paving the fluid road to flat holography

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Contents

Notations and Conventions

We report here a list of conventions used in this work:

- $(a, b, c, d, \dots)$ are ten-dimensional indices in Section 2, generic boundary $d{+}1$-dimensional indices in Appendix A and bulk four-dimensional indices in Appendix B. In every instance their role would be explicitly stated.
- $(M, N, P, Q, \dots)$ are $(d+2)$-dimensional indices of the bulk.
- $(m, n, p, q, \dots)$ are S^5 indices.
- $(\mu, \nu, \rho, \dots)$ are $(d+1)$-dimensional indices of the boundary. They are three-dimensional for four-dimensional bulks and two-dimensional for three-dimensional ones.
- g indicates the metric determinant, such that $\sqrt{-g} = \sqrt{-\det g}$.
- Vectors are always reported with an underline: for instance $\underline{u} = u^\mu \partial_\mu$.
- Forms are reported as $u = u_\mu dx^\mu$.
- The conformal factor in a Weyl transformation is spelled $\mathcal{B}$, and it is a function of all the coordinates.
- G_N is the Newton constant.
- k is the speed of light.
- For three-dimensional boundaries we define the transverse duality $\tilde{\eta}_{\mu\nu} = -\frac{u^\rho}{k}\eta_{\rho\mu\nu}$.
- In three dimensions: $\eta_{\sigma\lambda\mu} = \sqrt{-g}\epsilon_{\sigma\lambda\mu}$.
- In two dimensions: $\eta_{\mu\nu} = \sqrt{-g}\epsilon_{\mu\nu}$.
- ∇ is the relativistic Levi-Civita connection, except in Appendix A where it is the Weyl connection.
- $\hat{\nabla}$ is the Carroll-Levi-Civita connection.
- $\mathcal{D}_\mu$ is the gauged Weyl connection, which depends on the Weyl weight of the object it acts upon.
- $\mathcal{D}_i$ is the gauged Weyl-Carroll spatial connection.
- $\mathcal{D}_t$ is the gauged Weyl-Carroll temporal connection.
- $\frac{\mathrm{d}}{\mathrm{d}t}$ is the Galilean material derivative acting on scalars.
- $\frac{D}{\mathrm{d}t}$ is the Galilean material derivative acting on tensors.
- x refers generally to a set of coordinates, whereas its bold version $\boldsymbol{x}$ indicates spatial coordinates only. For instance $x = (t, \boldsymbol{x})$.
- BMS: Bondi-Metzner-Sachs. FG: Fefferman-Graham. WFG: Weyl-Fefferman-Graham. RT: Robinson-Trautman. AdS: Anti-de Sitter. CFT: conformal field theory. EH: Einstein-Hilbert. o.s.: on-shell. LL: Landau-Lifshitz.

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Résumé

L'objectif de cette thèse est l'étude de la correspondance fluide/gravité, réalisation macroscopique de la dualité AdS/CFT dans la limite où la constante cosmologique tend vers zéro (limite plate). La jauge de Fefferman-Graham, habituellement utilisée dans le dictionnaire holographique, est singulière dans la limite plate et cela constitue un obstacle dans le projet de formuler une théorie holographique pour des solutions asymptotiquement plates. Dans cette thèse, en passant par la formulation hydrodynamique de la théorie vivant au bord, nous construirons une jauge appelée jauge du développement en série dérivative où cette limite est bien définie. Cette jauge est construite en utilisant le symétrie de Weyl sur le bord, qui traduit la propriété de l'holographie de fournir une classe conforme de métriques plutôt qu'une métrique spécifique. Alors que la jauge de Fefferman-Graham est implémentée en coordonnée holographique radiale, le développement en série dérivative est construit sur des directions de genre lumière et c'est la raison pour laquelle la limite plate est bien définie dans cette jauge. En fait, alors que la théorie sur le bord pour des solutions asymptotiquement AdS est une hypersurface de genre temps sur laquelle la CFT vit, le bord d'une solution asymptotiquement plate est une hypersurface de genre lumiére.

Sur la géométrie du bord, la limite plate correspond à faire tendre la vitesse de la lumière vers zéro, situation connue sous le nom de limite carrollienne. Nous discuterons en détail cette limite et ses conséquences sur la géométrie et sur les difféomorphismes du bord. Un fluide relativiste admet une telle limite qui donne lieu à l'hydrodynamique carrollienne que l'on étudie ici en dimension arbitraire, parallèlement à son homologue galiléen qui est obtenu en faisant tendre la vitesse de la lumière vers l'infini.

Nous discuterons également du sort du tenseur énergie-impulsion relativiste dans la limite carrollienne et nous formulerons une théorie intrinsèquement carrollienne dans son ensemble. Cela nous permettra d'introduire les charges carrolliennes qui correspondent à des charges asymptotiques dans des exemples particuliers.

Ensuite, nous montrerons spécifiquement en dimensions quatre et trois du bulk qu'il est possible de construire des solutions asymptotiquement plates des équations d'Einstein en partant de systèmes hydrodynamiques conformes carrolliens du bord, définis ici sur l'hypersurface de genre lumière à l'infini.

En quatre dimensions, nous introduirons des conditions d'intégrabilité permettant de resommer la série dérivative sous forme fermée. Ces conditions restreignent la classe de solutions accessibles à celles qui sont algébriquement spéciales, grâce au théorème de Goldberg-Sachs. Nous développerons nos résultats dans des exemples précis et la solution de Robinson-Trautman sera utilisée plusieurs fois pour démontrer la puissance et l'universalité de notre formalisme.

En trois dimensions, toute configuration fluide du bord aboutit à une solution exacte des équations d'Einstein. Le développement en série dérivatif donne naissance à de nouvelles conditions de bord. Les solutions de Bañados sont un sous-ensemble des solutions obtenues et identifiées au moyen de leurs charges de surface.

La vitesse du fluide joue un rôle crucial dans le calcul des charges asymptotiques et en particulier, nous montrerons qu'il est impossible de la choisir de façon arbitraire. Nous accorderons donc une attention particulière au rôle du repère hydrodynamique, trop souvent ignoré en holographie.

Pour terminer, nous nous concentrerons sur la formulation de la correspondance AdS/CFT dans laquelle la symétrie de Weyl est explicite. Bien que cette symétrie soit un ingrédient incontournable de la correspondance fluide/gravité, elle n'est pas codée dans la formulation habituelle de l'holographie. Nous introduirons une nouvelle jauge et analyserons ses conséquences. Plus précisément, nous montrerons comment cette nouvelle jauge induit la métrique sur le bord ainsi qu'une connexion de Weyl, différente de la connexion de Levi-Civita habituelle. Enfin, nous étudierons les conséquences de ce résultat sur l'anomalie de Weyl, sur la procédure de renormalisation holographique et sur la théorie des champs du bord.

1 Introduction

This work is devoted to our recent results in flat holography. In order to contextualize it in the realm of theoretical physics, we start our road far away from the topic itself and drive ourselves toward it step by step.

In high energy physics, we nowadays refer to holography as a theory which presents two facets, a priori completely disentangled, but ultimately related via a so-called holographic dictionary. This dictionary is not only a way to relate quantities from one side with quantities from the other, but (and this is the power of holography) an identification of the dynamics. A holographic theory is therefore a duality between a theory and another. However we usually talk about holography when one of the two theories lives in dimensions higher than the other.

If we really want to go to the historical moment where, for the first time, a result indicating that gravity could be holographic has been found, we should go back to the main realization on the entropy of a black hole [1–3]. Indeed, the latter was found to scale as the area of the black hole horizon, whereas it is well-known that the entropy of a gas in a box scales like the volume – indicating therefore that gravity seems to be holographic [4, 5]. With almost half a century of developments separating us from this discovery, it is not surprising that a holographic theory has been found, where the degrees of freedom of a gravitational theory are translated into degrees of freedom of another theory living in less dimensions, as the entropy scaling law would suggest.

Despite this, what is still surprising is that this holographic theory is defined only for a particular ensemble of spacetimes, characterized by the presence of a negative cosmological constant. Even more cumbersome, the theory is fully understood and developed only in a limited number of circumstances. The holographic theory goes under the name AdS/CFT duality, discovered by Maldacena in [6] and promptly studied in (among others) [7–14].[1] In this holographic duality we have two seemingly unrelated theories combined. In the original and better understood formulation, on the one hand we have a ten-dimensional theory of gravity (type IIB string theory) for five-dimensional Anti-de Sitter spacetimes (i.e. spacetimes with negative curvature) times the five-dimensional sphere S^5. On the other hand we have a four-dimensional theory of matter called super conformal Yang-Mills. The latter is a conformal field theory, where gravity is non dynamical. We will discuss in detail the duality, both geometrically and dynamically. The message to retain is that there is a theory which predicts a correspondence between a gravitational theory and a theory of matter, the former living in the bulk while the latter on its boundary. One of the main motivations behind the community interest in holography is the effort to extend it toward a correspondence where the bulk has vanishing cosmological constant, which is a first step to describe the universe we live in – the value of the cosmological constant in nature is found to be extremely small but positive.

This correspondence can be conjectured to hold outside the realm of string theory and supersymmetric theories. It can be thought of as a general relationship between gravity and matter. Stated differently, it can be assumed to be valid in some limit of the parameters of the two theories. Even more generally, one may argue that holography is a property of gravity, in all its realizations. The AdS/CFT duality is a weak/strong coupling duality. This means that the more quantum effects are suppressed in the gravity side the more the boundary field theory is strongly coupled. In this setup the boundary theory cannot be studied perturbatively. Access to properties of this theory is thus very hard. We are here particularly interested in the limit where the bulk gravitational theory becomes pure Einstein general relativity. This means that we need to completely suppress quantum effects and break supersymmetry. The first task is achieved considering classical gravity duals of strongly coupled matter theory [17, 18], the second one needs more abstraction. Indeed, the fact that holography is still possible for non-supersymmetric theories is only conjectured and has as supporters only those who believe gravity itself is holographic. We will assume the bulk can be treated in its classical limit and in the absence of supersymmetry. On top of this, the fluid/gravity dictionary (the core of this thesis), treats the boundary theory using hydrodynamics. This limit is expected to be allowed in any field theory. It is a large distance, long wavelength (long time) approximation, [19–21]. It represents an effective description of the boundary CFT. The hydrodynamic limit is therefore a macroscopic limit where the field theory has been coarse-grained to the extend that only low frequency, long distance modes remain. The boundary theory is of course still strongly coupled, but we focus in this limit only on low frequency perturbations of this system. Fortunately, these modes are holographic, for they are also found in the analysis of black holes quasinormal modes. Thanks to these limits the fluid/gravity correspondence postulates a duality between a solution of Einstein classical equations and a relativistic conformal fluid living on the boundary. This is the setup we will use in this thesis.

There are two ways to use this kind of duality: either one obtains results for the boundary theory using the classical evolution of fields in the bulk, or one tries to find the holographic dual of a given fluid (and geometry) configuration in the boundary. The way to relate these two theories has been discussed in depth in [22–27], in an order-by-order expansion of the bulk line element. One of the novelty introduced in the fluid/gravity duality is

[1] See the reviews [15, 16] and references therein.

the gauge in which the bulk metric is implemented, called derivative expansion. The latter is strongly based on Weyl symmetry [28], which will be part of all this thesis and eventually arise in a self-contained discussion at the end of it. The derivative expansion is a bulk gauge inspired by hydrodynamics, where the expansion is performed in derivatives of a null-like congruence. We will study it in detail, explain how to derive it, and compare it to the Fefferman-Graham gauge, in which holography has been firstly defined.

We are interested here in a boundary-to-bulk approach, which is a sort of filling-in problem. The latter can be considered an ancestor of holography where, given some boundary data, a geometric reconstruction is performed [29]. We will see that the derivative expansion gives the correct evolution in the boundary to bulk expansion and the initial constraints will be encoded in the conservation of the boundary energy-momentum tensor, which encapsulates the dynamics of the boundary fluid. Recently a closed form of the bulk line element given fluid's data has been found, together with many properties of this particular duality [30–37]. Our first goal will be to review it in Section 2. To do so, we will need to discuss the properties of the boundary fluid, which is a relativistic fluid.[2] A particular property is that the fluid congruence, if the setup is relativistic, can be chosen at will. This allows for some internal freedom. For instance, one can choose a congruence such that the heat current is zero, reaching the so-called Landau-Lifschitz frame [41, 42]. We will see that this is not a wise choice within our formalism, for the boundary heat current is part of the data needed to describe the bulk dual.

So far we presented a brief introduction to holography. For us, after some limiting procedure, it boils down to be a duality between a $d+2$-dimensional solution of Einstein equations with negative cosmological constant and a conformal relativistic fluid living on its $d+1$-dimensional conformal boundary. We stressed that this scheme holds uniquely in the presence of a negative cosmological constant and that it is an important goal to try to extend this holographic construction to vanishing cosmological constant bulks. Within fluid/gravity, this is the result of this thesis, as we will shortly discuss.

We discuss the holographic construction for a fluid living on the three-dimensional boundary of a four-dimensional spacetime, the way the latter is written using data of the former, and the conditions one needs for obtaining that the conservation of the boundary energy-momentum tensor (boundary dynamics) translates into the bulk Einstein equations (bulk dynamics). Indeed, all the properties of a relativistic fluid, in the absence of additional conserved currents, are encapsulated in the energy-momentum tensor, its conservation being the dynamical equation of motion. We find that in four dimensions a particular class of bulk solutions can be achieved, due to the structure and imposition on the boundary system and to an application of the Goldberg-Sachs theorem[3] [44–47].[4] In particular these assumptions are consequences of the asymptotic structure of the bulk Weyl tensor [50, 51], involving an elegant relationship between the fluid dynamics and the geometry (encapsulated in the Cotton tensor, conformal in three boundary dimensions). The instructive example we decide to focus on is the Robinson-Trautman family of solutions [52–55]. These have been studied in holography [56–60], and showed to be a rich kind of non-stationary solutions, with fascinating boundary dynamics.

We then move to three-dimensional bulk. A simple calculation of degrees of freedom shows that in three dimensions gravity cannot propagate [61]. Therefore, solutions are characterized only in terms of their asymptotic charges [62–65].[5] The latter are computed given specific boundary conditions [67, 68]. Charges identify the bulk solution we are dealing with. We thus show that we can reconstruct using the boundary fluid at least all the known bulk solutions, known as Bañados solutions [69–72]. We then show that the hydrodynamic frame redefinition is broken here and setting the heat current to zero or not a priori changes the a posteriori result. The boundary two-dimensional fluid is far from trivial, also due to the presence of the conformal anomaly [72–75].

As advertised, the main result of this thesis is that the fluid/gravity AdS dictionary admits the zero cosmological constant limit [76]. We will refer to holography in this limit as flat holography. From the bulk, the finiteness is ensued by the choice of line element gauge – the derivative expansion – which in this respect is better suited than the Fefferman-Graham expansion [77, 78] (divergent as the cosmological constant is set to zero). From the boundary the result is at first rather odd: this limit corresponds to the limit where the speed of light (spelled k) tends to zero in the boundary matter theory. This limit is called a Carrollian limit, and represents the core of our work. In particular, the geometry in this limit becomes degenerate, passing from a time-like hypersurface (boundary of AdS) to a null-like hypersurface (null-like boundary of flat spacetimes). We would like to list here instances where Carrollian physics has entered the high energy physics world and the relevance it has for us, before showing its holographic implementation.

Although firstly introduced as a mathematical curiosity by Lévy-Leblond [79], the Carrollian limit is intensively

[2] For a recent discussion on relativistic fluids see [38–41]. References [42, 43] will also be intensively used.

[3] We show this explicitly in Appendix B, devoted also to the Petrov classification of the Weyl tensor.

[4] See also [48, 49] for a general analysis of Einstein solutions.

[5] Charges are computed in this thesis using the package [66].

making its way through high energy physics. In [79], the Carrollian limit of the Poincaré group is introduced as dual (speed of light to zero) to the well-known Galilean limit, where the speed of light k is sent to infinity. In this precursory work, this limit is from the group-theoretical viewpoint the Inönü-Wigner group contraction of the Poincaré group. A specific k-rescaling of the Poincaré algebra generators allows a well-defined $k \to \infty$ limit, which returns the Galilean algebra. Alternatively, one can rescales differently the generators and reach the Carroll group. While the Poincaré group treats on the same ground space and time transformations, the Galileo group does not, for time is absolute and space can be boosted. The Carroll group inverts the role played by time and space. In fact, in the latter space is absolute whereas time can be boosted. The three groups action on the spacetime under a Lorentz boost with speed v is schematically represented in the table below:

Group	time transformation	space transformation
Poincaré	$t' = \dfrac{t + v\boldsymbol{x}/k^2}{\sqrt{1 - v^2/k^2}}$	$\boldsymbol{x}' = \dfrac{\boldsymbol{x} + vt}{\sqrt{1 - v^2/k^2}}$
Galileo	$t' = t$	$\boldsymbol{x}' = \boldsymbol{x} + vt$
Carroll	$t' = t + b\boldsymbol{x}$	$\boldsymbol{x}' = \boldsymbol{x}$

Where we introduced the Carrollian inverse velocity $b = \frac{v}{k^2}$.

The previous exercise can be extended to the conformal group [80]. In this scenario the Carrollian contraction gives rise to the infinite-dimensional conformal Carroll group. The latter has been recently shown [81, 82] to realize the asymptotic symmetries of an asymptotically flat spacetime. Specifically, the conformal Carroll group of level 2 (dynamical exponent $z = 1$) is the group of asymptotic symmetries of any four-dimensional asymptotically flat spacetime, known as the BMS (Bondi-Metzner-Sachs) group [83–86], see also [87] for a recent interesting discussion of it. This result settles unquestionably the fate of holography in the flat limit, and shows that we are on the right track with our results. In a recent work we furthermore established, under suitable conditions, the presence of the BMS algebra on Carrollian spacetimes [88], independently of their embedding.[6]

The Carrollian limit is performed keeping the original metric as general as possible. Therefore, we do not only discuss geometries invariant under a particular Carrollian transformation, thus for which this transformation is an isometry. Instead, we are going to require our geometrical data to be covariant under what we defined as Carrollian diffeomorphisms

$$t' = t'(t, \vec{\boldsymbol{x}}) \qquad \vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}'(\vec{\boldsymbol{x}}). \tag{1}$$

These diffeomorphisms will be crucial in the following, and covariance under them will be a guideline along the way.

Our investigation paves the way toward a mathematical and microscopical formulation of dualities between matter theories and gravitational ones, without cosmological constant. This question has been already raised from the macroscopic point of view in [76, 95], and its relationship with the BMS underlying symmetries has been studied in [96, 97]. Furthermore, it has already received attention also from an algebraic perspective in [98–110]. In [111, 112], the importance of Carrollian physics is well underlined. These attempts followed the nominal formal discussion made in [113–116], where the question was first decrypted.

Inspired by the relativistic counterpart in the presence of a cosmological constant and based on the analysis [101, 117–119], in [120–122] a particular correspondence between the gravitational bulk and a CFT living on the d-dimensional spatial part of its null infinity has been developed. This correspondence is loosely based on the fact that the Lorentz part of the Poincaré group is both linearly realized in the bulk and non-linearly realized as the global conformal group of the d-dimensional celestial sphere. This would surely have to be included and retrievable in any supposed microscopical theory in the full null boundary.

Carroll physics has not only entered the realm of high energy physics through flat holography. Whenever a geometrical degeneracy presents itself, Carrollian geometry can play a crucial role. This has been noticed in tensionless strings, where the tensionless limit has been argued to be a Carrollian one, due to the degeneracy it infers on the worldsheet metric [123, 124]. Moreover, an analysis of Carrollian particles and superparticles appears in [125, 126]. The last sector where Carrollian physics has attracted attention is in electrodynamics, where, since photons behave like an ultra-relativistic gas, the Carrollian interpretation suits naturally [127].

To proceed any further, we need therefore to study in depth the effect of the Carrollian limit on hydrodynamics and geometry. The former has been dealt with in [128] and represents the core of Section 3. We noticed in [129] that it would have been too naive to take the Carrollian limit in full generality at the level of the energy-momentum itself, due to the richness of the geometrical background that intervenes in its equations of motion through the divergence. In some restricted cases where the geometry is simple it can be done, [130], but in general it leads to wrong Carrollian

[6] Similar study on physics on null structures can be found in [89–94].

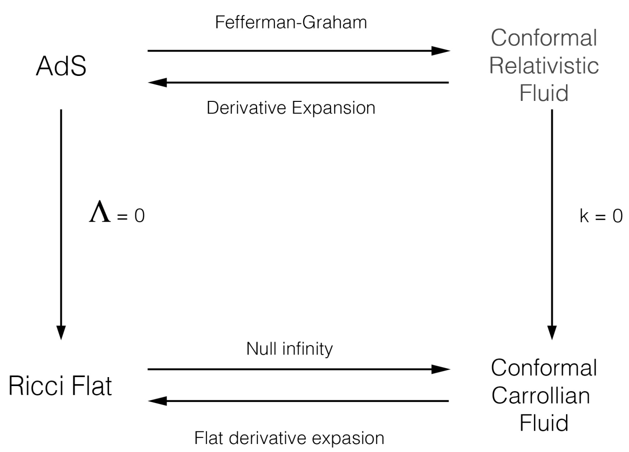
dynamics. We therefore introduce the intrinsic Carrollian counterpart of the energy-momentum tensor, the Carrollian momenta, and show that Carrollian covariance automatically implies their conservation. Furthermore, we construct Carrollian charges intrinsically for a Carrollian spacetime (like null infinity) and show that they match in some known cases the asymptotic charges [63]. We would like to remark at this point that the Carrollian limit of a geometry is intimately connected to the BMS symmetry. Every null hypersurface can be described within this formalism. This in particular applies to null infinity, but also to black hole horizons – which are null hypersurfaces. Furthermore, the idea of using fluid dynamics to describe the black hole horizon is not new and it is the building block of the membrane paradigm [131–133]. Nevertheless, it is only recently that the presence of BMS-like symmetries on the black hole horizon has been appreciated [134–147], together with its natural Carrollian geometrical interpretation [148, 149].

