respect to S) effective Cartier divisors.*

Assume that:

(1) $X \to S$ is CM,
(2) $Y \to S$ is Gorenstein,
(3) S is CM,
(4) X and Y have the same pure relative dimension over S,

(5) there are open sets $V \subset X$, $U \subset Y$ such that $f(V) \subset U$, U and V are smooth over S and $X \backslash V$ is of codimension 2 in X,
(6) the morphism f restricts to a surjective map from D_X to D_Y,
(7) $D_X \cap V$ and $D_Y \cap U$ are relative normal crossings divisors.

The fundamental class $\Theta : \mathscr{O}_X \to f^! \mathscr{O}_Y$ constructed in Proposition 2.6 induces a map:

$$\mathscr{O}_X(-D_X) \to f^! \mathscr{O}_Y(-D_Y).$$

Proof. We may assume that X and Y are smooth, D_X and D_Y are relative normal crossing divisors. In that case, we have a well defined differential map $\mathrm{df} : f^\star \Omega^1_{Y/S}(\log D_Y) \to \Omega^1_{X/S}(\log D_X)$. Taking the determinant yields $\det \mathrm{df} : f^\star \det \Omega^1_{Y/S}(D_Y) \to \det \Omega^1_{X/S}(D_X)$ or equivalently $\det \mathrm{df} : \mathscr{O}_X(-D_X) \to f^! \mathscr{O}_Y(-D_Y)$. □

2.4. **Cohomological correspondences.** Let X, Y be two S-schemes.

Definition 2.8. *A correspondence C over X and Y is a diagram of S-morphisms :*

$$\begin{tikzcd} & C \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X & & Y \end{tikzcd}$$

where X, Y, C have the same pure relative dimension over S and the morphisms p_1 and p_2 are projectively embeddable.

Remark 2.9. In practice, the maps p_1, p_2 will often be surjective and generically finite.

Let $\mathscr{F} \in \mathbf{D}^-_{qcoh}(\mathscr{O}_X)$ and $\mathscr{G} \in \mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$.

Definition 2.10. *A cohomological correspondence from $\mathscr{F}$ to $\mathscr{G}$ is the data of a correspondence C over X and Y and a map $T : R(p_1)_\star L p_2^\star \mathscr{F} \to \mathscr{G}$.*

The map T can be seen, by (2.1.A), as a map $Lp_2^\star \mathscr{F} \to p_1^! \mathscr{G}$. Note that if $\mathscr{F}$ and $\mathscr{G}$ are coherent sheaves (in degree 0), then $Lp_2^\star \mathscr{F}$ is concentrated in degrees ≤ 0 and $p_1^! \mathscr{G}$ is concentrated in degrees ≥ 0 (as follows from the construction in [Har66, III, Theorem 8.7]), so any map $Lp_2^\star \mathscr{F} \to p_1^! \mathscr{G}$ factors uniquely through the natural map $Lp_2^* \mathscr{F} \to p_2^* \mathscr{F}$. It gives rise to a map, still denoted by T, on cohomology:

$$\mathrm{R}\Gamma(X, \mathscr{F}) \xrightarrow{Lp_2^\star} \mathrm{R}\Gamma(C, p_2^\star \mathscr{F}) = \mathrm{R}\Gamma(Y, \mathrm{R}(p_1)_\star p_2^\star \mathscr{F}) \xrightarrow{T} \mathrm{R}\Gamma(Y, \mathscr{G}).$$

Example 2.11. Let C be a correspondence over X and Y. We assume that the map $p_1 : C \to Y$ satisfies the assumptions of Proposition 2.6, so there is a fundamental class $\mathscr{O}_C \to p_1^! \mathscr{O}_Y$. Let $\mathscr{F}$ and $\mathscr{G}$ be locally free sheaves of finite rank over X and Y. We assume that there is a map $p_2^\star \mathscr{F} \to p_1^\star \mathscr{G}$. Twisting the fundamental class by $p_1^\star \mathscr{G}$, we get a morphism $p_1^\star \mathscr{G} \to p_1^! \mathscr{O}_Y \otimes p_1^\star \mathscr{G}$ and using the isomorphism of Proposition 2.1, we get a map $p_1^\star \mathscr{G} \to p_1^! \mathscr{G}$. Composing everything we obtain $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{G}$.

3. The Satake isomorphism for unramified reductive groups

3.1. **The dual group.** Let K be a nonarchimedean local field, $\mathcal{O}_K$ its ring of integers, and $\pi \in \mathcal{O}_K$ a uniformizing element. Let q be the cardinality of the residue field $\mathcal{O}_K/(\pi)$. Let G be a reductive group over $\operatorname{Spec} \mathcal{O}_K$. We assume that G is quasi-split and splits over an unramified extension of K, and fix a Borel subgroup B and a maximal torus $T \subset B$. We denote by $X_\star(T)$ and $X^\star(T)$ the groups of cocharacters and characters of T. These groups carry an action of $\Gamma = \hat{\mathbb{Z}}$, the unramified Galois group. We denote by $T_d \subset T$ the maximal split torus inside T. Its cocharacter group is $X_\star(T)^\Gamma$ and we have $X^\star(T_d)_\mathbb{R} = (X^\star(T)_\mathbb{R})_\Gamma$.

We now introduce the Langlands dual of G, following [Bor79, Chapter I]. Let $\hat{G}$ be the dual group of G, defined over $\overline{\mathbb{Q}}$, and let $\hat{T}$ be a maximal torus in $\hat{G}$ (so that $X_\star(\hat{T}) = X^\star(T)$). We denote by Φ^+ the set of positive roots for B in $X^\star(T)$ and by $\hat{\Phi}^+ \subset X_\star(T) = X^\star(\hat{T})$ the corresponding set of positive coroots. This is the set of positive roots for a Borel subgroup $\hat{B}$ of $\hat{G}$ which contains $\hat{T}$.

We let W be the Weyl group of G and $\hat{G}$ and we denote by w_0 the longest element in W (which maps the Borel B to the opposite Borel). We let W_d be the subgroup of W which stabilizes T_d; this is also the fixed point set of the action of Γ on W [Bor79, §6.1].

The set of dominant weights for G is denoted by P^+. This is the cone in $X^\star(T)$ of characters λ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \hat{\Phi}^+$. We let $P^+_{\mathbb{R}} \subset X^\star(T)_{\mathbb{R}}$ be the positive Weyl chamber which is the $\mathbb{R}_{\geq 0}$-linear span of P^+. The cone $P^+_{\mathbb{R}}$ is a fundamental domain for the action of W on $X^\star(T) \otimes \mathbb{R}$. We have similar notations for $\hat{G}$. We denote by ρ the half sum of elements of Φ^+.

We choose a pinning to define an action of Γ on $\hat{G}$, preserving $\hat{B}$ and $\hat{T}$, and we then let ${}^LG = \hat{G} \rtimes \Gamma$.

3.2. **Hecke algebras.** Whenever we have a unimodular locally compact group $\mathcal{G}$ and a compact open subgroup $\mathcal{K}$, we denote by $\mathcal{H}(\mathcal{G}, \mathcal{K})$ the algebra of compactly supported left and right $\mathcal{K}$-invariant functions from $\mathcal{G}$ to $\mathbb{Z}$. The product in $\mathcal{H}(\mathcal{G}, \mathcal{K})$ is the convolution product (the Haar measure on $\mathcal{G}$ is normalized by $vol(\mathcal{K}) = 1$, so $\mathbf{1}_{\mathcal{K}}$ is the unit in $\mathcal{H}(\mathcal{G}, \mathcal{K})$).

We have an isomorphism of algebras $\mathcal{H}(T_d(K), T_d(\mathcal{O}_K)) = \mathbb{Z}[X_\star(T_d)]$. Moreover, the restriction map $\mathcal{H}(T(K), T(\mathcal{O}_K)) \to \mathcal{H}(T_d(K), T_d(\mathcal{O}_K))$ is an isomorphism of algebras because $T(K)/T(\mathcal{O}_K) = T_d(K)/T_d(\mathcal{O}_K)$ by [Bor79, §9.5].

Since G is quasi-split, the centralizer of T_d in G is T, so W_d is equal to $N(T_d)/T$. We have a Levi decomposition $B = TN$. Let δ be the modulus character for B.

We now study $\mathcal{H} := \mathcal{H}(G(K), G(\mathcal{O}_K))$, the spherical Hecke algebra for G. The characteristic functions $T_\lambda = \mathbf{1}_{G(\mathcal{O}_K)\lambda(\pi)G(\mathcal{O}_K)}$ for $\lambda \in (\hat{P}^+)^\Gamma$ form a $\mathbb{Z}$-basis of $\mathcal{H}$ by the Cartan decomposition.

We define the Satake transform:

$$\begin{aligned} \mathcal{S} : \mathcal{H} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \quad &\to \quad \mathcal{H}(T(K), T(\mathcal{O}_K)) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \\ f \quad &\mapsto \quad \delta(t)^{\frac{1}{2}} \int_{N(K)} f(tn)dn \end{aligned}$$

where the measure on $N(K)$ is normalized so that $N(K) \cap G(\mathcal{O}_K)$ has measure 1. It induces an isomorphism [Car79, Theorem 4.1]:

$$\mathcal{S} : \mathcal{H} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \simeq \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}][X_\star(T_d)]^{W_d}.$$

For any $\lambda \in (\hat{P}^+)^\Gamma$ let V_λ be the irreducible representation of $\hat{G}$ with highest weight λ. Since Γ preserves a pinning, the action of $\hat{G}$ on V_λ extends to an action of ${}^LG$. This extension is determined uniquely by the condition that Γ preserves a highest weight vector and any other extension differs from it by tensoring with a character of Γ. We continue to denote this extended representation by V_λ. If we consider (as we may) $\hat{G}$ and ${}^LG$ as being defined over $\mathbb{Q}$, then this extension of V_λ is also defined over $\mathbb{Q}$.

Let $[V_\lambda]$ be the the trace of the representation V_λ on $\hat{T} \rtimes \sigma$, where σ is the generator of Γ given by Frobenius. For $\lambda \in (\hat{P}^+)^\Gamma$, let $[\lambda]$ denote the formal sum of the elements in the W_d-orbit of λ, viewed as an element of $\mathbb{Z}[X_\star(T_d)]^{W_d}$.

Lemma 3.1. *The trace $[V_\lambda]$ belongs to $\mathbb{Z}[X_\star(T_d)]^{W_d}$. Moreover,*
$[V_\lambda] = [\lambda] + \sum_{\mu<\lambda} a_\mu[\mu]$ with $\mu \in (\hat{P}^+)^\Gamma$ and $a_\mu \in \mathbb{Z}$.[1]

[1] It can be shown that $a_\mu \geq 0$ but we do not need this.

Proof. We have a decomposition $V_\lambda = \oplus_{\mu \in X_\star(T)} V_\lambda^\mu$, where V_λ^μ is the weight μ eigenspace of V_λ for the action of $\hat{T}$. The action of σ permutes the spaces V_λ^μ, so if μ is not fixed by σ then the trace of any element $t \rtimes \sigma$ restricted to $\sum_{\mu' \in O_\Gamma(\mu)} V_\lambda^{\mu'}$ is zero. Here $O_\Gamma(\mu)$ denotes the Γ-orbit of μ. It follows that $[V_\lambda] \in \overline{\mathbb{Q}}[X_\star(T_d)]$. Since the weight spaces are permuted by W and σ acts trivially on W_d, $[V_\lambda] \in \overline{\mathbb{Q}}[X_\star(T_d)]^{W_d}$.

As noted earlier, we may consider $\hat{G}$, ${}^L G$ and V_λ as being defined over $\mathbb{Q}$. For each $\mu \in X_\star(T_d)$, the eigenspace V_λ^μ is defined over $\mathbb{Q}$ and σ acts on this space as a finite order operator, so its trace must be in $\mathbb{Z}$.

Finally, since the highest weight space is one dimensional, we have $[V_\lambda] = [\lambda] + \sum_{\mu<\lambda} a_\mu[\mu]$ $\mu \in (\hat{P}^+)^\Gamma$ with $a_\mu \in \mathbb{Z}$. □

Corollary 3.2. *The elements $[V_\lambda]_{\lambda \in (\hat{P}^+)^\Gamma}$ provide a $\mathbb{Z}$-basis of $\mathbb{Z}[X_\star(T_d)]^{W_d}$.*

Proof. This follows immediately from Lemma 3.1 and the fact that the set $\{[\lambda]\}_{\lambda \in (\hat{P}^+)^\Gamma}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[X_\star(T_d)]^{W_d}$. □

We may now relate the two bases $\{[V_\lambda]\}_{\lambda \in (\hat{P}^+)^\Gamma}$ and $\{T_\lambda\}_{\lambda \in (\hat{P}^+)^\Gamma}$ of the unramified Hecke algebra.

Proposition 3.3. *We have the following properties:*

(1) $\mathcal{S}(T_\lambda) = q^{\langle \lambda, \rho \rangle}[V_\lambda] + \sum_{\mu<\lambda} b_\lambda(\mu)(q^{\langle \mu, \rho \rangle}[V_\mu])$ for integers $b_\lambda(\mu)$,

(2) $q^{\langle \lambda, \rho \rangle}[V_\lambda] = \mathcal{S}(T_\lambda) + \sum_{\mu<\lambda} d_\lambda(\mu)\mathcal{S}(T_\mu)$ for integers $d_\lambda(\mu)$.

Proof. The formulae (1) and (2) are clearly equivalent. If G is split, these formulae are (3.9) and (3.12) of [Gro98], though a a complete proof is not given there so we provide the missing references:

By (6.8) of [Sat63], for λ, μ dominant coweights in $X_\star(T_d)$, $\mathcal{S}(T_\lambda)(\mu(\pi)) = \delta(\mu(\pi))a_\lambda(\mu)$, where $a_\lambda(\mu)$ is an integer. By [Sat63, Remark 2, p. 30], $a_\lambda(\lambda) = 1$ iff

$$(G(\mathcal{O}_K)\lambda(\pi)N(K)) \cap G(\mathcal{O}_K)\lambda(\pi)G(\mathcal{O}_K) = G(\mathcal{O}_K)\lambda(\pi). \tag{3.2.A}$$

This holds by part (ii) of the proposition in (4.4.4) of [BT72]. Part (i) of the same proposition implies that $a_\lambda(\mu) = 0$ unless $\mu \leq \lambda$. (We note that these results in the unramified quasi-split case follow from the case of split groups by base changing to an unramified extension over which the group splits, applying the results there, and then taking Galois invariants. Also, all cases that we actually use later are already proved in full in [Sat63].)

Since $\delta(\mu(\pi)) = q^{\langle \mu, \rho \rangle}$ (as B is a Borel subgroup), (1) now follows from Lemma 3.1. □

3.3. **Conjugacy classes.** We denote by $\hat{T}_d$ the torus with cocharacter group $X^\star(T_d)$. Note that there is a map $\hat{T} \to \hat{T}_d$. It follows from the Satake isomorphism that there is a bijection:

$$\mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}) = \hat{T}_d(\overline{\mathbb{Q}})/W_d$$

We have a surjective map $N_{\hat{G}}(\hat{T}) \to W$. Let us denote by $\widetilde{W_d}$ the inverse image of W_d. By Lemma 6.4 and 6.5 of [Bor79], we have natural bijections

$$\hat{T}(\overline{\mathbb{Q}}) \times \sigma / Int(\widetilde{W_d}) \to \hat{G}(\overline{\mathbb{Q}})^{ss}/\sigma - \mathrm{conj} \tag{3.3.A}$$

and

$$\hat{T}(\overline{\mathbb{Q}}) \times \sigma / Int(\widetilde{W_d}) \to \hat{T}_d(\overline{\mathbb{Q}})/W_d \tag{3.3.B}$$

so we deduce that there is a bijection

$$\mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}) \xrightarrow{\sim} \hat{G}(\overline{\mathbb{Q}})^{ss}/\sigma - \mathrm{conj}$$

which associates to any $\chi : \mathcal{H} \to \overline{\mathbb{Q}}$ a semi-simple σ-conjugacy class $c \in \hat{G}(\overline{\mathbb{Q}})^{ss}$ characterized by the property that

$$\chi([V_\lambda]) = \mathrm{Tr}(c \rtimes \sigma | V_\lambda)$$

for all $\lambda \in (\hat{P}^+)^\Gamma$.

3.4. **The Newton map.** Let us now fix a valuation v on $\overline{\mathbb{Q}}$ extending the p-adic valuation of $\mathbb{Q}$ for some prime number p, and normalized by $v(p) = 1$. The valuation v defines a homomorphism $\mathbb{G}_m(\overline{\mathbb{Q}}) \to \mathbb{R}$ and induces the Newton polygon map:

$$\mathrm{Newt}_v : \hat{G}(\overline{\mathbb{Q}})^{ss}/\sigma - \mathrm{conj} = \hat{T}_d(\overline{\mathbb{Q}})/W_d \xrightarrow{v} X^\star(T_d)_\mathbb{R}/W_d \xrightarrow{\phi} (P_\mathbb{R}^+)^\Gamma,$$

where the last map ϕ is defined as follows: We have a canonical identification of $(X^\star(T)_\mathbb{R})^\Gamma$ with $(X^\star(T)_\mathbb{R})_\Gamma = X^\star(T_d)_\mathbb{R}$ induced by the inclusion of $(X^\star(T)_\mathbb{R})^\Gamma$ in $X^\star(T)_\mathbb{R}$. Given $y \in X^\star(T_d)_\mathbb{R}/W_d$, let $x \in (X^\star(T)_\mathbb{R})^\Gamma$ be a lift of y, let $w \in W$ be such that $w(x) \in P_\mathbb{R}^+$ and then define $\phi(y)$ to be the Γ-average of $w(x)$, which is an element of $(P_\mathbb{R}^+)^\Gamma$. Namely,

$$\phi(y) = \frac{1}{|\Gamma/\mathrm{Stab}_\Gamma w(x)|} \sum_{\gamma \in \Gamma/\mathrm{Stab}_\Gamma w(x)} \gamma.w(x).$$

This is independent of the choice of the lift x of y, since any other lift is of the form $w'(x)$ with $w' \in W_d$.

Lemma 3.4. *Let $c \in \hat{G}(\overline{\mathbb{Q}})^{ss}/\mathrm{conj}$ and $\lambda \in (\hat{P}^+)^\Gamma$. Then the minimal valuation of an eigenvalue of $c \rtimes \sigma$ acting on V_λ is equal to $\langle w_0(\lambda), \mathrm{Newt}_v(c)\rangle$.*

Proof. Since the set of eigenvalues is invariant under conjugation, we may assume by (3.3.A) that $c \in \hat{T}(\overline{\mathbb{Q}})$. Let $\hat{T}^\Gamma$ be the largest subtorus of $\hat{T}$ on which Γ acts trivially. The map $\hat{T}^\Gamma \to \hat{T}_d$ induced by the inclusion of $\hat{T}^\Gamma$ in $\hat{T}$ is an isogeny, so the map $\hat{T}^\Gamma(\overline{\mathbb{Q}}) \to \hat{T}_d(\overline{\mathbb{Q}})$ is surjective. Thus, by (3.3.B) we may assume that $c \in \hat{T}^\Gamma(\overline{\mathbb{Q}})$.

Let μ be a weight of $\hat{T}$ occurring in V_λ. Then σ, hence also $c \rtimes \sigma$, preserves the subspace $\sum_{\mu' \in O_\Gamma(\mu)} V_\lambda^{\mu'}$ of V_λ. Since c is invariant under Γ, c acts on this space by the scalar $\mu(c)$. On the other hand σ acts on this space by a finite order automorphism. It follows that all the eigenvalues of $c \rtimes \sigma$ on this space have valuation equal to $v(\mu(c))$.

The choice of the element c gives an obvious lift $x \in (X^\star(T)_\mathbb{R})^\Gamma$ of the image of c in $X^\star(T_d)_\mathbb{R}/W_d$. Let $w \in W$ be such that $w(x) \in P_\mathbb{R}^+$. Then $v(\nu(w(c)) \geq 0$ for all positive coroots $\nu \in \hat{\Phi}^+$. The eigenvalue of smallest valuation of $w(c)$ on V_λ is then clearly the one corresponding to the lowest weight $w_0(\lambda)$, so it has valuation $\langle w_0(\lambda), w(x)\rangle$. Since λ, hence $w_0(\lambda)$, and also the pairing $\langle\ ,\ \rangle$, are invariant under Γ, we have

$$\langle w_0(\lambda), w(x)\rangle = \langle w_0(\lambda), \mathrm{Newt}_v(c)\rangle\ .$$

On the other hand, the set of eigenvalues of $w(c)$ and c on V_λ is the same. The lemma follows since we have seen that the set of valuations of the eigenvalues of c and $c \rtimes \sigma$ on V_λ is the same. □

Recall that for two elements $\nu_1, \nu_2 \in P_\mathbb{R}^+$, we write $\nu_1 \leq \nu_2$ if $\nu_2 - \nu_1$ is a linear combination with coefficients in $\mathbb{R}_{\geq 0}$ of elements of Φ^+. We use the same notation for the restriction of this ordering to $(P_\mathbb{R}^+)^\Gamma$.

Remark 3.5. Assume that G is GL_n, T is the diagonal torus and B is the upper triangular subgroup. We identify $X^\star(T)$ with $\mathbb{Z}^n$ and $P_\mathbb{R}^+ = \{(\lambda_n, \cdots, \lambda_1) \in \mathbb{R}^n,\ \lambda_n \geq \cdots \geq \lambda_1\}$. To any $\lambda \in P_\mathbb{R}^+$ we can associate a convex polygon in $\mathbb{R}^2$ whose vertices are the $(i, \lambda_1 + \cdots + \lambda_i)$ for $0 \leq i \leq n$. If $\mu, \nu \in P_\mathbb{R}^+$, then $\mu \leq \nu$ means that the polygon of μ is above the polygon of ν with the same ending point.

Lemma 3.6. *Let $\nu \in (P_{\mathbb{R}}^{+})^{\Gamma}$ and $c \in \hat{G}(\overline{\mathbb{Q}})^{ss}/\mathrm{conj}$. Then*

$$\mathrm{Newt}_v(c) \leq \nu$$

if and only if for all $\lambda \in (\hat{P}^{+})^{\Gamma}$

$$v(\mathrm{Tr}(c \rtimes \sigma | V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle.$$

Note that the second inequality in the lemma does not depend on the choice of extension of V_λ that we have made above, since any other choice differs from it by tensoring with a character of Γ.

Proof. When G is split, this is Lemme 1.3 of [Laf11]; we show that Lafforgue's proof extends to our setting without much difficulty. By (3.3.A) we may and do assume that $c \in \hat{T}(\overline{\mathbb{Q}})$.

We first assume that for all $\lambda \in (\hat{P}^{+})^{\Gamma}$, $v(\mathrm{Tr}(c \rtimes \sigma | V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle$. For $\lambda \in (\hat{P}^{+})^{\Gamma}$, let $r_\lambda = \dim(V_\lambda)$ and consider the representations $\wedge^i V_\lambda$ for $1 \leq i \leq r_\lambda$. The action of σ on V_λ maps V_λ^{μ} to $V_\lambda^{\sigma(\mu)}$ and so the analogous statement holds for the weight spaces of $\wedge^i V_\lambda$. As a representation of $\hat{G}$, this breaks up as a sum of representations V_μ with $\mu \in \hat{P}^{+}$ and $\mu \leq i\lambda$. If the line generated by the highest weight vector of such a representation is not preserved by σ, which always holds if $\mu \notin (\hat{P}^{+})^{\Gamma}$, then this summand of $\wedge^i V_\lambda$ is not preserved by σ and the trace of c on the sum of all such summands (which is preserved by c) is 0. If $\mu \in (\hat{P}^{+})^{\Gamma}$ and the line spanned by the highest weight vector is preserved by σ, then this $\hat{G}$-irreducible summand isomorphic to V_μ is preserved by ${}^{L}G$. It is not clear whether this representation is always isomorphic to our chosen extension but, as noted above, the condition in the lemma is independent of the choice.

Since the lowest weight occuring in all the representations above is $\geq w_0(i\lambda)$ and only the invariant μ contribute to the trace of c, we deduce from the assumption at the beginning that

$$v(\mathrm{Tr}(c \rtimes \sigma | \wedge^i V_\lambda)) \geq i \langle w_0(\lambda), \nu \rangle \tag{3.4.A}$$

for all $i \geq 0$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{r_\lambda}$ be the eigenvalues of c acting on V_λ ordered so that $v(\alpha_1) \leq \cdots \leq v(\alpha_{r_\lambda})$. Let $i \in \{1, \ldots, r_\lambda\}$ be such that $v(\alpha_1) = \cdots = v(\alpha_i) < v(\alpha_{i+1})$. Then $v(\mathrm{Tr}(c \rtimes \sigma | \wedge^i V_\lambda)) = iv(\alpha_1)$ so we deduce from (3.4.A) that $v(\alpha_1) \geq \langle w_0(\lambda), \nu \rangle$. On the other hand by Lemma 3.4, $v(\alpha_1) = \langle w_0(\lambda), \mathrm{Newt}_v(c) \rangle$ since $w_0(\lambda)$ is the lowest weight of V_λ. We thus have

$$\langle w_0(\lambda), \mathrm{Newt}_v(c) \rangle \geq \langle w_0(\lambda), \nu \rangle \tag{3.4.B}$$

for all $\lambda \in (P^{+})^{\Gamma}$. Since $\lambda \mapsto -w_0(\lambda)$ is a bijection of $(\hat{P}^{+})^{\Gamma}$ into itself and the cone generated by the elements of $(\hat{P}^{+})^{\Gamma}$ is dual to the Γ-invariants of the cone generated by the positive coroots of $\hat{G}$, we deduce that $\mathrm{Newt}_v(c) \leq \nu$.

The converse, not explicitly stated in [Laf11], is simpler: If $\mathrm{Newt}_v(c) \leq \nu$, then $\nu - \mathrm{Newt}_v(c)$ is a Γ-invariant element of the cone generated by the positive coroots of $\hat{G}$. The result then follows from the formula $v(\mathrm{Tr}(c \rtimes \sigma | V_\lambda)) = v(\alpha_1) = \langle w_0(\lambda), \mathrm{Newt}_v(c) \rangle$ used above and the fact that $\langle w_0(\lambda), \alpha \rangle \leq 0$ for any $\alpha \in \Phi^{+}$. □

4. Shimura varieties

4.1. **Shimura varieties in characteristic zero.** Let (G, X) be a Shimura datum [Del79, §1]. This means that G is a reductive group over $\mathbb{Q}$, and X is a $G(\mathbb{R})$-conjugacy class of morphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

satisfying the following conditions :

(1) Let $\mathfrak{g}$ be the Lie algebra of G. Then for any $h \in X$, the adjoint action determines on $\mathfrak{g}$ a Hodge structure of type $(-1,1)$, $(0,0)$, $(1,-1)$.
(2) For all $h \in X$, $\mathrm{Ad}h(i)$ is a Cartan involution on $G^{ad}(\mathbb{R})$.
(3) G^{ad} has no factor H defined over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact.

Example 4.1. The most fundamental example is the Siegel Shimura datum $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$. Let $g \in \mathbb{Z}_{\geq 1}$, let $V = \mathbb{Q}^{2g}$, and let ψ be the symplectic form with $\psi(e_i, e_{2g+1-i}) = 1$ if $1 \leq i \leq g$ and $\psi(e_i, e_j) = 0$ if $i + j \neq 2g + 1$. Let GSp_{2g} be the group of symplectic similitudes of (V, ψ). We take $\mathcal{H}_g^{\pm} = \{M \in \mathrm{M}_{g\times g}(\mathbb{C}),\ M = {}^tM,\ \mathrm{Im}(M) \text{ is definite}\}$ to be the Siegel space. Observe that $\mathcal{H}_g^{\pm} = \mathcal{H}_g^{+} \cup \mathcal{H}_g^{-}$, where $\mathcal{H}_g^{+}$ (resp. $\mathcal{H}_g^{-}$) is defined by the condition $\mathrm{Im}(M)$ is positive (resp. negative) definite. Then $\mathcal{H}_g^{\pm}$ is the $\mathrm{GSp}_{2g}(\mathbb{R})$-orbit of the morphism

$$h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}},\ a + ib \in \mathbb{C}^\times \mapsto \begin{pmatrix} a1_g & -bS_g \\ bS_g & a1_g \end{pmatrix}$$

where 1_g is the identity $g \times g$ matrix and S_g is the $g \times g$ matrix with 1's on the anti-diagonal and 0's otherwise.

Since $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, the choice of $h \in X$ determines, by projection to the first factor, a cocharacter $\mu : \mathbb{C}^\times \to G_{\mathbb{C}}$. We denote by $P_\mu \subset G_{\mathbb{C}}$ the parabolic defined by $P_\mu = \{g \in G_{\mathbb{C}},\ \lim_{t\to+\infty} \mathrm{Ad}(\mu(t))g \ \text{ exists}\}$ and by $M_\mu = \mathrm{Cent}_{G_{\mathbb{C}}}(\mu)$ its Levi factor.

Let $\mathrm{FL}_{G,X}$ be the Flag variety parametrizing parabolic subgroups in G of type P_μ. The Borel embedding is the map $X \hookrightarrow \mathrm{FL}_{G,X}$ given by $h \mapsto P_\mu$. Using this map, we endow X with a complex structure [Del79, Proposition 1.1.14].

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. We let $Sh_K(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$ be the associated Shimura variety. This is a complex manifold as soon as K is neat and in fact it has a structure of algebraic variety over $\mathbb{C}$.

The conjugacy classes of μ, P_μ and M_μ are defined over a number field $E = E(G,X)$ called the reflex field. We deduce that $\mathrm{FL}_{G,X}$ is defined over E. Moreover, it has been proven by Shimura, Deligne, Borovoi, Milne (see, e.g., [Mil83]) that $Sh_K(\mathbb{C})$ has a canonical model $Sh_K \to \mathrm{Spec}\ E$.

Example 4.2. In the Siegel case $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$, Sh_K has a moduli interpretation: it is the moduli space of abelian varieties of dimension g, with a polarization and a level structure (prescribed by K).

4.2. **Compactifications.** For any choice Σ of rational polyhedral cone decomposition, one can construct a toroidal compactification $Sh_{K,\Sigma}^{tor}$ of Sh_K. In general, this is a proper algebraic space over E and we have the property that $D_{K,\Sigma} = Sh_{K,\Sigma}^{tor} \setminus Sh_K$ is a Cartier divisor (see [AMRT10] and [Pin90]). Furthermore, for a suitable choice of Σ, $Sh_{K,\Sigma}^{tor}$ is smooth and projective.

4.3. **Automorphic vector bundles.** Let $Z_s(G)$ be the greatest sub-torus of the center of G which has no split subtorus over $\mathbb{Q}$ but which splits over $\mathbb{R}$. Let $G^c = G/Z_s(G)$.

Remark 4.3. If F is a totally real field and $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$, with center $Z = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$, then $Z_s(G)$ is the kernel of the norm map $Z \to \mathrm{GL}_1$.

One can define ([Mil90, III, §3]) an analytic space $P_K(\mathbb{C}) \to S_K(\mathbb{C})$, called the principal G^c-bundle by:

$$P_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/K.$$

There is a natural, $G(\mathbb{C})$-equivariant map $P_K(\mathbb{C}) \to \mathrm{FL}_{G,X}$ defined by sending $(x, g, g') \in G(\mathbb{Q})\backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/K$ to $g^{-1}x$. We therefore have a diagram of analytic spaces:

$$\begin{tikzcd} & P_K(\mathbb{C}) \arrow[dl] \arrow[dr] & \\ Sh_K(\mathbb{C}) & & \mathrm{FL}_{G,X} \end{tikzcd}$$

By [Mil90, III, Theorem 4.3 a)], $P_K(\mathbb{C})$ is the analytification of an algebraic variety P_K defined over E, and there is a diagram of schemes:

$$\begin{tikzcd} & P_K \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ Sh_K & & \mathrm{FL}_{G,X} \end{tikzcd}$$

where α is G-equivariant and β is a G^c-torsor.

Example 4.4. In the Siegel case, let $A \to Sh_K$ be the universal abelian scheme (only well defined up to quasi-isogenies, a representative of the isogeny class can be fixed by the choice of an integral PEL-datum). Then P_K is the GSp_{2g}-torsor of trivializations of $\mathcal{H}_{1,dR}(A/Sh_K)$ and the map α is given by the Hodge filtration on $\mathcal{H}_{1,dR}(A/Sh_K)$.

Let E' be a finite extension of E such that the conjugacy class of μ, M_μ and P_μ have representatives defined over E', and all finite dimensional algebraic representations of M_μ are defined over E'. We now consider all the schemes P_K, Sh_K, $\mathrm{FL}_{G,X} = G/P_\mu$ over E'. Denote by $VB_G(\mathrm{FL}_{G,X})$ the category of G-equivariant vector bundles on $\mathrm{FL}_{G,X}$ and by $\mathrm{Rep}_{E'}(M_\mu)$ the category of finite dimensional algebraic representations of M_μ on E'-vector spaces. There is a functor

$$\begin{array}{rcl} \mathrm{Rep}_{E'}(M_\mu) & \to & VB_G(\mathrm{FL}_{G,X}) \\ V & \mapsto & \mathcal{V} \end{array}$$

which is defined by $\mathcal{V} = G \times V/\sim$ where $\sim$ is the equivalence relation $(gx, v) \sim (g, xv)$ for all $(g, v, x) \in G \times V \times P_\mu$, and we let P_μ act on V through its projection $P_\mu \to M_\mu$.

Let $VB(Sh_K)$ be the category of vector bundles over Sh_K and $\mathrm{Rep}_{E'}(M_\mu/Z_s(G))$ the category of finite dimensional algebraic representations of $M_\mu/Z_s(G)$ on E'-vector spaces. We deduce that there is a functor

$$\begin{array}{rcl} \mathrm{Rep}_{E'}(M_\mu/Z_s(G)) & \to & VB(Sh_K) \\ V & \mapsto & \mathcal{V}_K \end{array}$$

which is defined as follows: a representation V of $M_\mu/Z_s(G)$ defines a representation of P_μ (by letting the unipotent radical act trivially) and therefore a G-equivariant vector bundle $\mathcal{V}$ over $\mathrm{FL}_{G,X}$. We can pull back this vector bundle by the map α to P_K and descend it to Sh_K using β.

4.4. **Automorphic vector bundles and compactifications.** By [Har89, Theorem 4.2], for a choice Σ of rational polyhedral cone decomposition, there is a G^c-torsor $P_{K,\Sigma} \to Sh^{tor}_{K,\Sigma}$ extending P_K, and a diagram:

$$\begin{tikzcd} & P_{K,\Sigma} \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ Sh^{tor}_{K,\Sigma} & & \mathrm{FL}_{G,X} \end{tikzcd}$$

where α is G-equivariant and β is a G^c-torsor. Therefore, the functor $V \mapsto \mathcal{V}_K$ extends to a functor

$$\begin{array}{rcl} \mathrm{Rep}_{E'}(M_\mu/Z_s(G)) & \to & VB(Sh^{tor}_{K,\Sigma}) \\ V & \mapsto & \mathcal{V}_{K,\Sigma}. \end{array}$$

and $\mathcal{V}_{K,\Sigma}$ is called the canonical extension of $\mathcal{V}_K$. Moreover, $\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma})$ is called the sub-canonical extension of $\mathcal{V}_K$.

4.5. **Cohomology of vector bundles and Hecke operators.** For any Σ, K and $V \in \mathrm{Rep}(M_\mu/Z_s(G))$, we identify the vector bundle $\mathcal{V}_{K,\Sigma}$ with its associated locally free sheaf of sections, and we consider the cohomology groups:

$$\mathrm{H}^\star(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}) \text{ and } \mathrm{H}^\star(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}(-D_{K,\Sigma})).$$

By [Har90b, Proposition 2.4] these groups are independent of the choice of Σ, and we therefore simplify notations and let $\mathrm{H}^i(K,V) = \mathrm{H}^i(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{K,\Sigma})$ and $\mathrm{H}^i_{cusp}(K,V) = \mathrm{H}^i(Sh^{tor}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}(-D_{K,\Sigma}))$.

Let $\mathcal{H}_K$ be the Hecke algebra of functions $G(\mathbb{A}_f) \to \mathbb{Z}$, which are compactly supported and bi K-invariant. By [Har90b, Proposition 2.6], the Hecke algebra $\mathcal{H}_K$ acts on the cohomology groups $\mathrm{H}^i(K,V)$ and $\mathrm{H}^i_{cusp}(K,V)$.

Observe that the Hecke algebra is generated by the characteristic functions $T_g = \mathbf{1}_{KgK}$ for $g \in G(\mathbb{A}_f)$. We spell out the action of T_g by writing the corresponding cohomological correspondence. For any $g \in G(\mathbb{A}_f)$, we have a correspondence (for suitable choices of polyhedral cone decomposition Σ, Σ' and Σ''):

$$\begin{tikzcd} & Sh^{tor}_{gKg^{-1}\cap K,\Sigma''} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ Sh^{tor}_{K,\Sigma'} & & Sh^{tor}_{K,\Sigma} \end{tikzcd}$$

where p_1 is simply the forgetful map (induced by the inclusion $gKg^{-1} \cap K \subset K$), and p_2 is the composite of the action map $g : Sh^{tor}_{gKg^{-1}\cap K,\Sigma''} \to Sh^{tor}_{K\cap g^{-1}Kg,\Sigma'''}$ and the forgetful map $Sh^{tor}_{K\cap g^{-1}Kg,\Sigma'''} \to Sh^{tor}_{K,\Sigma'}$ (induced by the inclusion $g^{-1}Kg \cap K \subset K$). There is a corresponding cohomological correspondence $T_g : p_2^\star \mathcal{V}_{K,\Sigma'} \to p_1^! \mathcal{V}_{K,\Sigma}$ which is simply obtained by composing the natural isomorphism $p_2^\star \mathcal{V}_{K,\Sigma'} \to p_1^\star \mathcal{V}_{K,\Sigma}$ (see [Har90b, §2.5]) and the map $p_1^\star \mathcal{V}_{K,\Sigma} \to p_1^! \mathcal{V}_{K,\Sigma}$ which is deduced from the fundamental class $p_1^\star \mathscr{O}_{Sh^{tor}_{K,\Sigma}} \to p_1^! \mathscr{O}_{Sh^{tor}_{K,\Sigma}}$ (see Proposition 2.6).

4.6. **The infinitesimal character.** Let again (G,X) be a Shimura datum. The reductive group G is defined over $\mathbb{Q}$. We have chosen an extension E' of the reflex field E over which all representations of M_μ are defined. This actually forces G to split over E'. Let S be a (split) maximal torus in $G_{E'}$ and let $X^\star(S)$ be its character group. We assume that $S \subset (M_\mu)_{E'}$. The roots for G that lie in the Lie algebra of M_μ are by definition the compact roots. The other roots are called non-compact. We make a choice of positive roots for M_μ. We also make a choice of positive roots for G by declaring that the non-compact positive roots are those corresponding to $\mathfrak{g}/\mathfrak{p}_\mu$. We denote by ρ the half sum of all the positive roots. Let κ be a highest weight for M_μ.

Definition 4.5. *We define the infinisitemal character of κ, denoted $\infty(\kappa)$, by the formula*

$$\infty(\kappa) = -\kappa - \rho \in X^\star(S)_{\mathbb{Q}}.$$

Remark 4.6. This is a representative of the infinitesimal character of automorphic representations contributing to the cohomology $\mathrm{H}^i(K, V_\kappa)$ or $\mathrm{H}^i_{cusp}(K, V_\kappa)$ [Har90b, Proposition 4.3.2].

We now choose a prime p such that the group $G_{\mathbb{Q}_p}$ is unramified at p. As in Section 3.1, we denote by $T \subset G_{\mathbb{Q}_p}$ a maximal torus, split over an unramified extension of $\mathbb{Q}_p$ and contained in a Borel subgroup $B \subset G_{\mathbb{Q}_p}$. We let $X^\star(T)$ be the character group of T and let $P^+ \subset X^\star(T)$ be the cone of dominant weights. We fix an embedding $\iota : E' \hookrightarrow \overline{\mathbb{Q}}_p$.

The tori $S \times_{\mathrm{Spec}\ E'} \mathrm{Spec}\ \overline{\mathbb{Q}}_p$ and $T \times_{\mathrm{Spec}\ \mathbb{Q}_p} \mathrm{Spec}\ \overline{\mathbb{Q}}_p$ are conjugated by some element g of $G(\overline{\mathbb{Q}}_p)$. Conjugation by g defines an isomorphism $X^\star(S) \to X^\star(T)$; this isomorphism depends on g, but the composite map $X^\star(S) \to X^\star(T) \to X^\star(T)/W \simeq P^+$ is independent of the choice of g.

We therefore get a canonical element $\infty(\kappa, \iota) \in P^+_{\mathbb{R}}$ (which only depends on ι), which is by definition the image of $\infty(\kappa)$ via the map $X^\star(S)_{\mathbb{R}} \to P^+_{\mathbb{R}}$.

4.7. **Newton and Hodge polygons.** We assume that $K = K_pK^p$ is neat and K_p is hyperspecial. Let V_κ be the irreducible representation of M_μ defined over E' with highest weight κ.

Let $\mathcal{H}_p = \mathcal{H}(G(\mathbb{Q}_p), K_p)$ be the Hecke algebra at p. It acts on the groups $\mathrm{H}^i(K, V_\kappa)$ and $\mathrm{H}^i_{cusp}(K, V_\kappa)$. We put $\mathrm{H}^i(K, V_\kappa)_{\overline{\mathbb{Q}}_p} = \mathrm{H}^i(K, V_\kappa) \otimes_{E',\iota} \overline{\mathbb{Q}}_p$ and $\mathrm{H}^i_{cusp}(K, V_\kappa)_{\overline{\mathbb{Q}}_p} = \mathrm{H}^i_{cusp}(K, V_\kappa) \otimes_{E',\iota} \overline{\mathbb{Q}}_p$.

Using the results of Sections 3.3 and 3.4, we have a Newton map:

$$\mathrm{Newt}_\iota : \mathrm{Hom}(\mathcal{H}_p, \overline{\mathbb{Q}}_p) = \hat{G}(\overline{\mathbb{Q}}_p)/\sigma - \mathrm{conj} \to (P^+_{\mathbb{R}})^\Gamma.$$

Conjecture 4.7. *Let $\chi : \mathcal{H}_p \to \overline{\mathbb{Q}}_p$ be a character occuring in $\mathrm{H}^i(K, V_\kappa)_{\overline{\mathbb{Q}}_p}$ or $\mathrm{H}^i_{cusp}(K, V_\kappa)_{\overline{\mathbb{Q}}_p}$. Then we have the inequality in $(P^+_{\mathbb{R}})^\Gamma$:*

$$\mathrm{Newt}_\iota(\chi) \leq \frac{1}{|\Gamma/\mathrm{Stab}_\Gamma(\infty(\kappa, \iota))|} \sum_{\gamma \in \Gamma/\mathrm{Stab}_\Gamma(\infty(\kappa,\iota))} -w_0(\gamma \cdot \infty(\kappa, \iota)),$$

where w_0 is the longest element of the Weyl group.

We now explain that this conjecture is compatible with existing conjectures on the existence and properties of Galois representations attached to automorphic representations. Our convention is that the Artin reciprocity law is normalized by sending uniformizing element to geometric Frobenius element and that the Hodge–Tate weight of the cyclotomic character is -1.

This inequality is to be viewed as an inequality between a Newton and a Hodge polygon. According to the work of [Jun18] (see also [Har90b]), all the cohomology $\mathrm{H}^i(K, V_\kappa)$ and $\mathrm{H}^i_{cusp}(K, V_\kappa)$ can be represented by automorphic forms. Let π be an automorphic representation contributing to these cohomology groups. We fix an embedding $E \to \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$ extending our embedding $\iota : E \hookrightarrow \overline{\mathbb{Q}}_p$. The automorphic representation π is C-algebraic with infinitesimal character $\infty(\kappa) = -\kappa - \rho$.

Let us assume that π is also L-algebraic (for simplicity). Then, according to [BG14, Conjecture 3.2.2], there should be a geometric Galois representation $\rho_{\pi,\iota} : G_{\mathbb{Q}} \to {}^L G(\overline{\mathbb{Q}}_p)$ satisfying a list of conditions. In particular, it should be crystalline at p (because π has spherical vectors at p) with Hodge–Tate weights $-\infty(\kappa, \iota)$. (Note that our sign convention concerning the Hodge–Tate weight of the cyclotomic character is opposite to [BG14]). Let $\chi_{\pi_p} : \mathcal{H}_p \to \mathbb{C}$ be the character describing the action of $\mathcal{H}_p$ on π_p. The conjugacy class of the crystalline Frobenius should be given by $\iota \circ \chi_{\pi_p}$ via the Satake isomorphism. Then the inequality in $(P_{\mathbb{R}}^{+})^{\Gamma}$:

$$\mathrm{Newt}_{\iota}(\iota\circ\chi_{\pi_p}) \le \frac{1}{|\Gamma/\mathrm{Stab}_{\Gamma}(\infty(\kappa,\iota))|}\sum_{\gamma\in\Gamma/\mathrm{Stab}_{\Gamma}(\infty(\kappa,\iota))} -w_0(\gamma\cdot\infty(\kappa,\iota))$$

is the Katz–Mazur inequality (see also [Laf11, Lemma 4.2], and Theorem 9.11).

Motivated by this conjecture, we can introduce a modified "integral structure" on the Hecke algebra $\mathcal{H}_p$:

Definition 4.8. *Let $\mathcal{H}_{p,\kappa,\iota}^{int}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{H}_p\otimes\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$ generated by the elements*

$$[V_\lambda]p^{\langle\lambda,\infty(\kappa,\iota)\rangle}$$

Lemma 4.9. *Conjecture 4.7 holds if and only if, for any character $\chi:\mathcal{H}_p\to\overline{\mathbb{Q}}_p$ occuring in $\mathrm{H}^i(K,V_\kappa)_{\overline{\mathbb{Q}}_p}$ or $\mathrm{H}^i_{cusp}(K,V_\kappa)_{\overline{\mathbb{Q}}_p}$, we have $\chi(\mathcal{H}_{p,\kappa,\iota}^{int})\subset\overline{\mathbb{Z}}_p$.*

Proof. This follows from Lemma 3.6. □

Lemma 4.10. *The algebra $\mathcal{H}_{p,\kappa,\iota}^{int}$ is a $\mathbb{Z}$-algebra of finite type.*

Proof. Let $\{\lambda_1,\cdots,\lambda_n\}$ be dominant weights of $\hat{G}$ which generate the cone $(\hat{P}^+)^{\Gamma}$ as a monoid and is closed under $\le$, in the sense that if $\lambda\in(\hat{P}^+)^{\Gamma}$ satisfies $\lambda\le\lambda_i$ for some $1\le i\le n$, then $\lambda\in\{\lambda_1,\cdots,\lambda_n\}$. We claim that the map $\mathbb{Z}[T_1,\cdots,T_n]\to\mathcal{H}_{p,\kappa}^{int}$ where T_i goes to $[V_{\lambda_i}]q^{\langle\lambda_i,\infty(\kappa,\iota)\rangle}$ is surjective. Let $\lambda\in(\hat{P}^+)^{\Gamma}$, and let us prove that $[V_\lambda]q^{\langle\lambda,\infty(\kappa,\iota)\rangle}$ is in the image. We argue by induction and assume that this holds for all $\lambda'\in(\hat{P}^+)^{\Gamma}$ with $\lambda'<\lambda$. We can write $\lambda=\sum_{i=1}^n k_i\lambda_i$ with $k_i\in\mathbb{Z}_{\ge0}$ and we have that

$$\begin{aligned}\prod_{i=1}^n T_i^{k_i} &= [V_\lambda]q^{\langle\lambda,\infty(\kappa,\iota)\rangle}+\sum_{\mu\in\hat{P}^+,\mu<\lambda}c_\mu[V_\mu]q^{\langle\lambda,\infty(\kappa,\iota)\rangle}\text{ with }c_\mu\in\mathbb{Z}_{\ge0}\\ &= [V_\lambda]q^{\langle\lambda,\infty(\kappa,\iota)\rangle}+\sum_{\mu\in\hat{P}^+,\ \mu<\lambda}c_\mu[V_\mu]q^{\langle\mu,\infty(\kappa,\iota)\rangle}q^{\langle\lambda-\mu,\infty(\kappa,\iota)\rangle}\end{aligned}$$

and we can conclude since $\langle\lambda-\mu,\infty(\kappa,\iota)\rangle\in\mathbb{Z}_{\ge0}$ because $\lambda-\mu$ is a finite sum of positive roots with non-zero integral coefficients for $\hat{G}$. □

Proposition 4.11. *Conjecture 4.7 holds if and only if both $\mathrm{H}^i(K,V_\kappa)_{\overline{\mathbb{Q}}_p}$ and $\mathrm{H}^i_{cusp}(K,V_\kappa)_{\overline{\mathbb{Q}}_p}$ contain $\overline{\mathbb{Z}}_p$-lattices which are stable under $\mathcal{H}_{p,\kappa,\iota}^{int}$.*

Proof. We deduce from Lemma 4.9, that we can find a basis for $\mathrm{H}^i(K,V_\kappa)_{\overline{\mathbb{Q}}_p}$, such that the elements $\mathcal{H}_{p,\kappa}^{int}$ act via upper triangular matrices with integral diagonal coefficients. After conjugating this basis by a diagonal matrix $\mathrm{diag}(p^{k_1},\cdots,p^{k_n})$ with $k_1\gg k_2\cdots\gg k_n$, and using that $\mathcal{H}_{p,\kappa,\iota}^{int}$ if a finite type algebra, we can suppose that it acts via integral matrices. □

4.8. **Shimura varieties of abelian type.** The theory of integral models of Shimura varieties produces (in many cases) integral structures on the coherent cohomology of automorphic vector bundles. In view of Proposition 4.11, it is natural to ask (see Conjectures 4.15 and 4.16 below) whether these integral structures are stable under the action of our integral Hecke algebras.

A Shimura datum (G,X) is of Hodge type if there is an embedding $(G,X)\hookrightarrow(\mathrm{GSp}_{2g},\mathcal{H}_g^{\pm})$. A Shimura datum (G,X) is of abelian type if there is a Shimura datum of Hodge type (G_1,X_1) and a central isogeny $G_1^{der}\to G^{der}$ which induces an isomorphism

$(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$, where X_1^{ad} is the $\mathrm{G}_1^{ad}(\mathbb{R})$-conjugacy class that contains X_1 (and similarly for X^{ad}).

Example 4.12. Here is an important example of abelian Shimura datum, that we call a Shimura datum of symplectic type. Let F be a totally real field. We let $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_{2g}$ and $X = (\mathcal{H}_g^{\pm})^{[F:\mathbb{Q}]}$. This datum is related to the following Hodge type datum (which is actually of PEL type): take $G_1 \subset G$ to be the subgroup of elements whose similitude factor is in GL_1 and $X_1 = (\mathcal{H}_g^{+})^{[F:\mathbb{Q}]} \cup (\mathcal{H}_g^{-})^{[F:\mathbb{Q}]}$.

We fix (G, X) a Shimura datum of abelian type and $K \subset G(\mathbb{A}_f)$ a neat compact open subgroup. Let p be a prime such that that G is unramified at p, $K = K^pK_p$ and K_p is hyperspecial. The group G has a reductive model over $\mathbb{Z}_{(p)}$. Let E' be a finite extension of $\mathbb{Q}$ unramified at p, which splits G (necessarily $E \hookrightarrow E'$). Let λ be a prime of $\mathcal{O}_E$ dividing p and λ' a prime of $\mathcal{O}_{E'}$ dividing λ. We denote by $\mathcal{O}_{E,\lambda}$ and $\mathcal{O}_{E',\lambda}$ the localizations of $\mathcal{O}_E$ and $\mathcal{O}_{E'}$ at λ and λ' respectively. The cocharacter μ is defined over E', and the parabolic P_μ as well as its Levi subgroup M_μ have a model over Spec $\mathcal{O}_{E',\lambda'}$. Moreover, the conjugacy class of P_μ is defined over Spec $\mathcal{O}_{E,\lambda}$. The flag variety $\mathrm{FL}_{G,X}$ therefore has a proper smooth canonical integral model over Spec $\mathcal{O}_{E,\lambda}$. To ease notations, we keep denoting by G the reductive model of G over $\mathbb{Z}_{(p)}$, by $\mathrm{FL}_{G,X} \to \mathrm{Spec}\ \mathcal{O}_{E,\lambda}$ the canonical integral model of $\mathrm{FL}_{G,X} \to \mathrm{Spec}\ E$, and by P_μ and M_μ the models over Spec $\mathcal{O}_{E',\lambda'}$ of P_μ and M_μ.

Theorem 4.13 ([Kis10], [KMP16])**.** *There is a canonical model* $\mathfrak{Sh}_K \to \mathrm{Spec}\ \mathcal{O}_{E,\lambda}$ *of* Sh_K.

Theorem 4.14. *Assume that (G, X) is a Shimura datum of Hodge type, or that (G, X) is a Shimura datum of symplectic type. Let Σ be a polyhedral cone decomposition.*

(1) There is a canonical integral model $\mathfrak{Sh}^{tor}_{K,\ \Sigma} \to \mathrm{Spec}\ \mathcal{O}_{E,\lambda}$ *for* $Sh^{tor}_{K,\Sigma}$ *which is smooth for suitable choices of Σ.*

(2) There is a canonical integral model $\mathfrak{P}_{K,\Sigma}$ for the principal G^c-torsor $P_{K,\Sigma}$ and there is a diagram :

$$\begin{tikzcd} & \mathfrak{P}_{K,\Sigma} \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \mathfrak{Sh}^{tor}_{K,\Sigma} & & \mathrm{FL}_{G,X} \end{tikzcd}$$

where α is G-equivariant and β is a G^c-torsor.

Proof. The Siegel case is [FC90], the PEL case is [Lan13] and [Lan12], and the Hodge case is [MP19]. To our knowledge, there is no reference for the abelian case in general. We will explain in Section 5 the proof for the case of a symplectic type Shimura datum by a simple reduction to the PEL case. This is a straightforward generalisation of the argument presented in [BCGP18, §3] in the case of the groups $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$. □

Let $\mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G))$ be the category of algebraic representations of $M_\mu/Z_s(G)$ ($M_\mu/Z_s(G)$ is viewed as a reductive group over Spec $\mathcal{O}_{E',\lambda'}$) over finite free $\mathcal{O}_{E',\lambda'}$-modules. Using Theorem 4.14 we get a functor

$$\begin{array}{rcl} \mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G)) & \to & VB(\mathfrak{Sh}^{tor}_{K,\Sigma}) \\ V & \mapsto & \mathcal{V}_{K,\Sigma}. \end{array}$$

which is an integral version of the functor of section 4.4. Depending on the context, $\mathcal{V}_{K,\Sigma}$ will mean the locally free sheaf over $\mathfrak{Sh}^{tor}_{K,\Sigma}$ attached to an object $V \in \mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G))$ or the locally free sheaf on $Sh^{tor}_{K,\Sigma}$ attached to $V_{E'} := V \otimes_{\mathcal{O}_{E',\lambda'}} E'$.

The cohomology complexes $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}^{int}_{K,\Sigma}(-D_{K,\Sigma}))$ are independent of Σ (this is a standard computation using the structure of the boundary, see [Lan17, Theorem 8.6] in the PEL case) and these are perfect complexes of $\mathcal{O}_{E',\lambda'}$-modules. We observe that

$$\mathrm{Im}\left(\mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma})\otimes_{\mathcal{O}_{E',\lambda'},\iota}\overline{\mathbb{Z}}_p\to \mathrm{H}^i(K,V_{E'})_{\overline{\mathbb{Q}}_p}\right)$$

and

$$\mathrm{Im}\left(\mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}^{int}_{K,\Sigma}(-D_{K,\Sigma}))\otimes_{\mathcal{O}_{E',\lambda'},\iota}\overline{\mathbb{Z}}_p\to \mathrm{H}^i_{cusp}(K,V_{E'})_{\overline{\mathbb{Q}}_p}\right)$$

are lattices that we denote respectively by $\mathrm{H}^i(K,V)_{\overline{\mathbb{Z}}_p}$ and $\mathrm{H}^i_{cusp}(K,V)_{\overline{\mathbb{Z}}_p}$.

4.9. **Conjectures on the action of the integral Hecke algebra.** Let κ be a dominant weight for $M_\mu/Z_s(G)$ and let V_κ be the corresponding Weyl representation, defined over $\mathcal{O}_{E',\lambda'}$. Namely, let $\kappa^\vee$ be the dominant weight $-w_{M_\mu}\kappa$ for w_{M_μ} the longest element of the Weyl group of M_μ. Let B_{M_μ} be the Borel of M_μ (corresponding to our choice of positive roots). We let V^{int}_κ be the space of functions $f: M_\mu\to\mathbb{A}^1$ with the transformation property: $f(mb)=\kappa^\vee(b)f(m)$ for all $b\in B_{M_\mu}$. The right action of G on itself given by $g\mapsto L(g^{-1})$ (where L is the left translation) induces a left action on the space V_κ, and $(V_\kappa)_{E'}$ is an irreducible highest weight representation of weight κ.

Conjecture 4.15. *The lattices $\mathrm{H}^i(K,V_\kappa)_{\overline{\mathbb{Z}}_p}$ and $\mathrm{H}^i_{cusp}(K,V_\kappa)_{\overline{\mathbb{Z}}_p}$ are stable under $\mathcal{H}^{int}_{p,\kappa,\iota}$.*

Conjecture 4.16. *The algebra $\mathcal{H}^{int}_{p,\kappa,\iota}$ acts on $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$.*

Observe that Conjecture 4.15 implies Conjecture 4.7 by Proposition 4.11, and Conjecture 4.16 obviously implies Conjecture 4.15.

5. Shimura varieties of symplectic type

This section is dedicated to Shimura varieties of symplectic type. We prove the missing part of Theorem 4.14, and we state in Theorem 5.9 some partial results towards Conjecture 4.16 which are proved in Section 6.