Once the formalism was ready to perform the Carrollian limit of relativistic hydrodynamics and the geometry was well-suited (using the so-called Randers-Papapetrou gauge), we implemented in [128] also the dual limit, where the speed of light k is now sent to infinity. To do this we chose an alternative parametrization, going under the name of Zermelo's ([150]), and we reached equations of motion for the most general Galilean fluid, covariant under Galilean diffeomorphisms. This work was motivated by attempts to find a unified framework, as in e.g. [151–156], fully covariant under Galilean transformations. From the geometrical viewpoint, the Galilean limit has been intensively studied, leading to the construction of Newton-Cartan structures [81, 90, 157–164].

With all this machinery at work we eventually present in Section 4 the missing link, which is the limit $k \to 0$ of the AdS derivative expansion. The bulk line element becomes a putative solution of Ricci-flat Einstein equations, while the boundary passes from a relativistic conformal fluid living on a time-like hypersurface to a conformal Carrollian fluid living on a null hypersurface [165, 166]. This allows to set the holographic dictionary between a Ricci-flat bulk and a Carrollian fluid. As already stated, the main realization is that the derivative expansion allows the vanishing cosmological constant limit. This limit has already been addressed in different fashions and scattered setups. For instance it has been considered on fixed time-like hypersurfaces near the conformal boundary in [167–171]. We believe our results help in achieving a comprehensive understanding on the topic, due to the solid AdS construction they are limit of. The final output of our process is a bulk line element (called flat derivative expansion), written exclusively as a function of Carrollian fluid and geometric data, which solves Einstein equations if the fluid is a solution of the Carrollian hydrodynamic equations.

We show that in four bulk dimensions we reach algebraically special bulk solutions, thanks to the Goldberg-Sachs theorem. On top of the flat Robinson-Trautman example, we explain also our scheme for the Kerr-Taub-NUT family of solutions. In three dimensions, we can recover all Barnich-Troessaert solutions [119], thanks to a careful inspection of the asymptotic charges and their algebras [63, 101, 102, 118, 172, 173]. Here again, the Carrollian heat current plays a special role and neglecting it would restrict the spectrum of solutions reached.

Let us trace the road done so far on the map. We started from the microscopic AdS/CFT duality, took its classical bulk limit and hydrodynamics boundary one, reaching the AdS fluid/gravity duality. We attacked the problem in a boundary-to-bulk approach, asking ourselves if, given a conformal relativistic fluid living on the time-like boundary of the asymptotically AdS bulk, one can reconstruct the dual Einstein solution. This is doable in four and three bulk dimensions using the derivative expansion gauge. We then proved that in the flat limit we reach a holographic duality between a solution of Ricci-flat Einstein equations and a conformal Carrollian fluid living on its null boundary. We unraveled thus a general picture that could be schematically represented as follows:



We will organize the structure of this thesis around this square of relationships. In particular, Section 2 covers the top side of the square, Section 3 the right and left ones and Section 4 the bottom one.

We marginally mentioned that in fluid/gravity the guideline to write the bulk line element in the derivative expansion gauge, given the fluid data on the boundary, is Weyl covariance. Due to the fact that the boundary metric is formally located at infinite distance, the boundary metric is defined up to a rescaling by a non-trivial function of the boundary coordinates. Id est, the boundary enjoys Weyl symmetry. The Fefferman-Graham gauge is not form invariant under such symmetry transformation.

Therefore, motivated by the importance of Weyl transformation in fluid/gravity, we go back to the microscopical AdS/CFT formulation in Section 5 and discuss an enhancement of the Fefferman-Graham (FG) gauge that allows to recover geometrically Weyl transformations [174]. The improved gauge, called Weyl-Fefferman-Graham (WFG), induces on the boundary a metric and a Weyl connection [175, 176] – instead of the usual picture where the Fefferman-Graham gauge induces a metric and its Levi-Civita connection. This is the first compelling result, showing how Weyl is geometrized in this picture.[7].

The FG gauge admits an expansion of the metric from the boundary to the bulk in powers of the holographic coordinate. Solving Einstein equations allows the extraction of the different terms of the expansion, all being determined by two terms in the expansion: the boundary conformal class of metrics and the vacuum expectation value of the energy-momentum tensor operator of the dual field theory, as originally discussed in [7, 11, 14]. It is a theorem that, given these two quantities, one can reconstruct, at least order by order, a bulk AdS spacetime in FG gauge, by imposing Einstein equations. The resolution of the latter for the WFG gauge leads to a modification of the subleading terms in this expansion: we will demonstrate that the modifications are such that each term is Weyl-covariant.

As already stated, the boundary metric is located at infinite distance. Thence, since the bulk action is on-shell proportional to the volume of the spacetime, divergences arise [10, 11, 13, 177]. While most of them can be counteracted adding local counterterms, in every odd bulk dimension there subsist some of them which cannot. These are interpreted as anomalies in the boundary Ward identity [178–184]. In our improved Weyl-Fefferman-Graham construction, the anomaly will be expressed uniquely as a function of Weyl-covariant tensors. We will furthermore present a cohomological interpretation of the Weyl anomaly, inspired by [185].

The presence of the anomaly is usually encoded in the fact that the boundary energy-momentum tensor acquires an anomalous trace [186–188]. Indeed in FG gauge, it is found that it must be a priori traceless. This boundary Ward identity is obtained by considering the boundary background as dictated by the induced metric. The latter is the only source usually considered. As such, there is only one sourced current. However, one finds that one must typically improve the energy-momentum tensor, as originally found in [189]. Here, we promote the Weyl connection to be part of the background data. From this perspective we are gauging the Weyl symmetry in the boundary [190–193], although more properly, we should view it as a local background symmetry. The holographic dictionary will return us directly the boundary Ward identity relating the trace of the energy-momentum with the divergence of the Weyl current. As a byproduct, our setup is also useful to analyze the profound relationship between Weyl invariance and conformal invariance, a subject which has been discussed for instance in [194,195]. We will present in the beginning of Section 5 a more technical introduction on the topic of Weyl holography, at the light of all the material presented in between.

We would like to conclude this introduction with a technical note for the reader.

- Section 2 is inspired by [60], [76] and [95].
- Section 3 is inspired by [128] and [129].
- Section 4 is inspired by [76], [129] and [95].
- Section 5 is inspired by [174].

2 Fluid-Gravity Correspondence in AdS

We review in this section the main features of the fluid/gravity duality. This duality is inspired by the microscopic AdS/CFT correspondence, which sets a link between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and super Yang-Mills in four dimensions. This duality is a very powerful tool, relating a theory of gravity to a matter theory without gravity itself. We do not want to digress here on the fascinating results of this duality and the massive research project it started. For us, it is enough to recall the main properties and the limits we will need to do in order to be able to

[7] Appendix A is devoted to the geometrical implementation of the Weyl connection in the boundary.

talk about fluids on one side and Einstein gravity on the other. Let therefore briefly remind us the dictionary for a scalar field and the correspondence at the level of the partition functions. This will be useful in particular in Section 5, where we will go back to more microscopic properties in AdS holography.

In $\mathrm{AdS}_5 \times S^5$ the metric factorizes: writing the ten-dimensional coordinates $\zeta^a = (x^M, y^m)$ with M coordinatization of AdS_5 and m of S^5

$$g_{ab} d\zeta^a d\zeta^b = g_{MN} dx^M dx^N + g_{mn} dy^m dy^n. \tag{2}$$

A massless scalar field $\phi(\zeta)$ can be decomposed using the spherical harmonics of S^5

$$\phi(\zeta) = \sum_i \varphi^{(i)}(x) Y_i(y). \tag{3}$$

In physical field theories, a state is associated with unitary irreducible representations of the symmetry group. For AdS_5 this is $SO(2,4)$[8] which has maximal compact subgroup $SO(2) \times SO(4)$. Using that $SO(4) \sim SU(2) \times SU(2)$, we can label states with representations of $SO(2) \times SU(2) \times SU(2)$, i.e. (Δ, J_1, J_2). The Casimir is then

$$C = \Delta(\Delta - 4) + 2J_1(J_1 + 1) + 2J_2(J_2 + 1). \tag{4}$$

For a scalar field $J_1 = 0 = J_2$.

By the Kaluza-Klein mechanism the $SO(6)$ isometry of S^5 becomes the gauge symmetry in five dimensions. The spherical harmonics on S^5 give an infinite tower of Kaluza-Klein particles on AdS_5. A consistent truncation of this spectrum can be made such that we can focus only on the ten-dimensional massless scalar field. The effect of the S^5 decomposition is, from the point of view of AdS_5, to infer a Kaluza-Klein mass on this field

$$\frac{m^2}{k^2} = \Delta(\Delta - 4), \tag{5}$$

where $k^2 = \frac{1}{l^2}$ with l^2 the AdS radius and Δ is the $SO(2)$ energy label for the field, which identifies its conformal dimension. This result is consistent with (4) for a scalar field. In other words, we have that a massless scalar field in ten dimensions on $\mathrm{AdS}_5 \times S^5$ reduces to a massive field on AdS_5 with its mass given by the quadratic Casimir of $SO(2,4)$, the symmetry group of AdS_5 itself. This can be shown taking the ten-dimensional Einstein-Hilbert action and proving that the Kaluza-Klein decomposition creates a kinetic term (in five dimensions) for the ten-dimensional massless scalar.

Therefore we consider a massive scalar field on AdS_5. Its action is given by

$$S = -\frac{1}{2} \int_{\mathrm{AdS}_5} d^5x \sqrt{-g} \Big(\partial_M \varphi \partial_N \varphi g^{MN} + m^2 \varphi^2 \Big). \tag{6}$$

Its Klein-Gordon equation can be explicitly solved, for it can be recast as a Bessel equation. Before discussing the result, we will elaborate on the geometrical structure of a $d+2$-dimensional AdS_{d+2} and its Poincaré coordinatization.

The space AdS_{d+2}, together with dS_{d+2} and Minkowski Mink_{d+2}, is a maximally symmetric spacetime (it has $\frac{1}{2}(d+2)(d+3)$ Killing vectors, the generators of $SO(2, d+1)$). It has negative constant curvature, corresponding to a hyperbolic geometry. It is indeed a spacetime with negative cosmological constant, $k^2 = -(d+1)\Lambda$. It is only for this sign of the cosmological constant that the holographic correspondence is best understood and developed. Writing $x^M = (z, x^\mu)$ the AdS metric reads

$$ds^2 = \frac{dz^2}{z^2 k^2} + \frac{\eta_{\mu\nu}}{z^2 k^2} dx^\mu dx^\nu. \tag{7}$$

The conformal boundary is located at $z \to 0$.[9] There the metric conformally diverges. We thence define the conformal boundary metric as (k has unit L^{-1} so zk is a dimensionless parameter)

$$ds^2_{\mathrm{bdy}} = \lim_{z \to 0} (z^2 k^2) ds^2. \tag{8}$$

The ambiguity in defining the boundary metric should make your hair curl. In fact, we usually refer to the boundary as a conformal class of metrics, since it is defined up to a conformal factor of the boundary coordinates

[8] The isometry group of Lorentzian AdS_n is $SO(2, n-1)$. Many results in this topic are obtain using the Euclidean continuation. Here, we work in Lorentzian signature unless otherwise stated.

[9] In fluid/gravity we mostly use $r = \frac{1}{z}$ and thus locate the conformal boundary at $r \to \infty$, due to multiple (debatable) reasons.

– neutralized by a redefinition of z. Nonetheless, this ambiguity is disregarded and the boundary metric is always fixed in practice. We postpone for later on (Section 5) an insightful treatment of this fact.

We are now ready to go back to (6) and solve it on the background (7), so assuming again five-dimensional bulk. This can be done analytically in the full spacetime but we are interested in the $z \to 0$ behavior. The full derivation of this solution is standard material on the topic [196–198]. From this point to the end of the section, we will assume Euclidean signature. The result is

$$\varphi(z,x) \sim (kz)^{\Delta_+}\xi^+(x)(1+O(z^2k^2)) + (kz)^{\Delta_-}\xi^-(x)(1+O(z^2k^2)), \tag{9}$$

where

$$\Delta_+(\Delta_+ - 4) = \frac{m^2}{k^2}, \qquad \Delta_-(\Delta_- - 4) = \frac{m^2}{k^2}. \tag{10}$$

The sum and difference of these two weights satisfy

$$\Delta_+ + \Delta_- = 4, \qquad \Delta_+ - \Delta_- = 2\sqrt{4 + \frac{m^2}{k^2}}. \tag{11}$$

We conclude that z^{Δ_-} is the most divergent term and thus defines the boundary value of the field

$$\lim_{z\to 0}(kz)^{-\Delta_-}\varphi(z,x) = \xi^-(x). \tag{12}$$

The conformal boundary hosts the advocated matter theory. The boundary value of the field gets the interpretation of a source for a local scalar operator $O(x)$ in the boundary theory, which is a conformal field theory. The generating functional is then

$$Z_{\text{CFT}}[\xi^-] = \langle e^{-\int d^4x \xi^-(x) O(x)} \rangle. \tag{13}$$

The space AdS_5 is the vacuum solution in the bulk. Its dual interpretation is the ground state of the dual CFT. Therefore $\xi^-(x)$ represents a deformation in the CFT, $\xi^-(x) = 0$ being the undeformed value. The holographic dictionary relates the partition function of the theory in the bulk with the boundary one (o.s. stands for on shell)

$$\frac{Z_{\text{CFT}}[\xi^-(x)]}{Z_{\text{CFT}}[0]} = Z^{\text{o.s.}}_{\text{Gravity}}[\xi^-(x)]. \tag{14}$$

This is the fundamental result of the gauge/gravity duality. Its domain of applicability spans from the well-understood $\text{AdS}_5 \times S^5$ vs four-dimensional super Yang-Mills duality to more conjectured dualities in various dimensions and boundary matter theory.

This equation is however not very handful unless we evaluate it in some limits. For instance, for α' and g_{string} small string theory reduces to supergravity where

$$Z^{\text{o.s.}}_{\text{Gravity}}[\xi^-(x)] = e^{-S^{\text{o.s.}}[\xi^-(x)]}. \tag{15}$$

Using this result one can compute the expectation value of the scalar operator sourced by $\xi^-(x)$. The final result (after appropriate renormalization) is that $\langle O(x) \rangle$ is proportional to $\xi^+(x)$. We thence have a nice interpretation of the bulk field expansion in terms of the boundary theory. The boundary value of a field is interpreted as a free source while the vev of the operator sourced by it is related to the other field in the expansion $\xi^+(x)$. As we will shortly see, this is a general feature, in particular also true for the bulk metric itself.

Keeping g_{string} small suppresses quantum corrections in the bulk. Therefore this double limit on string theory makes it become classical supergravity. If we moreover assume the dictionary being true also for non-supersymmetric theories than the bulk is nothing but Einstein general relativity at first order in all the various parameters. Consequently, the on shell action appearing in (15) is the Einstein-Hilbert action in five dimensions

$$S_{\text{EH}} = \frac{1}{16\pi G_N}\int d^5x\sqrt{-g}(R - 2\Lambda) \tag{16}$$

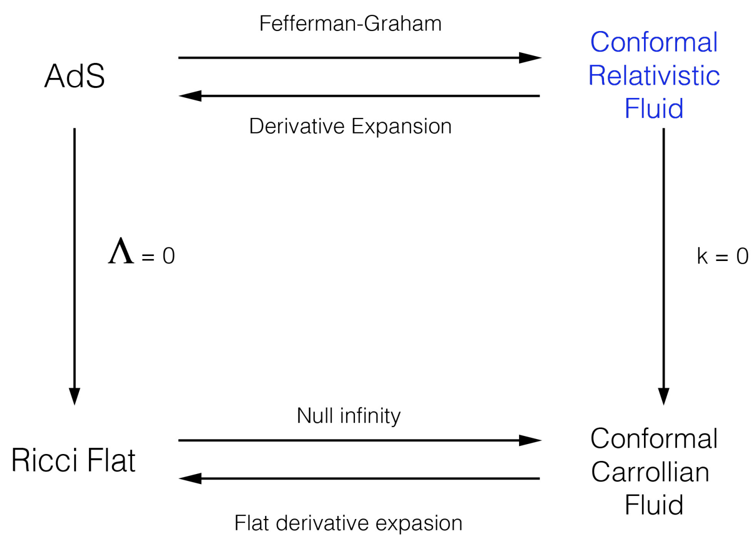
with G_N the Newton constant, plus contributions coming from the scalar field.

The boundary theory is a conformal field theory. The supergravity approximation in the bulk is dual to strong coupling approximation in the boundary theory. We will also consider the long-distance low-frequency approximation, as performed in [19, 20]. This limit corresponds to the relativistic hydrodynamic limit, where the energy-momentum tensor is decomposed according to a fluid congruence. This macroscopic limit coarse-grains the field theory in the boundary: n-point functions are replaced with fluid transport coefficients, related to the formers via Kubo formulas only in first order in the frequency. The hydrodynamic limit is focusing only on low frequency modes of the boundary field theory. That is its main drawback: we are loosing information on the boundary. However this limit enhances a formidable control on the system. It is especially useful to treat finite-temperature systems. We will see that computation-wise it organizes the theory in an elegant way.

Bringing together all different limits and approximations, fluid/gravity duality conjectures in its simpler formulation that Einstein gravity is dual to hydrodynamics. As anticipated, the fluid/gravity dictionary is written in the so-called derivative expansion gauge. The latter is inspired by the Hamiltonian temporal evolution of gravity, in which one solves initial constraints on a given space-like surface and then requires the temporal evolution to satisfy the remaining Einstein equations. In the derivative expansion the temporal evolution is replaced by a null-like evolution, from the boundary to the bulk. The explicit form of the gauge is tuned such that the in-falling evolution satisfies bulk Einstein equations. The parts of Einstein equations which encode the initial constraints are then encoded in the conservation of the boundary energy-momentum tensor, which puts the boundary fluid on shell. This boundary energy-momentum tensor can be decomposed along the fluid velocity and its orthogonal directions. It is the time-like boundary value of the null-like bulk congruence that defines the boundary fluid velocity. In hydrodynamics, the latter has a certain frame invariance, for it is possible to define it such that some dissipation phenomena can be included in the kinematic. We will discuss the possible hydrodynamic frames and the importance of working in the most general one in holography, to avoid constraints on relevant holographic data. We could have started the discussion by directly conjecturing the relationship between Einstein gravity and hydrodynamics. I believe this derivation of the duality, even though still hand waived, gives a nice glance of the story and contextualize it in a more general and fascinating picture.

2.1 Boundary Hydrodynamics

This section is devoted to boundary hydrodynamics, in the relativistic setup. With respect to our square-web of dualities, it is the blue sector below that we will discuss here



We will firstly discuss it in arbitrary dimensions and full generality (non necessarily conformal) and then specialize to three and two dimensions, relevant to the reconstruction of four and three gravitational bulks, respectively.

2.1.1 In Arbitrary Dimension

In this section we work on a generic $d+2$-dimensional bulk, i.e. a $d+1$-dimensional boundary. We denote the boundary metric $g_{\mu\nu}$ and we keep it as general as possible. As already anticipated, the bulk metric itself gives rise in the boundary theory to a source $g_{\mu\nu}$ and a vev. By construction, the latter is the energy-momentum tensor of the boundary theory

$$\langle T^{\mu\nu}\rangle = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{bdy}}}{\delta g_{\mu\nu}}. \tag{17}$$

The way this object is read-off from the boundary expansion of the bulk metric will be explained in the next section. Notice for the moment being that for empty AdS it is identically zero, which justifies why we think of the latter as dual to the CFT vacuum. Here we want to discuss the boundary hydrodynamics, so we interpret this tensor as the energy-momentum tensor of a fluid (we disregard from now on the expectation value $\langle\cdot\rangle$).

We now prove that, if the theory is covariant, this tensor is conserved. Consider the variation of the boundary action under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \rho^\mu$, we have (b.t. means possible disregarded boundary terms)

$$\delta_\rho S = \int d^{d+1}x \left(\frac{\delta S}{\delta g_{\mu\nu}}\delta_\rho g_{\mu\nu} + \frac{\delta S}{\delta\phi}\delta_\rho\phi\right) + \text{b.t.}, \tag{18}$$

where ϕ stands for the various other fields of the theory. We assume that we are on-shell so $\frac{\delta S}{\delta\phi} = 0$. Moreover, δ_ρ is the Lie derivative, which reads

$$\delta_\rho g_{\mu\nu} = \nabla_\mu\rho_\nu + \nabla_\nu\rho_\mu. \tag{19}$$

We thus obtain

$$\delta_\rho S = -\int d^{d+1}x\sqrt{-g}T^{\mu\nu}\nabla_\mu\xi_\nu = \int d^{d+1}x\sqrt{-g}\nabla_\mu T^{\mu\nu}\xi_\nu + \text{b. t.}. \tag{20}$$

If the theory is covariant, $\delta_\rho S = 0$ for all ρ. From this we deduce that $\nabla_\mu T^{\mu\nu}$ vanishes on shell, which is the usual conservation law of the energy-momentum tensor. As we will show this is related to bulk Einstein equations. In hydrodynamics the conservation of the energy-momentum tensor is not an identity of the theory but rather a dynamical equation for the fluid. This comes about because in the hydrodynamic regime we loose information on the microscopic action and as we will see $T^{\mu\nu}$ is now express in terms of the fluid variable, which are macroscopic quantities rather than fundamental fields. The interplay between micro and macro and fluid and geometry are at the heart of our construction and will arise many times.