5.1. **Group theoretic data.** Let F be a totally real field with $[F:\mathbb{Q}]=d$ and let (V,Ψ) be a symplectic F-vector space of dimension $2g$, with basis $e_1,\cdots,e_{2g}$ and $\Psi(e_i,e_j)=0$ if $j\neq 2g+1-i$ and $\Psi(e_i,e_{2g-i+1})=1$ if $1\le i\le g$. Let $V_0=\langle e_1,\cdots,e_g\rangle$ and $V_1=\langle e_{g+1},\cdots,e_{2g}\rangle$ be sub F-vector spaces of V. The pairing Ψ on V restricts to a perfect pairing between V_0 and V_1. Let G be the algebraic group over $\mathbb{Q}$ defined by

$$G(R)=\{(g,\nu)\in \mathrm{GL}_F(V\otimes_{\mathbb{Q}}R)\times(F\otimes_{\mathbb{Q}}R)^\times \mid \forall v,w\in V\otimes R,\ \Psi(gv,gw)=\nu\Psi(v,w)\}$$

for any $\mathbb{Q}$-algebra R. Let G_1 be the subgroup of G whose elements have their similitude factor ν in $\mathbb{G}_m$ embedded diagonaly in $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. Let also G^{der} be the derived subgroup of G defined by the condition $\nu=1$.

Let T^{der} be the diagonal maximal torus of G^{der}: $T^{der}=\{\mathrm{diag}(t_1,\cdots,t_g,t_g^{-1},\cdots,t_1^{-1})\}$ with $t_i\in\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. The center Z of G is the group $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ embedded diagonally in G. Let T be the maximal torus of G, which is generated by T^{der} and Z. Let Z_1 be the center of G_1, the image of $\mathbb{G}_m$ embedded diagonally in G_1. The maximal torus of G_1 is generated by Z_1 and T^{der}.

Let $X^\star(T)$ be the group of characters of T defined over $\overline{\mathbb{Q}}$, identified with tuples $\kappa=(k_{1,\sigma},\cdots,k_{g,\sigma};k_\sigma)_{\sigma\in\mathrm{Hom}(F,\overline{\mathbb{Q}})}\in\mathbb{Z}^{(g+1)d}$ satisfying the condition $k_\sigma=\sum_i k_{i,\sigma} \mod 2$, via the pairing:

$$\langle\kappa,(zt_1,\cdots,zt_g,zt_g^{-1},\cdots,zt_1^{-1})\rangle=\prod_\sigma\Big(\sigma(z)^{k_\sigma}\prod_{i=1}^g\sigma(t_i)^{k_{i,\sigma}}\Big)$$

We denote by $P \subset G$ the Siegel parabolic, i.e., the stabilizer of the Lagrangian subspace V_0. We denote by M its Levi quotient (which we identify with the standard Levi subgroup of P). Note that $M \simeq \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_g \times \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$.

The roots of G corresponding to the Lie algebra of M are by definition the compact roots, the other roots are called non-compact. We declare that a character of T is dominant for M if for all $\sigma \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$, $k_{1,\sigma} \geq \cdots \geq k_{g,\sigma}$. So our choice of Borel subgroup in M is the usual upper triangular Borel.

A character of T is dominant for G if for all $\sigma \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$, $0 \geq k_{1,\sigma} \geq \cdots \geq k_{g,\sigma}$ (so our choice of non-compact positive roots are those corresponding to $\mathfrak{g}/\mathfrak{p}$).

We have similar definitions for G_1 and we denote by T_1, M_1, … the intersections of T, M, … with G_1. Weights for T_1 are labelled by

$$\big((k_{1,\sigma}, \cdots, k_{g,\sigma})_{\sigma \in \mathrm{Hom}(F,\overline{\mathbb{Q}})}; k\big) \in \mathbb{Z}^{gd} \times \mathbb{Z}$$

with the condition that $k = \sum_{i,\sigma} k_{i,\sigma} \mod 2$ and a weight κ as above pairs with an element $t \in T_1$ via the formula:

$$\langle \kappa, (zt_1, \cdots, zt_g, zt_g^{-1}, \cdots, zt_1^{-1})\rangle = z^k \prod_\sigma \Big(\prod_{i=1}^g \sigma(t_i)^{k_{i,\sigma}}\Big)$$

5.2. **Shimura varieties of symplectic type in characteristic 0.** Let $\mathcal{H}_g^\pm$ be the Siegel space of symmetric matrices $M = A + iB \in \mathrm{M}_{g\times g}(\mathbb{C})$ with B definite (positive or negative). Let $X = (\mathcal{H}_g^\pm)^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ and $X_1 = (\mathcal{H}_g^+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}})} \cup (\mathcal{H}_g^-)^{\mathrm{Hom}(F,\overline{\mathbb{Q}})} \subset X$. The group $G(\mathbb{R})$ acts on X and its subgroup $G_1(\mathbb{R})$ stabilizes X_1. The pairs (G, X) and (G_1, X_1) are Shimura data. The Siegel parabolic P is a representative of the conjugacy class of P_μ.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. We assume that $K = \prod_\ell K_\ell$ and that K is neat. We also assume that p is unramified in F so $G(\mathbb{Q}_p) = \prod_{v|p} \mathrm{GSp}_{2g}(F_v)$ for unramified extensions F_v of $\mathbb{Q}_p$. We further assume that $K_p = \prod K_v \subset G(\mathbb{Z}_p)$ where K_v is either $\mathrm{GSp}_{2g}(\mathcal{O}_{F_v})$ or $\mathrm{Si}(v)$, the Siegel parahoric subgroup of elements with reduction mod p in $P(\mathcal{O}_{F_v}/p)$.[2]

We let $Sh_K(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$ be the quotient. Let $G(\mathbb{Q})^+ \subset G(\mathbb{Q})$ be the subgroup of elements whose similitude factor is totally positive.

By strong approximation, we can write

$$G(\mathbb{A}_f) = \coprod_c G(\mathbb{Q})^+ cK$$

where the elements $c \in G(\mathbb{A}_f)$ are such that the elements $\nu(c)$ range through a set of representatives of $F^{\times,+}\backslash(\mathbb{A}_f \otimes F)^\times/\nu(K)$ and we find that $Sh_K(\mathbb{C}) = \coprod_c \Gamma(c, K)\backslash\mathcal{H}_g^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ where $\Gamma(c, K) = G(\mathbb{Q})^+ \cap cKc^{-1}$.

This is an algebraic variety over $\mathbb{C}$, and it has a canonical model Sh_K over $\mathbb{Q}$. The Shimura variety is not of PEL type, and it is therefore useful to introduce another Shimura variety which is of PEL type. We begin by rewriting $Sh_K(\mathbb{C}) = (G(\mathbb{Q}) \cap K_p)\backslash(X \times G(\mathbb{A}_f^p)/K^p)$ and then consider the Shimura variety of PEL type (in fact an infinite union of such) $\widetilde{Sh}_K(\mathbb{C}) = (G_1(\mathbb{Q}) \cap K_p)\backslash(X_1 \times G(\mathbb{A}_f^p)/K^p)$.

By strong approximation, we find that

$$G(\mathbb{A}_f^p) = \coprod_c (G_1(\mathbb{Q})^+ \cap K_p)cK^p$$

[2] We could actually allow any parahoric level structure as in [BCGP18, Section 3.3.1], but in the sequel we shall only need to work at hyperspecial level or Siegel parahoric level.

where $c \in G(\mathbb{A}_f^p)$ ranges through a set of representatives of $\mathbb{Z}_{(p)}^{\times,+}\backslash(\mathbb{A}_f^p \otimes F)^\times/\nu(K^p)$ and we find that $\widetilde{Sh}_K(\mathbb{C}) = \coprod_c \Gamma_1(c,K)\backslash\mathcal{H}_g^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ where $\Gamma_1(c,K) = G_1(\mathbb{Q})^+ \cap cKc^{-1}$. It has a canonical model $\widetilde{Sh}_K \to \mathrm{Spec}\ \mathbb{Q}$ and we have a natural morphism:

$$\widetilde{Sh}_K \to Sh_K.$$

On the set of geometric connected components, this map is given by : $\mathbb{Z}_{(p)}^{\times,+}\backslash(\mathbb{A}_f^p \otimes F)^\times/\nu(K^p) \to \mathcal{O}_{F,p}^{\times,+}\backslash(\mathbb{A}_f^p \otimes F)^\times/\nu(K^p)$.

If we let $c \in G(\mathbb{A}_f^p)$ so that $\nu(c)$ defines geometrically connected components $(\widetilde{Sh}_K)_c$ of $\widetilde{Sh}_K$ and $(Sh_K)_c$ of Sh_K, then the corresponding map $(\widetilde{Sh}_K)_c \to (Sh_K)_c$ is a finite Galois cover with group

$$\Delta(K) = (\mathcal{O}_F^{\times,+} \cap \nu(K^p))/\nu(\mathcal{O}_F^\times \cap K^p),$$

where the first intersection is taken in $(\mathbb{A}_f^p \otimes F)^\times$ and the second in $G(\mathbb{A}_f^p)$, with $\mathcal{O}_F^\times$ embedded diagonally in the center of G.

5.3. **Integral models.** We observe that $\widetilde{Sh}_K$ has a canonical model $\widetilde{\mathfrak{Sh}}_K$ over $\mathbb{Z}_{(p)}$ which is a moduli space of abelian varieties of dimension gd with K-level structure, prime to p polarization, and an action of $\mathcal{O}_F$. More precisely, $\widetilde{\mathfrak{Sh}}_K$ represents the functor over $\mathbb{Z}_{(p)}$ that parametrizes equivalence classes of $(A, \iota, \lambda, \eta, \eta_p)$, where:

(1) $A \to \mathrm{Spec}\ R$ is an abelian scheme,
(2) $\iota : \mathcal{O}_F \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is an action,
(3) $\mathrm{Lie}(A)$ is a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}} R$-module of rank g,
(4) $\lambda : A \to A^t$ is a prime to p, $\mathcal{O}_F$-linear quasi-polarization,
(5) η is a K^p-level structure,
(6) η_p is a K_p-level structure.

Let us spell out the definition of K^p-level structure. We may assume without loss of generality that $S = \mathrm{Spec}\ R$ is connected, and we fix $\overline{s}$ a geometric point of S. The adelic Tate module $\mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p})$ carries a symplectic Weil pairing

$$<,>_\lambda\colon \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \times \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \to \mathrm{H}_1(\mathbb{G}_m|_{\overline{s}}, \mathbb{A}^{\infty,p})$$

or equivalently an F-linear symplectic pairing:

$$<,>_{1,\lambda}\colon \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \times \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \to \mathrm{H}_1(\mathbb{G}_m|_{\overline{s}}, \mathbb{A}^{\infty,p}) \otimes F.$$

The level structure η is a K^p-orbit of pairs of isomorphisms (η_1, η_2), where (with V the standard symplectic space defined above):

(1) An $\mathcal{O}_F$-linear isomorphism of $\Pi_1(S, \overline{s})$-modules $\eta_1 : V \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,p} \simeq \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p})$.
(2) An $\mathcal{O}_F$-linear isomorphism of $\Pi_1(S, \overline{s})$-modules $\eta_2 : F \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,p} \simeq F \otimes_{\mathbb{Z}} \mathrm{H}_1(\mathbb{G}_m|_{\overline{s}}, \mathbb{A}^{\infty,p})$.

We moreover impose that the following diagram is commutative:

$$\begin{tikzcd}
V \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,p} \times V \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,p} \arrow[r, "\eta_1 \times \eta_1"] \arrow[d, "<{,}>_1"] & \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \times \mathrm{H}_1(A|_{\overline{s}}, \mathbb{A}^{\infty,p}) \arrow[d, "<{,}>_{1,\lambda}"] \\
F \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,p} \arrow[r, "\eta_2"] & F \otimes_{\mathbb{Z}} \mathrm{H}_1(\mathbb{G}_m|_{\overline{s}}, \mathbb{A}^{\infty,p})
\end{tikzcd}$$

The K_p level structure η_p is the data, for each $v \mid p$ such that $K_v = \mathrm{Si}(v)$, of a maximal totally isotropic subgroup $H_v \subset A[v]$.

A map between quintuples $(A, \iota, \lambda, \eta, \eta_p)$ and $(A', \iota', \lambda', \eta', \eta'_p)$ is an $\mathcal{O}_F$-linear prime to p quasi-isogeny (in the sense of [Lan13, Definition 1.3.1.17]) $f : A \to A'$ such that

- $f^\star\lambda = r\lambda'$ for a locally constant function $r : S \to \mathbb{Z}_{(p)}^{\times,+}$,

- $f(\eta_p) = \eta'_p$, and
- $\mathrm{H}_1(f) \circ \eta = \eta'$.

This last condition means that η' is defined by $\mathrm{H}_1(f) \circ \eta_1 = \eta'_1$ and $\eta'_2 = r^{-1}\eta_2$. Also, we have denoted $\mathbb{Z}_{(p)}^{\times,+} = \mathbb{Q}_{>0}^{\times} \cap \mathbb{Z}_{(p)}^{\times}$.

Remark 5.1. Note that we allow the similitude factor in the level structure to be in $\mathbb{A}^{\infty,p} \otimes_{\mathbb{Q}} F(1)$, but we only allow quasi-isogenies with similitude factor in $\mathbb{A}^{\infty,p}(1)$.

We now define an action of $(\mathcal{O}_F)_{(p)}^{\times,+}$ on $\widetilde{Sh}_K$ by scaling the polarization. Namely, $x \in (\mathcal{O}_F)_{(p)}^{\times,+}$ sends $(A, \iota, \lambda, \eta, \eta_p)$ to $(A, \iota, x\lambda, x\eta, \eta_p)$ where $x\eta = (\eta_1, x\eta_2)$.

This action restricts to a trivial action on the subgroup $\nu(K^p \cap \mathcal{O}_{F,(p)}^{\times})$ (where $\mathcal{O}_{F,(p)}^{\times}$ is embedded diagonally in $G(\mathbb{A}_f)$), because for any $x \in K^p \cap \mathcal{O}_{F,(p)}^{\times}$, the multiplication by $x : A \to A$ identifies the points: $(A, \iota, \lambda, \eta, \eta_p)$ and $(A, \iota, x^{-2}\lambda, x^{-2}\eta, \eta_p)$.

We therefore get an action of $\Delta = \mathcal{O}_{F,(p)}^{\times,+}/\nu(K^p \cap \mathcal{O}_{F,(p)}^{\times})$. One can show that this action is free [BCGP18, Lemma 3.3.13]. The group Δ is infinite, but the stabilizer of any connected component of $\widetilde{Sh}_K$ is finite.

The étale surjective map $\widetilde{Sh}_K \to Sh_K$ identifies Sh_K as the quotient of $\widetilde{Sh}_K$ by the group Δ. The action of the group Δ extends to a free action on $\widetilde{\mathfrak{Sh}}_K$. We can form the quotient of $\widetilde{\mathfrak{Sh}}_K$ by Δ and this defines an integral model $\mathfrak{Sh}_K$ for Sh_K over $\mathbb{Z}_{(p)}$ (see [BCGP18, Section 3.3]).

5.4. **Compactifications.** We have smooth toroidal compactifications $\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor}$ for suitable choices of polyhedral cone decompositions. The action of Δ extends to an action on $\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor}$ and this action is free so we get a smooth integral toroidal compactification $\mathfrak{Sh}_{K,\Sigma}^{tor}$ of $\mathfrak{Sh}_K$. See Section 3.5 of [BCGP18].

5.5. **Integral automorphic sheaves.** Let E' be a galois closure of F, λ' be a place of E' above p and $\mathcal{O}_{E',\lambda'}$ the localization of $\mathcal{O}_{E'}$ at λ. Let $\mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_1)$ be the category of representations of M_1 over finite free $\mathcal{O}_{E',\lambda'}$-modules. Let $\mathrm{FL}_{G,X} = G/Q = G_1/Q_1$ be the flag variety.

By [Lan12, Proposition 6.9], over $\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor}$, the first relative de Rham homology group has a canonical extension $\mathcal{H}_{1,dR}(A/\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor})^{can}$, it carries the Hodge filtration

$$0 \to \omega_{A^t} \to \mathcal{H}_{1,dR}(A/\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor})^{can} \to \mathrm{Lie}(A) \to 0$$

and a pairing $\langle\ ,\ \rangle_\lambda$ induced by the polarization. We can consider the principal G_1-torsor $\widetilde{P}_{K,\Sigma} \to \widetilde{Sh}_{K,\Sigma}^{tor}$ of isomorphisms between $\mathcal{H}_{1,dR}(A/\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor})^{can}, \langle\ ,\ \rangle_\lambda$ and (V, Ψ).

We therefore obtain the following diagram:

$$\begin{tikzcd} & \widetilde{\mathfrak{P}}_{K,\Sigma} \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor} & & \mathrm{FL}_{G,X} \end{tikzcd}$$

where α is G_1-equivariant and β is a G_1-torsor. Using this G_1-torsor we can define a functor $\mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_1) \to VB(\widetilde{\mathfrak{Sh}}_{K,\Sigma}^{tor})$.

Let $\kappa = ((k_{1,\sigma}, \cdots, k_{g,\sigma})_\sigma); k)$ be a dominant weight for M_1. There is an associated Weyl representation V_κ and we denote by $\mathcal{V}_{\kappa,K,\Sigma}$ the locally free sheaf corresponding to it via the above functor.

Example 5.2. Over $\mathcal{O}_{E',\lambda}$ we have $\omega_A = \oplus_\sigma (\omega_A)_\sigma$. Let $\sigma_0 \in \mathrm{Hom}(F, E')$. Let $\kappa = ((k_{1,\sigma}, \cdots, k_{g,\sigma})_\sigma); k)$ with $k_{i,\sigma} = 0$ if $\sigma \neq \sigma_0$ and $(k_{1,\sigma_0}, \cdots, k_{g,\sigma_0}) = (0, \cdots, 0, -1)$ and $k = 1$, then $\mathcal{V}_{\kappa,K,\Sigma} = \mathrm{Lie}(A)_{\sigma_0}$. Therefore, if $\kappa = ((k_{1,\sigma}, \cdots, k_{g,\sigma})_\sigma); k)$ with $k_{i,\sigma} = 0$ if $\sigma \neq \sigma_0$ and $(k_{1,\sigma_0}, \cdots, k_{g,\sigma_0}) = (1, 0, \cdots, 0)$ and $k = -1$, then $\mathcal{V}_{\kappa,K,\Sigma} = (\omega_A)_{\sigma_0}$.

Let $Z_s(G)$ be the subgroup of the center of G equal to the kernel of the norm map and we let $G^c = G/Z_s(G)$. We now restate and prove the missing part of Theorem 4.14.

Proposition 5.3. *There is a natural commutative diagram:*

$$\begin{tikzcd} & \mathfrak{P}_{K,\Sigma} \arrow[dl, "\beta'"'] \arrow[dr, "\alpha'"] & \\ \mathfrak{Sh}^{tor}_{K,\Sigma} & & \mathrm{FL}_{G,X} \end{tikzcd}$$

where α' is G-equivariant and β' is a G^c-torsor.

Proof. We have a commutative diagram

$$\begin{tikzcd} & \widetilde{\mathfrak{P}}_{K,\Sigma} \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \widetilde{\mathfrak{Sh}}^{tor}_{K,\Sigma} & & \mathrm{FL}_{G,X} \end{tikzcd}$$

We consider the G^c-torsor $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G^c$. We claim that this torsor descends to $\mathfrak{Sh}^{tor}_{K,\Sigma}$. In order to prove this we must exhibit a descent datum for the action of Δ.

Let $(A, \iota, \lambda, \eta, \eta_p, \Psi)$ be an R-point of $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G$, where $\Psi : V \otimes R \to \mathcal{H}_{1,dR}(A/R)^{can}$ is a symplectic isomorphism up to a similitude factor in $(F \otimes R)^\times$. For any $x \in K^p \cap \mathcal{O}^\times_{F,(p)}$, the multiplication by $x^{-1} : A \to A$ induces a natural map:

$$\mathcal{H}_{1,dR}(A/R)^{can} \to \mathcal{H}_{1,dR}(A/R)^{can}$$

which is scalar multiplication by x^{-1} in the trivialization Ψ. We observe that $x \in Z_s(G)(\mathcal{O}_F)$, and therefore in $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G^c$ the multiplication by x^{-1} induces a canonical isomorphism between $(A, \iota, \lambda, \eta, \eta_p, \Psi)$ and $(A, \iota, x^2\lambda, \eta, \eta_p, \Psi)$. Therefore, we can simply define an action of $(\mathcal{O}_F)^{\times,+}_{(p)}$ on $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G^c$, by sending $(A, \iota, \lambda, \eta, \eta_p, \Psi)$ to $(A, \iota, x\lambda, x\eta, \eta_p, \Psi)$, and this action passes to the quotient to an action of Δ.

We can therefore descend the torsor $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G^c$ to a G^c-torsor $\mathfrak{P}_{K,\Sigma} \to \mathfrak{Sh}^{tor}_{K,\Sigma}$. Moreover, one descends similarly the P^c-reduction of $\widetilde{\mathfrak{P}}_{K,\Sigma} \times^{G_1} G^c$, and therefore get a map $\beta' : \mathfrak{P}_{K,\Sigma} \to \mathrm{FL}_{G,X}$. □

Corollary 5.4. *There is a functor* $\mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M/Z_s(G)) \to VB(\mathfrak{Sh}^{tor}_{K,\Sigma})$ *which makes the following diagram commute:*

$$\begin{tikzcd} \mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M_1) \arrow[r] & VB(\widetilde{\mathfrak{Sh}}^{tor}_{K,\Sigma}) \\ \mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M/Z_s(G)) \arrow[r] \arrow[u] & VB(\mathfrak{Sh}^{tor}_{K,\Sigma}) \arrow[u] \end{tikzcd}$$

Remark 5.5. Let $\kappa = (k_{1,\sigma}, \cdots, k_{g,\sigma}; k_\sigma)_{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ be a dominant weight for M, with associated Weyl representation V_κ. The representation V_κ belongs to $\mathrm{Rep}_{\mathcal{O}_{E',\lambda'}}(M/Z_s(G))$ if and only if $k_\sigma = k$ is independent of σ.

5.6. **Minuscule coweights and the main theorem in the symplectic case.**

5.6.1. *The general formula for minuscule coweights.* We take E' to be the Galois closure of F. We let $\iota : E' \to \overline{\mathbb{Q}}_p$. Let p be a prime unramified in F and denote by $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ the prime ideals above p in F. We let $I = \mathrm{Hom}(F, E')$ and for each $1 \leq i \leq n$, let

$$I_i = \{\sigma \in \mathrm{Hom}(F, E'),\ \iota \circ \sigma \text{ induces the } \mathfrak{p}_i\text{-adic valuation on } F\}.$$

We consider a weight $\kappa = ((k_{1,\sigma}, \cdots, k_{g,\sigma})_{\sigma \in I}; k)$ for G, where we assume that the parity of $\sum_i k_{i,\sigma}$ is independent of σ, and we choose $k \in \mathbb{Z}$ such that $\sum_i k_{i,\sigma} = k \mod 2$. We have, by Corollary 5.4, a sheaf $\mathcal{V}_{\kappa,K,\Sigma}$ on $\mathfrak{Sh}^{tor}_{K,\Sigma}$.

The spherical Hecke algebra at p, denoted $\mathcal{H}_p$, is a tensor product of the spherical Hecke algebras $\mathcal{H}_{\mathfrak{p}_i}$ at each prime $\mathfrak{p}_i$ dividing p. Each of the algebras $\mathcal{H}_{\mathfrak{p}_i}$ contains the following familiar characteristic functions of double cosets:

$$T^{naive}_{\mathfrak{p}_i} = \mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i^{-1}1_g, 1_g)\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})$$

and $S^{naive}_{\mathfrak{p}_i} = \mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i^{-1}, \mathfrak{p}_i^{-1})\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})$ and to each of these double cosets we can associate a cohomological correspondence over $\mathbb{Q}$ on the sheaf $\mathcal{V}_{\kappa,K,\Sigma}$ (see Section 4.5). We have added the superscript "naive" because the cohomological correspondence is not suitably normalized in general.

By definition $T^{naive}_{\mathfrak{p}_i} = T_\lambda$ (see Section 3.2) for the cocharacter $\lambda : t \mapsto (\prod_{\sigma \in I_i} \mathrm{diag}(t^{-1}1_g, 1_g)) \times (\prod_{\sigma \notin I_i} \mathrm{diag}(1_g, 1_g))$ which is given in coordinates by

$$\Big(\prod_{\sigma \notin I_i} (0, \cdots, 0; 0)_\sigma\Big) \times \Big(\prod_{\sigma \in I_i} (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}; -\tfrac{1}{2})\Big),$$

and $S^{naive}_{\mathfrak{p}_i} = T_\mu$ for the cocharacter $\mu : t \mapsto (\prod_{\sigma \in I_i} \mathrm{diag}(t^{-1}1_g, t^{-1}1_g)) \times (\prod_{\sigma \notin I_i} \mathrm{diag}(1_g, 1_g))$ which is given in coordinates by

$$\Big(\prod_{\sigma \notin I_i} (0, \cdots, 0; 0)_\sigma\Big) \times \Big(\prod_{\sigma \in I_i} (0, \cdots, 0; -1)\Big).$$

Remark 5.6. The goal of this remark is to justify the use of the double class

$$\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i^{-1}1_g, 1_g)\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})$$

rather than the double class

$$\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})\mathrm{diag}(\mathfrak{p}_i 1_g, 1_g)\mathrm{GSp}_{2g}(\mathcal{O}_{F_{\mathfrak{p}_i}})$$

which appears in some classical references. The difference can be explained as follows: the adelic points of G act on the right on the tower of Shimura varieties, and therefore on the left on the cohomology; in the classical theory of modular forms, one usually defines a right action of the Hecke algebra on the space of modular forms. Let us give some more details. To avoid complications with non-PEL Shimura varieties, we will assume that our totally real field is $\mathbb{Q}$. For simplicity, we also assume that $K \subset \mathrm{GSp}_4(\hat{\mathbb{Z}})$. Associated to the element $g = \mathrm{diag}(p^{-1}1_g, 1_g)$ we have a correspondence over $\mathbb{Q}$:

$$\begin{tikzcd} & Sh^{tor}_{gKg^{-1}\cap K,\Sigma''} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ Sh^{tor}_{K,\Sigma'} & & Sh^{tor}_{K,\Sigma} \end{tikzcd}$$

and the relation between $p_1^\star A$ and $p_2^\star A$ is given as follows (at least away from the boundary). There are symplectic isomorphisms $\psi_1 : \hat{\mathbb{Z}}^{2g} \to \mathrm{H}_1(p_1^\star A, \hat{\mathbb{Z}}) \mod gKg^{-1} \cap K$ and ψ_2 :

$\hat{\mathbb{Z}}^{2g} \to \mathrm{H}_1(p_2^\star A, \hat{\mathbb{Z}}) \mod g^{-1}Kg \cap K$ and a commutative diagram :

$$\begin{tikzcd} \hat{\mathbb{Z}}^{2g} \otimes \mathbb{Q} \arrow[r, "g"] \arrow[d, "\psi_2"] & \hat{\mathbb{Z}}^{2g} \otimes \mathbb{Q} \arrow[d, "\psi_1"] \\ \mathrm{H}_1(p_2^\star A, \hat{\mathbb{Z}}) \otimes \mathbb{Q} \arrow[r] & \mathrm{H}_1(p_1^\star A, \hat{\mathbb{Z}}) \otimes \mathbb{Q} \end{tikzcd}$$

Therefore, the lattice $\mathrm{H}_1(p_2^\star A, \hat{\mathbb{Z}})$ contains the lattice $\mathrm{H}_1(p_1^\star A, \hat{\mathbb{Z}})$. This means that $p_2^\star A$ appears to be the quotient of $p_1^\star A$ by a Lagrangian subgroup of $p_1^\star A[p]$. The Shimura variety $Sh_{gKg^{-1}\cap K}$ is therefore parametrizing all Lagrangian subgroups of $p_1^\star A$ and the projection p_2 is given by "taking the quotient by the Lagrangian subgroup". If we had made the choice of $g = \mathrm{diag}(1_g, p1_g)$, then we would have obtained the transposed correspondence.

Motivated by Definition 4.8, we now let

$$T_{\mathfrak{p}_i} = [V_\lambda] p^{\langle \lambda, \infty(\kappa,\iota)\rangle} = p^{\langle \lambda, \infty(\kappa,\iota)\rangle - \langle \lambda, \rho\rangle} T_{\mathfrak{p}_i}^{naive}$$

where the last equality follows from the fact that λ is minuscule. We now determine the value of the coefficient $\langle \lambda, \infty(\kappa, \iota))\rangle - \langle \lambda, \rho\rangle$.