The fluid lives in the boundary, it is a $d+1$-dimensional system. Its energy-momentum $T^{\mu\nu}$ can be geometrically decomposed as ($u^\mu = (u^0, u^i)$ with i running on the d spatial indices)

$$T^{\mu\nu} = (\varepsilon + p)\frac{u^\mu u^\nu}{k^2} + pg^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{k^2} + \frac{u^\nu q^\mu}{k^2}. \tag{21}$$

It is made of a perfect-fluid piece and terms resulting from friction and thermal conduction. It contains $d+2$ dynamical variables:

- energy per unit of proper volume (rest density) ε, and pressure p;
- d velocity-field components u^i (u^0 is determined by the normalization $\|u\|^2 = -k^2$).[10]

The dynamical equations of motion for a relativistic fluid are all encapsulated in the conservation of the energy-momentum tensor, which in the absence of external forces reads

$$\nabla_\mu T^{\mu\nu} = 0. \tag{22}$$

These are $d+1$ equations, a local-equilibrium thermodynamic equation of state[11] $p = p(T)$ is therefore needed for completing the system – T being the temperature of the system. We also have the usual Gibbs-Duhem relation for the grand potential $-p = \varepsilon - Ts$ with $s = \frac{\partial p}{\partial T}$.

For instance a conformal fluid would satisfy the equation of state

$$\varepsilon = dp, \tag{23}$$

which implies that the energy-momentum tensor is traceless $T^\mu_\mu = 0$. This would not be true in the presence of a conformal anomaly, which arises for even-dimensional boundary theories. We will touch upon this later on, where we will study two-dimensional fluids.

[10] k here is the velocity of light usually called c. It is a key quantity in the Carrollian limit discussed in next sections. The reason why we spell it k will become clear there. It is very often set to 1, we specifically do not want to do that.

[11] We omit here the chemical potential as we assume no independent conserved current.

The viscous stress tensor $\tau^{\mu\nu}$ and the heat current q^μ are purely transverse:

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \tfrac{1}{k^2} T_{\mu\nu} u^\mu u^\nu. \tag{24}$$

Hence, they are expressed in terms of u^i and their spatial components q_i and τ_{ij}. The quantities q_i and τ_{ij} capture the physical properties of the out of equilibrium state. They are usually expressed as expansions in temperature and velocity derivatives, the coefficients of which characterize the transport phenomena occurring in the fluid.

The transport coefficients can be determined either from the underlying microscopic theory, or phenomenologically. In first-order hydrodynamics

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta\Delta_{\mu\nu}\Theta, \tag{25}$$
$$q_{(1)\mu} = -\kappa\Delta_\mu{}^\nu\left(\partial_\nu T + \frac{T}{k^2}a_\nu\right), \tag{26}$$

where [12]

$$a_\mu = u^\nu\nabla_\nu u_\mu, \tag{27}$$
$$\Theta = \nabla_\mu u^\mu, \tag{28}$$
$$\sigma_{\mu\nu} = \nabla_{(\mu}u_{\nu)} + \frac{1}{k^2}u_{(\mu}a_{\nu)} - \frac{1}{d}\Theta\,\Delta_{\mu\nu}, \tag{29}$$
$$\omega_{\mu\nu} = \nabla_{[\mu}u_{\nu]} + \frac{1}{k^2}u_{[\mu}a_{\nu]}, \tag{30}$$

are the acceleration (transverse), the expansion, the shear and the vorticity of the velocity field (rank 2 transverse and traceless), with η, ζ the shear and bulk viscosities, and κ the thermal conductivity.

In the above expressions, $\Delta_{\mu\nu}$ is the projector onto the space transverse to the velocity field, and one similarly defines the longitudinal projector $U_{\mu\nu}$:

$$\Delta_{\mu\nu} = \frac{u_\mu u_\nu}{k^2} + g_{\mu\nu}, \quad U_{\mu\nu} = -\frac{u_\mu u_\nu}{k^2}. \tag{31}$$

We want to close this section with an important – often dismissed in holography – discussion on hydrodynamic field redefinitions. In relativistic fluids, the absence of sharp distinction between heat and matter fluxes leaves a freedom in setting the velocity field. Intuitively, this freedom reflects the idea that these two fluxes are just energy motion relativistically, so we could decide to orient the velocity along one flux only, the other, or a combination of them. Consequently, the macroscopic quantities $\{T, u, \mu\}$, with μ the chemical potential, can be redefined order by order in the hydrodynamic expansion.The guideline in this field redefinition is that microscopic quantities, such as the energy-momentum tensor and any other conserved currents, should be invariant. This comes about because only these objects have an interpretation in the microscopic field theory, and indeed we are discussing here their vevs.

The fluid-velocity ambiguity is well posed in the presence of an extra conserved current J [40, 42], naturally decomposed into a longitudinal perfect piece and a transverse part:

$$J^\mu = \varrho u^\mu + j^\mu. \tag{32}$$

Here j^μ encodes dissipation.

At equilibrium there is no redundancy in hydrodynamics, which translates the fact that there are no possible distinct energy flows to align along. Out of equilibrium the redundancy emerges in the heat current q and the non-perfect piece of the matter current j.

One may therefore set $j = 0$ and reach the so-called Eckart frame. Alternatively $q = 0$ defines the Landau-Lifshitz frame. These define the two extrema, a generic fluid frame have both j and q. In the absence of extra currents, setting $q = 0$ could possibly blur the physical phenomena occurring in the fluids under consideration.

Let us report explicitly some transformation rules between quantities in Landau-Lifshitz (LL) and Eckart (E) frame. Writing $\mathcal{Q}_{\mathsf{LL}} = \mathcal{Q}_{\mathsf{E}} + \delta\mathcal{Q}$ for any kinematical or thermodynamic quantity $\mathcal{Q}$, the displacements can be computed linearly, quadratically, and so on, based on the fundamental rule that the energy-momentum tensor T and the matter

[12] Our conventions for (anti-) symmetrization are $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

current J are frame-invariant. The variation in the velocity field is determined in terms of the heat current, non-zero in Eckart frame, vanishing in Landau-Lifshitz frame, by solving perturbatively the eigenvalue problem:

$$\delta u^{(1)} = \frac{q}{p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}}. \tag{33}$$

All other transformation rules are determined from the latter, using the quoted invariance and Gibbs-Duhem equation. The non-perfect matter-current component j is vanishing in Eckart and non-zero in Landau-Lifshitz, where its first-order value is

$$\delta j^{(1)} = -\frac{\varrho_{\mathsf{E}}}{p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}} q\,, \tag{34}$$

while

$$\delta\varepsilon^{(1)} = \delta\varrho^{(1)} = \delta s^{(1)} = \delta p^{(1)} = 0\,. \tag{35}$$

Similarly, we find

$$\delta\left(\frac{\mu}{T}\right)^{(1)} = \frac{q\cdot\tau_{\mathsf{E}}\cdot q}{\varrho_{\mathsf{E}} T_{\mathsf{E}} q^2}, \tag{36}$$

and using $\delta p = \varrho\delta\mu + s\delta T$ we can read off $\delta T^{(1)}$ and $\delta\mu^{(1)}$.

It should be noticed that the stress tensor τ_{E} is a correction with respect to the perfect fluid, of similar order than the heat current q. The first correction it receives is therefore of second order:

$$\delta\tau^{(2)\mu\nu} = \frac{q\cdot\tau_{\mathsf{E}}\cdot q}{(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}})\, q^2}\left(q^\mu u^\nu + q^\nu u^\mu\right) + \frac{tr\,\delta\tau^{(2)}}{d} h^{\mu\nu}. \tag{37}$$

In this expression, the trace of the correction, $tr\,\delta\tau^{(2)} = g_{\mu\nu}\,\delta\tau^{(2)\mu\nu}$, is left undetermined. This trace also appears in the second-order correction of the pressure,

$$\delta p^{(2)} = \frac{\delta\varepsilon^{(2)}}{d} - \frac{tr\,\delta\tau^{(2)}}{d}, \quad \delta\varepsilon^{(2)} = -\frac{q^2}{p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}}, \tag{38}$$

so that a freedom remains to reabsorb it or not in the latter (see discussion in [40]). The other second-order corrections from Eckart to Landau-Lifshitz frame read:

$$\delta u^{(2)} = \frac{1}{2\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)^2}\left(q^2 \mathsf{u}_{\mathsf{E}} - 2\tau_{\mathsf{E}}\cdot q_{\mathsf{E}}\right), \tag{39}$$

$$\delta j^{(2)} = -\frac{\varrho_{\mathsf{E}}}{\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)^2}\left(q^2 \mathsf{u}_{\mathsf{E}} - \tau_{\mathsf{E}}\cdot q_{\mathsf{E}}\right), \tag{40}$$

$$\delta s^{(2)} = \frac{q^2 s_{\mathsf{E}}}{2\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)^2} - \frac{q^2}{T_{\mathsf{E}}\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)}, \tag{41}$$

$$\delta\varrho^{(2)} = \frac{q^2 \varrho_{\mathsf{E}}}{2\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)^2}, \tag{42}$$

$$\delta\left(\frac{\mu}{T}\right)^{(2)} = -\frac{1}{\varrho_{\mathsf{E}} T_{\mathsf{E}}\left(p_{\mathsf{E}} + \varepsilon_{\mathsf{E}}\right)}\left(q^2 + \frac{q\cdot\tau_{\mathsf{E}}\cdot\tau_{\mathsf{E}}\cdot q}{q^2} - \left(\frac{q\cdot\tau_{\mathsf{E}}\cdot q}{q^2}\right)^2\right). \tag{43}$$

Finding the latter requires to analyse the eigenvalue problem of the energy-momentum tensor at third order. We can further combine (38) with (43) and $\delta p = \varrho\delta\mu + s\delta T$, and extract $\delta T^{(2)}$ and $\delta\mu^{(2)}$.

We can proceed similarly and obtain the above quantities at next order, or even further. Their expressions follow the pattern already visible in the first and second orders. It is readily seen that the expansions of all Landau-Lifshitz observables around their Eckart values are controlled by the parameter $\|q\|/p_{\mathsf{E}}+\varepsilon_{\mathsf{E}}$, i.e. basically the norm of the heat current. The magnitude of this quantity sets validity bounds on the frame transformation at hand. We also see that this hydrodynamic frame redefinition is based on the assumption that an extra independent current is available. Moving to the Landau-Lifshitz frame without such an extra current is therefore questionable, for it could incidentally constraint some degrees of freedom.

On more general footing we are sure that we are not making any assumption keeping q arbitrary. Consequently, we will keep the heat current as part of the physical data. Another reason why we decide to do so is because, although the boundary fluid can potentially be written in any frame, we will discuss resummation of bulk spacetimes using fluid data. It is not clear a priori that this procedure is insensitive to q.

A posteriori, we actually have the glance that the contrary is true, namely that this resummation procedure treats q as an important physical degree of freedom, necessary for the success of the reconstruction. This will be elucidated in particular in three-dimensional bulk reconstructions, at the level of the charges.

The punchline here is that, independently of the already deep question of whether in the absence of extra currents every fluid frame is achievable, we still do not know if the fluid/gravity correspondence is sensitive to the fluid field redefinition, so we work in the most general frame to avoid loss of universality.

2.1.2 In Dimension Three

So far we worked with generic dimension $d+1$, we now analyze some features of three-dimensional fluids. In three dimensions, the Hall viscosity appears as well in $\tau_{(1)\mu\nu}$:

$$-\zeta_{\mathrm{H}}\frac{u^{\sigma}}{k}\,\eta_{\sigma\lambda(\mu}\,\sigma_{\nu)\rho}\,g^{\lambda\rho}, \tag{44}$$

with $\eta_{\sigma\lambda\mu}=\sqrt{-g}\,\epsilon_{\sigma\lambda\mu}$.

It will be useful in the following to introduce the vorticity two-form

$$\omega=\frac{1}{2}\omega_{\mu\nu}\,dx^{\mu}\wedge dx^{\nu}=\frac{1}{2}\left(du+\frac{1}{k^2}u\wedge a\right), \tag{45}$$

where u and a are the one-forms $u=u_{\mu}dx^{\mu}$ and $a=a_{\mu}dx^{\mu}$. Its Hodge dual form is proportional to u in three dimensions:

$$k\gamma u=\star\omega\quad\Leftrightarrow\quad k\gamma u_{\mu}=\frac{1}{2}\eta_{\mu\nu\sigma}\omega^{\nu\sigma}, \tag{46}$$

In this expression γ is a scalar, that can also be expressed as

$$\gamma^2=\frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu}. \tag{47}$$

One can naturally define a fully antisymmetric two-index tensor as[13]

$$\tilde{\eta}_{\mu\nu}=-\frac{u^{\rho}}{k}\eta_{\rho\mu\nu}, \tag{48}$$

obeying

$$\tilde{\eta}_{\mu\sigma}\tilde{\eta}_{\nu}{}^{\sigma}=\Delta_{\mu\nu}. \tag{49}$$

With this tensor the vorticity reads:

$$\omega_{\mu\nu}=k^2\gamma\tilde{\eta}_{\mu\nu}. \tag{50}$$

All the introduced first-derivative object will play an important role in the following. An important final remark is that on a generic, possible time dependent, background these quantities are far from trivial even for an adapted fluid, i.e. a fluid with velocity $\underline{u}=\partial_t$.

Weyl Symmetry in 3-dimensional Fluids

As already discussed, holography does not furnish us a boundary metric but rather a conformal class of them. We will review the implications of this fact in full detail in section 5. The fluid/gravity picture is strongly based on Weyl covariance [23,26], and this Weyl transformation (defined precisely shortly) is a very powerful guideline for the setup. Weyl symmetry will be discussed in different fashions and contexts in this work. We limit here our attention to its importance for three-dimensional fluids.

The definition of the boundary metric is insensitive to a conformal rescaling. We call this rescaling a Weyl transformation and say that the boundary metric has weight -2:

$$ds^2_{\text{bdy}}\to\frac{ds^2_{\text{bdy}}}{\mathcal{B}(x)^2}. \tag{51}$$

[13] The $\sim$ is necessary to distinguish this object from the two-dimensional one $\eta_{\mu\nu}=\sqrt{-g}\epsilon_{\mu\nu}$ defined in the next section for two dimensions.

This scaling does not alter the definition of the boundary metric for it can be reabsorbed in a bulk redefinition of the holographic coordinate ($r \to \mathcal{B}(x)r$).[14]

At the level of hydrodynamics one should at the same time trade u_μ for $^{u_\mu}/_{\mathcal{B}}$ (velocity one-form), $\omega_{\mu\nu}$ for $^{\omega_{\mu\nu}}/_{\mathcal{B}}$ (vorticity two-form) and $T_{\mu\nu}$ for $\mathcal{B}T_{\mu\nu}$. As a consequence, the pressure and energy density have weight 3, the heat-current q_μ weight 2, and the viscous stress tensor $\tau_{\mu\nu}$ weight 1 under Weyl. These transformation rules for the fluid come from the fact that the energy-momentum $T_{\mu\nu}$ should have weight 1 (we will show it explicitly). Using its hydrodynamic decomposition one deduces the rule for all the other quantities.

Since we are looking for a holographic fluid, it is natural to package things in an explicitly Weyl covariant way. This requires to introduce a Weyl connection one-form:[15]

$$A = \frac{1}{k^2}\left(a - \frac{\Theta}{2}u\right), \tag{52}$$

which transforms as $A \to A - d\ln\mathcal{B}$.

Ordinary covariant derivatives ∇ are thus traded for Weyl covariant ones. The latter can be "gauged", given the weight w of the conformal tensor under consideration. Then for instance the Weyl covariant derivative of a weight-w tensor v_μ is

$$\mathcal{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho. \tag{53}$$

The Weyl covariant derivative is metric with non-vanishing commutator:

$$\begin{aligned} \mathcal{D}_\rho g_{\mu\nu} &= 0, && (54)\\ (\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu) f &= wfF_{\mu\nu}, && (55)\end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{56}$$

is the Weyl-invariant field strength.

Commuting the Weyl-covariant derivatives acting on vectors, as usual one defines the Weyl covariant Riemann tensor[16]

$$(\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu) V^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu}V^\sigma + wV^\rho F_{\mu\nu} \tag{57}$$

(V^ρ are weight-w) and the usual subsequent quantities. In three spacetime dimensions, the covariant Ricci (weight 0) and the scalar (weight 2) curvatures read:

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= R_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu + g_{\mu\nu}\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right) - F_{\mu\nu}, && (58)\\ \mathcal{R} &= R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu. && (59)\end{aligned}$$

Notice that the Weyl-Ricci tensor is not symmetric, due to the presence of $F_{\mu\nu}$.

The Weyl-invariant Schouten tensor[17] is

$$\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\mathcal{R}g_{\mu\nu} = S_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu - \frac{1}{2}A_\lambda A^\lambda g_{\mu\nu} - F_{\mu\nu}. \tag{60}$$

Other Weyl-covariant velocity-related quantities are

$$\begin{aligned} \mathcal{D}_\mu u_\nu &= \nabla_\mu u_\nu + \frac{1}{k^2}u_\mu a_\nu - \frac{\Theta}{2}\Delta_{\mu\nu}\\ &= \sigma_{\mu\nu} + \omega_{\mu\nu}, && (61)\\ \mathcal{D}_\nu \omega^\nu{}_\mu &= \nabla_\nu\omega^\nu{}_\mu, && (62)\\ \mathcal{D}_\nu \tilde{\eta}^\nu{}_\mu &= 2\gamma u_\mu, && (63)\\ u^\lambda\mathcal{R}_{\lambda\mu} &= \mathcal{D}_\lambda\left(\sigma^\lambda{}_\mu - \omega^\lambda{}_\mu\right) - u^\lambda F_{\lambda\mu}, && (64)\end{aligned}$$

of weights -1, 1, 0 and 1 (the scalar vorticity γ has weight 1).

[14] As remarked, in fluid/gravity we use holographic coordinate $r = 1/z$.
[15] The explicit form of A is obtained demanding $\mathcal{D}_\mu u^\mu = 0$ and $u^\lambda\mathcal{D}_\lambda u_\mu = 0$. See [23] for more details.
[16] In Appendix A we properly define these quantities in arbitrary dimension making use of the notion of Weyl connection.
[17] The ordinary Schouten tensor in three spacetime dimensions is given by $R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$.

In three dimensions the Weyl tensor is identically zero. All the information regarding the conformal structure of a given manifold are captured in the so-called Cotton tensor.[18] The Cotton tensor is generically a 3-index tensor with mixed symmetries. In three dimensions it can be dualized into a two-index, symmetric and traceless tensor. It is defined as

$$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma}\mathcal{D}_\rho\left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma}\right) = \eta_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right). \tag{65}$$

The Cotton tensor is Weyl-covariant of weight 1 (i.e. transforms as $C_{\mu\nu} \to \mathcal{B}\, C_{\mu\nu}$), and is identically conserved:

$$\mathcal{D}_\rho C^\rho{}_\nu = \nabla_\rho C^\rho{}_\nu = 0, \tag{66}$$

sharing thereby all properties of the energy-momentum tensor. This important fact will be relevant and suggests already that perhaps it is through these two objects that fluids and geometries are suppose to interact.

Following (21) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the fluid velocity u:

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}. \tag{67}$$

Such a decomposition naturally defines the weight-3 Cotton scalar density

$$c = \frac{1}{k^3}C_{\mu\nu}u^\mu u^\nu, \tag{68}$$

as the longitudinal component.

The symmetric and traceless Cotton stress tensor $c_{\mu\nu}$ and the Cotton current c_μ (weights 1 and 2, respectively) are purely transverse:

$$c_\mu{}^\mu = 0, \quad u^\mu c_{\mu\nu} = 0, \quad u^\mu c_\mu = 0, \tag{69}$$

and obey

$$c_{\mu\nu} = -k\Delta^\rho{}_\mu\Delta^\sigma{}_\nu C_{\rho\sigma} + \frac{ck^2}{2}\Delta_{\mu\nu}, \quad c_\nu = -cu_\nu - \frac{u^\mu C_{\mu\nu}}{k}. \tag{70}$$

One can use the definition (65) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$c = \frac{1}{k^2}u^\nu\tilde{\eta}^{\sigma\rho}\mathcal{D}_\rho\left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma}\right), \tag{71}$$

$$c_\nu = \tilde{\eta}^{\rho\sigma}\mathcal{D}_\rho\left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma}\right) - cu_\nu, \tag{72}$$

$$c_{\mu\nu} = -\Delta^\lambda{}_\mu\left(k\eta_\nu{}^{\rho\sigma} - u_\nu\tilde{\eta}^{\rho\sigma}\right)\mathcal{D}_\rho\left(\mathcal{S}_{\lambda\sigma} + F_{\lambda\sigma}\right) + \frac{ck^2}{2}\Delta_{\mu\nu}. \tag{73}$$

In section 2.2 we will discuss how, starting from the fluid boundary data, one can reconstruct a Einstein space in the bulk. There, we will see the crucial role played by the Cotton tensor in organizing this reconstruction.

2.1.3 In Dimension Two

We consider now two-dimensional fluids, dual to three-dimensional bulk geometries. In an abuse of notation, we still use greek indices μ to refer to the two boundary coordinates, such that $x^\mu = (x^0, x^1)$. Recall that in the presence of external force density f_ν the fluid satisfies:

$$\nabla^\mu T_{\mu\nu} = f_\nu. \tag{74}$$

Together with the equation of state (local thermodynamic equilibrium is assumed), this set of equations provide the hydrodynamic equations of motion. In two dimensions, the transverse direction with respect to u is entirely supported by the Hodge-dual $\star u$:[19]

$$\star\, u_\rho = u^\sigma\eta_{\sigma\rho}. \tag{75}$$

This dual congruence is space-like and normalized as $\|\star u\|^2 = k^2$.

Therefore we define

$$q = \chi \star u \quad \text{with} \quad \chi = -\frac{1}{k^2}\star u^\mu T_{\mu\nu}u^\nu, \tag{76}$$

[18] This tensor is non-zero in the boundary whenever the bulk is locally asymptotically AdS [32–34, 36, 76].