Lemma 5.7. *We have* $\langle \lambda, \infty(\kappa,\iota)\rangle - \langle \lambda, \rho\rangle =$

$$\sum_{\sigma \in I_i} \sup_{1\le j\le g} \left\{ \frac{\sum_{\ell=1}^{j} k_{\ell,\sigma} - \left(\sum_{\ell=j+1}^{g} k_{\ell,\sigma}\right) + k}{2} - \frac{j(j+1)}{2} \right\}$$

Proof. We have $\rho = (-1, -2, \cdots, -g; 0)_{\sigma\in I}$, so $(\kappa+\rho)_\sigma = (-1, -2, \cdots, -g; k)$ if $\sigma \notin I_i$ and $(\kappa+\rho)_\sigma = (k_{1,\sigma} - 1, k_{2,\sigma} - 2, \cdots, k_{g,\sigma} - g; k)$ if $\sigma \in I_i$. By definition, $\infty(\kappa,\iota)_\sigma$ is the Weyl translate of $-(\kappa+\rho)_\sigma$ in the dominant cone. We have $\lambda_\sigma = (0, \cdots, 0; 0)$ if $\sigma \notin I_i$, and $\lambda_\sigma = (-\frac{1}{2}, \cdots, -\frac{1}{2}; -\frac{1}{2})$. We clearly have

$$\langle \lambda, \infty(\kappa,\iota)\rangle - \langle \lambda, \rho\rangle = \sum_{\sigma\in I} \langle \lambda_\sigma, \infty(\kappa,\iota)_\sigma)\rangle - \langle \lambda_\sigma, \rho_\sigma\rangle$$

and the contribution to the sum of any $\sigma \notin I_i$ is 0.

Let us fix $\sigma \in I_i$ and compute the corresponding pairing at σ. We need to put each $-(\kappa+\rho)_\sigma$ in the dominant cone (i.e., the coordinates on the left of the ";" need to be non-positive and in decreasing order) and the pairing with λ_σ will amount to taking the sum of the coordinates and multiplying it by $-\frac{1}{2}$.

Now there will be some integer $0 \le j \le g$ such that the first j coordinates of $-(\kappa+\rho)_\sigma$ are non-positive and the next $g - j$ coordinates are non-negative. In that case, the weight will be put in the dominant form by first changing it to

$$(-k_{1,\sigma} + 1, \cdots, -k_{j,\sigma} + j, k_{j+1,\sigma} - j - 1, \cdots, k_{g,\sigma} - g; -k)$$

(so that all entries on the left of ";" are non-positive) and then applying an element of the Weyl group of the Levi to put it in the dominant form. Of course, this last operation is irrelevant for computing the pairing since we will take the sum of all coordinates.

We therefore find that in this case

$$\langle \lambda_\sigma, \infty(\kappa,\iota)_\sigma\rangle = \frac{\sum_{\ell=1}^{j} k_{\ell,\sigma} - \left(\sum_{\ell=j+1}^{g} k_{\ell,\sigma}\right) + k}{2} - \tfrac{1}{2}\sum_{\ell=1}^{j} \ell + \tfrac{1}{2}\sum_{\ell=j+1}^{g} \ell.$$

Now we note that for any $j' \neq j$, the corresponding sum on the RHS above, with j replaced by j', is less than or equal to the sum for j. The formula then follows from the observation that $-\langle \lambda_\sigma, \rho_\sigma\rangle = -\frac{1}{2}\sum_{\ell=1}^{g} \ell$. $\square$

Similarly, motivated by Definition 4.8, we also set

$$S_{\mathfrak{p}_i} = [V_\mu] p^{\langle \mu, \infty(\kappa,\iota)\rangle\rangle} = p^{\langle \lambda, \infty(\kappa,\iota)\rangle\rangle - \langle \lambda, \rho\rangle} S^{naive}_{\mathfrak{p}_i}$$

and it is elementary to check that $S_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i} k} S^{naive}_{\mathfrak{p}_i}$.

Remark 5.8. The coweights λ and μ (for varying primes $\mathfrak{p}_i$) are the only minuscule coweights (up to twist by a central coweight). Unfortunately, our techniques don't allow us to deal with non-minuscule coweights well, and this is why we do not consider the entire Hecke algebra.

5.6.2. *The main result in the symplectic case.* We have the following partial result towards Conjecture 4.16:

Theorem 5.9. *There are algebra morphisms*

$$\otimes_{0\leq i\leq m} \mathbb{Z}[T_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}^{-1}] \to \mathrm{End}\big(\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa_0,K,\Sigma})\big)$$

and

$$\otimes_{0\leq i\leq m} \mathbb{Z}[T_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}^{-1}] \to \mathrm{End}\big(\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa_0,K,\Sigma}(-D_{K,\Sigma}))\big)$$

extending the action over $\mathbb{Q}$ (see Section 4.5).

We now discuss some special cases of the theorem.

5.6.3. $G = \mathrm{GL}_2/\mathbb{Q}$. The sheaf of weight k modular forms corresponds to the weight $(k;-k) := \kappa$. From the formula of Lemma 5.7, we deduce that $T_p = p^{-\inf\{1,k\}} T^{naive}_p$ and $S_p = p^{-k} S^{naive}_p$ and that

$$\mathcal{H}^{int}_{p,\kappa} = \mathbb{Z}[T_p, S_p, S_p^{-1}].$$

It follows from Theorem 5.9, that $\mathcal{H}^{int}_{p,\kappa}$ acts on the cohomology complex of weight k modular forms. Conjecture 4.16 is thus proven in this case.

5.6.4. $G = \mathrm{GL}_2/F$. We let $\kappa = ((k_\sigma)_\sigma; k)_\sigma$ with the property that k and all the k_σ have the same parity. Then we find that

$$T_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i} \sup\{\frac{k_\sigma+k}{2}-1, \frac{k-k_\sigma}{2}\}} T^{naive}_{\mathfrak{p}_i}$$

and

$$S_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i} k} S^{naive}_{\mathfrak{p}_i}.$$

We deduce that $\mathcal{H}^{int}_{p,\kappa,\iota} = \otimes_i \mathcal{H}^{int}_{\mathfrak{p}_i,\kappa}$ where

$$\mathcal{H}^{int}_{\mathfrak{p}_i,\kappa} = \mathbb{Z}[T_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}, S_{\mathfrak{p}_i}^{-1}].$$

It follows from Theorem 5.9, that $\mathcal{H}^{int}_{p,\kappa,\iota}$ acts on the cohomology complex of weight κ modular forms. Conjecture 4.16 is thus also proven in this case.

5.6.5. $G = \mathrm{GSp}_{2g}/\mathbb{Q}$. We take $\kappa = (k_1, \cdots, k_g; -\sum k_i)$. Our choice of the central character is the standard choice in the theory of Siegel modular forms. Indeed, the sheaf $\mathcal{V}_{\kappa,K,\Sigma}$ has the following elementary description. First, on $\mathfrak{Sh}^{tor}_{K,\Sigma}$, we have a semi-abelian scheme A of dimension g, and we denote by ω_A the conormal sheaf. We denote by $\mathcal{T}$ the torsor of trivializations of ω_A. This is a GL_g-torsor and we let $\pi : \mathcal{T} \to \widetilde{Sh}^{tor}_{K,\Sigma}$ be the projection. Then, unravelling the definitions, we find that $\mathcal{V}_{\kappa,K,\Sigma} = \pi_\star \mathscr{O}_{\mathcal{T}}[\kappa^\vee]$ where $\kappa^\vee = (-k_g, \cdots, -k_1)$ (and $\pi_\star \mathscr{O}_{\mathcal{T}}[\kappa^\vee]$ is the subsheaf of $\pi_\star\mathscr{O}_{\mathcal{T}}$ of sections which transform by the character $\kappa^\vee$ under the action of the Borel). We finally obtain that

$$T_p = p^{\sup_{1\leq j\leq g}\{-\sum_{\ell=j+1}^g k_l - \frac{j(j+1)}{2}\}} T^{naive}_p$$

and $S_p = p^{-\sum k_i} S^{naive}_p$.

6. Local model in the symplectic case

In this section we will prove Theorem 5.9. The actions of the normalized Hecke operators will be defined using the results of Section 2, in particular the construction in Example 2.11. In order to do this, we need to understand the integrality properties of the Hecke correspondences with respect to automorphic vector bundles and also the pullback maps on differentials. This will be done by using the theory of local models of Shimura varieties. Variants of the local models we consider in this section were introduced in [CN92], [DP94], [dJ93] and studied further by Görtz [GÖ3]; for a general introduction to the theory of local models the reader may consult [PRS13].

After recalling the basic facts about the local models for symplectic groups, we prove two results, Proposition 6.5 (on the integrality properties of differentials) and Lemma 6.7 (on the integrality properties on automorphic bundles) which are used to construct normalized Hecke operators on the local model in Proposition 6.8. We then transport these computations to the Shimura variety side using Theorem 6.9 and thereby prove Theorem 5.9.

6.1. **Definition.** Let $g \in \mathbb{Z}_{\geq 1}$ be an integer. For any positive integer t, we let Id_t be the identity $t \times t$ matrix and we let K_t be the anti-diagonal matrix of size $t \times t$, with coefficients 1 on the anti-diagonal. When the context is clear, we sometimes write K instead of K_t and Id instead of Id_t.

Let $V_0 = \mathbb{Z}^{2g}$. We equip V_0 with the symplectic pairing ψ given by the matrix

$$J = \begin{pmatrix} 0 & K_g \\ -K_g & 0 \end{pmatrix}$$

We now consider modules $V_1, \cdots, V_{2g-1} = \mathbb{Z}^{2g}$ and the following chain:

$$V_\bullet : V_0 \to V_1 \to \cdots \to V_{2g-1} \to V_0$$

where the map from V_i to V_{i+1} is given in the canonical basis $(e_1, \cdots, e_{2g})$ of $\mathbb{Z}^{2g}$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto p e_{i+1}$. Whenever necessary, indices are taken modulo $2g$ so that $V_{2g} := V_0$.

This chain is self-dual. The pairing ψ and the maps in the chain induce pairings $\psi : V_r \times V_{2g-r} \to \mathbb{Z}$ which can be written as $p^2\psi'$, for a perfect pairing ψ'.

Let us fix a set $\emptyset \neq I \subset \{0, 1, \cdots, g\}$. We define the local model functor $\mathbf{M}_I$: $\mathbb{Z} - \mathbf{ALG} \to \mathbf{SETS}$ which associates to an object R of $\mathbb{Z} - \mathbf{ALG}$ the set of isomorphism classes of commutative diagrams

$$\begin{tikzcd} V_{i_0} \otimes_{\mathbb{Z}} R \arrow[r] & V_{i_1} \otimes_{\mathbb{Z}} R \arrow[r] & \cdots \arrow[r] & V_{i_m} \otimes_{\mathbb{Z}} R \\ F_{i_0} \arrow[u] \arrow[r] & F_{i_1} \arrow[u] \arrow[r] & \cdots \arrow[r] & F_{i_m} \arrow[u] \end{tikzcd}$$

where $i_0 < i_1 \cdots < i_m$ are such that $\{i_0, \cdots, i_m\} = I \cup \{2g - i \,|\, i \in I\}$, the modules F_{i_j}, for $0 \leq j \leq m$, are rank g locally direct factors of $V_{i_j} \otimes_{\mathbb{Z}} R$, and they are self dual in the sense that $F_{2g-i}^{\perp} = F_i$ for all $i \in I$ (with respect to the pairing ψ').

The functor $\mathbf{M}_I$ is represented by a projective scheme which we denote by M_I. It is a closed subscheme of a product of Grassmannians, the embedding being given by the vertical maps of diagrams as above.

When $\emptyset \neq J \subset I$, there is an obvious map $M_I \to M_J$ given by forgetting the modules F_j for $j \in I \setminus J$. There is a canonical isomorphism $M_{\{0\}} \simeq M_{\{g\}}$ given by taking $F_g \subset V_g \otimes_{\mathbb{Z}} R$ to be F_0, via the tautological identification of V_0 and V_g.

6.2. **The affine Grassmannian.** Let $\mathcal{V}_0 = \mathbb{F}_p[[t]]^{2g}$. We equip $\mathcal{V}_0$ with the symplectic pairing ψ given by the matrix

$$J = \begin{pmatrix} 0 & K_g \\ -K_g & 0 \end{pmatrix}$$

We now consider modules $\mathcal{V}_1, \cdots, \mathcal{V}_{2g-1} = \mathbb{F}_p[[t]]^{2g}$ and the chain:

$$\mathcal{V}_\bullet = \mathcal{V}_0 \to \mathcal{V}_1 \to \cdots \to \mathcal{V}_{2g-1} \to \mathcal{V}_0$$

where the map from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$ is given in the canonical basis $(e_1, \cdots, e_{2g})$ of $\mathbb{F}_p[[t]]^{2g}$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto te_{i+1}$. Whenever necessary, indices are taken modulo $2g$ so that $\mathcal{V}_{2g} := \mathcal{V}_0$. Observe that $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = V_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Let LG denote the loop group of GSp_{2g} over $\mathbb{F}_p$. The group LG acts naturally on $\mathcal{V}_0 \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$ and therefore it acts on the chain $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$.

Let $\emptyset \neq I \subset \{0, 1, \cdots, g\}$. Let $\mathcal{V}_\bullet^I$ be the subchain of $\mathcal{V}_\bullet$ where we keep only the modules indexed by elements $i \in I$ and $i' = 2g - i$ for $i \in I$. We denote by $\mathcal{P}_I$ the parahoric subgroup of LG of automorphisms of the chain $\mathcal{V}_\bullet^I$. For any I as above, we define the affine flag variety as the ind-scheme $\mathcal{F}_I := LG/\mathcal{P}_I$.

6.3. **Stratification of the local model.** It was observed by Goı̈rtz [GÖ3, §5] that the fibre $\overline{M}_I$ of M_I over Spec $\mathbb{F}_p$ embeds as a finite union of $\mathcal{P}_I$ orbits in $\mathcal{F}_I$. We recall the description of the map $\overline{M}_I \to \mathcal{F}_I$. Given a diagram

$$\begin{tikzcd}
V_{i_0} \otimes_{\mathbb{Z}} R \arrow[r] & V_{i_1} \otimes_{\mathbb{Z}} R \arrow[r] & \cdots \arrow[r] & V_{i_m} \otimes_{\mathbb{Z}} R \\
F_{i_0} \arrow[r] \arrow[u] & F_{i_1} \arrow[r] \arrow[u] & \cdots \arrow[r] & F_{i_m} \arrow[u]
\end{tikzcd}$$

corresponding to an R-point of $\overline{M}_I$, we can construct a new diagram:

$$\begin{tikzcd}
\mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R \arrow[r] & \mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R \arrow[r] & \cdots \arrow[r] & \mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R \\
\mathcal{F}_{i_0} \arrow[r] \arrow[u] & \mathcal{F}_{i_1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{F}_{i_m} \arrow[u] \\
t\mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R \arrow[r] \arrow[u] & t\mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R \arrow[r] \arrow[u] & \cdots \arrow[r] & t\mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R \arrow[u]
\end{tikzcd}$$

where all the vertical maps are inclusions and each $\mathcal{F}_{i_j}$ is determined by the property that $\mathcal{F}_{i_j}/t\mathcal{V}_{i_j} \otimes_{\mathbb{Z}} R = F_{i_j} \hookrightarrow (\mathcal{V}_{i_j}/t\mathcal{V}_{i_j}) \otimes_{\mathbb{Z}} R = V_{i_j} \otimes R$. The chain $\mathcal{F}_\bullet$ determines an R-point of $\mathcal{F}_I$.

We now recall the combinatorial description of the image of $\overline{M}_I$ in $\mathcal{F}_I$: Fix a Borel subgroup B of GSp_{2g} and a maximal torus $T \subset B$. This gives a base for the root datum of GSp_{2g} with a corresponding Dynkin diagram with g vertices and Weyl group W generated by reflections $s_1, s_2, \ldots, s_g$. Let $\widetilde{W}$ be the extended affine Weyl group of GSp_{2g}. This is the semi-direct product of the Weyl group W and the cocharacter group $\mathrm{X}_\star(T)$. It contains as a subgroup W_{af}, the affine Weyl group of GSp_{2g}, which is a Coxeter group with simple reflections $s_1, s_2, \ldots, s_g$ and one affine reflection s_0.

We give a concrete realization of W and $\widetilde{W}$ following [KR99]. The group W can be realized as the subgroup of $\mathcal{S}_{2g}$, the group of permutations of $\{1, 2, \ldots, 2g\}$, which satisfy the condition $w(i) + w(2g + i - 1) = 2g + 1$ for all $i = 1, \cdots, 2g$. The extended affine Weyl group of GL_{2g} is $\mathbb{Z}^{2g} \rtimes \mathcal{S}_{2g}$. It can be realized as a subgroup of the group of affine transformations of $\mathbb{Z}^{2g}$ with $\mathbb{Z}^{2g}$ acting by translation and $\mathcal{S}_{2g}$ by permutation of the coordinates. For an element $v \in \mathbb{Z}^{2g}$ we will denote the corresponding translation by t_v. The group $\widetilde{W}$ is realized as the centralizer in $\mathbb{Z}^{2g} \rtimes \mathcal{S}_{2g}$ of the element $(1,2g)(2,2g-1)\cdots(g,g+1) \in \mathcal{S}_{2g}$. We have $s_i = (i,i+1)(2g+1-i,2g-i)$ for $1 \leq i \leq g-1$, $s_g = (g,g+1)$ and $s_0 = \mathrm{t}_{(-1,0,\cdots,0,1)} \rtimes (1,2g)$. We let $\mu = (1,\cdots,1,0,\cdots 0) \in \mathbb{Z}^{2g}$ (with both 1 and 0 repeated g times) be the minuscule coweight corresponding to our situation.

For each I as above, we let W_I be the subgroup of $\widetilde{W}$ generated by the simple reflexions $s_i, i \notin I$. This is a finite group since $I \neq \emptyset$. The $\mathcal{P}_I$ orbits in $\mathcal{F}_I$ are parametrised by the double cosets $W_I \backslash \widetilde{W}/W_I$. For any $w \in \widetilde{W}$ we denote the orbit corresponding to the double coset $W_I w W_I$ by $U_{I,w}$ and the orbit closure by $X_{I,w}$. The orbits included in $\overline{M}_I$ are parametrized by the finite subset $Adm_I(\mu)$ of $W_I \backslash \widetilde{W}/W_I$ of μ-admissible elements as defined in [KR99, Introduction]. The open orbits—these are the only ones we will need to know explicitly—are parametrized by the double cosets corresponding to elements in the W-orbit of μ (viewed as translations in $\widetilde{W}$).

As noted earlier, whenever $J \subset I$ we have a map $M_I \to M_J$. This map is surjective and $Adm_J(\mu)$ is the image of $Adm_I(\mu)$ in $W_J \backslash \widetilde{W}/W_J$.

When $I = \{0,\cdots,g\}$, Kottwitz and Rapoport give a description of the set $Adm_I(\mu)$. This is a subset of $W_{af}\, c \subset \widetilde{W}$ where $c = \mathrm{t}_{(0,\cdots,0,1,\cdots,1)} \cdot (1,g+1)(2,g+2)\dots(g,2g)$. We can transport the Bruhat order and the length function from W_{af} to $W_{af}\, c$ via the bijection $W_{af} \simeq W_{af}\, c$ of multiplication by c on the right. We observe that $\mu \in W_{af}\, c$; indeed,

$$\mu = w_\mu c,$$

where $w_\mu = (s_g s_{g-1} \dots s_1)(s_g \dots s_2) \dots (s_g s_{g-1}) s_g \in W_{af}$. Observe also that the length of μ is $\frac{g(g+1)}{2}$. The set $Adm_I(\mu)$ is precisely the subset of $W_{af}\, c$ of elements which are $\leq$ a translation $\mathrm{t}_{w\cdot\mu}$ for some $w \in W$ [KR99, Theorem 4.5]. Furthermore, for each $w \in Adm_I(\mu)$, the stratum $U_{I,w}$ has dimension $\ell(w)$.

6.4. **Irreducible components.**

6.4.1. *The case that* $I = \{0,\cdots,g\}$. There are 2^g translations in the W orbit of μ and they correspond to the open stratum in each of the 2^g irreducible components of $\overline{M}_I$ when $I = \{0,\cdots,g\}$. These 2^g translations are parametrized by W/W_c where W_c is the subgroup of W of elements which stabilize μ. It identifies with the elements in $W \subset \mathcal{S}_{2g}$ which preserve the sets $\{1,\cdots,g\}$ and $\{g+1,\cdots,2g\}$, so this group is isomorphic to $\mathcal{S}_g$. The quotient W/W_c has a set of representatives in W, denoted W^c, and called Kostant representatives. An element $w \in W^c$ is characterized by the property that it is the element of minimal length in the coset wW_c. The elements in W^c are exactly the permuations $w \in W$ for which $w^{-1}(g) \geq \cdots \geq w^{-1}(1)$. Such representatives are in bijection with functions $s : \{1,\cdots,g\} \to \{1,\cdots,2g\}$ which are increasing and take exactly once one of the values $\{i, 2g+1-i\}$ for all $1 \leq i \leq g$. (Just set $s_w = w^{-1}(i)$). Moreover, the length of an element w with corresponding function s is $\frac{g(g+1)}{2} - \sum_{i \in \mathrm{Im}(s)\cap\{1,\cdots,g\}} g-i+1$.

We can concretely determine an element of each of the orbits in $\overline{M}_I$ corresponding to these $\mathrm{t}_{w\mu}$ for $I = \{0,\cdots,g\}$ as follows. The group $\widetilde{W}$ can be viewed as a subgroup of $\mathrm{GSp}_{2g}(\mathbb{F}_p((t)))$, with W begin represented by permutation matrices and the elements of $X_\star(T)$ as diagonal matrices by $\chi \in X_\star(T) \mapsto \chi(t)$. The element t_μ is thereby identified with $\mathrm{diag}(t,\cdots,t,1,\cdots 1)$.

We now consider the inclusion of chains:

$$t\mathcal{V}_\bullet \subset \mathrm{t}_{w\mu}(\mathcal{V}_\bullet) \subset \mathcal{V}_\bullet.$$

By reduction modulo t and using the identification $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = V_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$, we deduce that $\mathrm{t}_{w\mu}(\mathcal{V}_\bullet)/t\mathcal{V}_\bullet \hookrightarrow V_\bullet$ defines an $\mathbb{F}_p$ point of M_I, which represents the $w\mu$-orbit.

Remark 6.1. Let $F_\bullet \subset V_\bullet \otimes k$ be a k-point in the $w\mu$ orbit. Let $s : \{1, \cdots, g\} \to \{1, \dots, 2g\}$ be the associated function. We actually get that the map

$$F_{i-1} \to F_i$$

is an isomorphism if and only $V_{i-1}/F_{i-1} \to V_i/F_i$ has kernel and cokernel of dimension 1 and if and only if $i \in \mathrm{Im}(s)$.

6.4.2. *The case that* $I = \{0\}$. The special fiber $\overline{M}_{\{0\}}$ of $M_{\{0\}}$ is smooth and irreducible. Moreover, there is a single orbit

6.4.3. *The case that* $I = \{0, g\}$.

Lemma 6.2. *The special fiber $\overline{M}_{\{0,g\}}$ of $M_{\{0,g\}}$ has $g+1$ irreducible components. There are $g+1$ open strata for the KR stratification, indexed by the integers $0 \le s \le g$. For each $0 \le s \le g$, a representative of the s-stratum is given by taking*

$$F_0(s) = F_g(s) = \langle e_{s+1}, \cdots, e_g, e_{2g-s+1}, \cdots, e_{2g} \rangle$$

This corresponds to the element $\mathrm{diag}(t\mathrm{Id}_s, \mathrm{Id}_{g-s}, t\mathrm{Id}_{g-s}, \mathrm{Id}_s) \in LG$.

Proof. It is easy to see that for all $w \in W$, there exists a $w' \in W_I$ and $0 \le s \le g$ such that

$$w'\mathrm{t}_{w\mu} = \mathrm{diag}(t\mathrm{Id}_s, \mathrm{Id}_{g-s}, t\mathrm{Id}_{g-s}, \mathrm{Id}_s).$$

This simply follows from the fact that $\mathcal{S}_g = W_I$. It is easy to see that all the orbits corresponding to these elements are disjoint and of dimension $\frac{g(g+1)}{2}$. $\square$

Remark 6.3. We have the equality $\mathrm{diag}(t\mathrm{Id}_s, \mathrm{Id}_{g-s}, t\mathrm{Id}_{g-s}, \mathrm{Id}_s) = w(s)(t\mathrm{Id}_g, \mathrm{Id}_g)$ where $w(s)(i) = i$ if $i \le s$, $w(s)(i) = 2g+1-i$ if $s+1 \le i \le g$.

6.5. **Local geometry of the local model.** The local model $M_{\{0\}}$ is smooth of relative dimension $\frac{g(g+1)}{2}$. The other local models M_I for $I \neq \{0\}$ and $I \neq \{g\}$ are isomorphic to $M_{\{0\}}$ over Spec $\mathbb{Z}[1/p]$ but they have singular special fiber at p. Nevertheless, we have the following important result, the first part due to Görtz [Gö3, Theorem 2.1] and the second to He [He13, Theorem 1.2]:

Theorem 6.4. *The local models M_I are flat over $\mathbb{Z}$ and $\overline{M}_I$ is reduced. Furthermore, M_I is Cohen–Macaulay.*

6.6. **Hecke correspondence on the affine Grassmannians.** We consider the correspondence:

$$\begin{tikzcd} & \mathcal{F}_{\{0,g\}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{F}_{\{g\}} & & \mathcal{F}_{\{0\}} \end{tikzcd}$$

We now pick the element $w(s)\mu \in \widetilde{W}$ and restrict this map to a map:

$$\begin{tikzcd} & U_{\{0,g\},w(s)\mu} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ U_{\{g\},w(s)\mu} & & U_{\{0\},w(s)\mu} \end{tikzcd}$$

Proposition 6.5. *The map on differentials $\mathrm{d}p_1 : p_1^\star \Omega^1_{U_{\{0\},w(s)\mu}} \to \Omega^1_{U_{\{0,g\},w(s)\mu}}$ has kernel and cokernel a locally free sheaf of rank $\frac{(g-s)(g-s+1)}{2}$.*

Proof. We have

$$U_{\{0,g\},w(s)\mu} \simeq \mathcal{P}_{\{0,g\}}/\big(\mathcal{P}_{\{0,g\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0,g\}}\mathrm{t}_{w(s)\mu}^{-1}\big)$$

$$U_{\{0\},w(s)\mu} \simeq \mathcal{P}_{\{0\}}/\big(\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(s)\mu}^{-1}\big).$$

and p_1 is the obvious $\mathcal{P}_{(0,g)}$- equivariant projection:

$$\mathcal{P}_{\{0,g\}}/\big(\mathcal{P}_{\{0,g\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0,g\}}\mathrm{t}_{w(s)\mu}^{-1}\big) \to \mathcal{P}_{\{0\}}/\big(\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(s)\mu}^{-1}\big)$$

Because the map is $\mathcal{P}_{(0,g)}$-equivariant, it suffices to prove the claim in the tangent space at the identity.

We first determine the shape of $\mathcal{P}_{\{0\}}$ and $\mathcal{P}_{\{0,g\}}$. The group $\mathcal{P}_{\{0\}}$ is the hyperspecial subgroup of LG, whose R-points are $G(R[[t]])$. The group $\mathcal{P}_{\{0,g\}}$ is the Siegel parabolic group:

$$\mathcal{P}_{\{0,g\}}(R) = \big\{M = \left(\begin{smallmatrix} a & b \\ tc & d \end{smallmatrix}\right) \in LG(R)\big\}$$

where $a, b, c, d \in M_{g\times g}(R[[t]])$.

We now determine that $\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(s)\mu}^{-1}$ consists of matrices with the following shape (the $\star$ have integral values, the rows and columns are of size s, $g-s$, $g-s$, and s):

$$\begin{pmatrix} \star & t\star & \star & t\star \\ \star & \star & \star & \star \\ \star & t\star & \star & t\star \\ \star & \star & \star & \star \end{pmatrix}$$

We also determine that $\mathcal{P}_{\{0,g\}} \cap \mathrm{t}_{w(s)\mu}\mathcal{P}_{\{0,g\}}\mathrm{t}_{w(s)\mu}^{-1}$ consists of matrices with the following shape:

$$\begin{pmatrix} \star & t\star & \star & t\star \\ \star & \star & \star & \star \\ t\star & t^2\star & \star & t\star \\ t\star & t\star & \star & \star \end{pmatrix}$$

Passing to the Lie algebras, we easily see that the kernel of the map:

$$\mathfrak{p}_{\{0,g\}}/\big(\mathfrak{p}_{\{0,g\}} \cap \mathrm{t}_{w(s)\mu}\mathfrak{p}_{\{0,g\}}\mathrm{t}_{w(s)\mu}^{-1}\big) \to \mathfrak{p}_{\{0\}}/\big(\mathfrak{p}_{\{0\}} \cap \mathrm{t}_{w(s)\mu}\mathfrak{p}_{\{0\}}\mathrm{t}_{w(s)\mu}^{-1}\big)$$

is the set of matrices of the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & tA & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A \in M_{g-s\times g-s}(\mathbb{F}_p)$ satisfies $K_{g-s}^t A K_{g-s} = A$ (the symplectic condition). $\square$

6.7. **The Hecke correspondence on the local model.** We consider the correspondence:

$$\begin{tikzcd} & M_{\{0,g\}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ M_{\{g\}} & & M_{\{0\}} \end{tikzcd}$$

6.7.1. *Sheaves on the local model.* Let X be a scheme and L be a locally free sheaf of rank g over X. We let $T_L = \mathrm{Isom}_X(\mathscr{O}_X^g, L)$ be the associated torsor. We let ω be the universal trivialization. The group GL_g acts on the right by $\omega\gamma = \omega \circ \gamma$. Let T be the standard diagonal torus in GL_g and let B be the upper triangular Borel.