[19] Our conventions in 2 dimensions are: $\eta_{\sigma\rho} = \sqrt{-g}\epsilon_{\sigma\rho}$ with $\epsilon_{01} = +1$. Hence $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$.

as the local heat density. Similarly, the viscous stress tensor has a unique component encoded in the viscous stress scalar τ:

$$\tau_{\mu\nu} = \tau h_{\mu\nu} \quad \text{with} \quad \Delta_{\mu\nu} = \frac{1}{k^2} \star u_\mu \star u_\nu \tag{77}$$

the projector onto the space transverse to the velocity field.

The energy-momentum trace reads:

$$T^\mu{}_\mu = p - \varepsilon + \tau. \tag{78}$$

The pressure p and the viscous stress scalar τ appear in the fully transverse component of the energy-momentum tensor. Their sum is therefore the total stress.

If the system is free and at global equilibrium, τ vanishes and the stress is given by the thermodynamic pressure p alone. Hence, the viscous stress scalar τ is usually expressed as an expansion in temperature and velocity gradients, and this distinguishes it from p. The same holds for the heat current q. The coefficients of these expansions characterize the transport phenomena occurring in the fluid.

The shear and the vorticity vanish identically in two spacetime dimensions. The only non-vanishing first-derivative tensors of the velocity are the acceleration and the expansion

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{79}$$

and one defines similarly the expansion of the dual congruence as[20]

$$\Theta^\star = \nabla_\mu \star u^\mu, \tag{80}$$

which enables us expressing the acceleration:

$$a_\mu = \Theta^\star \star u_\mu. \tag{81}$$

In first-order hydrodynamics[21]

$$\tau_{(1)} = -\zeta\Theta, \tag{82}$$

$$\chi_{(1)} = -\frac{\kappa}{k^2}\left(\star \underline{u}(T) + T\Theta^\star\right). \tag{83}$$

As usual, ζ is the bulk viscosity and κ is the thermal conductivity – assumed constant in this expression.

It is convenient to use the orthonormal Cartan frame $\{u/k, \star u/k\}$. Then the metric reads ($u^2 = u_\mu u_\nu dx^\mu dx^\nu$):

$$ds^2 = \frac{1}{k^2}\left(-u^2 + \star u^2\right), \tag{84}$$

while the energy-momentum tensor takes the form:

$$T = T_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2k^2}\left((\varepsilon + \chi)(u + \star u)^2 + (\varepsilon - \chi)(u - \star u)^2\right) + \frac{1}{k^2}(p - \varepsilon + \tau) \star u^2. \tag{85}$$

Weyl Symmetry in Two-dimensional Fluids

Let us see the role played by Weyl symmetry in two dimensions. Under Weyl transformations

$$ds^2 \to \frac{ds^2}{\mathcal{B}^2}, \tag{86}$$

the velocity form components u_μ are traded for $u_\mu/\mathcal{B}$, the energy and heat densities have weight 2, and the local-equilibrium equation of state is conformal

$$\varepsilon = p, \tag{87}$$

which is accompanied by Stefan's law (σ is the Stefan-Boltzmann constant):

$$\varepsilon = \sigma T^2. \tag{88}$$

[20] The hodge-dual of a scalar is a two-form and would spell with a suffix star. Instead, $\Theta^\star$ is just another scalar.

[21] For any vector $\underline{v}$ and function f, $\underline{v}(f)$ stands for $v^\mu \partial_\mu f$. We remind the following identities: $d^\dagger df = -\Box f$ with $d^\dagger w = \star d \star w = -\nabla^\mu w_\mu$ and $df = \frac{1}{k^2}\left(\star \underline{u}(f) \star u - \underline{u}(f) u\right)$, $\star df = \frac{1}{k^2}\left(\star \underline{u}(f) u - \underline{u}(f) \star u\right)$.

Hence, the trace of the energy-momentum tensor is τ. In the absence of anomalies it vanishes and $T_{\mu\nu}$ is invariant under (86).[22] If τ is non-vanishing, the fluid is not conformal and τ is an anomalous weight-2 quantity. We can now proceed in the exact same way as we did in the three-dimensional case, and introduce Weyl-covariant derivatives and curvature objects. We still report here the analysis since some differences persist due to the different dimensionality, and the fact that we can package here the transverse direction in a very elucidating manner.

We thus introduce a Weyl connection one-form

$$A = \frac{1}{k^2}\left(a - \Theta u\right) = \frac{1}{k^2}\left(\Theta^\star \star u - \Theta u\right), \tag{89}$$

which transforms as $A \to A - d\ln\mathcal{B}$. Ordinary covariant derivatives ∇ are traded for Weyl covariant ones $\mathcal{D} = \nabla + w\,A$, w being the conformal weight of the tensor under consideration. On tensor v_μ and of scalar function Φ of weight w it acts as:

$$\mathcal{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho, \tag{90}$$

$$\mathcal{D}_\nu \Phi = \partial_\nu \Phi + w A_\nu \Phi. \tag{91}$$

As before, this covariant derivative is metric-compatible with commutator:[23]

$$\mathcal{D}_\rho g_{\mu\nu} = 0, \tag{92}$$

$$\left(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu\right) f = w f F_{\mu\nu}, \tag{93}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{94}$$

is the Weyl-invariant field strength.

In two dimensions we can dualize it to a weight-2 scalar

$$F = \star dA = \eta^{\mu\nu}\partial_\mu A_\nu = \frac{1}{k^2}\left(\star \underline{u}(\Theta) - \underline{u}(\Theta^\star)\right). \tag{95}$$

Like in 3 dimensions, one can extract the various curvature tensors.

With respect to the Levi-Civita ones, the covariant Ricci tensor (weight-0) and the scalar (weight-2) curvatures read:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda - F_{\mu\nu}, \tag{96}$$

$$\mathcal{R} = R + 2\nabla_\mu A^\mu. \tag{97}$$

It turns out that $R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda$ vanishes identically. Hence

$$\mathcal{R} = 0 \Leftrightarrow R = 2d^\dagger A \quad \text{and} \quad \mathcal{R}_{\mu\nu} = -F_{\mu\nu}. \tag{98}$$

The ordinary scalar curvature has a weight-2 anomalous transformation

$$R \to \mathcal{B}^2\left(R + 2\Box \ln \mathcal{B}\right) \tag{99}$$

(the box operator is here referring to the metric before the Weyl transformation).

Using these tools as well as the identity

$$\nabla^\mu T_{\mu\nu} = \mathcal{D}^\mu T_{\mu\nu} - A_\nu T^\mu{}_\mu, \tag{100}$$

the general fluid equations (74) with $\varepsilon = p$, projected on the light-cone directions $u \pm \star u$ acquires a simple form:[24]

$$\left(u^\mu + * u^\mu\right)\mathcal{D}_\mu\left(\varepsilon + \chi\right) + \left(u^\mu - \star u^\mu\right) f_\mu = -\Theta\tau - \Theta^\star\tau - \star\underline{u}(\tau), \tag{101}$$

$$\left(u^\mu - \star u^\mu\right)\mathcal{D}_\mu\left(\varepsilon - \chi\right) + \left(u^\mu + \star u^\mu\right) f_\mu = -\Theta\tau + \Theta^\star\tau + \star\underline{u}(\tau). \tag{102}$$

Equivalently:

$$d\left(\sqrt{\varepsilon + \chi + \tau/2}(u + \star u)\right) + \frac{1}{2\sqrt{\varepsilon + \chi + \tau/2}}(u - \star u)\wedge\star\left(f - \frac{1}{2}d\tau\right) = 0, \tag{103}$$

$$d\left(\sqrt{\varepsilon - \chi + \tau/2}(u - \star u)\right) - \frac{1}{2\sqrt{\varepsilon - \chi + \tau/2}}(u + \star u)\wedge\star\left(f - \frac{1}{2}d\tau\right) = 0. \tag{104}$$

[22] In general $T_{\mu\nu}$ has weight $d-1$ under Weyl. That is why it has weight 1 in 3 boundary dymensions ($d = 2$) and weight 0 in 2 boundary dimensions ($d = 1$).

[23] We remind that useful informations on the Weyl geometry are stored in Appendix A.

[24] Notice that any congruence with $w = -1$ in two dimensions obeys $\mathcal{D}_\mu u_\nu = \nabla_\mu u_\nu + \frac{1}{k^2}u_\mu a_\nu - \Theta\Delta_{\mu\nu} = 0$ due to the absence of shear and vorticity, and similarly $\mathcal{D}_\mu \star u_\nu = 0$.

Hydrodynamic Frames

In two dimensions, thanks to the fact that the orthogonal subspace to the fluid velocity is again a one-dimensional space, we have a very powerful control on the theory. In particular, changing hydrodynamic frame, i.e. fluid velocity field while keeping T unchanged, amounts just to perform an arbitrary local Lorentz transformation on the Cartan frame

$$\begin{pmatrix} u' \\ \star u' \end{pmatrix} = \begin{pmatrix} \cosh\psi(x) & \sinh\psi(x) \\ \sinh\psi(x) & \cosh\psi(x) \end{pmatrix} \begin{pmatrix} u \\ \star u \end{pmatrix}, \tag{105}$$

or for the null directions $u' \pm \star u' = (u \pm \star u)\, e^{\pm\psi}$.

This affects the Weyl connection and Weyl curvature scalar

$$A' = A - \star d\psi \tag{106}$$
$$F' = F + \Box\psi. \tag{107}$$

By construction, the transformation (105) has to keep the energy-momentum tensor invariant. This happens provided the energy density and the heat density transform appropriately. Imposing also that in the new frame $\varepsilon' = p'$, we conclude that

$$\begin{pmatrix} \varepsilon' \\ \chi' \end{pmatrix} = \begin{pmatrix} \cosh 2\psi(x) & -\sinh 2\psi(x) \\ -\sinh 2\psi(x) & \cosh 2\psi(x) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \chi \end{pmatrix} + \tau \sinh\psi(x) \begin{pmatrix} \sinh\psi(x) \\ -\cosh\psi(x) \end{pmatrix}, \tag{108}$$

while, due to the invariance of the trace,

$$\tau' = \tau. \tag{109}$$

Equivalently one can use $\sqrt{\left(\varepsilon' \pm \chi' + \frac{\tau'}{2}\right)} = \sqrt{\left(\varepsilon \pm \chi + \frac{\tau}{2}\right)}\, e^{\mp\psi}$.

The energy-momentum tensor can be diagonalized with a specific local Lorentz transformation. This means that in this frame there is no heat dissipation. By definition, this is the Landau-Lifshitz frame, where the heat current χ_{LL} is vanishing. We find

$$T = \frac{\varepsilon_{\mathsf{LL}}}{k^2} u_{\mathsf{LL}}^2 + \frac{\varepsilon_{\mathsf{LL}} + \tau}{k^2} \star u_{\mathsf{LL}}^2 \tag{110}$$

since $\tau_{\mathsf{LL}} = \tau$ and $\chi_{\mathsf{LL}} = 0$.

The latter condition allows to find the local boost towards the Landau-Lifshitz frame

$$e^{4\psi_{\mathsf{LL}}} = \frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}. \tag{111}$$

With this, one finds the Landau-Lifshitz energy density

$$\varepsilon_{\mathsf{LL}} = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} - \frac{\tau}{2}. \tag{112}$$

It exhibits an upper bound for χ^2, $\chi^2_{\mathsf{max}} = (\varepsilon + \tau/2)^2$. We interpret it as a translation of causality and unitarity properties of the underlying microscopic field theory.

The eigenvalue[25] $\varepsilon_{\mathsf{LL}}$ is supported by the time-like eigenvector

$$u_{\mathsf{LL}} = \frac{1}{2}\left(\left(\frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}\right)^{1/4}(u + \star u) + \left(\frac{\varepsilon - \chi + \tau/2}{\varepsilon + \chi + \tau/2}\right)^{1/4}(u - \star u)\right), \tag{113}$$

whereas

$$\varepsilon^\star_{\mathsf{LL}} = \varepsilon_{\mathsf{LL}} + \tau = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} + \frac{\tau}{2} \tag{114}$$

is the eigenvalue along the space-like eigenvector $\star u_{\mathsf{LL}}$.

The fluid equations (103) and (104) are recast as follows

$$2\sqrt{\varepsilon_{\mathsf{LL}}}\, d^\dagger\left(\sqrt{\varepsilon_{\mathsf{LL}}}\, u_{\mathsf{LL}}\right) - u_{\mathsf{LL}} \cdot f - \Theta_{\mathsf{LL}}\tau = 0\,, \tag{115}$$
$$2\sqrt{\varepsilon^\star_{\mathsf{LL}}}\, d^\dagger\left(\sqrt{\varepsilon^\star_{\mathsf{LL}}} \star u_{\mathsf{LL}}\right) + \star u_{\mathsf{LL}} \cdot f + \Theta^\star_{\mathsf{LL}}\tau = 0\,. \tag{116}$$

[25] We make for simplicity the implicit assumption that the energy density is positive. This needs not be true, however, and the holographic fluid dual to global AdS_3 has indeed negative energy.

For a non-anomalous conformal fluid, and at zero external force $f = 0$, the forms $\sqrt{\varepsilon \pm \chi}(u \pm \star u)$ are closed, and can be used to define a privileged light-cone coordinate system, adapted to the fluid configuration. In this specific case, the on-shell Weyl scalar curvature reads

$$F = -\frac{1}{2}\Box \ln \sqrt{\frac{\varepsilon + \chi}{\varepsilon - \chi}}. \tag{117}$$

In this case the frame transformation (105) acts on the energy and heat densities as a spin-two electric-magnetic boost, the energy being electric and the heat magnetic.

Light-Cone vs Randers-Papapetrou Parametrizations

Light-Cone Every two-dimensional metric is amenable by diffeomorphisms to a conformally flat form:

$$ds^2 = e^{-2\omega} dx^+ dx^-. \tag{118}$$

With this choice and our conventions $g_{+-} = {}^1\!/\!{}_2 e^{-2\omega}$, $\eta_{+-} = {}^1\!/\!{}_2 e^{-2\omega}$, $\eta^{+-} = -2e^{2\omega}$, $\eta_+{}^+ = 1$, $\eta_-{}^- = -1$. Notice also that $\star (dx^+ \wedge dx^-) = \eta^{+-} = -2e^{2\omega}$. Time and space are defined as $x^\pm = \boldsymbol{x} \pm kt$. The conformal factor ω is an arbitrary function of x^+ and x^-.

Any normalized congruence has the following form:

$$u = u_+ dx^+ + u_- dx^- \quad \Leftrightarrow \quad \star u = -u_+ dx^+ + u_- dx^-, \tag{119}$$

where $u_\pm$, functions of x^+ and x^-, are related by the normalization condition

$$u_+ u_- = -\frac{k^2}{4} e^{-2\omega}. \tag{120}$$

Without loss of generality, we can parameterize the velocity field as

$$u_+ = -\frac{k}{2} e^{-\omega} \sqrt{\xi}, \quad u_- = \frac{k}{2} e^{-\omega} \frac{1}{\sqrt{\xi}}, \tag{121}$$

where $\xi = \xi(x^+, x^-)$ is defined as the ratio

$$\xi = -\frac{u_+}{u_-}. \tag{122}$$

The choice $\xi = 1$ corresponds to a co-moving fluid because in this case $u = -k^2 e^{-\omega} dt$.

For the congruence at hand

$$\Theta \pm \Theta^\star = \pm 2k e^{2\omega} \partial_\pm e^{-\left(\omega \pm \ln \sqrt{\xi}\right)}. \tag{123}$$

Moreover:

$$A = -d\omega + \star d \ln \sqrt{\xi} \quad \text{and} \quad F = -\Box \ln \sqrt{\xi} = -2e^{2\omega} \partial_+ \partial_- \ln \xi, \tag{124}$$

whereas the Levi-Civita scalar curvature reads

$$R = 2\Box\omega = 8e^{2\omega} \partial_+ \partial_- \omega. \tag{125}$$

In this frame $\{dx^+, dx^-\}$, the components of a general energy-momentum tensor with $\epsilon = p$, are

$$\begin{gathered} T_{++} = \frac{\xi}{2}\left(\varepsilon - \chi + \frac{\tau}{2}\right) e^{-2\omega}, \quad T_{--} = \frac{1}{2\xi}\left(\varepsilon + \chi + \frac{\tau}{2}\right) e^{-2\omega}, \\ T_{+-} = T_{-+} = \frac{\tau}{4} e^{-2\omega}. \end{gathered} \tag{126}$$

For a conformal fluid $\tau = 0$, thus $T_{+-} = 0 = T_{-+}$ and

$$(\varepsilon + \chi)(\varepsilon - \chi) = 4e^{4\omega} T_{++} T_{--}, \quad \frac{\varepsilon + \chi}{\varepsilon - \chi} = \frac{T_{--}}{T_{++}} \xi^2. \tag{127}$$

In the latter case, and in the absence of external forces, the forms (103) and (104) are closed, which implies $(\varepsilon-\chi)e^{-2\omega}\xi$ being locally a function of x^+ and $(\varepsilon+\chi)\frac{e^{-2\omega}}{\xi}$ of x^-. Observe that in the Landau-Lifshitz frame ($\chi_{\text{LL}}=0$)

$$\xi_{\text{LL}}^2=\frac{T_{++}}{T_{--}}, \quad \varepsilon_{\text{LL}}^2=4e^{4\omega}T_{++}T_{--}. \tag{128}$$

In this frame, on-shell, F vanishes.

Moving from a given hydrodynamic frame to another by a local Lorentz boost, amounts to perform the following transformation on the function ξ

$$\xi(x^+,x^-)\to\xi'(x^+,x^-)=e^{-2\psi(x^+,x^-)}\xi(x^+,x^-). \tag{129}$$

Randers-Papapetrou The light-cone frame is not well suited for the Carrollian limit, which is the subject of Section 3. Carrollian fluid dynamics is elegantly reached in the Randers-Papapetrou frame, where (here we work with coordinates t and x. The latter is not reported in bold for it is one-dimensional here)

$$ds^2=-k^2\left(\Omega dt-b_x dx\right)^2+a dx^2 \tag{130}$$

with all three functions of the coordinates t and x.

A generic velocity vector field $\underline{u}$ reads:

$$\underline{u}=\gamma\left(\partial_t+v^x\partial_x\right). \tag{131}$$

It is convenient to parametrize the velocity v^x as[26]

$$v^x=\frac{k^2\Omega\beta^x}{1+k^2\beta\cdot b}\Leftrightarrow\beta^x=\frac{v^x}{k^2\Omega\left(1-\frac{v^x b_x}{\Omega}\right)} \tag{132}$$

with Lorentz factor

$$\gamma=\frac{1+k^2\beta\cdot b}{\Omega\sqrt{1-k^2\beta^2}}. \tag{133}$$

The velocity form and its dual read:

$$u=-\frac{k^2}{\sqrt{1-k^2\beta^2}}\left(\Omega dt-\left(b_x+\beta_x\right)dx\right), \quad \star u=k\sqrt{a}\Omega\gamma\left(dx-v^x dt\right), \tag{134}$$

while the corresponding vector is

$$\star\,\underline{u}=\frac{k}{\sqrt{a}\sqrt{1-k^2\beta^2}}\left(\frac{b_x+\beta_x}{\Omega}\partial_t+\partial_x\right). \tag{135}$$

We can determine the form of the heat current q, which must be proportional to $\star u$, in terms of a single component q_x. We find

$$\chi=\frac{q_x}{k\sqrt{a}\Omega\gamma}=\frac{q^x\sqrt{a}\sqrt{1-k^2\beta^2}}{k}. \tag{136}$$

Similarly, for the viscous stress tensor

$$\tau=\frac{\tau_{xx}}{a\Omega^2\gamma^2}=\tau^{xx}a\left(1-k^2\beta^2\right). \tag{137}$$

Performing a local Lorentz boost (105) on the hydrodynamic frame does not affect the geometric objects Ω, b_x or a, and is thus entirely captured by the transformation of the vector $\underline{\beta}=\beta^x\partial_x$. This is expected, for $\underline{\beta}$ is the kinetic quantity, as we will fully unravel in Section 3.

Parameterizing the boost in terms of a spatial vector $\underline{B}=B^x\partial_x$ as

$$\cosh\psi=\Gamma=\frac{1}{\sqrt{1-k^2B^2}}, \quad \sinh\psi=\Gamma k\sqrt{a}B^x=\frac{k\sqrt{a}B^x}{\sqrt{1-k^2B^2}}, \tag{138}$$

[26] Notice that $\beta_x+b_x=-\frac{\Omega u_x}{ku_0}$. We define as usual $b^x=a^{xx}b_x$, $\beta_x=a_{xx}\beta^x$, $v_x=a_{xx}v^x$ with $a_{xx}={}^1\!/_{a^{xx}}=a$, $b^2=b_xb^x$, $\beta^2=\beta\cdot\beta=\beta_x\beta^x$ and $b\cdot\beta=b_x\beta^x$.

we get:

$$\underline{\beta}' = \frac{\underline{\beta} + \underline{B}}{1 + k^2 \underline{\beta} \cdot \underline{B}}, \tag{139}$$

as expected from the velocity rule composition in special relativity. Moreover

$$\varepsilon' = \frac{1}{1 - k^2 B^2} \left(\left(1 + k^2 B^2\right) \varepsilon - k\sqrt{a} B^x 2\chi + k^2 B^2 \tau \right), \tag{140}$$

$$\chi' = \frac{1}{1 - k^2 B^2} \left(\left(1 + k^2 B^2\right) \chi - k\sqrt{a} B^x (2\varepsilon + \tau) \right). \tag{141}$$

Using (136) and (137), we eventually reach

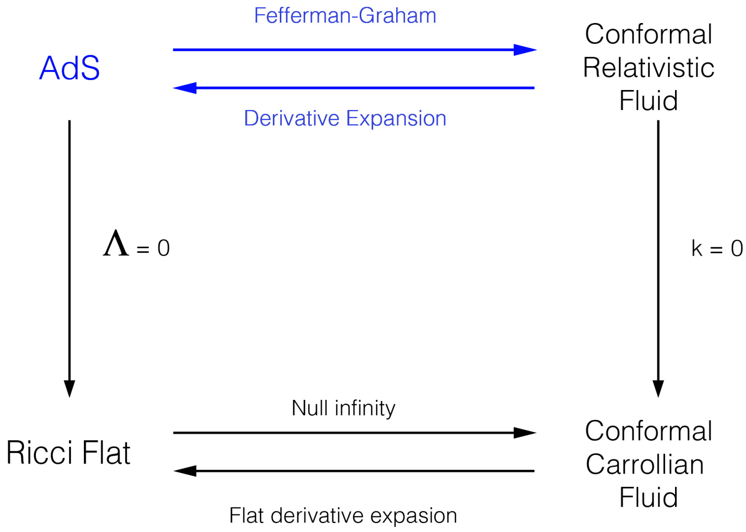
$$\frac{q'_x}{\sqrt{a}} = \left(\left(1 + k^2 B^2\right) \chi - k\sqrt{a} B^x (2\varepsilon + \tau) \right) k \frac{\left(1 + k^2 \left(\beta \cdot B + (\beta + B) \cdot b\right)\right)}{\left(1 - k^2 \beta^2\right)^{1/2} \left(1 - k^2 B^2\right)^{3/2}}, \tag{142}$$

$$\frac{\tau'_{xx}}{a} = \tau \frac{\left(1 + k^2 \left(\beta \cdot B + (\beta + B) \cdot b\right)\right)^2}{\left(1 - k^2 \beta^2\right) \left(1 - k^2 B^2\right)}. \tag{143}$$

These quantities will become useful when trying to reach Carrollian hydrodynamics in two dimensions.

2.2 Bulk Gravity

We are now ready to discuss the properties of the bulk spacetime and how to reconstruct a given solution of Einstein equations from the boundary fluid data, which is the blue sector of our square:



We will first of all review the Fefferman-Graham (FG) expansion, then explain the salient features of the derivative expansion and justify why it is more useful from the hydrodynamic viewpoint. We will then specialize in four and three bulk dimensions, where we will be able to fully characterize bulk solutions from the boundary. This latter task remains an open question in bulk dimension five and higher.

2.2.1 Fefferman-Graham vs Derivative Expansion

The FG expansion is at the core of holography. It is based on a theorem by Fefferman and Graham [77, 78] stating that the metric of a locally asymptotically AdS$_{d+2}$ geometry can be put in the form (k is the inverse of the AdS radius)

$$ds^2 = \frac{dz^2}{k^2 z^2} + h_{\mu\nu}(z; x) dx^\mu dx^\nu. \tag{144}$$

The conformal boundary is a constant-z hypersurface at $z = 0$ in these coordinates. To obtain this form, one has used up all of the diffeomorphism invariance, apart from residual transformations of the $x^\mu \to x'^\mu(x)$, which of course would change the components of $h_{\mu\nu}$ in general. Near $z = 0$, $h_{\mu\nu}(z; x)$ may be expanded

$$h_{\mu\nu}(z;x) = \frac{1}{k^2 z^2}\left[g^{(0)}_{\mu\nu}(x) + k^2 z^2 g^{(2)}_{\mu\nu}(x) + k^4 z^4 g^{(4)}_{\mu\nu}(x) + ...\right] + (kz)^{d-1}\left[T^{(0)}_{\mu\nu}(x) + k^2 z^2 T^{(2)}_{\mu\nu}(x) + ...\right]. \quad (145)$$

A particularly interesting feature of this gauge is that every bulk solution can be obtained by specifying the boundary metric $g^{(0)}_{\mu\nu}$ and energy-momentum tensor $T^{(0)}_{\mu\nu}$. These two objects are for the bulk metric the same as the source and the vev were for the bulk scalar field we saw at the beginning of Section 2. All the subleading objects in the two series are written on-shell as a function of these 2 objects and their derivative. In this sense, $g^{(0)}$ and $T^{(0)}$ can be interpreted has initial position and momentum for gravity. Einstein equations in the bulk then express all the subleading terms in the expansions.

The drawbacks of this gauge are mainly three. Firstly, it is always (except for empty AdS) an infinite expansion. Secondly – and fundamentally for hydrodynamics – this gauge is not suitable for Weyl transformations (see section 5). Lastly, it does not admit a smooth $k \to 0$ limit. This last remark is crucial in our work. Indeed, we will see that a Carrollian limit in hydrodynamics corresponds to a $k \to 0$ limit in the gravitational bulk. We therefore need to choose a gauge that smoothly allows such a limit.

These weakness will be cured in the derivative expansion. Nonetheless, the very merit of the FG expansion is its mathematical robustness, guaranteed by the fact that it is a proved theorem that every locally asymptotically AdS metric can be written in this way.

More recently, fluid/gravity correspondence has provided an alternative to FG, known as derivative expansion [24–27]. It is inspired from the fluid derivative expansion, and is implemented in Eddington-Finkelstein (EF) coordinates. The metric of an Einstein spacetime is expanded in a lightlike direction and the information on the boundary fluid is made available in a slightly different manner, involving explicitly a velocity field whose derivatives set the order of the expansion. Conversely, the boundary fluid data, including the fluid congruence, allow to reconstruct an exact bulk Einstein spacetime.

This reconstruction is heavily based on Weyl invariance. Indeed, it treats the null coordinate (and derivatives) as an expansion parameter and associates at every order in the expansion the possible Weyl covariant boundary terms with compatible Weyl weight. Although less robust mathematically, the derivative expansion has several advantages over FG:

- it can be resummed leading to algebraically special Einstein spacetimes in a closed form,
- boundary geometrical terms appear packaged at specific orders in the derivative expansion, which makes their classification easier
- the spacetime metric is expanded along a null rather than a spatial direction. This is ultimately the reason why it admits a consistent limit of vanishing scalar curvature. Having a null holographic direction is therefore crucial.

Hence, it appears to be applicable to Ricci-flat spacetimes and emerges as a valuable tool for setting up flat holography. Such a smooth behavior is not generic, as in most coordinate systems switching off the scalar curvature for a Einstein space leads to plain Minkowski spacetime.

The velocity field of a relativistic fluid is not a physical observable, and therefore, as already discussed, it can be redefined by a hydrodynamic field redefinition. Nonetheless it appears explicitly in the derivative expansion. This is in contrast to what happens in the FG expansion, where the Einstein bulk reconstruction is solely based on the boundary metric and the boundary energy-momentum tensor.

As we will see, the derivative expansion moves the different degrees of freedom in different places. In fact, the energy-momentum tensor will satisfy integrability conditions that relates it to the boundary geometry. Hence, the fluid velocity, although immaterial in the boundary theory, represents for the bulk reconstruction an important piece of information, for it appears explicitly in the derivative expansion (it actually organizes the latter).

Following the above logic, it is clear that when writing the derivative expansion, some implicit gauge choice may be made, partly locking the form of the velocity. A frame redefinition would change the fluid velocity. If altogether allowed from the bulk point of view, this is expected to be reabsorbed by some appropriate bulk diffeomorphism.

Analyzing the role of the velocity field in the fluid/gravity derivative expansion is not an easy task. In particular, the integrability conditions that we mentioned set a relationship that involves the latter. This could blur the boundary frame redefinition freedom. We should therefore be prudent and do not assume any specific a priori fluid gauge. The exact way in which a boundary fluid redefinition affects the bulk reconstruction procedure is still an open question and argument of recent investigation.

Another important open question is to write down the bulk reconstructed metric in five dimensions and higher, in the derivative expansion gauge. Some progress have been recently made [199], but it remains an unsolved issue so far. Our reconstruction works perfectly in four and three bulk dimensions, which is the argument of the next sections.

2.2.2 Bulk Reconstruction in Four Dimensions

As mentioned, the fluid/gravity correspondence is historically based on the holographic coordinate $r = {}^1\!/_z$, which therefore places the boundary at $r \to \infty$. The logic here is to write all the possible Weyl covariant term with the correct weight at a given order in the r expansion.

This exercise, in four dimensions, results in:

$$\begin{aligned} ds^2_{\text{bulk}} &= 2\frac{u}{k^2}(dr + rA) + r^2 ds^2 + \frac{S}{k^4} \\ &+\frac{u^2}{k^4 r^2}\left(1 - \frac{1}{2k^4 r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{8\pi G_N T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^4}\right) \\ &+ \text{terms with } \sigma,\, \sigma^2,\, \nabla\sigma,\, \ldots + O\left(\mathcal{D}^4 u\right). \end{aligned} \tag{146}$$

In this expression

- S is a Weyl-invariant tensor:

$$S = S_{\mu\nu}dx^\mu dx^\nu = -2u\mathcal{D}_\nu \omega^\nu{}_\mu dx^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu} dx^\mu dx^\nu - u^2\frac{\mathcal{R}}{2}, \tag{147}$$

  compatible with the fact that it appears at order 1 in the r expansion;

- the boundary metric is parametrized à la Randers-Papapetrou:

$$ds^2 = -k^2\left(\Omega dt - b_i dx^i\right)^2 + a_{ij}dx^i dx^j, \tag{148}$$

  this parametrization is important in the following and will be intensively discussed, for the moment all what matters is that every metric can be parametrized in this way;

- the boundary conformal fluid velocity field and the corresponding one form are

$$\underline{u} = \frac{1}{\Omega}\partial_t \quad \Leftrightarrow \quad u = -k^2\left(\Omega dt - b_i dx^i\right), \tag{149}$$

  i.e. the fluid is at rest in the frame associated with the coordinates in (148) – this is not a limitation, as one can always choose a local frame where the fluid is at rest, in which the metric reads (148) (with Ω, b_i and a_{ij} functions of all coordinates);

- $\omega_{\mu\nu}$ is the vorticity of u as given in (30), which reads:

$$\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{k^2}{2}\left(\partial_i b_j + \frac{1}{\Omega}b_i\partial_j\Omega + \frac{1}{\Omega}b_i\partial_t b_j\right)dx^i \wedge dx^j; \tag{150}$$

- using this result

$$\frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta} = \gamma^2 = \frac{1}{2}a^{ik}a^{jl}\left(\partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]}\right)\left(\partial_{[k}b_{l]} + \frac{1}{\Omega}b_{[k}\partial_{l]}\Omega + \frac{1}{\Omega}b_{[k}\partial_t b_{l]}\right); \tag{151}$$

- the expansion (28) and acceleration (27) are

$$\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \tag{152}$$

$$a = k^2\left(\partial_i \ln\Omega + \frac{1}{\Omega}\partial_t b_i\right)dx^i, \tag{153}$$

leading to the Weyl connection (52)

$$A = \frac{1}{\Omega}\left(\partial_i\Omega + \partial_t b_i - \frac{1}{2}b_i\partial_t \ln\sqrt{a}\right)dx^i + \frac{1}{2}\partial_t \ln\sqrt{a}dt\,, \tag{154}$$

with a the determinant of a_{ij};

- $\frac{1}{k^2}T_{\mu\nu}u^\mu u^\nu$ is the energy density ε of the fluid, and in Randers-Papapetrou $q_0 = \tau_{00} = \tau_{0i} = \tau_{i0} = 0$ due to (24) and (149);
- $\frac{1}{2k^4}C_{\lambda\mu}u^\lambda\eta^{\mu\nu\sigma}\omega_{\nu\sigma} = c\gamma$, where we have used (46) and (68), and similarly $c_0 = c_{00} = c_{0i} = c_{i0} = 0$;
- σ, σ^2, $\nabla\sigma$ stand for the shear of u and combinations of it, as computed from (29):

$$\sigma = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right)dx^i dx^j. \tag{155}$$

We close this paragraph stressing again the importance of Weyl covariance in the reconstruction of the bulk line element.

Resummation and Exact Einstein Spacetimes in Closed Form

In order to further probe the derivative expansion (146), we will impose the fluid velocity congruence shearless. This choice has the virtue of reducing considerably the number of terms compatible with conformal invariance in (146), and potentially making this expansion resummable, thus leading to an Einstein metric written in a closed form.

Nevertheless, this shearless condition together with integrability conditions, reduce the class of Einstein spacetimes that can be reconstructed holographic to the algebraically special ones [33, 34, 36].[27] Going beyond this class is an open problem. This result should be stressed: we are constraining the boundary theory, but a posteriori we know that the subclass of Einstein spaces we are reaching (algebraically special), is general enough to include, for instance, all known black hole solutions. Notice that this shearless condition is also potentially not harmless regarding hydrodynamic frames. In fact, a hydrodynamic frame transformation may return a shearfull velocity starting from a shearless one.

By direct inspection (following e.g. [24]), it is tempting to try a resummation of (146) using the following substitution:

$$1 - \frac{\gamma^2}{r^2} \to \frac{r^2}{\rho^2} \tag{156}$$

with γ defined in (151) and

$$\rho^2 = r^2 + \gamma^2. \tag{157}$$

The success of this resummation is not a priori guaranteed. It is a posteriori confirmed thanks to the fact that we are able to write a large spectrum of bulk solutions in this form.

With this procedure, we postulate the resummed expansion

$$\boxed{ds^2_{\text{res. Einstein}} = 2\frac{u}{k^2}(dr + rA) + r^2 ds^2 + \frac{S}{k^4} + \frac{u^2}{k^4\rho^2}\left(8\pi G_N \varepsilon r + c\gamma\right),} \tag{158}$$

which is indeed written in a closed form, since all subleading piece in r-expansion have been resummed.

It is evident that the shearless condition was a very powerful tool. Under the conditions listed below, the metric (158) defines the line element of an exact Einstein space with $\Lambda = -3k^2$.

1. The congruence u is shearless. This requires (see (155))

$$\partial_t a_{ij} = a_{ij}\partial_t \ln\sqrt{a}. \tag{159}$$

It is equivalent to ask that the two-dimensional spatial section defined at every time t and equipped with the metric $d\ell^2 = a_{ij}dx^i dx^j$ is conformally flat. This may come as a surprise because every two-dimensional metric is conformally flat. However, a_{ij} generally depends on space $\boldsymbol{x}$ and time t, and the transformation

[27] Appendix B touches upon Petrov classification and Goldberg-Sachs theorem. We explain there the meaning of algebraically special.

required to bring it in a form proportional to the flat-space metric might depend on time. This would spoil the three-dimensional structure (148) and alter the a priori given u. Hence, $d\ell^2$ is conformally flat within the three-dimensional spacetime (148) under the condition that the transformation used to reach the explicit conformally flat form be of the type $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$. This exists if and only if (159) is satisfied.

Under this condition, one can always choose $\zeta = \zeta(\boldsymbol{x})$, $\bar{\zeta} = \bar{\zeta}(\boldsymbol{x})$ such that

$$d\ell^2 = a_{ij}\, dx^i dx^j = \frac{2}{P^2} d\zeta d\bar{\zeta} \tag{160}$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function. Even though this does not hold for arbitrary $\underline{u} = \frac{\partial_t}{\Omega}$, one can show that there exists always a congruence for which it does [45].[28]

2. The heat current of the boundary fluid (21) is identified with the transverse-dual of the Cotton current defined in (67) and (70), via the u-transverse duality defined in (48):

$$q_\mu = \frac{1}{8\pi G_N} \tilde{\eta}^\nu{}_\mu c_\nu = \frac{1}{8\pi G_N} \tilde{\eta}^\nu{}_\mu \tilde{\eta}^{\rho\sigma} \mathcal{D}_\rho \left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma}\right), \tag{161}$$

where we used (72) in the last expression. In holomorphic coordinates[29]

$$q = \frac{i}{8\pi G_N}\left(c_\zeta d\zeta - c_{\bar{\zeta}} d\bar{\zeta}\right). \tag{162}$$

3. The viscous stress tensor of the boundary conformal fluid (21) is identified with the transverse-dual of the Cotton stress tensor defined in (67) and (70). Following the same pattern as for the heat current, we obtain:

$$\begin{aligned} \tau_{\mu\nu} &= -\frac{1}{8\pi G_N k^2} \tilde{\eta}^\rho{}_\mu c_{\rho\nu} \\ &= \frac{1}{8\pi G_N k^2}\left(-\frac{1}{2} u^\lambda \tilde{\eta}_{\mu\nu} \tilde{\eta}^{\rho\sigma} + \tilde{\eta}^\lambda{}_\mu \left(k\eta_\nu{}^{\rho\sigma} - u_\nu \tilde{\eta}^{\rho\sigma}\right)\right) \mathcal{D}_\rho \left(\mathcal{S}_{\lambda\sigma} + F_{\lambda\sigma}\right), \end{aligned} \tag{163}$$

where we also used (73) in the last equality. In complex coordinates:

$$\tau = -\frac{i}{8\pi G_N k^2}\left(c_{\zeta\zeta} d\zeta^2 - c_{\bar{\zeta}\bar{\zeta}} d\bar{\zeta}^2\right). \tag{164}$$

4. The energy-momentum tensor defined in (21) with $p = \varepsilon/2$, heat current as in (161) and viscous stress tensor as in (163) must be conserved

$$\nabla_\mu T^{\mu\nu} = 0 \tag{165}$$

These are differential constraints that from a bulk perspective can be thought of as a generalization of the Gauss law.

Identifying parts of the energy-momentum tensor with the Cotton tensor may be viewed as setting integrability conditions, similar to the electric-magnetic duality conditions in electromagnetism, or in Euclidean gravitational dynamics. As opposed to the latter, it is here implemented in a rather unconventional manner, on the conformal boundary, via the transverse-to-u duality $\tilde{\eta}_{\mu\nu}$. Notice that in the FG gauge $T_{\mu\nu}$ and the boundary metric encode all the informations. Here however the derivative expansion shuffles the degrees of freedom differently, and thus integrability conditions arise.

As examples demonstrate, the Cotton tensor contains the gravitational magnetic part of the informations, like the NUT charge, while the energy-momentum tensor the electric one, like the black hole mass. These integrability conditions strikingly resemble the gravitational counterpart of electric-magnetic duality. The exact form of this duality is settle to guarantee the a posteriori success of the reconstruction. A deeper investigation is under study on these relationships and what they infer e.g. on the charges.

It is important to emphasize that the conservation equations concern all boundary data. On the fluid side the only remaining unknown piece is the energy density $\varepsilon(x)$, whereas for the boundary metric $\Omega(x)$, $b_i(x)$ and $a_{ij}(x)$

[28] This should again ring a bell about our discussion on the hydrodynamic frame: there is no reason for this $\underline{u}$ being the fluid velocity in a particular gauge and asking to be in a specific gauge to begin with could be incompatible.

[29] Orientation is chosen such that in the coordinate frame $\eta_{0\zeta\bar{\zeta}} = \sqrt{-g}\epsilon_{0\zeta\bar{\zeta}} = \frac{i\Omega}{P^2}$, where $x^0 = kt$. Thus $\tilde{\eta}^\zeta{}_\zeta = i$ and $\tilde{\eta}^{\bar{\zeta}}{}_{\bar{\zeta}} = -i$.

are available and must obey (165), together with $\varepsilon(x)$. Given these ingredients, (165) turns out to be precisely the set of equations obtained by demanding bulk Einstein equations be satisfied with the metric (158). This observation is at the heart of our analysis. We want to conclude the analysis with a recap of the road done here. The pattern to follow is the following:

- Parametrize the boundary metric in Randers-Papapetrou (148), the specific form of $\{\Omega, b_i, a_{ij}\}$ characterizes the solution.
- Choose the fluid velocity to be $\underline{u} = \frac{1}{\Omega}\partial_t$.
- Impose that the spatial part of the boundary satisfies the shearless condition (159).
- Compute the bulk metric (158).
- Impose boundary integrability for q (161) and τ (163).
- Build with these data the boundary energy-momentum tensor (21). At this point we are left with a bulk line element and an energy-momentum tensor written as a function of $\{\Omega, b_i, a_{ij}, \varepsilon\}$.
- Require the energy-momentum conservation (165). These are the bulk Einstein equations for the reconstructed metric with $\Lambda = -3k^2$.

Therefore, starting from a boundary metric and an adapted fluid, we reconstruct a bulk solution of AdS Einstein equations following the steps just depicted. A natural question arise: what is the domain of applicability of this procedure? Stated differently, we next wonder which bulk solution can be reconstructed in this way. The answer will turn out to be every algebraically special bulk solution, thanks to the Goldberg-Sachs theorem (see Appendix B for a recap on the latter and the proof of this statement).

The Bulk Algebraic Structure

We would like to return again on the crucial identification of the non-perfect energy-momentum tensor pieces with the corresponding Cotton components by transverse dualization. What does motivate these choices? The answer to this question is rooted to the Weyl tensor and to the remarkable integrability properties its structure can provide to the system [50, 51].

Let us firstly notice that from the bulk perspective the vector $\underline{u}$, which is timelike in the boundary, is a manifestly null congruence associated with the vector ∂_r. One can show that this bulk congruence is also geodesic and shearfree. Therefore, accordingly to the generalizations of the Goldberg-Sachs theorem, if the bulk metric (158) is an Einstein space, then it is algebraically special, i.e. of Petrov type II, III, D and N.[30]

Owing to the close relationship between the algebraic structure and the integrability properties of Einstein equations, it is clear why the absence of shear in the fluid congruence plays such an instrumental role in making the resummed expression (158) an exact Einstein space.