Let $X^\star(T)$ be the character group of T. We have $X^\star(T) \simeq \mathbb{Z}^g$ via $(k_1, \cdots, k_g) \mapsto [\mathrm{diag}(t_1, \cdots, t_g) \mapsto \prod t_i^{k_i}]$ and P^+, the cone of dominant weights, is given by $k_1 \geq k_2 \geq \cdots \geq k_g$.

For all $\kappa \in X^+(T)$ we denote by $L_\kappa = \pi_\star \mathscr{O}_{T_L}[\kappa^\vee]$ where $\pi_\star \mathscr{O}_{T_L}[\kappa^\vee]$ is the subsheaf of $\pi_\star \mathscr{O}_{T_L}$ of sections $f(\omega)$ (ω a trivialization of L) which satisfy $f(\omega b) = \kappa^\vee(b) f(\omega)$ and $\kappa^\vee = (-k_g, \cdots, -k_1)$ and $b \in B$. This is a locally free sheaf over X.

We can in particular apply this construction to the sheaves $L_0 = (V_0/F_0)^\vee$ on $M_{\{0\}}$ and $L_g = (V_g/F_g)^\vee$ on $M_{\{g\}}$ to obtain sheaves $L_{0,\kappa}$ on $M_{\{0\}}$ and $L_{g,\kappa}$ on $M_{\{g\}}$.

Remark 6.6. We have isomorphisms $L_g \simeq F_g$ and $L_0 \simeq F_0$ using the pairing, but these isomorphisms are not equivariant for the action of the center of the group G (there is a "Tate" twist).

6.7.2. *The map $L_{g,\kappa} \to L_{0,\kappa}$.* The natural map $V_0 \to V_g$ induces a map $p_1^\star V_0/F_0 \to p_2^\star V_g/F_g$ over $M_{\{0,g\}}$ and by duality a map $p_2^\star L_g \to p_1^\star L_0$ that we denote by α. The map α is an isomorphism on the generic fibre of $M_{\{0,g\}}$, so it induces an isomorphism $\alpha^\star : p_2^\star L_{g,\kappa} \to p_1^\star L_{0,\kappa}$ on the generic fibre for each κ. We now investigate the integral properties of this map.

Lemma 6.7. *Let $\kappa = (k_1, \cdots, k_g)$. Let $0 \leq s \leq g$. Let ξ be the generic point of $U_{\{0,g\}}(s)$. The map $\alpha^\star$ induces a map $\alpha^\star : (p_2^\star L_{g,\kappa})_\xi \to p^{k_g + \cdots + k_{g-s+1}} (p_1^\star L_{0,\kappa})_\xi$ over the local ring $\mathscr{O}_{M_{0,g},\xi}$.*

Proof. We first check that over $U_{\{0,g\}}(s)$, the map α has kernel and cokernel a locally free sheaf of rank s. Indeed, it is enough to check this at the point corresponding to $\mathrm{t}_{w(s)\mu}$, in which case the corresponding diagram is:

$$\begin{tikzcd} \mathcal{V}_0 \arrow[r, "\mathrm{diag}(t1_g, 1_g)"] & \mathcal{V}_g \\ \mathcal{V}_0 \arrow[u, "\mathrm{t}_{w(s)\mu}"] \arrow[r, "\mathrm{diag}(t1_g, 1_g)"] & \mathcal{V}_g \arrow[u, "\mathrm{t}_{w(s)\mu}"'] \end{tikzcd}$$

and our claim is simply that the map $\mathcal{V}_0/\mathrm{t}_{w(s)\mu}\mathcal{V}_0 \to \mathcal{V}_g/\mathrm{t}_{w(s)\mu}\mathcal{V}_g$ has kernel of rank s. This is obvious.

We can work over the completion R of $\mathscr{O}_{M_{0,g},\xi}$, which has uniformizing element p. We also denote by v the p-adic valuation on R normalized by $v(p) = 1$. We fix isomorphisms $L_0 \simeq R^g$ and $L_g \simeq R^g$ such that $\alpha = \mathrm{diag}(p1_s, 1_{g-s})$ in these bases. We have

$$GL_g(R) = \coprod_{w \in \mathcal{S}_g} \mathrm{Iw} w \mathrm{U}(R)$$

the Iwahori decomposition with Iw the matrices which are upper triangular mod p and U the unipotent radical of B. Let $f \in L_{r,\kappa}$. Then for $i \in \mathrm{Iw}$ and $w \in \mathcal{S}_g$,

$$\begin{aligned} \alpha^\star f(iw) &= f(\alpha^{-1} iw) \\ &= f(\alpha^{-1} i\alpha w w^{-1} \alpha^{-1} w) \\ &= f(\alpha^{-1} i \alpha w) \kappa^\vee(w^{-1}\alpha^{-1} w) \end{aligned}$$

Since $\alpha^{-1} i \alpha w \in \mathrm{GL}_g(R)$, we deduce that $v\big(f(\alpha^{-1} i\alpha w)\kappa^\vee(w^{-1}\alpha^{-1}w)\big) \geq v(\kappa^\vee(w^{-1}\alpha^{-1}w)) \geq k_g + k_{g-1} + \cdots + k_{(g-s+1)}$ for all w since $k_1 \geq k_2 \geq \cdots \geq k_g$. $\square$

6.7.3. *The cohomological correspondence.* We may now construct a cohomological correspondence. By Proposition 2.6 and Theorem 6.4, we have a fundamental class $p_1^\star \mathscr{O}_{M_{\{0\}}} \to p_1^! \mathscr{O}_{M_{\{0\}}}$. Moreover, the sheaf $p_1^! \mathscr{O}_{M_{\{0\}}}$ is a CM sheaf.

There is also a map $\alpha : p_2^\star L_{g,\kappa} \dashrightarrow p_1^\star L_{0,\kappa}$ defined on the generic fibre, so that putting everything together, we have a generically defined map (the naive cohomological correspondence):

$$T^{naive} : p_2^\star L_{g,\kappa} \dashrightarrow p_1^! L_{0,\kappa}$$

We may now normalize this correspondence.

Proposition 6.8. *Let* $T = p^{-\inf_j \left\{ \sum_{\ell=j+1}^{g} k_\ell + \frac{j(j+1)}{2} \right\}} T^{naive}$. *Then T is a true cohomological correspondence:*

$$T : p_2^\star L_{g,\kappa} \to p_1^! L_{0,\kappa}$$

Proof. Because $p_1^! L_{0,\kappa}$ is a CM sheaf, any generically defined map from a locally free sheaf into $p_1^! L_{0,\kappa}$ is defined globally if it is defined in codimension 1. So it is enough to check that T is defined at the generic points of all the components of $\overline{M}_{\{0,g\}}$. Let $0 \leq s \leq g$ and let ξ be the generic point of the stratum $U_{\{0,g\}}(s)$. At this point, we see that $T^{naive} : (p_2^\star L_{g,\kappa})_\xi \to p^{\sum_{\ell=g-s+1}^{g} k_\ell + \frac{(g-s)(g-s+1)}{2}} (p_1^! L_{0,\kappa})_\xi$ by combining Lemma 6.7 and Proposition 6.5. $\square$

6.7.4. *Proof of Theorem 5.9.* The main point of the theorem is to construct the action of $T_{\mathfrak{p}_i}$. The action of $S_{\mathfrak{p}_i}$ is by automorphisms (they are some generalized diamond operators) and the commutativity of the various operators is rather formal.

We fix some prime $\mathfrak{p}_i$. We let $K(\mathfrak{p}_i) = K \cap t_i K t_i^{-1}$ for $t_i = \mathrm{diag}(\varpi_i^{-1} 1_g, 1_g) \subset G(\mathbb{A}_f)$ where ϖ_i is the finite adèle which is 1 at all places different from $\mathfrak{p}_i$ and p at $\mathfrak{p}_i$.

We claim that there is a Hecke correspondence:

$$\begin{tikzcd}
 & \mathfrak{Sh}^{tor}_{K(\mathfrak{p}_i),\Sigma''} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
\mathfrak{Sh}^{tor}_{K,\Sigma'} & & \mathfrak{Sh}^{tor}_{K,\Sigma}
\end{tikzcd}$$

which extends the usual Hecke correspondence on the generic fiber. In order to construct this correspondence we use the PEL Shimura variety. We claim that there is a diagram :

(6.7.A)

$$\begin{tikzcd}
 & \widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)} \arrow[dl, "\tilde{p}_2"'] \arrow[d] \arrow[dr, "\tilde{p}_1"] & \\
\widetilde{\mathfrak{Sh}}_K \arrow[d] & \mathfrak{Sh}_{K(\mathfrak{p}_i)} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \widetilde{\mathfrak{Sh}}_K \arrow[d] \\
\mathfrak{Sh}_K & & \mathfrak{Sh}_K
\end{tikzcd}$$

where the lower hat is simply the quotient of the top hat by the action of Δ. The right square is cartesian by definition. We explain how to define $\tilde{p}_2$ in order to make the left square cartesian. We take a totally positive element $x_i \in F^{\times,+}$ which has the property that its $\mathfrak{p}_i$-adic valuation is exactly 1, but that its $\mathfrak{p}_j$-adic valuation is 0 for all $j \neq i$. Given a point $(A, \iota, \lambda, \eta, \eta_p)$ in $\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)}$ where η_p corresponds to a maximal totally isotropic subgroup H of $A[\mathfrak{p}_i]$ we define $\tilde{p}_2((A, \iota, \lambda, \eta, \eta_p)) = (A', \iota', \lambda', \eta')$ where $A' = A/H$, ι' and η'

have the obvious definitions, and λ' is defined by descending the polarization $x_i\lambda$ to A/H. This indeed defines a prime to p polarization.

The vertical maps in the above diagram are étale and surjective. We note that $\tilde{p}_2$ is not canonical (because of the ambiguity in the choice of x_i), but that p_2 is canonical.

Let $\kappa = ((k_{1,\sigma}, \cdots, k_{g,\sigma}); k)_{\sigma\in I}$ be a dominant weight for $M_\mu/Z_c(G)$: $k_{1,\sigma} \geq \cdots \geq k_{g,\sigma}$, and both k and $\sum_i k_{i,\sigma}$ have the same parity. After inverting p, there is a map (denoted T_{t_i} in section 4.5) $T^{naive}_{\mathfrak{p}_i} : p_2^\star \mathcal{V}_{\kappa,\Sigma} \to p_1^! \mathcal{V}_{\kappa,K,\Sigma}$ respecting the cuspidal subsheaves and therefore inducing $p_2^\star\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\ \Sigma}) \to p_1^!\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})$.

We will prove that $p_1^!\mathcal{V}_{\kappa,K,\Sigma}$ and $p_1^!\mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})$ are CM sheaves over $\mathcal{O}_{E',\lambda'}$ and that

$$T_{\mathfrak{p}_i} = p^{\sum_{\sigma\in I_i}\sup_{1\leq j\leq g}\left\{\frac{\sum_{\ell=1}^{j}k_{\ell,\sigma}-\sum_{\ell=j+1}^{g}k_{\ell,\sigma}+k}{2}-\frac{j(j+1)}{2}\right\}}T^{naive}_{\mathfrak{p}_i}$$

is a well defined map integrally.

Actually, once we prove that $p_1^!\mathcal{V}_{\kappa,K,\Sigma}$ is a CM sheaf, it will be enough to check that the map $T_{\mathfrak{p}_i}$ is defined in codimension 1. Since it is well defined in characteristic 0, and the boundary is flat over $\mathbb{Z}_p$, it will be enough to check that the map $T_{\mathfrak{p}_i}$ is defined on the interior $\mathfrak{Sh}_{K(\mathfrak{p}_i)}$ of $\mathfrak{Sh}^{tor}_{K(\mathfrak{p}_i),\Sigma''}$.

The main idea is to reduce everything to local model computations. This is slightly delicate since our Shimura datum is only of abelian type, but we can reduce to working with a PEL Shimura datum.

We can pull back the cohomological correspondence $T^{naive}_{\mathfrak{p}_i}$ over $\mathbb{Q}$ to a cohomological correspondence $\widetilde{T^{naive}_{\mathfrak{p}_i}} : \tilde{p}_2^\star\mathcal{V}_\kappa \to \tilde{p}_1^!\mathcal{V}_\kappa$. It is enough to prove everything for the latter correspondence.

Now we have a local model diagram of correspondences:

(6.7.B)
$$\begin{tikzcd}
 & \widetilde{\mathfrak{P}}_{K(\mathfrak{p}_i)} \arrow[dl, "f"'] \arrow[d, shift right, "q_2"'] \arrow[d, shift left, "q_1"] \arrow[dr, "g"] & \\
\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)} \arrow[d, shift right, "\tilde{p}_2"'] \arrow[d, shift left, "\tilde{p}_1"] & \widetilde{\mathfrak{P}}_K \arrow[dl, "h"'] \arrow[dr, "e"] & M^{loc}_{K(\mathfrak{p}_i)} \arrow[d, shift right, "t_2"'] \arrow[d, shift left, "t_1"] \\
\widetilde{\mathfrak{Sh}}_K & & M^{loc}_K
\end{tikzcd}$$

By definition, $M^{loc}_K = \prod_{\sigma\in I} M_{\{0\}}$ and $M^{loc}_{K(\mathfrak{p}_i)} = \prod_{\sigma\in I_i} M_{\{0,g\}}\prod_{\sigma\notin I_i} M_{\{0\}}$. The projection t_1 is the product of the projections $p_1 : M_{\{0,g\}} \to M_{\{0\}}$ at places $\sigma\in I_i$ and the identity otherwise. The projection t_2 is the product of the projections $p_2 : M_{\{0,g\}} \to M_{\{g\}} \simeq M_{\{0\}}$ at $\sigma\in I_i$ and the identity if $\sigma\notin I_i$. The map h is the torsor of symplectic trivialisations of $\mathcal{H}_{1,dR}(A/\widetilde{Sh}_K)$ for A the universal abelian scheme. The map f is the torsor of symplectic trivialisations of the chain $\mathcal{H}_{1,dR}(A/\widetilde{Sh}_K) \to \mathcal{H}_{1,dR}((A/H)/\widetilde{Sh}_K) \to \mathcal{H}_{1,dR}(A/\widetilde{Sh}_K)$ (i.e., isomorphisms with the chain $\prod_{\sigma\in I_i}(V_0 \to V_g \to V_0) \times \prod_{\sigma\notin I_i}(V_0 \stackrel{\mathrm{Id}}{\to} V_0 \stackrel{p\mathrm{Id}}{\to} V_0))$. The maps g and e are given by the Hodge filtration.

This diagram is commutative, the diagonal maps are smooth and the diagonal maps going to the left are surjective, but the squares are not cartesian! We have the following theorem:

Theorem 6.9. *Let $\bar{x} : \mathrm{Spec}(k) \to \tilde{\mathfrak{P}}_{K(\mathfrak{p_i})}$. Let $\bar{y} = f(\bar{x})$, $\bar{z} = \tilde{p}_1(\bar{y})$, $\bar{y}' = g(\bar{x})$, $\bar{z}' = t_1(\bar{y}')$. Then there are isomorphisms between the strict henselizations:*

$$\mathscr{O}_{\widetilde{\mathfrak{Sh}}_K,\bar{z}} \simeq \mathscr{O}_{M^{loc}_K,\bar{z}'}$$

and

$$\mathscr{O}_{\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)},\bar{y}} \simeq \mathscr{O}_{M^{loc}_{K(\mathfrak{p}_i)},\bar{y}'}.$$

Moreover, there is a commutative diagram between the maps on Zariski cotangent spaces at $\bar{y}, \bar{y}'$ and $\bar{z}, \bar{z}'$:

$$\begin{tikzcd}
\mathfrak{m}_{\mathscr{O}_{\widetilde{\mathfrak{Sh}}_K,\bar{z}}}/\mathfrak{m}^2_{\mathscr{O}_{\widetilde{\mathfrak{Sh}}_K,\bar{z}}} \arrow[r, "\sim"] \arrow[d, "\mathrm{d}\tilde{p}_1"] & \mathfrak{m}_{\mathscr{O}_{M^{loc}_K,\bar{z}'}}/\mathfrak{m}^2_{\mathscr{O}_{M^{loc}_K,\bar{z}'}} \arrow[d, "\mathrm{d}t_1"] \\
\mathfrak{m}_{\mathscr{O}_{\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)},\bar{y}}}/\mathfrak{m}^2_{\mathscr{O}_{\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)},\bar{y}}} \arrow[r, "\sim"] & \mathfrak{m}_{\mathscr{O}_{M^{loc}_{K(\mathfrak{p}_i)},\bar{y}'}}/\mathfrak{m}^2_{\mathscr{O}_{M^{loc}_{K(\mathfrak{p}_i)},\bar{y}'}}
\end{tikzcd}$$

Proof. The first point is the main result of local model theory. The second point is an immediate consequence of Grothendieck–Messing deformation theory (see [dJ93, Thm. 2.1] for a precise statement of this theory). □

Corollary 6.10. *The map $\widetilde{\mathfrak{Sh}}^{tor}_{K,\Sigma} \to \operatorname{Spec} \mathcal{O}_{E',\lambda'}$ is smooth and the map $\widetilde{\mathfrak{Sh}}^{tor}_{K(\mathfrak{p}_i),\Sigma} \to \operatorname{Spec} \mathcal{O}_{E',\lambda'}$ is a CM map.*

Proof. Over the interior of the moduli space, this follows from the previous theorem. The description of the integral toroidal compactification in [MP19] shows that the property holds everywhere. □

It follows that the cohomological correspondence can be extended to a rational map from a locally free sheaf to a CM sheaf $\widetilde{T^{naive}_{\mathfrak{p}_i}} : p_2^\star \mathcal{V}_{\kappa,K,\Sigma} \dashrightarrow p_1^! \mathcal{V}_{\kappa,K,\Sigma}$. In particular, this corollary implies that it is enough to work with the interior of the Shimura variety.

For any $\mu = (\mu_1, \cdots, \mu_g)$, with $\mu_1 \geq \cdots \geq \mu_g$, we have already defined two sheaves $p_1^\star L_{0,\mu}$ and $p_2^\star L_{g,\mu}$ over $M_{\{0,g\}}$, and a rational map $\alpha^\star : p_2^\star L_{g,\mu} \dashrightarrow p_1^\star L_{g,\mu}$.

Let us write $\kappa_\sigma = (k_{1,\sigma}, \cdots, k_{g,\sigma})$ for all σ. We define sheaves $L_{0,\kappa} = \boxtimes_\sigma L_{0,\kappa_\sigma}$ and $L_{g,\kappa} = \boxtimes_{\sigma \in I_i} L_{g,\kappa_\sigma} \boxtimes_{\sigma \notin I_i} L_{0,\kappa_\sigma}$ on M^{loc}_K and a map

$$\beta^\star = \boxtimes_{\sigma \in I_i} \alpha^\star \boxtimes_{\sigma \notin I_i} \mathrm{Id} : t_2^\star L_{g,\kappa} \to t_1^\star L_{0,\kappa}.$$

Lemma 6.11. *Over $\widetilde{\mathfrak{P}}_{K(\mathfrak{p}_i)}$, we have a commutative diagram:*

$$\begin{tikzcd}
g^\star t_2^\star L_{g,\kappa} \arrow[r, "p^{-\sum_{\sigma\in I_i} \frac{k+\sum_j k_{j,\sigma}}{2}} g^\star \beta^\star"] \arrow[d, "\sim"] & g^\star t_1^\star L_{0,\kappa} \arrow[d, "\sim"] \\
f^\star p_2^\star \mathcal{V}_{\kappa,K} \arrow[r] & f^\star p_1^\star \mathcal{V}_{\kappa,K}
\end{tikzcd}$$

Proof. The sheaf $(V_0/L_0)^\vee$ on the σ-component of the local model M^{loc}_K corresponds to the sheaf $\omega_{A,\sigma}$ by definition. Therefore, the sheaves $L_{0,\kappa}$ and $L_{g,\kappa}$ correspond to the representations of M_μ of highest weight $(k_{1,\sigma}, \cdots, k_{g,\sigma}; -\sum_i k_{i,\sigma})_{\sigma \in I}$. There are isomorphisms of sheaves over $\widetilde{\mathfrak{P}}_{K(\mathfrak{p}_i)}$: $g^\star t_2^\star L_{g,\kappa} \simeq f^\star p_2^\star \mathcal{V}_{\kappa,K}$ and $g^\star t_1^\star L_{0,\kappa} \simeq f^\star p_1^\star \mathcal{V}_{\kappa,K}$ but these are not G-equivariant isomorphisms. We can make them G-equivariant as follows. Over $M_{\{0\}} = M_{\{g\}} = G/P$ we have a G-equivariant sheaf $\mathcal{L}$ corresponding to the similtude character of G (viewed as a P-representation). This sheaf has a trivialization (given by the similitude character of G), but its G-equivariant structure is not trivial. There is a canonical map $p_2^\star \mathcal{L} \to p_1^\star \mathcal{L}$ over $M_{\{0,g\}}$ which is multiplication by p^{-1} in the trivializations.

We can twist $L_{0,\kappa} = \boxtimes_\sigma L_{0,\kappa_\sigma}$ to $L'_{0,\kappa} = \boxtimes_\sigma L_{0,\kappa_\sigma} \otimes \mathcal{L}^{\frac{k+\sum_\ell k_{\ell,\sigma}}{2}}$ and $L_{g,\kappa}$ to

$$L'_{g,\kappa} = \boxtimes_{\sigma\in I_i} L_{g,\kappa_\sigma} \otimes \mathcal{L}^{\frac{k+\sum_\ell k_{\ell,\sigma}}{2}} \boxtimes_{\sigma\notin I_i} L_{0,\kappa_\sigma} \otimes \mathcal{L}^{\frac{k+\sum_\ell k_{\ell,\sigma}}{2}}.$$

Therefore, we have a commutative diagram over $M^{loc}_{K(\mathfrak{p}_i)}$:

$$\begin{tikzcd} t_2^\star L_{g,\kappa} \arrow[r, "p^{-\sum_{\sigma\in I_i}\frac{k+\sum_j k_{j,\sigma}}{2}}\beta^\star"] \arrow[d, "\sim"] & t_1^\star L_{0,\kappa} \arrow[d, "\sim"] \\ t_2^\star L'_{g,\kappa} \arrow[r, "(\beta')^\star"] & t_1^\star L_{0,\kappa} \end{tikzcd}$$

for $(\beta')^\star$ the natural map coming from the G-equivariant structure. After twisting we have a commutative diagram:

$$\begin{tikzcd} g^\star t_2^\star L'_{g,\kappa} \arrow[r, "g^\star(\beta')^\star"] \arrow[d, "\sim"] & g^\star t_1^\star L'_{0,\kappa} \arrow[d, "\sim"] \\ f^\star p_2^\star \mathcal{V}_{\kappa,K} \arrow[r] & f^\star p_1^\star \mathcal{V}_{\kappa,K} \end{tikzcd}$$

□

We can now conclude the proof of Theorem 5.9. Let ξ be a generic point of the special fiber of $\widetilde{\mathfrak{Sh}}_{K(\mathfrak{p}_i)}$. It corresponds on the local model $M^{loc}_{K(\mathfrak{p}_i)}$ to a point in the stratum $\prod_{\sigma\in I_i} U_{\{0,g\},w(s_\sigma)\mu} \times \prod_{\sigma\notin I_i} U_{\{0\},\mu}$.

Using the definition of β in terms of α, Lemma 6.7, Theorem 6.9 and Lemma 6.11, we deduce that on the local ring at ξ, we have a map

$$\widetilde{T^{naive}_{\mathfrak{p}_i}} : (\tilde{p}_2^\star \mathcal{V}_\kappa)_\xi \to p^{\sum_{\sigma\in I_i} \frac{(g-s_\sigma)(g-s_\sigma+1)}{2} + \frac{-\sum_{\ell=1}^{g-s_\sigma} k_{\ell,\sigma} + \sum_{\ell=g-s_\sigma+1}^{g} k_{\ell,\sigma}+k}{2}} (\tilde{p}_1^! \mathcal{V}_\kappa)_\xi.$$

We conclude using the CM property in Corollary 6.10 that we have a well defined cohomological correspondence

$$\widetilde{T_{\mathfrak{p}_i}} := p^{\sum_{\sigma\in I_i} \sup_{1\le j\le g}\left\{\frac{\sum_{\ell=1}^{j} k_{\ell,\sigma} - \sum_{\ell=j+1}^{g} k_{\ell,\sigma}+k}{2} - \frac{j(j+1)}{2}\right\}} \widetilde{T^{naive}_{\mathfrak{p}_i}} : \tilde{p}_2^\star \mathcal{V}_\kappa \to \tilde{p}_1^! \mathcal{V}_\kappa$$

which, using the diagram (6.7.A), shows that we have a cohomological correspondence

$$T_{\mathfrak{p}_i} := p^{\sum_{\sigma\in I_i} \sup_{1\le j\le g}\left\{\frac{\sum_{\ell=1}^{j} k_{\ell,\sigma} - \sum_{\ell=j+1}^{g} k_{\ell,\sigma}+k}{2} - \frac{j(j+1)}{2}\right\}} T^{naive}_{\mathfrak{p}_i} : p_2^\star \mathcal{V}_\kappa \to p_1^! \mathcal{V}_\kappa\,.$$

7. Unitary Shimura varieties

This section is dedicated to unitary Shimura varieties.

7.1. **The Shimura datum.** Let F be a totally real field, and let L be a totally imaginary quadratic extension of F. We denote by $c \in \mathrm{Gal}(L/F)$ the complex conjugation. We let $I = \mathrm{Hom}(F, \overline{\mathbb{Q}})$ and for all $\sigma \in I$ we chose an extension $\tau : L \to \overline{\mathbb{Q}}$ of σ to L. Therefore, $\mathrm{Hom}(L, \overline{\mathbb{Q}}) = I \coprod I \circ c$.

Let V be a K vector space of dimension n, together with a hermitian form $\langle\,,\rangle$. We assume that this form is not definite at at least one real place of F. Let G be the reductive group over $\mathbb{Q}$ of similitudes of $(V, \langle\,,\rangle)$. Namely:

$$G = \{(g,c) \in \mathrm{Res}_{L/\mathbb{Q}}\mathrm{GL}(V) \times \mathbb{G}_m,\ \langle gv, gw\rangle = c\langle v, w\rangle\ \forall v, w \in V\}$$

We have natural isomorphisms $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^I$ and $L \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{C}^I$ given by $x \otimes y \mapsto (\tau(x)y)_{\tau \in I}$. We let $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R} = \oplus V_{\mathbb{R},\tau}$ where $V_{\mathbb{R},\tau} = V \otimes_{F,\tau} \mathbb{R}$. We chose an isomorphism $V_{\mathbb{R},\tau} \simeq \mathbb{C}^n$ where L acts on $\mathbb{C}^n$ via τ and the hermitian pairing induced on $\mathbb{C}^n$, denoted by $\langle\, ,\rangle_\tau$, is of signature (p_τ, q_τ) and is in the standard form $\sum_{i=1}^{p_\tau} z_i \bar{z}_i - \sum_{i=p_\tau+1}^{n} z_i \bar{z}_i$.

We deduce that $G_{\mathbb{R}} = \mathrm{G}(\prod_{\tau \in I} U(p_\tau, q_\tau))$. The natural action of $G_{\mathbb{R}}$ on $V_{\mathbb{R}} \simeq \oplus_{\tau \in I} \mathbb{C}^n$ gives an embedding $G_{\mathbb{R}} \subset \prod_{\tau \in I} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_n$.

Our Shimura datum is (G, X) where X is the $G(\mathbb{R})$-orbit of the homomorphism $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ given by $h_0(z) = \prod_\tau z_{p_\tau, q_\tau}$ where $z_{p_\tau, q_\tau} \in \mathrm{GL}_n(\mathbb{C})$ is the diagonal matrix $\mathrm{diag}(z 1_{p_\tau}, \bar{z} 1_{q_\tau})$.

The centralizer of h_0 is $K_\infty \times \mathbb{R}^{\times,+}$ where $K_\infty = \prod_\tau U(p_\tau)(\mathbb{R}) \times U(q_\tau)(\mathbb{R})$ is a maximal compact subgroup and $\mathbb{R}^{\times,+}$ is the connected component of the identity in the center of $G(\mathbb{R})$.