The structure of the bulk Weyl tensor makes it possible to go deeper in foreseeing how the boundary data should be tuned in order for the resummation to be successful. Indeed the Weyl tensor, if packaged using the Atiyah-Singer decomposition, can be expanded for large-r, and the dominant term ($1/r^3$) gives the following combination of the boundary energy-momentum and Cotton tensors:

$$T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{i}{8\pi G_N k} C_{\mu\nu}, \tag{166}$$

satisfying a conservation equation, analogue to (165)

$$\nabla^\mu T^{\pm}_{\mu\nu} = 0. \tag{167}$$

For algebraically special spaces, these complex conjugate tensors simplify considerably, and this suggests the transverse duality enforced between the Cotton and the energy-momentum non-perfect components. Using (162) and (164), we find indeed for the tensor T^+ in complex coordinates:

$$T^+ = \left(\varepsilon + \frac{ic}{8\pi G_N}\right)\left(\frac{u^2}{k^2} + \frac{1}{2}d\ell^2\right) + \frac{i}{4\pi G_N k^2}\left(2c_\zeta d\zeta u - c_{\zeta\zeta} d\zeta^2\right), \tag{168}$$

[30] Appendix B will be useful throughout all this section.

and similarly for T^- obtained by complex conjugation with

$$\varepsilon_\pm = \varepsilon \pm \frac{ic}{8\pi G_N}. \tag{169}$$

The bulk Weyl tensor and consequently the Petrov class of the bulk Einstein space are encoded in the three complex functions of the boundary coordinates: ε_+, c_ζ and $c_{\zeta\zeta}$. The proposed resummation procedure, based on boundary relativistic fluid dynamics of non-perfect fluids with heat current and stress tensor designed from the boundary Cotton tensor, allows to reconstruct all algebraically special four-dimensional Einstein spaces. We explain how in Appendix B.

The simplest correspond to a Cotton tensor of the perfect form [33]. The complete class of Plebański-Demiański family requires non-trivial b_i with two commuting Killing fields [37], while vanishing b_i without isometry leads to the Robinson-Trautman Einstein spaces [36, 60], which is the example we decide to treat in detail in the next section, to familiarize with the procedure previously outlined.

2.2.3 The Robinson-Trautman Example

Reconstruction

Consider the boundary metric

$$ds^2 = -k^2 dt^2 + \frac{2}{P^2} d\zeta d\bar{\zeta}. \tag{170}$$

This metric has $\Omega = 1$ and $b = 0$.

The vector ∂_t is hypersurface-orthogonal, and the normal hypersurfaces are constant-t sections. The Gaussian curvature of the latter is

$$K = \Delta \ln P \tag{171}$$

with $\Delta = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta$.

The Cotton tensor, computed using (65), reads:

$$C = i \begin{pmatrix} dt & d\zeta & d\bar{\zeta} \end{pmatrix} \begin{pmatrix} 0 & -\frac{k}{2}\partial_\zeta K & \frac{k}{2}\partial_{\bar{\zeta}} K \\ -\frac{k}{2}\partial_\zeta K & -\partial_t \left(\frac{\partial_\zeta^2 P}{kP} \right) & 0 \\ \frac{k}{2}\partial_{\bar{\zeta}} K & 0 & \partial_t \left(\frac{\partial_{\bar{\zeta}}^2 P}{kP} \right) \end{pmatrix} \begin{pmatrix} dt \\ d\zeta \\ d\bar{\zeta} \end{pmatrix}, \tag{172}$$

which is a real tensor.

Notice that we have no control on the frame in which the fluid is described, as the velocity field is the shearless congruence read off directly from the boundary metric (170) (see (149)):

$$u = -k^2 dt, \tag{173}$$

which has no vorticity, no acceleration but is expanding at a rate

$$\Theta = -2\partial_t \ln P. \tag{174}$$

We should stress that in this frame, the holographic fluid exhibits a finite number of corrections with respect to a perfect fluid, as the energy-momentum tensor is basically third-order in derivatives of geometric quantities. This is not surprising and it is a rather general feature of exact Einstein bulk spaces to lead to holographic fluid configurations which do not trigger all transport coefficients. Still, the kinematic state is non-trivial, and the absence of certain series of corrections in the energy-momentum tensor is really the signature of vanishing of the corresponding transport coefficients.

With respect to our general procedure, we have already defined the boundary metric and the velocity field. Incidentally, we readily see that our metric has a shearless spatial part. The next step is the computation of the bulk metric (158). For this, notice that $c = 0$ here. We obtain

$$ds^2_{\text{res. Einstein}} = -2dt(dr + H dt) + 2\frac{r^2}{P^2} d\zeta d\bar{\zeta} \tag{175}$$

with

$$2H = k^2 r^2 - 2r\partial_t \ln P + K - \frac{8\pi G_N \varepsilon}{r}. \tag{176}$$

Notice at this point that ε is an arbitrary function of the boundary coordinates, we have not yet imposed integrability conditions and Einstein equations, and this is the next step in our procedure. Using (161) and (163) we compute the dissipative tensors of the boundary energy-momentum tensor to be

$$\begin{aligned} q &= -\frac{1}{16\pi G_N}\left(\partial_\zeta K d\zeta + \partial_{\bar\zeta} K d\bar\zeta\right), && (177)\\ \tau &= \frac{1}{8\pi G_N k^2 P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)d\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)d\bar\zeta^2\right). && (178)\end{aligned}$$

We are converging toward the end of our analysis. We now have all the ingredients to write the energy-momentum tensor (21)

$$T = \frac{1}{16\pi G_N k}\begin{pmatrix} dt & d\zeta & d\bar\zeta\end{pmatrix}\begin{pmatrix} 16\pi G_N \varepsilon k^2 & \partial_\zeta K & \partial_{\bar\zeta} K\\ \partial_\zeta K & \frac{2}{k^2}\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) & \frac{2M}{P^2}\\ \partial_{\bar\zeta}K & \frac{2M}{P^2} & \frac{2}{k^2}\partial_t\left(\frac{\partial_{\bar\zeta}^2 P}{P}\right)\end{pmatrix}\begin{pmatrix} dt\\ d\zeta\\ d\bar\zeta\end{pmatrix}. \tag{179}$$

We then focus on its conservation. Indeed our general analysis shows that these equations would furnish us bulk Einstein equations. We first of all identify (the reason will become clear shortly)

$$M = 4\pi G_N \varepsilon \tag{180}$$

and then impose (22):

$$\nabla\cdot T = 0 \quad\Longleftrightarrow\quad \begin{cases}\Delta K + 12 M\partial_t \ln P = 4\partial_t M,\\ \partial_\zeta M = 0, \quad \partial_{\bar\zeta} M = 0.\end{cases} \tag{181}$$

The first equation is the celebrated bulk Robinson-Trautman equation, here expressed in terms of $M(t) = 4\pi G_N\varepsilon(t)$, which are indeed the Einstein equations for the line element (175). The boundary fluids emerging in the systems considered here have a specific physical behavior. This behavior is inherited from the boundary geometry, since their excursion away from perfection is encoded in the Cotton tensor via the transverse duality. In the hydrodynamic frame at hand, this implies in particular that the derivative expansion of the energy-momentum tensor terminates at third order. As repeatedly remarked, holography sets a deep relationship between the boundary fluid and the geometry on which it lies. We have therefore concluded our ensemble of steps and obtained, starting simply from a boundary metric, a bulk highly non-trivial solution of Einstein equations.

An important remark is that the hydrodynamic frame at hand, as stressed, is not the Landau-Lifshitz frame, because the fluid has a heat current. One could move to the Landau-Lifshitz frame by redefening the fluid velocity order by order to remove this current [56, 57, 59]. The drawbacks of this fluid frame redefinition are easily understandable. Firstly, the finite order expansions here would be traded with infinite expansions, harder to handle. Secondly, holography in the way we constructed it is sensible to the heat current, setting it to zero breaks down our reconstruction procedure. Lastly, starting from the bulk in the Fefferman-Graham gauge, the boundary energy-momentum is found already a fluid frame which possesses a heat current.

Before concluding this section we want to discuss the bulk Petrov classes reached with this particular solution, and how to tune it.

Petrov Classification

The Robinson-Trautman equation has been obtained from purely boundary considerations, by imposing the conservation of the boundary energy-momentum tensor, and we can similarly tune the boundary data in order to control the bulk Petrov type of the bulk Einstein space. Generically the latter is type II because we can prove that the bulk congruence ∂_r is null, geodesic and shearless, and using thus the extensions of Goldberg-Sachs theorem, the reconstructed bulk space is algebraically special (we prove it in Appendix B).

To further analyze the algebraic properties, consider the reference tensors $T^\pm$ as in (166), which we generally write in the form

$$8\pi G_N k i\, \mathrm{Im} T^+ = \begin{pmatrix} dt & d\zeta & d\bar\zeta\end{pmatrix}\begin{pmatrix} 0 & -\frac{3M\alpha^+}{2P^2}+\frac{\beta}{2} & \frac{3M\alpha^-}{2P^2}-\frac{\bar\beta}{2}\\ -\frac{3M\alpha^+}{2P^2}+\frac{\beta}{2} & \frac{3M(\alpha^+)^2}{2P^4k^2}+\frac{\gamma}{k^2} & 0\\ \frac{3M\alpha^-}{2P^2}-\frac{\bar\beta}{2} & 0 & -\frac{3M(\alpha^-)^2}{2P^4k^2}-\frac{\bar\gamma}{k^2}\end{pmatrix}\begin{pmatrix} dt\\ d\zeta\\ d\bar\zeta\end{pmatrix}, \tag{182}$$

and

$$8\pi G_N k\,\text{Re}T^+ =$$
$$\begin{pmatrix} dt & d\zeta & d\bar\zeta \end{pmatrix}\begin{pmatrix} 2k^2M & -\frac{3M\alpha^+}{2P^2}+\frac{\beta}{2} & -\frac{3M\alpha^-}{2P^2}+\frac{\bar\beta}{2} \\ -\frac{3M\alpha^+}{2P^2}+\frac{\beta}{2} & \frac{3M(\alpha^+)^2}{2P^4k^2}+\frac{\gamma}{k^2} & \frac{M}{P^2} \\ -\frac{3M\alpha^-}{2P^2}+\frac{\bar\beta}{2} & \frac{M}{P^2} & \frac{3M(\alpha^-)^2}{2P^4k^2}+\frac{\bar\gamma}{k^2} \end{pmatrix}\begin{pmatrix} dt \\ d\zeta \\ d\bar\zeta \end{pmatrix}. \tag{183}$$

The reference tensor at hand depends on M and three complex arbitrary functions of t, ζ and $\bar\zeta$: α^+, β and γ. The functions $\{\alpha^+, \beta, \gamma\}$ are not explicit in the energy-momentum tensor and Cotton tensor if they satisfy the equations

$$3M\frac{\alpha^+}{P^2} + \partial_\zeta K = \beta \quad \text{and} \quad \text{c.c.}\,, \tag{184}$$

and

$$\frac{3}{2}M\frac{(\alpha^+)^2}{P^4} + \gamma = \partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) \quad \text{and} \quad \text{c.c.}\,. \tag{185}$$

By tuning all these functions ($M(t), \alpha^\pm(t,\zeta,\bar\zeta)$, $\beta(t,\zeta,\bar\zeta)$, $\bar\beta(t,\zeta,\bar\zeta)$, $\gamma(t,\zeta,\bar\zeta)$ and $\bar\gamma(t,\zeta,\bar\zeta)$) we can scan different classes:

- If $M = 0$, $\alpha^\pm$ are immaterial and $\beta(t,\zeta,\bar\zeta)$ and $\gamma(t,\zeta,\bar\zeta)$ are fully determined by (184) and (185):

$$\beta = \partial_\zeta K \quad \text{and} \quad \text{c.c.}\,, \tag{186}$$
$$\gamma = \partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) \quad \text{and} \quad \text{c.c.}\,. \tag{187}$$

  Furthermore, the Robinson-Trautman equation guarantees holomorphicity for β, function of (t,ζ) only. Hence, the bulk is generically Petrov type III. When $\beta = 0$, it becomes type N, where now $K = K(t)$, following (186). The most general $P(t,\zeta,\bar\zeta)$ such that its curvature is a function of time only was found in [200], and reads:

$$P(t,\zeta,\bar\zeta) = \frac{1+\frac{\epsilon}{2}h(t,\zeta)\,\bar h(t,\bar\zeta)}{\sqrt{2f(t)\,\partial_\zeta h(t,\bar\zeta)\,\partial_{\bar\zeta}\bar h(t,\bar\zeta)}} \tag{188}$$

  with $\epsilon = 0, \pm 1$ and arbitrary functions $f(t)$ and $h(t,\zeta)$.

- If $\beta = \gamma = 0$, $\alpha^\pm$ are read-off from (184):

$$\alpha^+ = -\frac{P^2}{3M}\partial_\zeta K \quad \text{and} \quad \text{c.c.}\,, \tag{189}$$

  and the geometry is subject to a further constraint[31] obtained by combining (185) and (189):

$$6M\,\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) = (\partial_\zeta K)^2 \quad \text{and} \quad \text{c.c.}\,. \tag{190}$$

  The bulk is still type II, but choosing holomorphic $\alpha^- = \alpha^-(t,\zeta)$, i.e.

$$\partial_\zeta\left(P^2\partial_\zeta K\right) = 0 \quad \text{and} \quad \text{c.c.}\,, \tag{191}$$

  together with the constraint (190), makes it type D. There are two independent type D solutions:

  1. The Schwarzschild, reached with $P = 1 + \frac{\epsilon}{2}\zeta\bar\zeta$ and $K = \epsilon$, which is asymptotically anti-de Sitter.
  2. The C-metric, which requires $P^2\partial_\zeta K = h(\bar\zeta) \neq 0$ and is asymptotically locally anti-de Sitter due to a non-vanishing boundary Cotton tensor.

[31] Notice a useful identity: $\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) = \frac{1}{P^2}\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)$.

We would like to end the current section with some general comments regarding the bulk Einstein spaces under consideration.

With the exception of the Petrov-D solutions quoted above, Robinson-Trautman spacetimes are time-dependent and carry gravitational radiation. Once this radiation is emitted, the spacetime settles down generically to an anti-de Sitter Schwarzschild black hole.[32] The general features of this evolution are captured by the Robinson-Trautman equation, which, following [201], is a parabolic equation describing a Calabi flow on a two-surface. As long as $M \neq 0$, these spacetimes exhibit a past singularity at $r = 0$, past-trapped two-surfaces and a future horizon, which is the anti-de Sitter Schwarzschild horizon at late times. Unfortunately, singularities are often developed on this horizon and no smooth extension is possible beyond, in the interior region.

Irregularities of the two-surface $\mathcal{S}$ time-dependent metric

$$d\ell^2 = \frac{2}{P(t,\zeta,\bar{\zeta})^2} d\zeta d\bar{\zeta}, \tag{192}$$

possibly present at early times, are washed out by the evolution, as usual with geometric flows. The flow at hand, governed by the Robinson-Trautman equation, has the following salient properties:

$$\frac{d}{dt}\int_{\mathcal{S}} \frac{d^2\zeta}{P^2} = 0, \tag{193}$$

$$\frac{d}{dt}\int_{\mathcal{S}} \frac{d^2\zeta}{P^2} K = 0, \tag{194}$$

where $d^2\zeta = -i\, d\zeta \wedge d\bar{\zeta}$. Hence, the area of $\mathcal{S}$ and its average curvature are preserved along the flow, which, at late times, brings the metric into a symmetric geometry compatible with the original topology. From the spacetime perspective, this situation corresponds indeed to the evolution towards an anti-de Sitter Schwarzschild black hole.

2.2.4 Bulk Reconstruction in Three Dimensions

Gravity in three dimensions cannot propagate. Einstein equations are therefore solved by empty AdS locally [61].

Nonetheless, global issues and identifications of points make the story richer than it seems. The absence of propagating gravitational degrees of freedom implies that the asymptotic charges are integrable [63]. They characterize the bulk solution under consideration. Even if two bulk solutions will again locally be AdS$_3$, they differ if their asymptotic charges differ. These charges are eventually the most important thing we should care about in three dimensions, for they distinguishes uniquely the various bulk solutions.

As previously mentioned, the FG expansion for empty AdS is a finite expansion. This is thus always the case in here. Additionally, also the usual derivative expansion terminates at finite order. The reason is that most geometric and fluid tensors vanish (like the shear or the vorticity), reducing the number of available terms compatible with conformal invariance.

As opposed to higher dimension, where its conformal weight forbids it, the heat current enters directly in the resummation formula. It morally replaces the role played by the Cotton density. In fact, in two boundary dimensions the Cotton tensor is identically zero.

Specifically, the exercise of writing compatible terms with Weyl covariance in three bulk dimensions results in:

$$\boxed{ds^2_{\text{Einstein}} = 2\frac{u}{k^2}\left(dr + rA\right) + r^2 ds^2 + \frac{8\pi G_N}{k^4} u \left(\varepsilon u + \chi \star u\right),} \tag{195}$$

where A is displayed in (89), ε and χ being the energy and heat densities of the fluid. These enter the fluid energy-momentum tensor (85) together with τ, which carries the anomaly:

$$\tau = \frac{R}{8\pi G_N} = \frac{1}{4\pi G_N k^2}\left(\Theta^2 - \Theta^{\star 2} + \underline{u}(\Theta) - \star\underline{u}(\Theta^\star)\right) \tag{196}$$

(we keep the conformal state equation $\varepsilon = p$). For a flat boundary this anomaly is absent, but Weyl transformations bring it back.

[32] This is the reason why (180) has been imposed: M then is the Schwarzschild black hole mass once the solution settles down.

The metric (195) provides an exact Einstein, asymptotically AdS spacetime with $\Lambda = -k^2$, under the necessary and sufficient condition that the non-conformal fluid energy-momentum tensor (85) obeys[33]

$$\nabla^\mu \left(T_{\mu\nu} + D_{\mu\nu}\right) = 0, \tag{197}$$

where $D_{\mu\nu}$ is a symmetric and traceless tensor which reads:

$$D_{\mu\nu}dx^\mu dx^\nu = \frac{1}{8\pi G_N k^4}\left(\left(\underline{u}(\Theta) + \star\underline{u}(\Theta^\star) - \frac{k^2}{2}R\right)\left(u^2 + \star u^2\right) - 4\star\underline{u}(\Theta)u\star u\right). \tag{198}$$

On the one hand, the holographic energy-momentum tensor is the sum $T_{\mu\nu} + D_{\mu\nu}$, and this can be shown following the Balasubramanian-Kraus method [11]. On the other hand, the holographic fluid is subject to an external force with density

$$f_\nu = -\nabla^\mu D_{\mu\nu}. \tag{199}$$

Its longitudinal and transverse components are (F is given in (95))

$$u^\mu f_\mu = -\frac{1}{4\pi G_N}\left(\star\underline{u}(F) + 2\Theta^\star F + \frac{1}{2}\Theta R\right), \tag{200}$$

$$\star u^\mu f_\mu = \frac{1}{8\pi G_N}\left(\star\underline{u}(R) + \Theta^\star R\right). \tag{201}$$

Combining these with (101), (102) and (196) we find

$$\left(u^\mu + \star u^\mu\right)\mathcal{D}_\mu\left(\varepsilon + \chi\right) = \frac{1}{4\pi G_N}\star u^\mu \mathcal{D}_\mu F, \tag{202}$$

$$\left(u^\mu - \star u^\mu\right)\mathcal{D}_\mu\left(\varepsilon - \chi\right) = \frac{1}{4\pi G_N}\star u^\mu \mathcal{D}_\mu F. \tag{203}$$

Notice that eventually these equations are Weyl-covariant (weight-3) despite the conformal anomaly.

An important remark is in order regarding the holographic fluid. Rather than $T_{\mu\nu}$, we could have adopted $T_{\mu\nu} + D_{\mu\nu}$ as its energy-momentum tensor. The latter would have been decomposed as in (85), with $\tilde{\varepsilon} = \tilde{p}$ and $\tilde{\chi}$ though ($\tilde{\tau} = \tau$ since $D_{\mu\nu}$ has vanishing trace):

$$\tilde{\varepsilon} = \varepsilon + \frac{1}{8\pi G_N k^2}\left(\underline{u}(\Theta) + \star\underline{u}(\Theta^\star)\right) - \frac{R}{16\pi G_N}, \tag{204}$$

$$\tilde{\chi} = \chi - \frac{1}{4\pi G_N k^2}\star\underline{u}(\Theta). \tag{205}$$

We did not make this choice for two reasons: (i) in (195) we used ε and χ rather than $\tilde{\varepsilon}$ and $\tilde{\chi}$ for reconstructing the bulk; (ii) ε and χ/k are finite in the limit of vanishing k, whereas $\tilde{\varepsilon}$ and $\tilde{\chi}/k$ are not. This last fact is not an obstruction per se. However, we will present later on the Carrollian limit of relativistic fluids that have finite leading order in k for both these terms. The output with ε and χ is the foreseeable one, whereas there is no guarantee that with $\tilde{\varepsilon}$ and $\tilde{\chi}/k$ things will eventually work out.

The metric (195) is the most general locally AdS spacetime in Eddington-Finkelstein coordinates. The corresponding gauge (falloffs) includes but does not always coincide with BMS. From this perspective, this result is new although it may not contain any new solutions compared to Bañados, all captured either in BMS or in Fefferman-Graham gauge [102, 119]. Charges computation is in order to give a definite answer to these wondering.

The bonus here is the hydrodynamic interpretation: the corresponding fluid is defined on a generally curved boundary and has an arbitrary velocity field. This should be contrasted with the treatment of three-dimensional fluid/gravity correspondence worked out previously [22, 24], where the host geometry was flat, avoiding the issue of conformal anomaly. Furthermore the fluid has been very often assumed perfect by hydrodynamic frame choice, which gives rise to a holographic dual that overlaps only partially with the Bañados solutions, as we will shortly see by computing charges.

[33] Here we resum all bulk spacetimes, as the charges computation will confirm. We thence do not need any integrability condition. Notice also that we constantly refer to the conservation of the energy-momentum tensor in the absence of external forces. When this kind of statements is made, we consider the holographic tensor, here $T_{\mu\nu} + D_{\mu\nu}$.