7.2. **The flag variety.** The embedding $G_{\mathbb{R}} \to \prod_{\tau \in I} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_n$ induces, after extending scalars to $\mathbb{C}$ and projecting $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_n \times_{\mathrm{Spec}\, \mathbb{R}} \mathrm{Spec}\, \mathbb{C} = \mathrm{GL}_n \times \mathrm{GL}_n$ onto the first factor, a morphism $G_{\mathbb{C}} \to (\prod_{\tau \in I} \mathrm{GL}_n)$ of algebraic groups over $\mathrm{Spec}\, \mathbb{C}$. We thus get an isomorphism $G_{\mathbb{C}} \to (\prod_{\tau \in I} \mathrm{GL}_n) \times \mathbb{G}_m$ whose second component is the similitude factor. The cocharacter μ_0 attached to h_0 is given by $\mu_0(z) = \prod_\tau \mathrm{diag}(z 1_{p_\tau}, 1_{q_\tau}) \times z$.

We deduce that a representative of P_μ is given by the group $(\prod_\tau P_{p_\tau, q_\tau}) \times \mathbb{G}_m \subset G_{\mathbb{C}}$ with P_{p_τ, q_τ} the standard parabolic subgroup of GL_n of lower triangular matrices with blocks of size p_τ and q_τ, with Levi $\mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$.

The Borel embedding is the map

$$X \to \mathrm{FL}_{G,X}$$

sending h to the Hodge filtration $\mathrm{Fil}_h = \oplus \mathrm{Fil}_{h,\tau}$ (i.e., the subspace stabilized by P_μ) on

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \oplus_\tau V_{\mathbb{R},\tau} \otimes_{\mathbb{R}} \mathbb{C}.$$

We have $V_{\mathbb{R},\tau} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} \simeq \oplus_\tau (\mathbb{C}^n \oplus \mathbb{C}^n)$ where the last map is given for each τ by

$$\begin{array}{ccc} \mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} & \to & \mathbb{C}^n \oplus \mathbb{C}^n \\ v \otimes x & \mapsto & (vx, \overline{v}x) \end{array}$$

We denote by $V_{\mathbb{C},\tau,+}$ and $V_{\mathbb{C},\tau,-}$ the two factors in this isomorphism. The pairing $\langle\ ,\rangle_\tau$ induces a perfect pairing between $V_{\mathbb{C},\tau,+}$ and $V_{\mathbb{C},\tau,-}$. We can therefore think of $\mathrm{FL}_{G,X}$ as a product of Grassmannians parametrizing for each τ a direct summand $\mathrm{Fil}_\tau = \mathrm{Fil}_{\tau,+} \oplus \mathrm{Fil}_{\tau,-} \subset V_{\mathbb{C},\tau,+} \oplus V_{\mathbb{C},\tau,-}$, where $\mathrm{Fil}_{\tau,+}$ has rank q_τ, $\mathrm{Fil}_{\tau,-}$ has rank p_τ, and they are orthogonal with each other for the pairing $\langle\ ,\rangle_\tau$ (therefore Fil_τ is determined by $\mathrm{Fil}_{\tau,+}$).

The filtration at the point h_0, is given by $< e_{p_\tau+1}, \cdots, e_n > \oplus < e_1, \cdots, e_{p_\tau} > \subset \mathbb{C}^n \oplus \mathbb{C}^n$ for each τ (in the canonical basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$).

We now choose the diagonal maximal torus $S \subset G_{\mathbb{R}}$. Then $S = (\prod_\tau \mathrm{U}(1)^n) \times \mathbb{G}_m / \mu_2$ and its character group is the subgroup of $(\mathbb{Z}^n)^I \times \mathbb{Z}$ of elements $((a_{1,\tau}, \cdots, a_{n,\tau})_\tau; k)$ with the condition that $\sum_{i,\tau} a_{i,\tau} = k \mod 2$. We have $\mathrm{U}(1) \times_{\mathrm{Spec}\, \mathbb{R}} \mathrm{Spec}\, \mathbb{C} = \{(z_1, z_2) \in \mathbb{G}_m \times \mathbb{G}_m,\ z_1 z_2 = 1\}$, and the projection on the first coordinate induces an isomorphism $\mathrm{U}(1) \times_{\mathbb{R}} \mathbb{C} \simeq \mathbb{G}_m$. We have

$$S_{\mathbb{C}} \simeq (\prod_\tau \mathbb{G}_m^n) \times \mathbb{G}_m / \mu_2 \hookrightarrow G_{\mathbb{C}} \simeq (\prod_{\tau \in I} \mathrm{GL}_n) \times \mathbb{G}_m$$

and this map is given explicitly by $((x_{1,\tau}, \cdots, x_{n,\tau})_\tau; t) \mapsto \prod_\tau \mathrm{diag}(t x_{1,\tau}, \cdots, t x_{n,\tau})_\tau \times t^2$.

We choose the upper triangular Borel in $G_{\mathbb{C}}$ (this choice is compatible with our conventions in Section 4.6) and the corresponding dominant cone in $X^\star(S_{\mathbb{C}})$ is given by the condition $a_{1,\tau} \geq \cdots \geq a_{n,\tau}$ for all τ. There is also an associated dominant cone for the

Levi $\prod_\tau(\mathrm{GL}_{p_\tau}\times\mathrm{GL}_{q_\tau})\times\mathbb{G}_m$ which is given by the conditions $a_{1,\tau}\geq\cdots\geq a_{p_\tau,\tau}$ and $a_{p_\tau+1,\tau}\geq\cdots\geq a_{n,\tau}$ for all τ.

The (trivial) vector bundle $V_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$ over $\mathrm{FL}_{G,X}$ is associated with a representation of $G_\mathbb{C}$. This representation is the direct sum of the standard n-dimensional representation and its complex conjugate. For any $\tau_0\in I$, the direct factors $V_{\mathbb{C},\tau_0,+}$ and $V_{\mathbb{C},\tau_0,-}$ correspond respectively to the representations of $G_\mathbb{C}$ with highest weight $((1,\cdots,0)_{\tau_0},(0,\cdots,0)_{\tau\neq\tau_0};1)$ and $((0,\cdots,-1)_{\tau_0},(0,\cdots,0)_{\tau\neq\tau_0};1)$.

The vector bundles $V_{\mathbb{C},\tau_0,+}/\mathrm{Fil}_{\tau_0,+}$ and $V_{\mathbb{C},\tau_0,-}/\mathrm{Fil}_{\tau_0,-}$ correspond respectively to the two irreducible representations of the Levi $\prod_\tau(\mathrm{GL}_{p_\tau}\times\mathrm{GL}_{q_\tau})\times\mathbb{G}_m$ with highest weights: $((1,\cdots,0)_{\tau_0},(0,\cdots,0)_{\tau\neq\tau_0};1)$ and $((0,\cdots,-1)_{\tau_0},(0,\cdots,0)_{\tau\neq\tau_0};1)$.

7.3. **Integral model.** The reflex field E is the fixed field in $\overline{\mathbb{Q}}$ of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of all elements acting trivially on the set $\{(p_\tau,q_\tau)\}_\tau$.

We now let p be a prime unramified in L. We choose an $\mathcal{O}_K$-lattice $V_\mathbb{Z}\subset V$ and we assume that $V_\mathbb{Z}\otimes\mathbb{Z}_{(p)}$ is self dual for the pairing $\langle\,,\rangle$. The choice of this lattice gives an integral model for G, and this model is reductive over $\mathbb{Z}_{(p)}$. Let $K=K^pK_p\subset G(\mathbb{A}_f)$ be a compact open subgroup with K_p hyperspecial. Let λ be a finite place of E over p.

The Shimura variety $\mathfrak{Sh}_K$ represents the functor on the category of noetherian $\mathcal{O}_{E,\lambda}$-algebras that sends a noetherian $\mathcal{O}_{E,\lambda}$-algebra to the set of equivalence classes of (A,ι,λ,η), where:

(1) $A\to\mathrm{Spec}\,R$ is an abelian scheme,
(2) $\lambda:A\to A^t$ is a prime to p quasi-polarization,
(3) $\iota:\mathcal{O}_L\to\mathrm{End}(A)\otimes\mathbb{Z}_{(p)}$ is a homomorphism of algebras with involution,
(4) $\mathrm{Lie}(A)$ satisfies the determinant condition (see Remark 7.1),
(5) η is a K^p-level structure.

Remark 7.1. The determinant condition is the following. We let E' be an extension of E which contains the Galois closure of L. We let λ' be a place of E' above λ. It is enough to spell out the condition when R is an $\mathcal{O}_{E',\lambda'}$-algebra. In that situation, we have

$$\mathrm{Lie}(A)=\bigoplus_\tau\mathrm{Lie}(A)_{\tau,+}\oplus\mathrm{Lie}(A)_{\tau,-}$$

where $\mathrm{Lie}(A)_{\tau,+}=\mathrm{Lie}(A)\otimes_{\mathcal{O}_L\otimes R,\tau}R$ and $\mathrm{Lie}(A)_{\tau,-}=\mathrm{Lie}(A)\otimes_{\mathcal{O}_L\otimes R,\tau\circ c}R$. The condition is that $\mathrm{Lie}(A)_{\tau,+}$ is a locally free R-module of rank p_τ and $\mathrm{Lie}(A)_{\tau,-}$ is locally free of rank q_τ.

7.4. **The vector bundle dictionary.** This section follows [Gol14, Section 5.5]. We now work over $\mathcal{O}_{E',\lambda}$ as in the preceeding remark. We have $\omega_A=\bigoplus_\tau\omega_{A,\tau,+}\oplus\omega_{A,\tau,-}$. It follows from the discussion towards the end of Section 7.2 that $\omega_{A,\tau_0,+}$ corresponds to the irreducible representation of $(\prod_\tau P_{p_\tau,q_\tau})\times\mathbb{G}_m$ of highest weight $\kappa=((a_{1,\tau},\cdots,a_{n,\tau})_\tau;k)$ with $(a_{1,\tau_0},\cdots,a_{n,\tau_0})=(0,\cdots,0,-1,0,\cdots,0)$ (with -1 in position p_τ), $k=-1$, $(a_{1,\tau},\cdots,a_{n,\tau})=(0,\cdots,0)$ if $\tau\neq\tau_0$. We deduce similarly that $\omega_{A,\tau_0,-}$ corresponds to the weight $\kappa=((a_{1,\tau},\cdots,a_{n,\tau})_\tau;k)$ with $(a_{1,\tau_0},\cdots,a_{n,\tau_0})=(0,\cdots,0,1,0,\cdots,0)$ (with 1 in position $p_\tau+1$), $k=-1$, $(a_{1,\tau},\cdots,a_{n,\tau})=(0,\cdots,0)$ if $\tau\neq\tau_0$.

We now choose integers $a_{1,\tau}\geq\cdots\geq a_{p_\tau,\tau}$ and $b_{1,\tau}\geq\cdots\geq b_{q_\tau,\tau}$ for all $\tau\in I$ and consider the representation of the Levi M_μ of the parabolic P_μ with highest weight $\kappa=((-a_{p_\tau,\tau},\cdots,-a_{1,\tau},b_{1,\tau},\cdots,b_{q_\tau,\tau});-\sum_{\ell,\tau}a_{\ell,\tau}-\sum_{\ell,\tau}b_{\ell,\tau})$. The sheaf $\mathcal{V}_{\kappa,K}$ associated with this weight has the following description. We denote by $\pi_{\tau,+}:\mathcal{T}_{\tau,+}\to\mathfrak{Sh}_K$ and $\pi_{\tau,-}:\mathcal{T}_{\tau,-}\to\mathfrak{Sh}_K$ the GL_{p_τ} and GL_{q_τ} torsors of trivialization of $\omega_{A,\tau,+}$ and $\omega_{A,\tau,-}$ We let $\omega_{A,\tau,+}^{(a_\tau)}=(\pi_{\tau,+})_\star(\mathscr{O}_{\mathcal{T}_{\tau,+}})[a_{1,\tau},\cdots,a_{p_\tau,\tau}]$ and $\omega_{A,\tau,-}^{(b_\tau)}=(\pi_{\tau,+})_\star(\mathscr{O}_{\mathcal{T}_{\tau,+}})[-b_{q_\tau,\tau},\cdots,-b_{1,\tau}]$ (where the bracket means the isotypic part for the representation of the upper triangular Borel given by the characters $(a_{1,\tau},\cdots,a_{p_\tau,\tau})$ and $(-b_{q_\tau,\tau},\cdots,-b_{1,\tau})$ respectively).

Then we find that $\mathcal{V}_{\kappa,K}$ is simply $\otimes_\tau \omega_{A,\tau,+}^{(a_\tau)} \otimes \omega_{A,\tau,-}^{(b_\tau)}$ that is usually considered in the theory of modular forms on unitary groups (we mean that our normalization of the central character is the usual one).

7.5. **Dual group.** Recall that we have $G_{\mathbb{C}} \simeq (\prod_\tau \mathrm{GL}_n) \times \mathbb{G}_m$ with diagonal torus $S_{\mathbb{C}} = (\prod_\tau \mathbb{G}_m) \times \mathbb{G}_m/\mu_2$ embedded via $((x_{1,\tau}, \cdots, x_{n,\tau})_\tau; t) \mapsto \prod_\tau \mathrm{diag}(tx_{1,\tau}, \cdots, tx_{n,\tau}) \times t^2$. Moreover, our choice of positive roots is given by the upper triangular Borel. The character group $X^\star(S_{\mathbb{C}})$ is the subgroup of $(\mathbb{Z}^n)^I \times \mathbb{Z}$ of elements $((a_{1,\tau}, \cdots, a_{n,\tau})_\tau; k)$ with the condition that $\sum_{i,\tau} a_{i,\tau} = k \mod 2$. We have $((a_{1,\tau}, \cdots, a_{n,\tau})_\tau; k).((x_{1,\tau}, \cdots, x_{n,\tau})_\tau; t) = \prod_{i,\tau} x_{i,\tau}^{a_{i,\tau}} t^k$.

The cocharacter group $X_\star(S_{\mathbb{C}})$ identifies with the subgroup of $(\frac{1}{2}\mathbb{Z}^n)^I \times \frac{1}{2}\mathbb{Z}$ of elements $((r_{1,\tau}, \cdots, r_{n,\tau}); r)$ with $r + r_{i,\tau} \in \mathbb{Z}$ for all (i,τ). To an element $((r_{1,\tau}, \cdots, r_{n,\tau}); r)$ we associate the cocharacter

$$t \mapsto ((t^{r_{1,\tau}}, \cdots, t^{r_{n,\tau}})_\tau; t^r) = \prod_\tau \mathrm{diag}((t^{r_{1,\tau}+r}, \cdots, t^{r_{n,\tau}+r})) \times t^{2r}.$$

The pairing between characters and cocharacters is given

$$\langle ((a_{1,\tau}, \cdots, a_{n,\tau})_\tau; k), ((r_{1,\tau}, \cdots, r_{n,\tau})_\tau; r) \rangle = \sum_{i,\tau} a_{i,\tau} r_{i,\tau} + rk.$$

We have the usual identification $\hat{G} \simeq G_{\mathbb{C}} = \prod_\tau (\mathrm{GL}_n) \times \mathbb{G}_m$ with standard torus $\hat{S} = S_{\mathbb{C}}$. We use the standard pinning. The complex conjugation acts on $\hat{S}$ by

$$((x_{1,\tau}, \cdots, x_{n,\tau})_\tau; t) \mapsto ((x_{1,\tau}^{-1}, \cdots, x_{n,\tau}^{-1})_\tau; t)$$

and on the full $\hat{G}$ by $(g, c) \mapsto (\Phi_N\, {}^t g^{-1} \Phi_N^{-1} c, c)$ where Φ_N is the anti-diagonal matrix with alternating 1 and -1's on the anti-diagonal.

7.6. **Formulas for the minuscule coweights.** We let $\iota : E' \hookrightarrow \overline{\mathbb{Q}}_p$ be an embedding continuous for the λ'-adic topology on the source and p-adic topology on the target. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ be the primes above p in F. We assume that they all split in L. We have a partition $I = \coprod I_i$ where I_i is the set of embeddings $\tau : F \to E'$ for which $\iota \circ \tau$ induces the $\mathfrak{p}_i$-adic topology. The local L-group at p is simply a product $\prod_i (\mathrm{GL}_n)^{I_i} \times \mathbb{G}_m \rtimes \Gamma$, where $\Gamma = \hat{\mathbb{Z}}$ is the absolute Galois group of $\mathbb{F}_p$. It acts by permutation on each of the sets I_i.

We fix $0 \leq i \leq m$ and $1 \leq j \leq n$. We denote by $V_{j,i}$ the representation of $\hat{G}$ whose underlying vector space is $\otimes_{\tau \in I_i} (\Lambda^j \mathbb{C}^n)^\vee$, with action of $\hat{G} = (\prod_\tau \mathrm{GL}_n) \times \mathbb{G}_m$ given by the dual of the j-th exterior standard action of the τ-factor for all $\tau \in I_i$, and the inverse scalar multiplication by $\mathbb{G}_m$. The corresponding highest weight $\lambda_{i,j}$ is: $((a_{1,\tau}, \cdots, a_{n,\tau}); -2 - \sharp I_i)$ with $(a_{1,\tau}, \cdots, a_{n,\tau}) = (0, \cdots, 0, -1, \cdots, -1)$ (with j many -1s) if $\tau \in I_i$, $(a_{1,\tau}, \cdots, a_{n,\tau}) = (0, \cdots, 0)$ if $\tau \notin I_i$. This representation extends to a representation of the local L-group.

The corresponding cocharacter of $G_{\mathbb{C}}$ is

$$t \mapsto \Big(\prod_{\tau \notin I_i} 1_n \Big) \times \Big(\prod_{\tau \in I_i} \mathrm{diag}(1_{n-j}, t^{-1} 1_j) \Big) \times t^{-1}$$

which is given in coordinates by $((r_{1,\tau}, \cdots, r_{n,\tau}); -\frac{1}{2})$ with $(r_{1,\tau}, \cdots, r_{n,\tau}) = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2})$ (with $n-j$ many $\frac{1}{2}$s) if $\tau \in I_i$, $(a_{1,\tau}, \cdots, a_{n,\tau}) = (\frac{1}{2}, \cdots, \frac{1}{2})$ if $\tau \notin I_i$.

We let $T_{\mathfrak{p}_i,j}^{naive}$ be the Hecke operator associated to $T_{\lambda_{i,j}}$. We have an embedding $G(\mathbb{Q}_p) \hookrightarrow \prod_i \mathrm{GL}_n(F_{\mathfrak{p}_i}) \times \mathrm{GL}_n(F_{\mathfrak{p}_i})$, and the operator $T_{\mathfrak{p}_i,j}^{naive}$ is associated to the double

coset
$$\mathrm{G}(\mathbb{Z}_p)\Big(\prod_{l\neq i} 1_n \times \mathfrak{p}_l^{-1}1_n\Big) \times (\mathrm{diag}(1_{n-j}, \mathfrak{p}_i^{-1}1_j) \times \mathrm{diag}(\mathfrak{p}_i^{-1}1_{n-j}, 1_j))\mathrm{G}(\mathbb{Z}_p).$$

Remark 7.2. Our definition of the Hecke operators $T^{naive}_{\mathfrak{p}_i,j}$ depends on the choice of a prime above each $\mathfrak{p}_k$ in L (. This means that some symmetry has been broken. For an explanation of the use of the double class
$$\mathrm{G}(\mathbb{Z}_p)\Big(\prod_{l\neq i} 1_n \times \mathfrak{p}_l^{-1}1_n\Big) \times (\mathrm{diag}(1_{n-j}, \mathfrak{p}_i^{-1}1_j) \times \mathrm{diag}(\mathfrak{p}_i^{-1}1_{n-j}, 1_j))\mathrm{G}(\mathbb{Z}_p)$$
rather than its inverse, we refer to Remark 5.6.

As in Section 7.4, we choose integers $a_{1,\tau} \geq \cdots \geq a_{p_\tau,\tau}$ and $b_{1,\tau} \geq \cdots \geq b_{q_\tau,\tau}$ for all $\tau \in I$ and we consider the representation of the Levi M_μ of the parabolic P_μ with highest weight $\kappa = ((-a_{p_\tau,\tau}, \cdots, -a_{1,\tau}, b_{1,\tau}, \cdots, b_{q_\tau}); -\sum_{\ell,\tau} a_{\ell,\tau} - \sum_{\ell,\tau} b_{\ell,\tau})$.

Remark 7.3. In the rest of this section, we work with the weight κ, and therefore we fix a particular normalization of the central character, as justified in Section 7.4. We leave it to the reader to formulate Lemma 7.4 for another choice of central character. Theorem 7.5 holds for any normalization of the central character.

Motivated by Definition 4.8, we let $T_{\mathfrak{p}_i,j} = p^{\langle\lambda_{i,j},\infty(\kappa,\iota)\rangle - \langle\lambda_{i,j},\rho\rangle} T^{naive}_{\mathfrak{p}_i,j}$. We now find a formula for the coefficient $\langle\lambda_{i,j},\infty(\kappa,\iota)\rangle - \langle\lambda_{i,j},\rho\rangle$.

Lemma 7.4. *We have* $\langle\lambda_{i,j},\infty(\kappa,\iota)\rangle - \langle\lambda_{i,j},\rho\rangle =$
$$\sum\nolimits_{\tau\notin I_i}\Big(-\sum_{\ell=1}^{q_\tau} b_{\ell,\tau}\Big) + \sum_{\tau\in I_i}\Big(\sup_{0\leq r_\tau\leq p_\tau, 0\leq s_\tau\leq q_\tau,\ s_\tau+r_\tau=n-j}\Big\{-\sum_{\ell=r_\tau+1}^{p_\tau} a_{\ell,\tau} - \sum_{\ell=q_\tau-s_\tau+1}^{q_\tau} b_{\ell,\tau} - r_\tau(q_\tau-s_\tau)\Big\}\Big)$$

Proof. We observe that $\rho = ((\frac{n-1}{2}, \cdots, \frac{1-n}{2}); 0)$ so $\kappa+\rho$ is equal to
$$((-a_{p_\tau,\tau}+\tfrac{n-1}{2}, \cdots, -a_{1,\tau}+\tfrac{n-1}{2}-p_\tau+1, b_{1,\tau}-\tfrac{n-1}{2}+q_\tau-1, \cdots, b_{q_\tau}-\tfrac{n-1}{2}); -\sum_{\ell,\tau} a_{\ell,\tau} - \sum_{\ell,\tau} b_{\ell,\tau}).$$

Now we need to compute the pairing $\langle\lambda_{i,j},\infty(\kappa,\iota)\rangle - \langle\lambda_{i,j},\rho\rangle$. The product $\langle\lambda_{i,j},\infty(\kappa,\iota)\rangle$ decomposes as
$$\sum_\tau \langle(\lambda_{i,j})_\tau, \infty(\kappa,\iota)_\tau\rangle_\tau - \Big(\frac{1}{2}\sum_{\ell,\tau} a_{\ell,\tau} + \sum_{\ell,\tau} b_{\ell,\tau}\Big).$$

For all $\tau \notin I_i$, we have
$$\langle(\lambda_{i,j})_\tau, \infty(\kappa,\iota)_\tau\rangle_\tau = \frac{1}{2}\Big(\sum_\ell a_{\ell,\tau} - \sum_\ell b_{\ell,\tau}\Big).$$

For all $\tau \in I_i$, there will be integers $0 \leq r_\tau \leq p_\tau$ and $0 \leq s_\tau \leq q_\tau$ such that $s_\tau + r_\tau = n - j$ and such that the first $n-j$ coordinates of $\infty(\kappa,\iota)_\tau$ put in the dominant form will be (possibly not in this order)
$$\{-a_{1,\tau} + \tfrac{n-1}{2} - p_\tau + 1, \cdots, -a_{r,\tau} + \tfrac{n-1}{2} - p_\tau + r_\tau, b_{q_\tau} - \tfrac{n-1}{2}, b_{q_\tau-s_\tau+1} - \tfrac{n-1}{2} + s_\tau - 1\}.$$

In this case, an easy computation shows that
$$\langle(\lambda_{i,j})_\tau, \infty(\kappa,\iota)_\tau\rangle_\tau = \frac{1}{2}\Big(\sum_{\ell=1}^{r_\tau} a_{\ell,\tau} - \sum_{\ell=r_\tau+1}^{p_\tau} a_{\ell,\tau} + \sum_{\ell=1}^{q_\tau-s_\tau} b_{\ell,\tau} - \sum_{\ell=q_\tau-s_\tau+1}^{q_\tau} b_{\ell,\tau}\Big) +$$
$$-\tfrac{n-1}{2}r_\tau + \tfrac{r_\tau(r_\tau-1)}{2} + (p_\tau - r_\tau)r_\tau + \tfrac{n-1}{2}s_\tau - \tfrac{s_\tau(s_\tau-1)}{2}.$$

Moreover, we have the pairing $\langle\lambda_{i,j},\rho\rangle_\tau = \frac{j(n-j)}{2}$ if $\tau \in I_i$ and $\langle\lambda_{i,j},\rho\rangle_\tau = 0$ otherwise. So we finally get that for all $\tau \in I_i$

$$\begin{aligned}&\langle(\lambda_{i,j})_\tau, \infty(\kappa,\iota)_\tau\rangle_\tau - \langle\lambda_{i,j},\rho\rangle_\tau \\ &\qquad= \frac{1}{2}\Big(\sum_{\ell=1}^{r} a_{\ell,\tau} - \sum_{\ell=r_\tau+1}^{p_\tau} a_{\ell,\tau} + \sum_{\ell=1}^{q_\tau-s_\tau} b_{\ell,\tau} - \sum_{\ell=q_\tau-s_\tau+1}^{q_\tau} b_{\ell,\tau}\Big) - r_\tau(q_\tau - s_\tau)\end{aligned}$$

while for all $\tau \notin I_i$, we have

$$\langle(\lambda_{i,j})_\tau, \infty(\kappa,\iota)_\tau\rangle_\tau - \langle\lambda_{i,j},\rho\rangle_\tau = \frac{1}{2}\Big(\sum_{\ell} a_{\ell,\tau} - \sum_{\ell} b_{\ell,\tau}\Big).$$

It remains to add the factor $\sum_\tau -(\frac{1}{2}\sum_{\ell,\tau} a_{\ell,\tau} + \sum_{\ell,\tau} b_{\ell,\tau})$ to conclude. □

7.6.1. *The main theorem for unitary groups.* We have that $\mathcal{H}^{int}_{p,\kappa,\iota} = \otimes_{0\le i\le m}\mathbb{Z}[T_{\mathfrak{p}_i,j},\ 0 \le j \le n, T^{-1}_{\mathfrak{p}_i,0}, T^{-1}_{\mathfrak{p}_i,n}]$ and the following result gives some evidence towards Conjecture 4.16.

Theorem 7.5. *The operators $T_{\mathfrak{p}_i,j},\ 0 \le j \le n,\ 1 \le i \le m$, act on $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma})$ and $\mathrm{R}\Gamma(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}))$.*

Remark 7.6. We do not know how to prove the commutativity of the operators $T_{\mathfrak{p}_i,j}$ in general. The main point is that although the operators $T_{\mathfrak{p}_i,j}$ are associated to minuscule coweights, the composition of two such operators will not in general be associated to a minuscule coweight.

Corollary 7.7. *Conjecture 4.15 holds in this case.*

Proof. Conjecture 4.15 asserts that

$$\mathrm{Im}\big(\mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}) \to \mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,K,\Sigma}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\big)$$

is a lattice in $\mathrm{H}^i(\mathfrak{Sh}^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ stable under the action of $\mathcal{H}^{int}_{p,\kappa,\iota}$. This follows from Theorem 7.5. □

7.6.2. *Example: the group* $\mathrm{GU}(2,1)$. Let us assume that $F = \mathbb{Q}$, and that the signature $(p_\tau, q_\tau) = (2,1)$ for a chosen embedding $\tau : L \to \overline{\mathbb{Q}}$. Let p be a prime that splits in L. We choose an embedding $\iota : L \hookrightarrow \overline{\mathbb{Q}}_p$ which corresponds to a prime $\mathfrak{p}$ above p. We have $\mathfrak{p}\mathfrak{p}^c = p$. We have an isomorphism $G_{\mathbb{Q}_p} = \mathrm{GL}_3 \times \mathbb{G}_m$

The spherical Hecke algebra at p is the polynomial algebra generated by the characteristic functions (identified with double cosets):

(1) $T^{naive}_0 = K_p(\mathrm{diag}(1,1,1) \times p^{-1})K_p$, $(T^{naive}_0)^{-1}$,
(2) $T^{naive}_1 = K_p(\mathrm{diag}(1,1,p^{-1}) \times p^{-1})K_p$,
(3) $T^{naive}_2 = K_p(\mathrm{diag}(1,p^{-1},p^{-1}) \times p^{-1})K_p$,
(4) $T^{naive}_3 = K_p(\mathrm{diag}(p^{-1},p^{-1},p^{-1}) \times p^{-1})K_p$, $(T^{naive}_3)^{-1}$,

There are associated representations of the dual group $V_j = (\Lambda^j\mathbb{C}^3)^\vee$ (and action of $\mathbb{G}_m$ by the inverse character), for $0 \le j \le 3$. The relation given by the Satake transform is (observe that all these representations are minuscule):

(1) $T^{naive}_0 = [V_{0,\tau}]$,
(2) $T^{naive}_1 = p[V_{1,\tau}]$,
(3) $T^{naive}_2 = p[V_{2,\tau}]$,
(4) $T^{naive}_3 = [V_{3,\tau}]$.