For practical purposes, we can work in light-cone coordinates, introduced in (118). Solving the fluid equations (202), (203), we obtain the fluid densities ε and χ in terms of two arbitrary chiral functions $\ell_\pm$

$$\varepsilon = \frac{e^{2\omega}}{4\pi G_N}\left(\frac{\ell_+}{\xi}+\xi\ell_- - \frac{3\left(\partial_+\xi\right)^2}{4\xi^3}+\frac{\partial_+^2\xi}{2\xi^2}+\frac{\left(\partial_-\xi\right)^2}{4\xi}-\frac{\partial_-^2\xi}{2}\right), \tag{206}$$

$$\chi = \frac{e^{2\omega}}{4\pi G_N}\left(-\frac{\ell_+}{\xi}+\xi\ell_- + \frac{3\left(\partial_+\xi\right)^2}{4\xi^3}-\frac{\partial_+^2\xi}{2\xi^2}+\frac{\left(\partial_-\xi\right)^2}{4\xi}-\frac{\partial_-^2\xi}{2}+\frac{\partial_+\xi\partial_-\xi}{\xi^2}-\frac{\partial_+\partial_-\xi}{\xi}\right). \tag{207}$$

Gathering these data inside (195) provides, in the gauge at hand, the general class of locally AdS three-dimensional spacetime with curved conformal boundary. The conformal factor $\exp 2\omega$ plays actually no role because, as one readily sees from the above expressions, it can be reabsorbed with the redefinition of r into $r\exp\omega$, bringing (195) to its flat-boundary form.[34]

As we will shortly see, the arbitrary function $\xi(x^+,x^-)$ is more insidious regarding the charges. A specific example of curved boundary with $\Omega = \exp 2\beta$, $b_x = 0$, $a = 1$ and fluid velocity $u = -k^2 e^{2\beta} dt$ (comoving) was investigated in [107], outside of the fluid/gravity framework, and the output agrees with our general results.

Flatness requirements are equivalent to $R = 0$ and $F = 0$. In light-cone frame, this amounts to (see (124) and (125))

$$\omega = 0 \quad \text{and} \quad \xi(x^+,x^-) = -\frac{\xi^-(x^-)}{\xi^+(x^+)}, \tag{208}$$

where the minus sign is conventional.

Trading the chiral functions $\ell_\pm$ for $L_\pm$ defined as (the prime indicates total derivative with respect to the only argument of the functions $\xi^\pm$)

$$\ell_\pm = \frac{1}{\left(\xi^\pm\right)^2}\left(L_\pm - \frac{(\xi^{\pm\prime})^2 - 2\xi^\pm\xi^{\pm\prime\prime}}{4}\right), \tag{209}$$

we finally obtain the following metric:

$$\begin{aligned} ds^2_{\text{Einstein}} = & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}dx^+ - \sqrt{-\frac{\xi^+}{\xi^-}}dx^-\right)dr \\ & + \left(\frac{L_+}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{+\prime}\right)\left(\frac{dx^+}{\xi^+}\right)^2 + \left(\frac{L_-}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{-\prime}\right)\left(\frac{dx^-}{\xi^-}\right)^2 \\ & + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime}+\xi^{-\prime}\right) + \frac{L_+ + L_-}{k^2\xi^+\xi^-}\right)dx^+dx^-. \end{aligned} \tag{210}$$

This metric depends on four arbitrary functions: $\xi^+(x^+)$ and $\xi^-(x^-)$ carrying information about the holographic fluid velocity, and $L_+(x^+)$, $L_-(x^-)$, which together with $\xi^+(x^+)$ and $\xi^-(x^-)$ shape the energy-momentum tensor – here traceless due to the absence of anomaly for flat boundaries.

Indeed we have

$$\varepsilon = -\frac{1}{4\pi G_N}\frac{L_+ + L_-}{\xi^+\xi^-}, \quad \chi = \frac{1}{4\pi G_N}\frac{L_+ - L_-}{\xi^+\xi^-}, \tag{211}$$

and in turn

$$T_{\pm\pm} = \frac{L_\pm}{4\pi G_N(\xi^\pm)^2}. \tag{212}$$

In three dimensions, any Einstein spacetime is locally anti-de Sitter. Hence, there exists always a coordinate transformation that can be used to bring it into a canonical AdS_3 form. This is a large gauge transformation whenever the original Einstein spacetime has non-trivial conserved charges. The determination of the latter is therefore crucial for a faithful identification of the solution under consideration. It allows to evaluate the precise role played by the above arbitrary functions.

The charge computation requires a complete family of asymptotic Killing vectors, determined according to the r-falloffs. The metric (210) does not fit into the BMS gauge, unless $\xi^\pm$ are constant. This is equivalent to saying that

[34] This should be contrasted with the more intricate situation regarding this conformal factor inside the analogous formula in FG gauge, see (2.21) of [119].

the fluid has a uniform velocity, and can therefore be set at rest by an innocuous global Lorentz boost tuning $\xi^+ = 1$ and $\xi^- = -1$.

We will first focus on this case, where the asymptotic Killing vectors are known, and move next to the other extreme, demanding the fluid be perfect, i.e. in Landau-Lifshitz hydrodynamic frame. In the latter instance we will have to determine this family of vectors beforehand, as the gauge will no longer be BMS. Investigating the general situation captured by (210) is the next natural step, and is indeed current investigation.

Dissipative Static Fluid

As anticipated, this class of solutions is reached by demanding $\xi^\pm = \pm 1$, while keeping $L^\pm$ arbitrary.

We obtain

$$ds^2_{\text{Einstein}} = -\frac{1}{k}\left(dx^+ - dx^-\right) dr + r^2 dx^+ dx^- + \frac{1}{k^2}\left(L_+ dx^+ - L_- dx^-\right)\left(dx^+ - dx^-\right), \tag{213}$$

which is the canonical expression of Bañados solutions in BMS gauge. Following (211), the boundary fluid energy and heat densities are $\varepsilon = {}^{1}\!/_{4\pi G_N}\left(L_+ + L_-\right)$ and $\chi = -{}^{1}\!/_{4\pi G_N}\left(L_+ - L_-\right)$. Therefore the heat current is not vanishing, and in the present hydrodynamic frame the fluid is at rest and dissipative.

The metric (213) is form-invariant under the action of this diffeomorphism

$$\underline{\zeta} = \zeta^r \partial_r + \zeta^+ \partial_+ + \zeta^- \partial_- \tag{214}$$

with

$$\zeta^r = -\frac{r}{2}\left(Y^{+\prime} + Y^{-\prime}\right) + \frac{1}{2k}\left(Y^{+\prime\prime} - Y^{-\prime\prime}\right) - \frac{1}{2k^2 r}\left(L_+ - L_-\right)\left(Y^{+\prime} - Y^{-\prime}\right), \tag{215}$$

$$\zeta^\pm = Y^\pm - \frac{1}{2kr}\left(Y^{+\prime} - Y^{-\prime}\right), \tag{216}$$

for arbitrary chiral functions $Y^+(x^+)$ and $Y^-(x^-)$.

These vector fields generate a diffeomorphism that alters the various functions in the metric according to (MN are three-dimensional bulk indices)

$$-\mathcal{L}_{\underline{\zeta}} g_{MN} = \delta_{\underline{\zeta}} g_{MN} = \frac{\partial g_{MN}}{\partial L_+}\delta_{\underline{\zeta}} L_+ + \frac{\partial g_{MN}}{\partial L_-}\delta_{\underline{\zeta}} L_- \tag{217}$$

with

$$\delta_{\underline{\zeta}} L_\pm = -Y^\pm L_\pm' - 2 L_\pm Y^{\pm\prime} + \frac{1}{2} Y^{\pm\prime\prime\prime}. \tag{218}$$

The last term in this expression is responsible for the emergence of a central charge in the surface-charge algebra. These vectors obey an algebra for the modified Lie bracket (see e.g. [119]):

$$\underline{\zeta}_3 = \left[\underline{\zeta}_1, \underline{\zeta}_2\right]_{\text{M}} = \left[\underline{\zeta}_1, \underline{\zeta}_2\right] - \delta_{\underline{\zeta}_2}\underline{\zeta}_1 + \delta_{\underline{\zeta}_1}\underline{\zeta}_2 \tag{219}$$

with[35] $\underline{\zeta}_a = \underline{\zeta}\left(Y_a^+, Y_a^-\right)$ and

$$Y_3^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm. \tag{220}$$

The surface charges are computed for an arbitrary metric g of the type (213) with empty AdS_3 as reference background. The latter has metric $\bar{g}$ with $L_+ = L_- = -\frac{1}{4}$ i.e. $\varepsilon = -\frac{1}{8\pi G_N}$ and $\chi = 0$. The final integral is performed over the compact spatial boundary coordinate $x \in [0, 2\pi]$:

$$Q_Y\left[g - \bar{g}, \bar{g}\right] = \frac{1}{8\pi k G_N}\int_0^{2\pi} dx \left(Y^+\left(L_+ + \frac{1}{4}\right) - Y^-\left(L_- + \frac{1}{4}\right)\right). \tag{221}$$

These charges are in agreement with the quoted literature,[36] and their algebra is determined as usual:

$$\left\{Q_{Y_1}, Q_{Y_2}\right\} = \delta_{\underline{\zeta}_1} Q_{Y_2} = -\delta_{\underline{\zeta}_2} Q_{Y_1}. \tag{222}$$

[35]Here $\delta_{\underline{\zeta}_2}\underline{\zeta}_1$ stands for the variation produced on $\underline{\zeta}_1$ by $\underline{\zeta}_2$, and this is not vanishing because $\underline{\zeta}_1$ depends explicitly on $L_\pm$: $\delta_{\underline{\zeta}_2}\underline{\zeta}_1 = \left(\frac{\partial \zeta_1^\alpha}{\partial L_+}\delta_{\underline{\zeta}_2} L_+ + \frac{\partial \zeta_1^\alpha}{\partial L_-}\delta_{\underline{\zeta}_2} L_-\right)\partial_\alpha$.

[36]Some relative-sign differences are due to different conventions used for the light-cone coordinates, here defined as $x^\pm = \boldsymbol{x} \pm kt$.

Introducing the modes

$$L_m^\pm = \frac{1}{8\pi k G_N}\int_0^{2\pi} d\boldsymbol{x} e^{imx^\pm}\left(L_\pm + \frac{1}{4}\right) \tag{223}$$

the algebra reads:

$$i\left\{L_m^\pm, L_n^\pm\right\} = (m-n)L_{m+n}^\pm + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}\,, \quad \left\{L_m^\pm, L_n^\mp\right\} = 0. \tag{224}$$

This double realization of Virasoro algebra with Brown-Henneaux central charge $c = \frac{3}{2kG_N}$ is the expected result for Bañados solutions (213).

Perfect Fluid with Arbitrary Velocity

In Landau-Lifshitz frame the heat current vanishes ($\chi = 0$) and the boundary conformal fluid is perfect. Equation (211) returns

$$L_+ = L_- = \frac{M}{2}, \tag{225}$$

with M constant, while it gives the energy density $\varepsilon = -\frac{M}{4\pi G_N \xi^+\xi^-}$.

The reconstructed bulk family of metrics

$$\begin{aligned} ds^2_{\text{Einstein}} &= -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}dx^+ - \sqrt{-\frac{\xi^+}{\xi^-}}dx^-\right)dr + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime}+\xi^{-\prime}\right) + \frac{M}{k^2\xi^+\xi^-}\right)dx^+dx^- \\ &\quad + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{+\prime}\right)\left(\frac{dx^+}{\xi^+}\right)^2 + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{-\prime}\right)\left(\frac{dx^-}{\xi^-}\right)^2 \end{aligned} \tag{226}$$

is not in BMS gauge, unless $\xi^\pm$ are constant. Again this latter subset is entirely captured by $\xi^\pm = \pm 1$, and the resulting solution is BTZ together with all non-spinning zero-modes of Bañados family:

$$ds^2_{\text{Einstein}} = -\frac{1}{k}\left(dx^+ - dx^-\right)dr + r^2 dx^+dx^- + \frac{M}{2k^2}\left(dx^+ - dx^-\right)^2. \tag{227}$$

The asymptotic structure rising in (226) is now respected by the following family of asymptotic Killing vectors

$$\underline{\eta} = \eta^r\partial_r + \eta^+\partial_+ + \eta^-\partial_-, \tag{228}$$

expressed in terms of two arbitrary chiral functions $\epsilon^\pm(x^\pm)$

$$\eta^r = -\frac{r}{2}\left(\epsilon^{+\prime} + \epsilon^{-\prime}\right), \quad \eta^\pm = \epsilon^\pm. \tag{229}$$

These vectors, slightly different from those found for the dissipative boundary fluids, appear as the result of an exhaustive analysis of (226). They do not support subleading terms, and since they do not depend on the functions $\xi^\pm$, they form an algebra for the Lie bracket:

$$\left[\underline{\eta}_1, \underline{\eta}_2\right] = \underline{\eta}_3 \tag{230}$$

with $\underline{\eta}_a = \underline{\eta}\left(\epsilon_a^+, \epsilon_a^-\right)$ and

$$\epsilon_3^\pm = \epsilon_1^\pm\epsilon_2^{\pm\prime} - \epsilon_2^\pm\epsilon_1^{\pm\prime}. \tag{231}$$

They induce the exact transformation (MN are 3-dimensional bulk indices)

$$-\mathcal{L}_{\underline{\eta}}g_{MN} = \delta_{\underline{\eta}}g_{MN} = \frac{\partial g_{MN}}{\partial\xi^+}\delta_{\underline{\eta}}\xi^+ + \frac{\partial g_{MN}}{\partial\xi^{+\prime}}\delta_{\underline{\eta}}\xi^{+\prime} + \frac{\partial g_{MN}}{\partial\xi^-}\delta_{\underline{\eta}}\xi^- + \frac{\partial g_{MN}}{\partial\xi^{-\prime}}\delta_{\underline{\eta}}\xi^{-\prime} \tag{232}$$

with

$$\delta_{\underline{\eta}}\xi^\pm = \xi^\pm\epsilon^{\pm\prime} - \epsilon^\pm\xi^{\pm\prime}. \tag{233}$$

Following the customary pattern, we can determine the conserved charges, with AdS_3 as reference background, now reached with $\xi^\pm = \pm 1$ and $M = -1/2$ (again $\varepsilon = -\frac{1}{8\pi G_N}$ and $\chi = 0$):

$$Q_\epsilon\left[g - \bar{g}, \bar{g}\right] = \frac{1}{16\pi k G_N}\int_0^{2\pi} dx\left(\epsilon^+\left(\frac{1}{\xi^{+2}} - 1\right) - \epsilon^-\left(\frac{1}{\xi^{-2}} - 1\right)\right), \tag{234}$$

as well as their algebra:

$$\{Q_{\epsilon_1}, Q_{\epsilon_2}\} = \delta_{\underline{\eta}_1} Q_{\epsilon_2} = -\delta_{\underline{\eta}_2} Q_{\epsilon_1}. \tag{235}$$

Defining now

$$Z^{\pm}_m = \frac{1}{16\pi k G_N} \int_0^{2\pi} dx e^{imx^{\pm}} \left(\frac{1}{\xi^{\pm 2}} - 1 \right) \tag{236}$$

we find

$$i\left\{Z^{\pm}_m, Z^{\pm}_n\right\} = (m-n) Z^{\pm}_{m+n} + \frac{m}{4kG_N}\delta_{m+n,0}\,, \quad \left\{Z^{\pm}_m, Z^{\mp}_n\right\} = 0. \tag{237}$$

The central extension of this algebra can be reabsorbed in the following redefinition of the modes $Z^{\pm}_m$

$$\tilde{Z}^{\pm}_m = Z^{\pm}_m + \frac{1}{8kG_N}\delta_{m,0}. \tag{238}$$

Therefore, (237) becomes

$$i\left\{\tilde{Z}^{\pm}_m, \tilde{Z}^{\pm}_n\right\} = (m-n)\tilde{Z}^{\pm}_{m+n}, \quad \left\{\tilde{Z}^{\pm}_m, \tilde{Z}^{\mp}_n\right\} = 0. \tag{239}$$

The algebra at hand (239) is de Witt rather than Virasoro, and this outcome demonstrates the already advertised result: the family of locally AdS spacetimes obtained in holography from two-dimensional fluids in the Landau-Lifshitz frame overlaps only partially the space of Bañados solutions. This overlap encompasses the non-spinning BTZ and excess or defects geometries provided in (227).

We eventually reach the important conclusion that bulk reconstruction, within our framework, is sensitive to boundary hydrodynamic frame. Setting the heat current to zero a priori is not a natural choice, and limits the resummable solutions in the bulk.

Our analysis has been very fruitful in three and four bulk dimensions, where we managed to gain very powerful control on the bulk theory starting from boundary fluid data. As stressed, it would be interesting to try to extend this dictionary to higher dimensional bulks. Additionally, although we achieved all (known) bulk solutions in three dimensions, in four dimensions we saw that, due to integrability, we can obtain only a limited (still very large) class of bulk solutions. Another natural direction is to try to release our assumptions and look for a complete reconstruction.

Lastly, we discussed so far AdS bulks only. In such a situation the whole microscopic AdS/CFT dictionary is at work. We took a long road toward hydrodynamics to address questions that we could have at least be posed directly from a field theoretical viewpoint. The advantage of this was to have better control, and indeed most of the solutions we resummed do not have a fully understood microscopic boundary theory. Our detour has also a very insightful consequence. The fluid/gravity dictionary unraveled a bulk gauge better suited for hydrodynamics, the derivative expansion. It is in this gauge that we will show how to implement a flat limit $\Lambda = -\frac{d(d+1)}{2}k^2 \to 0$.

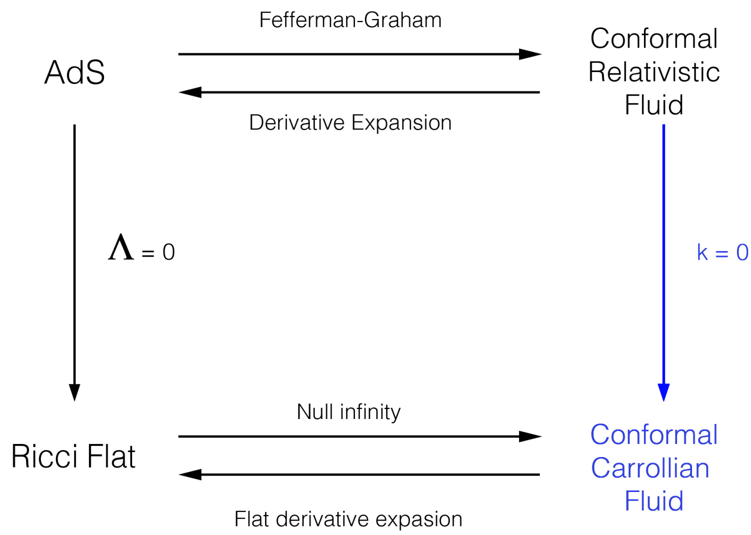
The FG expansion trivially diverges in such a limit, while the derivative expansion will miraculously be finite, leading to a holographic dictionary between asymptotically flat solutions in the bulk and the $k \to 0$ boundary theory. What is this boundary theory? k in the boundary plays the role of the speed of light, what does it mean to take $k \to 0$ in a fluid? This limit degenerates the boundary metric, what is happening to the geometry in such a limit? All these questions are the subject of the next chapters, and the core of this project.

3 Carrollian Limit in Hydrodynamics

So far we have worked with a $d+1$-dimensional relativistic boundary and a corresponding $d+2$ bulk solution of Einstein equations with negative cosmological constant. This is the best understood framework. The $d+1$-dimensional holographic fluid living on the boundary enjoys spectacular properties: it is a conformal fluid with dissipative tensors dictated by the surrounding geometry. The Randers-Papapetrou parametrization of the boundary metric was a successful choice and the bulk cosmological constant was found to be $\Lambda = -\frac{d(d+1)}{2}k^2$.

We discussed two gauges of the bulk metric, both with advantages and drawbacks. The FG gauge allows to extract the boundary metric and energy-momentum tensor. It is mathematically well-defined, and implies in most cases an infinite expansion. To go from the bulk to the boundary is by far the best instrument we have. An alternative frame inspired by fluid dynamics, known as derivative expansion, was introduced as well. This gauge is based on Weyl-covariance, an important symmetry of the boundary theory.

It is in this framework that we achieved, starting from boundary data only, a set of full solution of bulk Einstein equations. It is therefore this latter the best way to move from the boundary to the bulk. In the boundary, k is the speed of light. Therefore, the fundamental result we observe is that the bulk flat limit $\Lambda \to 0$ (which is well understood and always achievable)[37] corresponds to a boundary where the speed of light is sent to zero $k \to 0$. We call this limit a Carrollian limit, the fluid we reach a Carrollian fluid and in general we refer to this theory as a Carrollian theory. We will explain in great detail the reason why we call the $k = 0$ theory in this way. Regarding the general picture, we are dealing in this chapter with the blue part:



The vanishing speed of light limit is at first very cumbersome: how do we make sense out of this limit? The latter is indeed degenerate both physically and geometrically. Two main wonders arise, the limit at the geometrical level and at the fluid dynamical one. We will discuss here both of them and show that there is a way to extrapolate insightful and meaningful informations from this limit. Eventually we will be able to obtain a boundary theory dual to asympotically flat bulk solutions (Ricci flat). This is a first major step toward what is now referred to as flat holography, i.e. a holographic duality between solutions of Einstein equations with vanishing cosmological constant and matter theory living on its boundary. We are going to show where and how our theory makes contact with other works and previous result on the topic. In particular we will see, at least from the geometrical and group theoretical viewpoints, that our results extend previous attempts and generalize them. We would like to recall that many efforts in scattered directions have been made in understanding the holographic dictionary for flat spacetime.