We now pick integers (k_1, k_2, k_3) and consider the automorphic vector bundle:

$$\mathrm{Sym}^{k_1-k_2}\omega_{A,\tau,+} \otimes \overset{k_2}{\det}\,\omega_{A,\tau,+} \otimes \overset{k_3}{\det}\,\omega_{A,\tau,-}.$$

It is the sheaf $\mathcal{V}_{\kappa,K,\Sigma}$ for $\kappa = (-k_2, -k_1, k_3; -k_2 - k_1 - k_3)$. It follows from the construction that the sheaf $\det \mathcal{H}_{1,dR}(A)_{\tau,+}$ is tivial (viewed as a sheaf, the equivariant action is not trivial). It is therefore harmless to assume that k_3 is constant and we assume that $k_3 = 1$.

We conclude that:

$$\mathcal{H}^{int}_{p,\kappa,\iota} = \mathbb{Z}\big[(p^{-1}T_0^{naive})^{\pm 1}, p^{-1-\inf\{1,k_2\}}T_1^{naive}, p^{-1-\inf\{k_2,k_1+k_2\}}T_2^{naive}, (p^{-k_1-k_2}T_3^{naive})^{\pm 1}\big]$$

8. Local models in the linear case

In this section we will prove Theorem 7.5. As in the symplectic case, the actions of the normalized Hecke operators will be defined using the results of Section 2, in particular the construction in Example 2.11. In order to do this, we need to understand the integrality properties of the Hecke correspondences with respect to automorphic vector bundles and also the pullback maps on differentials. This will be done by using the theory of local models of Shimura varieties.

After recalling the basic facts about the local models for general linear groups, we prove two results, Proposition 8.3 (on the integrality properties of differentials) and Lemma 8.4 (on the integrality properties on automorphic bundles) which are used to construct normalized Hecke operators on the local model in Proposition 8.5. These computations imply Theorem 7.5 in the same way as in the proof of Theorem 5.9.

8.1. **Definition.** Let $n, p, q \in \mathbb{Z}_{\geq 1}$ be integers such that $n = p + q$. We now consider modules $V_0, \cdots, V_{n-1} = \mathbb{Z}^n$ and the following chain:

$$V_\bullet : V_0 \to V_1 \to \cdots \to V_{n-1} \to V_0$$

where the map from V_i to V_{i+1} is given in the canonical basis $(e_1, \cdots, e_n)$ of $\mathbb{Z}^n$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto pe_{i+1}$. Whenever necessary, indices are taken modulo n so that $V_n := V_0$ and the chain $V_\bullet$ can be seen as an infinite chain.

Fix a set $\emptyset \neq I \subset \{0, 1, \cdots, n-1\}$. We define the local model functor $\mathbf{M}_I : \mathbb{Z}-\mathbf{ALG} \to \mathbf{SETS}$ which associates to an object R of $\mathbb{Z}-\mathbf{ALG}$ the set of isomorphism classes of commutative diagrams

$$\begin{array}{ccccccccc}
\longrightarrow & V_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & V_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow \cdots \longrightarrow & V_{i_m} \otimes_{\mathbb{Z}} R & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & F_{i_0} & \longrightarrow & F_{i_1} & \longrightarrow \cdots \longrightarrow & F_{i_m} & \longrightarrow
\end{array}$$

where $i_0 < i_1 \cdots < i_m$ are such that $\{i_0, \cdots, i_m\} = I$ and the modules F_{i_j} for $0 \leq j \leq m$ are rank q locally direct factors.

The functor $\mathbf{M}_I$ is represented by a projective scheme M_I which is a closed subscheme of a product of Grassmannians.

When $\emptyset \neq J \subset I$, there is an obvious map $M_I \to M_J$ given by forgetting the modules F_j for $j \in I \setminus J$.

8.2. **The affine Grassmannian.** We now consider modules $\mathcal{V}_0, \cdots, \mathcal{V}_{n-1} = \mathbb{F}_p[[t]]^n$ and the chain:

$$\mathcal{V}_\bullet = \mathcal{V}_0 \to \mathcal{V}_1 \to \cdots \to \mathcal{V}_{n-1} \to \mathcal{V}_0$$

where the map from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$ is given in the canonical basis $(e_1, \cdots, e_n)$ of $\mathbb{F}_p[[t]]^n$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto te_{i+1}$. Whenever necessary, indices are taken modulo n so that $\mathcal{V}_n := \mathcal{V}_0$ and the chain $\mathcal{V}_\bullet$ is made infinite. Observe that $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = V_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Let LG denote the loop group of GL_n over $\mathbb{F}_p$. The group LG acts naturally on $\mathcal{V}_0 \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$ and therefore it acts on the chain $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$.

Let $\emptyset \neq I \subset \{0, 1, \cdots, n-1\}$ and let $\mathcal{V}_\bullet^I$ be the subchain of $\mathcal{V}_\bullet$ where we keep only the modules indexed by elements $i \in I$ (modulo n).

We denote by $\mathcal{P}_I$ the parahoric subgroup of LG of automorphisms of the chain $\mathcal{V}_\bullet^I$. For any I, we define the flag variety as the ind-scheme $\mathcal{F}_I := LG/\mathcal{P}_I$.

8.3. **Stratification of the local model.** As in the case of symplectic groups, the special fibre $\overline{M}_I$ of M_I at p embeds as a finite union of $\mathcal{P}_I$ orbits in $\mathcal{F}_I$.

We recall the description of the map $\overline{M}_I \to \mathcal{F}_I$. Given a diagram

$$\begin{array}{ccccccc} V_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & V_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow \cdots \longrightarrow & V_{i_m} \otimes_{\mathbb{Z}} R \\ \uparrow & & \uparrow & & \uparrow \\ F_{i_0} & \longrightarrow & F_{i_1} & \longrightarrow \cdots \longrightarrow & F_{i_m} \end{array}$$

corresponding to an R-point of $\overline{M}_I$, we can construct a new diagram:

$$\begin{array}{ccccccc} \mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & \mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow \cdots \longrightarrow & \mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}_{i_0} & \longrightarrow & \mathcal{F}_{i_1} & \longrightarrow \cdots \longrightarrow & \mathcal{F}_{i_m} \\ \uparrow & & \uparrow & & \uparrow \\ t\mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & t\mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow \cdots \longrightarrow & t\mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R \end{array}$$

where all the vertical maps are inclusions and each $\mathcal{F}_{i_j}$ is determined by the property that $\mathcal{F}_{i_j}/t\mathcal{V}_{i_j} \otimes_{\mathbb{Z}} R = F_{i_j} \hookrightarrow (\mathcal{V}_{i_j}/t\mathcal{V}_{i_j}) \otimes_{\mathbb{Z}} R = V_{i_j} \otimes R$. The chain $\mathcal{F}_\bullet$ determines an R-point of $\mathcal{F}_I$.

We now recall the combinatorial description of the image of $\overline{M}_I$ in $\mathcal{F}_I$: Fix a Borel subgroup B of GL_n and a maximal torus $T \subset B$. This gives a base for the root datum of GL_n. Let $\widetilde{W}$ be the extended affine Weyl group of GL_n. This is the semi-direct product of the Weyl group $W = \mathcal{S}_n$ and the cocharacter group $\mathrm{X}_\star(T) = \mathbb{Z}^n$. It can be realized as a subgroup of the group of affine transformations of $\mathbb{Z}^n$ with $\mathbb{Z}^n$ acting by translation and $\mathcal{S}_n$ by permutation of the coordinates. Let $s_i = (i, i+1)$ for $1 \leq i \leq g-1$, and $s_0 = \mathrm{t}_{(-1,0,\cdots,0,1)} \rtimes (1, n)$ be the usual choice of simple reflections. We let $\mu = (1, \cdots, 1, 0, \cdots 0) \in \mathbb{Z}^n$ (where 1 occurs p times) be the minuscule coweight corresponding to our situation.

For each $\emptyset \neq I$ as above, we let W_I be the subgroup of $\widetilde{W}$ generated by the simple reflexions $s_i, i \notin I$. This is a finite group. The $\mathcal{P}_I$ orbits in $\mathcal{F}_I$ are parametrised by the double cosets $W_I \backslash \widetilde{W} / W_I$. For any $w \in \widetilde{W}$ we denote the orbit corresponding to the double coset $W_I w W_I$ by $U_{I,w}$ and the orbit closure by $X_{I,w}$. The orbits included in $\overline{M}_I$ are parametrized by the finite subset $Adm_I(\mu)$ of $W_I \backslash \widetilde{W} / W_I$ of μ-admissible elements.

Whenever $J \subset I$ we have a map $M_I \to M_J$. This map is surjective and $Adm_J(\mu)$ is the image of $Adm_I(\mu)$ in $W_J \backslash \widetilde{W} / W_J$.

When $I = \{0, \cdots, n-1\}$, Kottwitz and Rapoport give a description of the set $Adm_I(\mu)$. The set $Adm_I(\mu)$ is precisely the subset of $\widetilde{W}$ of elements which are $\leq$ (in the Bruhar order) a translation $\mathrm{t}_{w\mu}$ for an element $w \in W$. In particular, the orbits that are open in $\overline{M}_I$ correspond to the translations $\mathrm{t}_{w\mu}$.

8.4. **Irreducible components.**

8.4.1. *The case that* $I = \{0, \cdots, n-1\}$. There are actually $\frac{n!}{p!q!}$ translations $w \cdot \mu$ and they correspond to the open stratum in each of the $\frac{n!}{p!q!}$ irreducible components of $\overline{M}_I$ when $I = \{0, \cdots, n-1\}$.

These $\frac{n!}{p!q!}$ translations are parametrized by W/W_c where W_c is the subgroup of W of elements which stabilize μ. It identifies with the elements in $W \subset \mathcal{S}_n$ which preserve the sets $\{1, \cdots, p\}$ and $\{p+1, \cdots, n\}$. This group is isomorphic to $\mathcal{S}_p \times \mathcal{S}_q$.

We can concretely determine an element of each of the orbits corresponding to these $\mathrm{t}_{w\mu}$ in M_I for $I = \{0, \cdots, n-1\}$ as follows. The group $\widetilde{W}$ can be viewed as a subgroup of $\mathrm{GL}_n(\mathbb{F}_p((t)))$. The element t_μ is identified with $\mathrm{diag}(t, \cdots, t, 1, \cdots 1)$.

We now consider the inclusion of chains:

$$t\mathcal{V}_\bullet \subset \mathrm{t}_{w\mu}\mathcal{V}_\bullet \subset \mathcal{V}_\bullet.$$

By reduction modulo t and using the identification $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = V_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$, we deduce that $\mathrm{t}_{w\mu}\mathcal{V}_\bullet/t\mathcal{V}_\bullet \hookrightarrow V_\bullet$ defines an $\mathbb{F}_p$-point of M_I, which represents the $w\mu$-orbit.

8.4.2. *The case that* $\sharp I = 1$. For any $j, j' \in I$, the spaces $M_{\{j\}}$ and $M_{\{j'\}}$ are canonically isomorphic. The special fiber $\overline{M}_{\{0\}}$ of $M_{\{0\}}$ is smooth and irreducible of dimension pq. Moreover, there is a single orbit.

8.4.3. *The case that* $I = \{0, j\}$.

Lemma 8.1. *The special fiber* $\overline{M}_{\{0,j\}}$ *of* $M_{\{0,j\}}$ *has* $\inf\{p, j\} - \sup\{0, j-q\} + 1$ *irreducible components, indexed by integers* $\sup\{0, j-q\} \leq r \leq \inf\{p, j\}$. *For each* $\sup\{0, j-q\} \leq r \leq \{p, j\}$, *a representative of the* r*-stratum is given by taking*

$$F_0(r) = F_j(r) = \langle e_{r+1}, \cdots, e_p, e_{n-j+r+1}, \cdots, e_n \rangle$$

This corresponds to the element $\mathrm{t}_{w(r)\mu} = \mathrm{diag}(t\mathrm{Id}_r, \mathrm{Id}_{j-r}, t\mathrm{Id}_{p-r}, \mathrm{Id}_{q-j+r}) \in LG$.

8.5. **Local geometry of the local model.** The local model $M_{\{0\}}$ is smooth of relative dimension pq. The other local models M_I are isomorphic to $M_{\{0\}}$ over Spec $\mathbb{Z}[1/p]$ but they have singular special fiber at p. Nevertheless, we have the following analogue of Theorem 6.4, the first part due to Görtz [Gö01] and the second to He [He13, Theorem 1.2]

Theorem 8.2. *The local models* M_I *are flat over* $\mathbb{Z}$ *and* $\overline{M}_I$ *is reduced. Furthermore,* M_I *is Cohen–Macaulay.*

8.6. **Hecke correspondence of the affine Grassmannians.** We consider the correspondence:

$$\begin{tikzcd} & \mathcal{F}_{\{0,j\}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{F}_{\{j\}} & & \mathcal{F}_{\{0\}} \end{tikzcd}$$

We now pick the element $w(r)\mu \in \widetilde{W}$ and restrict this diagram to get maps:

$$\begin{tikzcd} & U_{\{0,j\},w(r)\mu} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ U_{\{j\},w(r)\mu} & & U_{\{0\},w(r)\mu} \end{tikzcd}$$

Proposition 8.3. *The map on differentials* $\mathrm{d}p_1 : p_1^\star \Omega^1_{U_{\{0\},w(r)\mu}/\mathbb{F}_p} \to \Omega^1_{U_{\{0,j\},w(r)\mu}/\mathbb{F}_p}$ *has kernel and cokernel a locally free sheaf of rank* $(j-r)(p-r)$.

Proof. We have

$$U_{\{0,j\},w(r)\mu} \simeq \mathcal{P}_{\{0,j\}}/\big(\mathcal{P}_{\{0,j\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0,j\}}\mathrm{t}_{w(r)\mu}^{-1}\big)$$

$$U_{\{0\},w(r)\mu} \simeq \mathcal{P}_{\{0\}}/\big(\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(r)\mu}^{-1}\big),$$

and p_1 is the obvious $\mathcal{P}_{(0,j)}$- equivariant projection:

$$\mathcal{P}_{\{0,j\}}/\big(\mathcal{P}_{\{0,j\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0,j\}}\mathrm{t}_{w(r)\mu}^{-1}\big) \to \mathcal{P}_{\{0\}}/\big(\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(r)\mu}^{-1}\big).$$

Because the map is $\mathcal{P}_{(0,j)}$-equivariant, it suffices to prove the claim in the tangent space at the identity.

We first determine the shape of $\mathcal{P}_{\{0\}}$ and $\mathcal{P}_{\{0,j\}}$. The group $\mathcal{P}_{\{0\}}$ is the hyperspecial subgroup of LG, whose R-points are $G(R[[t]])$. The group $\mathcal{P}_{\{0,j\}}$ is the parahoric group with shape:

$$\mathcal{P}_{\{0,g\}}(R) = \{M = \begin{pmatrix} a & b \\ tc & d \end{pmatrix} \in LG(R)\}$$

where $a \in M_{j\times j}(R[[t]])$, $d \in M_{n-j\times n-j}(R[[t]])$, $c \in M_{n-j\times j}(R[[t]])$ and $d \in M_{j\times n-j}(R[[t]])$.

One computes that $\mathcal{P}_{\{0\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0\}}\mathrm{t}_{w(s)\mu}^{-1}$ consists of matrices of the following shape (the $\star$ have integral values, the rows and columns are of size r, $j-r$, $p-r$, and $q-j+r$):

$$\begin{pmatrix} \star & t\star & \star & t\star \\ \star & \star & \star & \star \\ \star & t\star & \star & t\star \\ \star & \star & \star & \star \end{pmatrix}$$

One also computes that $\mathcal{P}_{\{0,j\}} \cap \mathrm{t}_{w(r)\mu}\mathcal{P}_{\{0,j\}}\mathrm{t}_{w(r)\mu}^{-1}$ consists of matrices with the following shape:

$$\begin{pmatrix} \star & t\star & \star & t\star \\ \star & \star & \star & \star \\ t\star & t^2\star & \star & t\star \\ t\star & t\star & \star & \star \end{pmatrix}$$

Passing to the Lie algebras, we easily see that the kernel of the map

$$\mathfrak{p}_{\{0,j\}}/\big(\mathfrak{p}_{\{0,j\}} \cap \mathrm{t}_{w(r)\mu}\mathfrak{p}_{\{0,j\}}\mathrm{t}_{w(r)\mu}^{-1}\big) \to \mathfrak{p}_{\{0\}}/\big(\mathfrak{p}_{\{0\}} \cap \mathrm{t}_{w(r)\mu}\mathfrak{p}_{\{0\}}\mathrm{t}_{w(r)\mu}^{-1}\big)$$

is the set of matrices of the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & tA & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A \in M_{p-r\times j-r}(\mathbb{F}_p)$.

□

8.7. **The Hecke correspondence on the local model.** We consider the correspondence:

$$\begin{tikzcd} & M_{\{0,j\}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ M_{\{j\}} & & M_{\{0\}} \end{tikzcd}$$

8.7.1. *Sheaves on the local model.* Let $\kappa = (a_1, \cdots, a_p, b_1, \cdots, b_p)$, with $a_1 \geq \cdots \geq a_p$ and $b_1 \geq \cdots \geq b_q$, be a weight. We have a locally free rank p sheaf $(V_0/F_0)^\vee$, and a locally free rank q sheaf F_0 over M_0, and similarly over M_j. We let $L_0 = (V_0/F_0)^\vee \oplus F_0$ and $L_j = (V_j/F_j)^\vee \oplus F_j$. To the weight κ we can naturally attach locally free sheaves $L_{0,\kappa}$ and $L_{j,\kappa}$ using the procedure described in Section 6.7.1.

8.7.2. *The map $L_{j,\kappa} \to L_{0,\kappa}$.* The natural map $V_0 \to V_j$ induces a map $V_0/F_0 \to V_j/F_j$ and by duality a map $V_j/F_j \to V_0/F_0$ that we denote by α_1. The map $V_j \to V_0$ induces a map $F_j \to F_0$ denoted by α_2. We let $\alpha = (\alpha_1, \alpha_2) : L_j \to L_0$. The map α is an isomorphism over $\mathbb{Z}[1/p]$ and it induces an isomorphism $\alpha^\star : p_2^\star L_{j,\kappa} \to p_1^\star L_{0,\kappa}$. We now investigate the integral properties of this map.

Lemma 8.4. *Let $\kappa = (k_1, \cdots, k_n)$. Let $\sup\{0, j-q\} \leq r \leq \inf\{p, j\}$. Let ξ be the generic point of $U_{\{0,j\}}(r)$. The map $\alpha^\star$ induces a map*

$$\alpha^\star : (p_2^\star L_{j,\kappa})_\xi \to p^{a_p + \cdots + a_{p-r+1}} p^{b_q + \cdots + b_{j-r}} (p_1^\star L_{0,\kappa})_\xi$$

over the local ring $\mathscr{O}_{M_{0,j},\xi}$.

Proof. We first check that over $U_{\{0,j\}}(r)$, the map α_1 has kernel and cokernel a locally free sheaf of rank r and the map α_2 has kernel and cokernel a locally free sheaf of rank $q-j+r$. Indeed, it is enough to check this at the point corresponding to $\mathrm{t}_{w(r)\mu}$, in which case the corresponding diagram is:

$$\begin{tikzcd}
\mathcal{V}_0 \arrow[r, "\mathrm{diag}(t1_j, 1_{n-j})"] & \mathcal{V}_j \arrow[r, "\mathrm{diag}(1_j, t1_{n-j})"] & \mathcal{V}_0 \\
\mathcal{V}_0 \arrow[u, "\mathrm{t}_{w(r)\mu}"] \arrow[r, "\mathrm{diag}(t1_j, 1_{n-j})"] & \mathcal{V}_j \arrow[u, "\mathrm{t}_{w(r)\mu}"] \arrow[r, "\mathrm{diag}(1_j, t1_{n-j})"] & \mathcal{V}_0 \arrow[u, "\mathrm{t}_{w(r)\mu}"]
\end{tikzcd}$$

and our claim is simply that the map $\mathcal{V}_0/\mathrm{t}_{w(r)\mu}\mathcal{V}_0 \to \mathcal{V}_j/\mathrm{t}_{w(r)\mu}\mathcal{V}_j$ has kernel of rank r and $\mathrm{t}_{w(r)\mu}\mathcal{V}_j/t\mathcal{V}_j \to \mathrm{t}_{w(r)\mu}\mathcal{V}_0/t\mathcal{V}_0$ has kernel of rank $q-j+r$. This is obvious. We now conclude the proof as in Lemma 6.7.

□

8.7.3. *The cohomological correspondence.* We may now construct a cohomological correspondence. By Proposition 2.6 and Theorem 8.2, we have a fundamental class $p_1^\star \mathscr{O}_{M_{\{0\}}} \to p_1^! \mathscr{O}_{M_{\{0\}}}$. Moreover, the sheaf $p_1^! \mathscr{O}_{M_{\{0\}}}$ is a CM sheaf.

There is also a rational map $\alpha : p_2^\star L_{j,\kappa} \dashrightarrow p_1^\star L_{0,\kappa}$, so that putting everything together, we have a rational map (the naive cohomological correspondence):

$$T^{naive} : p_2^\star L_{j,\kappa} \dashrightarrow p_1^! L_{0,\kappa}$$

We may now normalize this correspondence.

Proposition 8.5. *Let $T = p^{-\inf\left\{\sum_{\ell=p-r+1}^{p} a_\ell + \sum_{\ell=j-r}^{q} b_q + (j-r)(p-r)\right\}} T^{naive}$. Then T is a true cohomological correspondence:*

$$T : p_2^\star L_{j,\kappa} \to p_1^! L_{0,\kappa}$$

Proof. Because $p_1^! L_{0,\kappa}$ is a *CM* sheaf, any rational map from a locally free sheaf into $p_1^! L_{0,\kappa}$ is well defined if it is well defined in codimension 1. So it is enough to check that it is well defined on the generic points of all the components in the special fiber. Let $\sup\{0, j-q\} \leq r \leq \inf\{j, p\}$. Let ξ be the generic point of the stratum $U_{\{0,j\}}(r)$. We then see that $T^{naive} : (p_2^\star L_{j,\kappa})_\xi \to p^{\sum_{\ell=p-r+1}^{p} a_\ell + \sum_{\ell=j-r}^{q} b_q + (j-r)(p-r)} (p_1^! L_{0,\kappa})_\xi$ by combining Lemma 8.4 and Proposition 8.3. □

8.7.4. *Proof of Theorem 7.5.* Given Proposition 8.5, this is very similar to the proof of the main theorem in the symplectic case (see Section 6.7.4), and left to the reader.

9. Weakly regular automorphic representations and Galois representations

In this section we study a class of automorphic forms that realize in the coherent cohomology of Shimura varieties and have associated Galois representations.

9.1. **Weakly regular, odd, essentially (conjugate) self dual algebraic cuspidal automorphic representations.** Let L be a CM or totally real field, F the maximal totally real subfield of L, and c the complex conjugation (trivial if $L = F$ is a totally real field). We let $I = \mathrm{Hom}(F, \mathbb{C})$ and $J = \mathrm{Hom}(L, \mathbb{C})$.

Let π be a cuspidal automorphic representation of GL_n/L (we do not assume that the central character of π is unitary). We define below certain properties (1), (2), (3) and (4) of such representations. We will say that a π satisfying these properties is a weakly regular, algebraic, odd, essentially (conjugate) self dual, cuspidal automorphic representation.

1. **Essentially (conjugate) self dual.** We say that π is essentially self dual when L is totally real and essentially conjugate self dual when L is CM if:
$$\pi^c = \pi^\vee \otimes \chi,$$
where $\chi_0 : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is a character such that $\chi_0(-1_v)$ is independent of v for all $v \mid \infty$, and $\chi = \chi_0 \circ \mathrm{N}_{L/F} \circ \det$. We note that in the CM case, the character χ_0 is not unique because we can multiply χ_0 by the character associated to the quadratic extension L/F without changing χ. This will however change the sign of $\chi_0(-1_v)$.
2. **C-algebraic.** We say that π is C-algebraic if the infinitesimal character $\lambda = ((\lambda_{1,\tau}, \cdots, \lambda_{n,\tau})_{\tau\in J})$ of π_∞ lies in $\left((\mathbb{Z}^n + \frac{n-1}{2}\mathbb{Z}^n)/\mathcal{S}_n\right)^J$.
3. **Weakly regular.** We say that a C-algebraic π is weakly regular if for each $\tau \in J$, after applying a permutation in $\mathcal{S}_n$ to the indices, we have $\lambda_{1,\tau} > \cdots > \lambda_{[n/2],\tau}$ and $\lambda_{[n/2]+1,\tau} > \cdots > \lambda_{n,\tau}$. We say that a C-algebraic π is regular if for each $\tau \in J$, after applying a permutation in $\mathcal{S}_n$ we have $\lambda_{1,\tau} > \cdots > \lambda_{n,\tau}$.
4. **Odd.** We consider an oddness condition on a C-algebraic, essentially (conjugate) self dual (π, χ) which is given by the existence of a pole at $s = 1$ for a certain L-function.

   If L is CM, the oddness condition is that $L(s, \mathrm{Asai}^{(-1)^{n-1}\epsilon(\chi_0)}(\pi) \otimes \chi_0^{-1})$ has a pole at $s = 1$.

   If $L = F$ is totally real, the oddness condition is that $L(s, \Lambda^2\pi \otimes \chi^{-1})$ has a pole at $s = 1$ if $\epsilon(\chi_0) = 1$ and n is even, and that $L(s, \mathrm{Sym}^2\pi \otimes \chi^{-1})$ has a pole at $s = 1$ is $\epsilon(\chi_0) = -1$ and n is even or $\epsilon(\chi_0) = 1$ and n is odd. We will see below that $\epsilon(\chi_0) = 1$ when n is odd and L is totally real.

Let us explain the definition of the above L-functions and of the sign $\epsilon(\chi_0)$. We first recall the definition of the Asai representation. Assume that L is CM. Consider the group $(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \mathrm{Gal}(L/F)$ with $\mathrm{Gal}(L/F)$ acting by permutation of the factors (in other words, this is (a finite form of) the L-group over F of $\mathrm{Res}_{L/F}\mathrm{GL}_n$).

The tensor product of the standard representations give a representation $\mathbb{C}^n \otimes \mathbb{C}^n$ of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. For $\epsilon \in \{-1, 1\}$, it extends to representations $\mathrm{Asai}^\epsilon : (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \mathrm{Gal}(L/F) \to \mathrm{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ by putting $\mathrm{Asai}^\epsilon(c)(v \otimes w) = \epsilon w \otimes v$.

We now define the sign of χ_0: It follows from the C-algebraicity of π that χ_0 is algebraic. Therefore, there is an integer q such that $\chi_0 = \chi_0^f|\,.\,|^q$ where χ_0^f is a finite order character. We let $\epsilon(\chi_0) = \chi_0(-1_v)(-1)^q$ for any place $v \mid \infty$.

We observe that when n is odd and L is totally real, $\epsilon(\chi_0) = 1$. Indeed, if c is the central character of π, we find that $c^2 = \chi_0^n$, and since c is an algebraic Hecke character, we deduce that q is even and $\chi_0(-1_v) = 1$.

One may also verify that in the CM case, the oddness condition only depends on χ and not on χ_0.

The following shows that the oddness condition is often implied by the other assumptions:

Theorem 9.1. *Let π be a weakly regular, algebraic, essentially (conjugate) self dual, cuspidal automorphic representation of GL_n/L. Then π is odd unless possibly when n is even and for all $\tau \in J$, there is an ordering of the infinitesimal character $\lambda = ((\lambda_{1,\tau}, \cdots, \lambda_{n,\tau})_{\tau \in J})$ of π_∞ such that $\lambda_{i,\tau} = \lambda_{i+n/2,\tau}$ for all $1 \leq i \leq n/2$ and $\tau \in J$.*

Proof. Lemma 9.5, reduces the proof of the theorem to the CM case. The CM case follows from Theorem 9.7. □

Remark 9.2. Let π_1 and π_2 be two cuspidal automorphic representations of $\mathrm{GL}_2/\mathbb{Q}$, both L-algebraic and with trivial infinitesimal character. Assume that $(\pi_1)_\infty$ is a limit of discrete series (and therefore corresponds to a non-trivial parameter $W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ via the local Langlands correspondence), while $(\pi_2)_\infty$ corresponds to the trivial parameter $W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$. Then $\pi_1 \otimes |\det|^{\frac{1}{2}}$ is C-algebraic, arises from a weight one modular form and is odd, while $\pi_2 \otimes |\det|^{\frac{1}{2}}$ arises from a Maass form of Galois type, and is not odd. Thus, π_1 is an automorphic form which has a realization in the coherent cohomology of modular curves, while π_2 has no such realization. We observe that we cannot distinguish $(\pi_1)_\infty$ from $(\pi_2)_\infty$ by looking at the infinitesimal character, it is the oddness property of π_1 or π_2 that distinguishes them.

9.1.1. *Oddness and base change.* In this section we let L be a CM field and F its maximal totally real subfield. If π is a cuspidal automorphic representation of GL_n/F, we let $Res_L(\pi)$ be the base change lift of π to an automorphic representation of GL_n/L (see [AC89]).

Proposition 9.3. *Let L be a CM field and F its maximal totally real subfield. Let $\chi_{L/F}$ be the associated quadratic character. Let π be a weakly regular, algebraic, odd, essentially self dual cuspidal automorphic representation of GL_n/F. Assume that $\pi \neq \pi \otimes \chi_{L/F}$. Then $Res_L(\pi)$ is a weakly regular, algebraic, odd, essentially conjugate self dual, cuspidal automorphic representation of GL_n/L.*

Proof. This follows from Lemma 9.4 and Lemma 9.5. □

Lemma 9.4. *Let L be a CM field and F be its maximal totally real subfield. Let $\chi_{L/F}$ be the associated quadratic character. Let π be a weakly regular, algebraic, essentially self dual, cuspidal automorphic representation of GL_n/F. Assume that $\pi \neq \pi \otimes \chi_{L/F}$. Then $Res_L(\pi)$ is a weakly regular, algebraic, essentially conjugate self dual, cuspidal automorphic representation of GL_n/L.*

Proof. Since $\pi \neq \pi \otimes \chi_{L/F}$, we deduce from [AC89, Theorem 4.2] that $Res_L(\pi)$ is cuspidal. Since $Res_L(\pi) = Res_L(\pi)^c$, we deduce that $Res_L(\pi)$ is essentially conjugate self dual. Let $v \mid \infty$ be a place of F. Let w be a place of L above v. Then $Res_L(\pi)_w$ and π_v have the same infinitesimal character which is C-algebraic and weakly regular. □

Lemma 9.5. *Let π be a cuspidal automorphic representation of GL_n/F satisfying the assumptions of Lemma 9.4, with base change $Res_L(\pi)$. Then π is odd if and only if $Res_L(\pi)$ is odd.*

Proof. We have an L-group "diagonal" embedding $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{Gal}(L/F) \hookrightarrow (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \mathrm{Gal}(L/F)$, where $\mathrm{GL_n}(\mathbb{C}) \times \mathrm{Gal}(L/F)$ is (a finite form of) the L-group of GL_n/F. We have the decomposition

$$\mathrm{Asai}^+|_{\mathrm{GL}_n(\mathbb{C})\times\mathrm{Gal}(L/F)} = \Lambda^2\mathbb{C}^n \otimes \chi_{L/F} \bigoplus \mathrm{Sym}^2\mathbb{C}^n$$

and

$$\mathrm{Asai}^-|_{\mathrm{GL}_n(\mathbb{C})\times\mathrm{Gal}(L/F)} = \Lambda^2\mathbb{C}^n \bigoplus \mathrm{Sym}^2\mathbb{C}^n \otimes \chi_{L/F}.$$

It follows that $L(s, \mathrm{Asai}^{(-1)^{n-1}\epsilon(\chi_0)}(Res_L(\pi)) \otimes \chi_0^{-1}) =$

$$L(s, \Lambda^2\pi \otimes \chi_{L/F}^{\epsilon(\chi_0)(-1)^n} \otimes \chi_0^{-1})L(s, \mathrm{Sym}^2\pi \otimes \chi_{L/F}^{\epsilon(\chi_0)(-1)^{n+1}} \otimes \chi_0^{-1}).$$

By [Sha97, Theorem 4.1], neither of the L-functions $L(s, \Lambda^2\pi \otimes \chi_{L/K}^{\epsilon(\chi_0)(-1)^n} \otimes \chi_0^{-1})$ and $L(s, \mathrm{Sym}^2\pi \otimes \chi_{L/F}^{\epsilon(\chi_0)(-1)^{n+1}} \otimes \chi_0^{-1})$ vanishes at $s = 1$. We therefore deduce that $Res_L(\pi)$ is odd if π is odd. Conversely, if we assume that $Res_L(\pi)$ is odd, then at least one of $L(s, \Lambda^2\pi \otimes \chi_{L/K}^{\epsilon(\chi_0)(-1)^n} \otimes \chi_0^{-1})$ or $L(s, \mathrm{Sym}^2\pi \otimes \chi_{L/F}^{\epsilon(\chi_0)(-1)^{n+1}} \otimes \chi_0^{-1})$ has a pole at $s = 1$. But notice that $L(s, \pi \otimes \pi \otimes \chi_{L/K} \otimes \chi_0^{-1})$ is holomorphic at $s = 1$ since $\pi \neq \pi \otimes \chi_{L/K}$. It is now easy to deduce that π is odd (by examining the parity of n and the sign $\epsilon(\chi_0)$). □

9.1.2. *Descent to a unitary group.* In this section we consider a CM field L. We say that π on GL_n/L is conjugate self dual (as opposed to essentially conjugate self dual) if $\pi^c = \pi^\vee$. It is not hard to prove that if π is essentially conjugate self dual, there exists an algebraic Hecke character $\psi : \mathbf{A}_L^\times/L^\times \to \mathbb{C}^\times$ such that $\pi \otimes \psi$ is conjugate self dual.

Theorem 9.6. *Let L be a CM-field and π a weakly regular, algebraic, odd, conjugate self dual, cuspidal automorphic representation of GL_n/L. Then there exists a cuspidal automorphic representation $\tilde{\pi}$ of the quasi-split unitary group $\mathrm{U}(n)/F$ such that π is the transfer of $\tilde{\pi}$ for the standard base change embedding and $\tilde{\pi}_\infty$ is a non degenerate limit of discrete series.*

Proof. This follows from the main results of [Mok15]. We give some explanations.

Let $\mathrm{U}(n)/F$ be the quasi-split unitary group. Its L-group ${}^L\mathrm{U}(n)/F$ (over F) is isomorphic to $\mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(L/F)$ with the complex conjugation c acting by $g \mapsto \Phi_n\, {}^tg^{-1}\Phi_n^{-1}$ where Φ_n is the anti-diagonal matrix with alternating 1 and -1 on the anti-diagonal.

Let v be a place of F and let W_{F_v} be the Weil group of F_v. First assume that v does not split in L. The local L-group ${}^L\mathrm{U}(n)/F_v$ is isomorphic to $\mathrm{GL}_n(\mathbb{C}) \rtimes W_{F_v}$ where an element of $W_{F_v} \setminus W_{L_v}$ acts via $g \mapsto \Phi_n\, {}^tg^{-1}\Phi_n^{-1}$. If v splits, the local L-group ${}^L\mathrm{U}(n)/F_v$ is isomorphic to $\mathrm{GL}_n(\mathbb{C}) \times W_{F_v}$.

Let us denote by $G = \mathrm{Res}_{L/F}\mathrm{GL}_n$. The L-group of G is isomorphic to $(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \mathrm{Gal}(L/F)$ with the complex conjugation acting by permuting the factors. We similarly have local L-groups ${}^LG/F_v = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes W_{F_v}$ if v does not split and ${}^LG/F_v = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \times W_{F_v}$ if v splits.

We have the standard base change embedding of L-groups $\xi :\ {}^L\mathrm{U}(n) \to {}^LG$ which sends $g \rtimes 1$ to $(g,\ {}^tg^{-1}) \rtimes 1$ and $1 \rtimes c$ to $(\Phi_n, \Phi_n^{-1}) \rtimes c$. We have similar local versions $\xi_v :\ {}^L\mathrm{U}(n)/F_v \to {}^LG/F_v$ for any place v of F.

Let v be a place of F. Denote by L_{F_v} the local Langlands group of F_v. We let $\Phi(n)_v$ be the set of isomorphism classes of parameters $L_{F_v} \to {}^LG/F_v(\mathbb{C})$ and we let $\Phi(\mathrm{U}(n))_v$ be the set of isomorphism classes of parameters $L_{F_v} \to {}^L\mathrm{U}(n)/F_v(\mathbb{C})$.

In the case that v does not split in L, we recall how one can identify parameters $L_{L_v} \to \mathrm{GL}_n(\mathbb{C})$ with parameters $L_{F_v} \to {}^LG/F_v(\mathbb{C})$. We choose an element $w_c \in L_{F_v} \setminus L_{L_v}$.

To $\rho : L_{L_v} \to GL_n(\mathbb{C})$, we associate the parameter $\rho' : L_{F_v} \to {}^L G/F_v(\mathbb{C})$ defined by $\rho'(\sigma) = (\rho(\sigma), \rho(w_c^{-1}\sigma w_c)) \rtimes 1$ if $\sigma \in L_{L_v}$ and $\rho'(w_c) = (\rho(w_c^2), 1) \rtimes c$.

By [Mok15, Lemma 2.2.1], if v does not split, the natural map $\xi_v^\star : \Phi(\mathrm{U}(n))_v \to \Phi(n)_v$, given by $\phi \mapsto \xi_v \circ \phi$, induces a bijection between parameters $\phi : L_{F_v} \to {}^L\mathrm{U}(n)/F_v(\mathbb{C})$, and parameters $\rho : L_{L_v} \to GL_n(\mathbb{C})$ for which there exists an invertible matrix A with:

(1) ${}^t\rho^c A \rho = A$, where $\rho^c(\sigma) = \rho(w_c^{-1}\sigma w_c)$.
(2) ${}^tA = (-1)^{n-1} A\rho(w_c^2)$.

Indeed, to such a parameter ρ we associate the parameter ϕ defined by $\phi(\sigma) = \rho(\sigma) \rtimes 1$ if $\sigma \in L_{L_v}$ and $\phi(w_c) = C \rtimes c$ where $C = A^{-1}\Phi_n$.

Now let us consider a place v of F that splits in L. Let w and $\bar{w}$ be the two places in L above v. We have canonical isomorphisms $L_{L_w} = L_{F_v}$ and $L_{L_{\bar{w}}} = L_{F_v}$. A parameter $\rho : L_{F_v} \to {}^L G/F_v(\mathbb{C})$ can be written as $\rho(\sigma) = (\rho_w(\sigma), \rho_{\bar{w}}(\sigma)) \times \sigma$ for all $\sigma \in L_{F_v}$, where $\rho_w : L_{L_w} \to \mathrm{GL}_n(\mathbb{C})$ and $\rho_{\bar{w}} : L_{L_{\bar{w}}} \to \mathrm{GL}_n(\mathbb{C})$.

In this case, the natural map $\xi_v^\star : \Phi(\mathrm{U}(n))_v \to \Phi(n)_v$, given by $\phi \mapsto \xi_v \circ \phi$, induces a bijection between parameters $\phi : L_{F_v} \to {}^L\mathrm{U}(n)/F_v(\mathbb{C})$, and parameters ρ satisfying $\rho_w \simeq \rho_{\bar{w}}^\vee$.

For any place v of F there is associated to π_v a local Langlands parameter $\phi_{\pi_v} : L_{F_v} \to {}^L G/F_v(\mathbb{C})$ ([HT01], [Hen00]). Since π is conjugate self dual and odd, it follows from [Mok15, Theorems 2.4.10, 2.5.4] that this parameter arises from a unique parameter $\tilde{\phi}_{\pi_v} : L_{F_v} \to {}^L\mathrm{U}(n)/F_v(\mathbb{C})$. By [Mok15, Theorem 2.5.1], there is a local packet Π_v associated to $\tilde{\phi}_{\pi_v}$.

By [Mok15, Theorem 2.5.1], since π is cuspidal automorphic, any representation $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ with $\tilde{\pi}_v \in \Pi_v$ is cuspidal automorphic on $\mathrm{U}(n)/F$.

Let us describe in more details the parameter $\tilde{\phi}_{\pi_v}$ at a place $v \mid \infty$. Associated to π_v there is a parameter

$$\rho_v : L_{L_v} = \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$$

which is conjugated to

$$z \mapsto \mathrm{diag}((z/\bar{z})^{\lambda_{1,v}}, \cdots, (z/\bar{z})^{\lambda_{n,v}}).$$

This parameter is conjugate self dual with respect to the standard orthogonal form given by $A = \mathrm{Id}$.

Using the recipe of [Mok15, Lemma 2.2.1] (observe that $(-1)^{n-1}\rho_v(w_c^2) = 1$), we deduce that the parameter $\tilde{\phi}_{\pi_v}$ corresponding to Π_v is the non degenerate limit of discrete series parameter given by:

$$L_{F_v} = W_{\mathbb{R}} \to {}^L\mathrm{U}(n)/\mathbb{R}(\mathbb{C})$$

with $\phi(z) = \mathrm{diag}((z/\bar{z})^{\lambda_{1,v}}, \cdots, (z/\bar{z})^{\lambda_{n,v}}) \rtimes 1$ and $\phi(j) = \Phi_n \rtimes c$ where $j \in W_{\mathbb{R}} \setminus \mathbb{C}^\times$ satisfies $j^2 = -1$.

□

Theorem 9.7. *Let π be a weakly regular, algebraic, conjugate self dual, cuspidal automorphic representatioin π of GL_n/L. Let $\lambda = (\lambda_{i,\tau})$ be its infinitesimal character. Then π is automatically odd unless possibly when n is even and for all τ, $\lambda_{i,\tau} = \lambda_{i+n/2,\tau}$ for some ordering of the infinitesimal character.*

Proof. This again follows from the results of [Mok15]. If π is not odd, we deduce that $L(s, \mathrm{Asai}^{(-1)^n}(\pi))$ has a pole at $s = 1$. Let us fix a character $\chi_- : \mathbb{A}_L^\times/L^\times \to \mathbb{C}^\times$ verifying: $\chi_-^c = \chi_-^{-1}$ and $\chi_-|\mathbb{A}_F^\times$ corresponds to the quadratic character of L/F. We can define a twisted L-group embedding $\xi_- : {}^L\mathrm{U}(n) \to {}^L G$ (for the Weil group form of the L-groups) which sends $g \rtimes 1$ to $(g, {}^tg^{-1}) \rtimes 1$, $1 \rtimes \sigma$ to $(\chi_-(\sigma), \chi_-^{-1}(\sigma)) \rtimes \sigma$ if $\sigma \in W_F$, and $1 \rtimes c$ to $(-\Phi_n, \Phi_n^{-1}) \rtimes w_c$ for $w_c \in W_L \backslash W_F$. A similar argument as in the proof of theorem 9.6 shows that π descends via the twisted L-group embedding to an automorphic representation of $\mathrm{U}(n)$. For each place $v \mid \infty$ the parameter of π_v, $\rho_v : L_{L_v} = \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$ is conjugated to $z \mapsto \mathrm{diag}((z/\bar{z})^{\lambda_{1,v}}, \cdots, (z/\bar{z})^{\lambda_{n,v}})$. For π_v to descend via the twisted L-group embedding, there should exist a non degenerate symplectic form A on $\mathbb{C}^n$ such that ${}^t\rho_v(\bar{z})A\rho_v(z) = A$ for all $z \in \mathbb{C}^\times$. It is easy to see that there is no such symplectic form, unless when n is even and $\lambda_{i,v} = \lambda_{i+n/2,v}$ for some ordering of the $(\lambda_{i,v})$. □

9.2. **Automorphic Galois representations.** In this section we let again L be a CM or totally real field. We first recall the following result concerning Galois representations attached to regular, essentially (conjugate) self dual, cuspidal automorphic representations of GL_n/L (see, e.g., [CH13], [BLGGT14]).

Theorem 9.8 (Bellaiche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, Taylor, ...)**.** *Let π be a regular, algebraic, (essentially) conjugate self dual cuspidal automorphic representation of GL_n/L. In particular $\pi^c = \pi^\vee \otimes \chi$ and the infintesimal character of π is $\lambda = ((\lambda_{1,\tau}, \cdots, \lambda_{n,\tau})_{\tau\in J})$ with $\lambda_{1,\tau} > \cdots > \lambda_{n,\tau}$. Let $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. There is a continuous Galois representation $\rho_{\pi,\iota} : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:*

(1) $\rho^c_{\pi,\iota} \simeq \rho^\vee_{\pi,\iota} \otimes \epsilon_p^{1-n} \otimes \chi_\iota$ where χ_ι is the p-adic realization of χ and ϵ_p is the cyclotomic character,
(2) $\rho_{\pi,\iota}$ is pure,
(3) $\rho_{\pi,\iota}$ is de Rham at all places dividing p, with $\iota^{-1} \circ \tau$-Hodge-Tate weights: $(-\lambda_{n,\tau} + \frac{n-1}{2}, \cdots, -\lambda_{1,\tau} + \frac{n-1}{2})$,
(4) for all finite place v one has:

$$WD(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

Remark 9.9. Our convention is that the reciprocity law is normalized by sending geometric Frobenius to a uniformizing element. Moreover, the cyclotomic character has Hodge-Tate weight -1.

In the situation that regular is replaced by the weaker assumption of being weakly regular and odd, we have the following weaker theorem:

Theorem 9.10. *Let π be a weakly regular, algebraic, odd, (essentially) conjugate self dual, cuspidal automorphic representation of GL_n/L. In particular, $\pi^c = \pi^\vee \otimes \chi$. There is a continuous Galois representation $\rho_{\pi,\iota} : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:*

(1) $\rho^c_{\pi,\iota} \simeq \rho^\vee \otimes \epsilon_p^{1-n} \otimes \chi_\iota$ where χ_ι is the p-adic realization of χ,
(2) $\rho_{\pi,\iota}$ is unramified at all finite places $v \nmid p$ for which π_v is unramified and one has:

$$WD(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

Proof. By the patching technique of [Sor], we may reduce to the case of a CM field. Using Theorem 9.6, there is a $\tilde{\pi}$ on the quasi-split unitary group $\mathrm{U}(n)/F$, which transfers to π. Moreover, $\tilde{\pi}_\infty$ is a non degenerate limit of discrete series and therefore realizes in the coherent cohomology of a unitary Shimura variety. We can then apply the main result of [PS16] or [GK19] to conclude. The point is that the Hecke eigensystem of $\tilde{\pi}$ is a p-adic limit of Hecke eigensystems of regular, essentially conjugate self dual, automorphic representations to which Theorem 9.8 applies. □

One conjectures that $\rho_{\pi,\iota}$ in Theorem 9.10 satisfies the stronger properties of Theorem 9.8. In particular, it should be de Rham with Hodge–Tate weights $(-\lambda_{n,\tau}+\frac{n-1}{2}, \cdots, -\lambda_{1,\tau}+\frac{n-1}{2})$, for $\lambda = ((\lambda_{1,\tau}, \cdots, \lambda_{n,\tau})_{\tau\in J})$ the infinitesimal character of π_∞, and $WD(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} =$

$\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}})$ for all finite place. This has been verified in some special cases (e.g., for weight 1 modular forms).

We can prove the following very weak instance of local-global compatibility at places dividing p:

Theorem 9.11. *Let π be a weakly regular, algebraic, odd, essentially (conjugate) self dual, cuspidal automorphic representation of GL_n/L with infinitesimal character $\lambda = (\lambda_{i,\tau}, 1 \leq i \leq n, \tau \in \mathrm{Hom}(L, \overline{\mathbb{Q}}))$ and $\lambda_{1,\tau} \geq \cdots \geq \lambda_{n,\tau}$. Let p be a prime unramified in L and let $w \mid p$ be a finite place in L. Assume also that π_w is spherical, and corresponds to a semi-simple conjugacy class $\mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ by the Satake isomorphism. We let $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ be an embedding and v the associated p-adic valuation normalized by $v(p) = 1$. After permuting we assume that $v(a_1) \leq \cdots \leq v(a_n)$. Let $I_w \subset \mathrm{Hom}(L, \overline{\mathbb{Q}})$ be the set of embeddings τ such that $\iota \circ \tau$ induces the w-adic valuation on L. Then we have*

$$\sum_{i=1}^{k} v(a_i) \geq \sum_{\tau \in I_w} \sum_{\ell=1}^{k} -\lambda_{\ell,\tau}$$

for $1 \leq k \leq n$, with equality if $k = n$.

Proof. We first reduce to the CM case because if L is totally real, we can consider a CM quadratic extension L' of L such that w splits in L' and $Res_{L'}(\pi)$ is cuspidal (for example, it suffices to take L' to be ramified over L at some finite place v_0 of L where π_{v_0} is unramified). Now assume L is CM and let F be its maximal totally real subfield. We next explain how one can reduce to the case where all primes $v \mid p$ in F split in L. We can construct a totally real quadratic extension F' of F with the property that p is unramified in F' and for all places $w \mid p$ of F, w splits in F' if and only if w splits in L. We may also impose the additional condition that F' ramifies at some place v_0 of F which is inert in L and such that π_{v_0} is unramified. We now set $L' = F'L$. It is clearly sufficient to prove the result for the base change of π to L'. Note that this base change is still weakly regular, algebraic, odd, essentially conjugate self dual, cuspidal. The cuspidality follows from our choice of L', it being ramified at v_0. The oddness is another application of Shahidi's theorem that $L(s, \mathrm{Asai}^{(-1)^{n-1}\epsilon(\chi_0)}(\pi) \otimes \chi_0^{-1} \otimes \chi_{L'/L})$ does not vanish at $s = 1$.

We have thus reduced to the situation that L is a CM field and all primes $v \mid p$ in F split in L. It is useful to make the further restriction that there exists a quadratic imaginary extension L_0 of $\mathbb{Q}$ such that $L = L_0 F$. We can achieve this by considering a quadratic imaginary extension L_0 of $\mathbb{Q}$ such that p splits in L_0 and ramifies at a prime q such that there is a place v_0 of L above q which is unramified over $\mathbb{Q}$ and such that π_{v_0} is spherical. We may now replace L by LL_0 which is a CM field containing a quadratic imaginary field and π by $Res_{LL_0}(\pi)$.

By Theorem 9.6, there is a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{U}(n)/F$ which transfers to π for the standard base change embedding and is a limit of discrete series at infinity (with infinitesimal character given by λ, see the end of the proof of Theorem 9.6). The result now essentially follows from Corollary 7.7, but there is a subtlety in that Corollary 7.7 applies to the group $\mathrm{GU}(n)$ and not $\mathrm{U}(n)$. We will reduce to this case.

Let $\tilde{Z}$ be the center of $\mathrm{GU}(n)$. We observe that $\tilde{Z} \times \mathrm{U}(n) \to \mathrm{GU}(n)$ is a surjective map of algebraic groups ($\mathrm{U}(n)$ is viewed as an algebraic group over $\mathbb{Q}$ by Weil restriction). Let c be the central character of $\tilde{\pi}$. Let c' be an extension of c to an algebraic automorphic character of Z which is unramified at all places dividing p. We claim that the cuspidal automorphic representation $\tilde{\pi}$ admits an "extension" to a cuspidal automorphic representation $\tilde{\pi}'$ of $\mathrm{GU}(n)$ with central character c'. We explain the meaning of this "extension". By definition $\tilde{\pi}'$ will realize in the space of cusp forms with central character c' on $\mathrm{GU}(n)$, say $\mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c'}$. There is a well defined restriction map $res : \mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c'} \to \mathfrak{A}_{cusp}(\mathrm{U}(n))_c$ and we say that $\tilde{\pi}'$ extends $\tilde{\pi}$ if $\tilde{\pi}$ is a constituent of $res(\tilde{\pi}')$. Observe that the very general result [HS12, Theorem 4.14] implies that π admits an extension for some choice of c'. We will prove the slightly stronger result that in our case, for any choice of c' lifting c, the map res is surjective. Our argument follows [Che18] which proves a similar result under slightly different hypotheses. We take $\phi \in \mathfrak{A}_{cusp}(\mathrm{U}(n))_c$. One first extends ϕ to a function ϕ_1 on $\tilde{Z}(\mathbb{A}_\mathbb{Q})\mathrm{U}(n)(\mathbb{A}_\mathbb{Q}) \subset \mathrm{GU}(n)(\mathbb{A}_\mathbb{Q})$ satisfying $\phi_1(zg) = c'(z)\phi(g)$ for all $(z, g) \in \tilde{Z}(\mathbb{A}_\mathbb{Q}) \times \mathrm{U}(n)(\mathbb{A}_\mathbb{Q})$. We then extend ϕ_1 to a function ϕ_2 on $\mathrm{GU}(n)(\mathbb{Q})\tilde{Z}(\mathbb{A}_\mathbb{Q})\mathrm{U}(n)(\mathbb{A}_\mathbb{Q}) \subset \mathrm{GU}(n)(\mathbb{A}_\mathbb{Q})$ by letting $\phi_2(\gamma g) = \phi_1(g)$ for all $(\gamma, g) \in \mathrm{GU}(n)(\mathbb{Q}) \times \tilde{Z}(\mathbb{A}_\mathbb{Q})\mathrm{U}(n)(\mathbb{A}_\mathbb{Q})$. To check that ϕ_2 is well defined, it suffices to prove that $\mathrm{GU}(n)(\mathbb{Q}) \cap \tilde{Z}(\mathbb{A}_\mathbb{Q})\mathrm{U}(n)(\mathbb{A}_\mathbb{Q}) \subseteq \tilde{Z}(\mathbb{Q})\mathrm{U}(n)(\mathbb{Q})$. This actually amounts to proving that $\mathbb{Q}^\times \cap \nu(\tilde{Z}(\mathbb{A}_\mathbb{Q})) \subseteq \nu(\tilde{Z}(\mathbb{Q}))$, which follows from Hasse's theorem that in a quadratic extension any local norm is a global norm. Finally, we claim that $\mathrm{GU}(n)(\mathbb{Q})\tilde{Z}(\mathbb{A}_\mathbb{Q})\mathrm{U}(n)(\mathbb{A}_\mathbb{Q})$ is of finite index in $\mathrm{GU}(n)(\mathbb{A}_\mathbb{Q})$ (the quotient is dominated by $(\mathbb{Q}^\times \mathrm{N}_{L_0/\mathbb{Q}}\mathbb{A}_{L_0}^\times)\backslash\mathbb{A}_\mathbb{Q}^\times$). We can therefore extend ϕ_2 by zero to a function ϕ_3 defined on $\mathrm{GU}(n)(\mathbb{A}_\mathbb{Q})$. The verification that $\phi_3 \in \mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c'}$ is made in [Che18]. By construction, $res(\phi_3) = \phi$.

Since all places $v \mid p$ in F split in L, we have that $\mathrm{GU}(n)(\mathbb{Q}_p) = \mathrm{U}(n)(\mathbb{Q}_p)Z(\mathbb{Q}_p)$, so we deduce that $\tilde{\pi}'_v$ is spherical at all places $v \mid p$. Moreover, the Satake parameters of $\tilde{\pi}'_v$ and $\tilde{\pi}_v$ are related via the map on dual groups $\widehat{\mathrm{GU}(n)} = \mathbb{G}_m \times \mathrm{GL}_n^{[F:\mathbb{Q}]} \to \widehat{\mathrm{U}(n)} = \mathrm{GL}_n^{[F:\mathbb{Q}]}$ (which forgets the $\mathbb{G}_m$). There is a similar story at archimedean places and we deduce that $\tilde{\pi}'_\infty$ is a limit of discrete series. We may apply Corollary 7.7 to $\tilde{\pi}'$, and we deduce the validity of Conjecture 4.7 in this case, which is the inequality:

$$\mathrm{Newt}_\iota(\chi) \leq \frac{1}{|\Gamma/\mathrm{Stab}_\Gamma(\infty(\kappa,\iota))|} \sum_{\gamma \in \Gamma/\mathrm{Stab}_\Gamma(\infty(\kappa,\iota))} -w_0(\gamma \cdot \infty(\kappa, \iota)).$$

We project this identity under the map $\widehat{\mathrm{GU}(n)} \to \widehat{\mathrm{U}(n)}$ and then further via the map $\mathrm{GL}_n^{[F:\mathbb{Q}]} \to \mathrm{GL}_n^{[F_v:\mathbb{Q}_p]}$ for a chosen prime $v \in F$ below the prime $w \in L$. We can now unravel the meaning of this identity.

The local L group of $\mathrm{Res}_{F_v/\mathbb{Q}_p}\mathrm{GL}_n$ is $\mathrm{GL}_n^{[F_v:\mathbb{Q}_p]} \rtimes \Gamma$ where $\Gamma = \hat{\mathbb{Z}}$ acts by permutation of the factors.

The Satake parameter of $\pi_w = \tilde{\pi}_v$ is given by the conjugacy class of $\mathrm{diag}(a_1, \cdots, a_n)$, and via the Newton map it goes to

$$\frac{1}{[F_v : \mathbb{Q}_p]}(v(a_n), \cdots, v(a_1))^{[F_v:\mathbb{Q}_p]}$$

in $(P_\mathbb{R}^+)^\Gamma$ where $(P_\mathbb{R}^+)^\Gamma$ is the subset of Γ-invariants in the cone of dominant weights $P_\mathbb{R}^+ \subset (\mathbb{R}^n)^{[F_v:\mathbb{Q}_p]}$. The projection of the infinitesimal character is $(\lambda_{i,\tau})_{\tau \in I_w}$ and the average $\sum_{\gamma \in \Gamma/\mathrm{Stab}_\Gamma(\infty(\kappa,\iota))} -w_0(\gamma \cdot \infty(\kappa, \iota))$ gives:

$$\frac{1}{[F_v : \mathbb{Q}_p]}\Big(\sum_{\tau \in I_w} -\lambda_{n,\tau}, \cdots, \sum_{\tau \in I_w} -\lambda_{1,\tau}\Big)^{[F_v:\mathbb{Q}_p]}.$$

The theorem is proven.

□

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