Here we make a step in this direction, but our theory has important limitations. Firstly, we cannot address the microscopic structure of the boundary theory since we take the limit at the hydrodynamic level. In AdS we know it is a CFT, here we suspect it is a BMS field theory (we will talk about it shortly) but many things remain to be understood. Secondly, we are reconstructing a classical bulk so we do not have any control on the effect of our

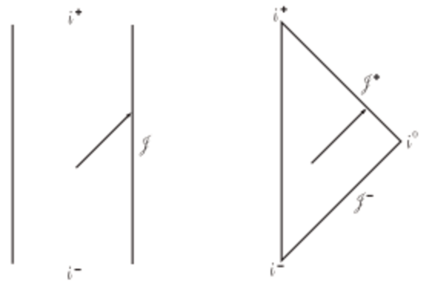
[37] We should clarify this point: when we say that the flat limit in the bulk is always achievable we mean that there exists given a bulk solution a gauge such that this limit trivially applies. This does not mean that every gauge is suited for such a limit.

limiting procedure in string theory. Lastly, we loose the powerful tool of the FG gauge in this limit. This gauge becomes indeed singular. We therefore loose a completely mastered way of gong from the bulk to the boundary in the flat case. The derivative expansion is smooth in the limit, and this is its best feature and our major discovery. Altogether, these limitations could be addressed by further inspection. In this respect, they are intriguing future directions of research. We already mentioned some of these observations in the introduction. We will, at the light of this chapter, return to this discussion in the conclusions.

In this chapter we begin with an analysis of the geometry in the $k \to 0$ limit. We then study the limit at the level of the equations of motion for the relativistic fluid. We do all this in arbitrary dimension and for generic fluid. We then specialize to holographic fluids in three and two dimensions. An important question that arises in the limit concerns the fate of the energy-momentum tensor. We will show how to introduce its Carrollian counterparts and compute Carrollian charges. Our formalism for the limit of the fluid equations of motion is so general and powerful that we conclude showing how to take the well-known dual limit $k \to \infty$ and reach Galilean fluids starting from relativistic ones, on completely general background and in a Galilean covariant fashion.

3.1 Geometrical Setup

We saw that the Ricci-flat limit is achieved at vanishing k. The role played by the conformal boundary is in this case replaced by null infinity, dubbed $\mathscr{I}$. As described in [101], the Penrose diagram before and after the limit is given by



The boundary $\mathscr{I}$ in the limit becomes in fact a degenerate $d+1$ manifold of rank d. Geometrically, we therefore recover it starting from our Randers-Papapetrou relativistic parametrization:

$$ds^2 = -k^2 \left(\Omega dt - b_i dx^i\right)^2 + a_{ij} dx^i dx^j. \tag{240}$$

and consistently send $k \to 0$. [38]

As remarked, since the derivative expansion is performed along null tubes, it provides the appropriate arena for studying both the nature of the boundary and the dynamics of the degrees of freedom it hosts as holographic duals to the bulk Ricci-flat spacetime. This comes about because the boundary of asymptotically flat solutions is null, so describing the bulk in null tubes is better suited for this situation. We see thence that for vanishing k, time decouples in the boundary geometry. There exist two decoupling limits, associated with two distinct contractions of the Poincaré group: the Galilean, reached at infinite velocity of light, and the Carrollian, emerging at zero velocity of light. In the metric above, k plays effectively the role of velocity of light and $k \to 0$ is indeed a Carrollian limit [79].

This very elementary observation sets precisely and unambiguously the fate of asymptotically flat holography: the reconstruction of four-dimensional Ricci-flat spacetimes is based on Carrollian boundary geometry. The appearance of Carrollian symmetry, or better, conformal Carrollian symmetry at null infinity of asymptotically flat spacetimes has already attracted attention in the framework of flat holography [80–82, 108, 119].

The novelties we bring in the present work are mainly two. On the one hand, the Carrollian geometry emerging at null infinity is generally non-flat, i.e. it is not isometric under the Carroll group, but under a more general group associated with a time-dependent positive-definite spatial metric and a Carrollian time arrow, this general Carrollian geometry being covariant under a subgroup of the diffeomorphisms called Carrollian diffeomorphisms. On the other hand, the Carrollian surface is the natural host for a Carrollian fluid, zero-k limit of the relativistic boundary fluid dual to the original Einstein space of which we consider the flat limit. This Carrollian fluid must be considered as the holographic dual of a Ricci-flat spacetime, and its dynamics as the dual of gravitational bulk dynamics at zero

[38] Another possibility would be consider the double scaling where Ωk^2 is finite. This would result in a non-degenerate limit. However, it will not be smooth at the level of the bulk line element.

cosmological constant. We recently discussed in [88] hot to construct a Carrollian structure in general. This is based on the seminal works [90,93]. For the sake of clarity and fluidity of our discussion, we will not report on these results here, and refer the reader to the aforementioned papers for further informations.

Connection and Curvature

The Carrollian geometry consists of a spatial surface $\mathcal{S}$ endowed with a positive-definite metric

$$d\ell^2 = a_{ij} dx^i dx^j, \tag{241}$$

and a Carrollian time $t \in \mathbb{R}$.[39]

The metric on $\mathcal{S}$ is generically time-dependent: $a_{ij} = a_{ij}(t, \boldsymbol{x})$. Much like a Galilean space is observed from a spatial frame moving with respect to a local inertial frame with velocity $\underline{w}$, a Carrollian frame is described by a form $b = b_i(t, \boldsymbol{x})\, dx^i$. The latter is an inverse velocity, describing a temporal frame. It can be interpreted as a Ehresman connection, dictating how the null direction is fibrated [88]. A scalar $\Omega(t, \boldsymbol{x})$ is also introduced, as it naturally arises from the $k \to 0$ limit. It plays a rule analogous to the lapse in the ADM decomposition.

We define Carrollian diffeomorphisms as

$$t' = t'(t, \boldsymbol{x}) \quad \text{and} \quad \boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x}) \tag{242}$$

with Jacobian functions

$$J(t, \boldsymbol{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \boldsymbol{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i{}_j(\boldsymbol{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{243}$$

Those are the diffeomorphisms adapted to the Carrollian geometry since under such transformations, $d\ell^2$ remains a positive-definite metric (it does not produce terms involving dt'). Indeed,

$$a'_{ij} = a_{nl} J^{-1n}{}_i J^{-1l}{}_j, \quad b'_j = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}{}_j, \quad \Omega' = \frac{\Omega}{J}, \tag{244}$$

whereas the time and space derivatives become

$$\partial'_t = \frac{1}{J}\partial_t, \quad \partial'_j = J^{-1i}{}_j \left(\partial_i - \frac{j_i}{J}\partial_t\right). \tag{245}$$

We will show in a short while that the Carrollian fluid equations are precisely covariant under this particular set of diffeomorphisms. Expression (245) shows that the ordinary exterior derivative of a scalar function does not transform as a form. To overcome this issue we introduce a Carrollian derivative as

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t, \tag{246}$$

transforming as

$$\hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j. \tag{247}$$

Acting on scalars this provides a form, whereas for any other tensor it must be covariantized by introducing a new connection for Carrollian geometry, called Levi-Civita-Carroll connection, whose coefficients are the Christoffel-Carroll symbols,

$$\hat{\gamma}^i_{jn} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{ln} + \hat{\partial}_n a_{lj} - \hat{\partial}_l a_{jn}\right) = \gamma^i_{jn} + c^i_{jn}. \tag{248}$$

The Levi-Civita-Carroll covariant derivative acts symbolically as $\hat{\nabla} = \hat{\partial} + \hat{\gamma}$. It is metric and torsionless: $\hat{\nabla}_i a_{jk} = 0$, $\hat{t}^k{}_{ij} = 2\hat{\gamma}^k_{[ij]} = 0$. There is however a non-zero field strength, since the derivatives $\hat{\nabla}_i$ do not commute, even when acting of scalar functions Φ – where they are identical to $\hat{\partial}_i$:

$$[\hat{\nabla}_i, \hat{\nabla}_j]\Phi = \frac{2}{\Omega}\varpi_{ij}\partial_t\Phi. \tag{249}$$

[39] We are genuinely describing a spacetime $\mathbb{R} \times \mathcal{S}$ endowed with a Carrollian structure, and this is actually how the boundary geometry should be spelled.

Here ϖ_{ij} is a 2-form identified as the Carrollian vorticity defined using the Carrollian acceleration one-form φ_i:

$$\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right) = \partial_t \frac{b_i}{\Omega} + \hat{\partial}_i \ln \Omega, \tag{250}$$

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]} = \frac{\Omega}{2}\left(\hat{\partial}_i \frac{b_j}{\Omega} - \hat{\partial}_j \frac{b_i}{\Omega}\right). \tag{251}$$

Since in our holographic setup the original relativistic fluid is at rest, the kinematical inverse-velocity variable potentially present in the Carrollian limit vanishes, see [95] for further details and the physical interpretation of this inverse velocity. A Carrollian fluid is always at rest, but could generally be obtained from a relativistic fluid moving at $v^i = k^2\beta^i + O\left(k^4\right)$. In this case, the inverse velocity β^i would contribute to the kinematics and the dynamics of the fluid, as we will see in the next section. Here, $v^i = 0$ before the limit $k \to 0$ is taken, so $\beta^i = 0$. Hence the various kinematical quantities such as the vorticity and the acceleration are purely geometric and originate from the temporal Carrollian frame used to describe the surface $\mathcal{S}$. As we will see later, they turn out to be $k \to 0$ counterparts of their relativistic ancestors defined in (27), (28), (29) and (30).

The time derivative transforms as in (245), and acting on any tensor under Carrollian diffeomorphisms, it provides another tensor. This ordinary time derivative has nonetheless an unsatisfactory feature: its action on the metric does not vanish. One is tempted therefore to set a new time derivative $\hat{\partial}_t$ such that $\hat{\partial}_t a_{jk} = 0$, while keeping the transformation rule under Carrollian diffeomorphisms: $\hat{\partial}'_t = \frac{1}{J}\hat{\partial}_t$.

This is achieved by introducing a temporal Carrollian connection

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega} a^{ik}\partial_t a_{kj}, \tag{252}$$

which allows us to define the time covariant derivative on a vector field:

$$\frac{1}{\Omega}\hat{\partial}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \tag{253}$$

while on a scalar the action is as the ordinary time derivative: $\hat{\partial}_t \Phi = \partial_t \Phi$.

Leibniz rule allows extending the action of this derivative to any tensor. Calling $\hat{\gamma}^i_j$ a connection is actually misleading because it transforms as a genuine tensor under Carrollian diffeomorphisms: $\hat{\gamma}'^k{}_j = J^k_n J^{-1m}{}_j \hat{\gamma}^n{}_m$. Its trace and traceless parts have a well-defined kinematical interpretation, as the expansion and shear, completing the acceleration and vorticity introduced earlier:

$$\theta = \hat{\gamma}^i{}_i = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \quad \xi^i{}_j = \hat{\gamma}^i{}_j - \frac{1}{d}\delta^i_j\theta. \tag{254}$$

We can define the curvature associated with a connection, by computing the commutator of covariant derivatives acting on a vector field. We find

$$\left[\hat{\nabla}_k, \hat{\nabla}_l\right] V^i = \hat{r}^i{}_{jkl}V^j + \varpi_{kl}\frac{2}{\Omega}\partial_t V^i, \tag{255}$$

where

$$\hat{r}^i{}_{jnl} = \hat{\partial}_n \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{nj} + \hat{\gamma}^i_{nm}\hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm}\hat{\gamma}^m_{nj} \tag{256}$$

is a genuine tensor under Carrollian diffeomorphisms, the Riemann-Carroll tensor.

As usual, the Ricci-Carroll tensor is

$$\hat{r}_{ij} = \hat{r}^k{}_{ikj}. \tag{257}$$

It is not symmetric in general ($\hat{r}_{ij} \neq \hat{r}_{ji}$) and carries d^2 independent components:

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K}a_{ij} + \hat{A}\eta_{ij}. \tag{258}$$

If we specialize to three boundary dimensions, we can write[40]

$$\hat{K} = \frac{1}{2}a^{ij}\hat{r}_{ij} = \frac{1}{2}\hat{r}, \quad \hat{A} = \frac{1}{2}\tilde{\eta}^{ij}\hat{r}_{ij} = \star\varpi\theta \tag{259}$$

[40] We use $\tilde{\eta}_{ij} = \sqrt{a}\epsilon_{ij}$, which matches, in the zero-k limit, with the spatial components of the $\tilde{\eta}_{\mu\nu}$ introduced in (48). To avoid confusion we also quote that $\tilde{\eta}^{il}\tilde{\eta}_{jl} = \delta^i_j$ and $\tilde{\eta}^{ij}\tilde{\eta}_{ij} = 2$.

which are the scalar-electric and scalar-magnetic Gauss-Carroll curvatures, with

$$\star\varpi = \frac{1}{2}\tilde{\eta}^{ij}\varpi_{ij}. \tag{260}$$

We now go back to arbitrary dimension. Since time and space are intimately related in Carrollian geometry, curvature extends also in time. This can be seen by computing the covariant time and space derivatives commutator:

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\nabla}_i\right] V^i = -2\hat{r}_i V^i + \left(\theta\delta^j_i - \hat{\gamma}^j{}_i\right)\varphi_j V^i + \left(\varphi_i \frac{1}{\Omega}\hat{\partial}_t - \hat{\gamma}^j{}_i \hat{\nabla}_j\right) V^i. \tag{261}$$

A Carroll curvature one-form emerges thus as

$$\hat{r}_i = \frac{1}{d}\left(\hat{\nabla}_j \xi^j{}_i + \frac{1-d}{d}\hat{\partial}_i\theta\right). \tag{262}$$

Again in three dimensions we will show that the Ricci-Carroll curvature tensor $\hat{r}_{ij}$ and the Carroll curvature one-form $\hat{r}_i$ are actually the Carrollian vanishing-k contraction of the ordinary Ricci tensor $R_{\mu\nu}$ associated with the original four-dimensional pseudo-Riemannian AdS boundary, of Randers-Papapetrou type (148). The identification of the various pieces is however a subtle task because in this kind of limit, where the size of one dimension shrinks, the curvature usually develops divergences. From the perspective of the final Carrollian geometry this does not produce any harm because the involved components decouple.

The metric (241) of the Carrollian geometry on $\mathcal{S}$ may or may not be recast in conformally flat form (160) using Carrollian diffeomorphisms (242). A necessary and sufficient condition is the vanishing of the Carrollian shear ξ_{ij}, displayed in (254). Assuming this holds, one proves that the traceless and symmetric piece of the Ricci-Carroll tensor is zero,

$$\hat{s}_{ij} = 0. \tag{263}$$

The absence of shear will be imposed later on, where it plays the same crucial role in the resummation of the derivative expansion that it played for AdS.

The Conformal Carrollian Geometry

In the present set-up, the spatial surface $\mathcal{S}$ appears as the co-dimension two surface at null infinity of the resulting Ricci-flat geometry. This is a subspace of null infinity $\mathcal{I}$. The latter is the result of the $k \to 0$ limit of the time-like AdS boundary. The bulk congruence tangent to ∂_r is lightlike. Hence the holographic limit $r \to \infty$ is lightlike, already at finite k, which is a well known feature of the derivative expansion, expressed by construction in Eddington-Finkelstein coordinates.

What is specific about $k = 0$ is the decoupling of time. The geometry of $\mathcal{I}$ is equipped with a conformal class of metrics rather than with a metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to study conformal Carrollian geometry as opposed to plain Carrollian geometry. The action of Weyl transformations on the elements of the Carrollian geometry is inherited from (86):

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}, \quad b_i \to \frac{b_i}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \tag{264}$$

where $\mathcal{B} = \mathcal{B}(t, \boldsymbol{x})$ is an arbitrary function.

The Carrollian vorticity (251) and shear (254) transform covariantly under (264): $\varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}$, $\xi_{ij} \to \frac{1}{\mathcal{B}}\xi_{ij}$. However, the Levi-Civita-Carroll covariant derivatives $\hat{\nabla}$ and $\hat{\partial}_t$ defined previously for Carrollian geometry are not covariant under (264). We then replace them with Weyl-Carroll covariant spatial and time derivatives built on the Carrollian acceleration φ_i (250) and the Carrollian expansion (254), which transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{d}{\Omega}\partial_t \mathcal{B}. \tag{265}$$

In particular, these can be combined in[41]

$$\alpha_i = \varphi_i - \frac{\theta}{d} b_i, \tag{266}$$

[41] Contrary to φ_i, α_i is not a Carrollian one-form, i.e. it does not transform covariantly under Carrollian diffeomorphisms (242).

transforming under Weyl rescaling as:

$$\alpha_i \to \alpha_i - \partial_i \ln \mathcal{B}. \tag{267}$$

The Weyl-Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ are defined according to the pattern (52), (53). They obey

$$\hat{\mathscr{D}}_j a_{kl} = 0, \quad \hat{\mathscr{D}}_t a_{kl} = 0. \tag{268}$$

For a weight-w scalar function Φ, or a weight-w vector V^i, i.e. scaling with $\mathcal{B}^w$ under (264), we introduce

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w\varphi_j \Phi, \quad \hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i, \tag{269}$$

which leave the weight unaltered.

Similarly, we define

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \Phi = \frac{1}{\Omega}\hat{\partial}_t \Phi + \frac{w}{d}\theta\Phi = \frac{1}{\Omega}\partial_t \Phi + \frac{w}{d}\theta\Phi, \tag{270}$$

and

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\hat{\partial}_t V^l + \frac{w-1}{d}\theta V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta V^l + \xi^l{}_i V^i, \tag{271}$$

where $\frac{1}{\Omega}\hat{\mathscr{D}}_t$ increases the weight by one unit. The action of $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ on any other tensor is obtained using the Leibniz rule.

The Weyl-Carroll connection is torsion-free because

$$\left[\hat{\mathscr{D}}_i, \hat{\mathscr{D}}_j\right]\Phi = \frac{2}{\Omega}\varpi_{ij}\hat{\mathscr{D}}_t\Phi + w\left(\varphi_{ij} - \varpi_{ij}\theta\right)\Phi \tag{272}$$

does not contain terms of the type $\hat{\mathscr{D}}_k\Phi$. Here $\varphi_{ij} = \hat{\partial}_i\varphi_j - \hat{\partial}_j\varphi_i$ is a Carrollian two-form, not conformal though. The connection (272) is accompanied with its own curvature tensors, which emerge in the commutation of Weyl-Carroll covariant derivatives acting e.g. on vectors

$$\left[\hat{\mathscr{D}}_k, \hat{\mathscr{D}}_l\right]V^i = \left(\hat{\mathscr{R}}^i{}_{jkl} - 2\xi^i{}_j\varpi_{kl}\right)V^j + \varpi_{kl}\frac{2}{\Omega}\hat{\mathscr{D}}_t V^i + w\left(\varphi_{kl} - \varpi_{kl}\theta\right)V^i. \tag{273}$$

The combination $\varphi_{kl} - \varpi_{kl}\theta$ forms a weight-0 conformal two-form.

Moreover

$$\begin{aligned}\hat{\mathscr{R}}^i{}_{jkl} \quad = \quad & \hat{r}^i{}_{jkl} - \delta^i_j\varphi_{kl} - a_{jk}\hat{\nabla}_l\varphi^i + a_{jl}\hat{\nabla}_k\varphi^i + \delta^i_k\hat{\nabla}_l\varphi_j - \delta^i_l\hat{\nabla}_k\varphi_j \\ & + \varphi^i\left(\varphi_k a_{jl} - \varphi_l a_{jk}\right) - \left(\delta^i_k a_{jl} - \delta^i_l a_{jk}\right)\varphi_m\varphi^m + \left(\delta^i_k\varphi_l - \delta^i_l\varphi_k\right)\varphi_j\end{aligned} \tag{274}$$

is the Riemann-Weyl-Carroll weight-0 tensor, from which we define

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^k{}_{ikj}. \tag{275}$$

We also quote

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]\Phi = w\hat{\mathscr{R}}_i\Phi - \xi^j{}_i\hat{\mathscr{D}}_j\Phi \tag{276}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]V^i = (w-d)\hat{\mathscr{R}}_i V^i - V^i\hat{\mathscr{D}}_j\xi^j{}_i - \xi^j{}_i\hat{\mathscr{D}}_j V^i, \tag{277}$$

with

$$\hat{\mathscr{R}}_i = \hat{r}_i + \frac{1}{\Omega}\hat{\partial}_t\varphi_i - \frac{1}{d}\hat{\nabla}_j\hat{\gamma}^j{}_i + \xi^j{}_i\varphi_j = \frac{1}{\Omega}\partial_t\varphi_i - \frac{1}{d}\left(\hat{\partial}_i + \varphi_i\right)\theta. \tag{278}$$

This is a Weyl-covariant weight-1 curvature one-form, where $\hat{r}_i$ is given in (262).

The Ricci-Weyl-Carroll tensor (275) is not symmetric in general: $\hat{\mathscr{R}}_{ij} \neq \hat{\mathscr{R}}_{ji}$. Using (257) we can recast it as

$$\hat{\mathscr{R}}_{ij} = \hat{s}_{ij} + \hat{\mathscr{K}}a_{ij} + \hat{\mathscr{A}}\eta_{ij}. \tag{279}$$

In three dimensions we can rewrite the Weyl-covariant scalar-electric and scalar-magnetic Gauss-Carroll curvatures as

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathscr{A}} = \frac{1}{2}\tilde{\eta}^{ij}\hat{\mathscr{R}}_{ij} = \hat{A} - \star\varphi, \tag{280}$$

with $\hat{K}$ and $\hat{A}$ defined for 3 dimensions in (259).

Before closing the present section, it is desirable to make a clarification, useful for the three-dimensional theory, to which we specify here. Weyl transformations (264) should not be confused with the action of the conformal Carroll group, which is a subset of Carrollian diffeomorphisms defined as [82]

$$\text{CCarr}\left(\mathbb{R}\times\mathcal{S}, d\ell^2, \underline{u}\right) = \left\{\phi \in \text{Diff}(\mathbb{R}\times\mathcal{S}), \quad d\ell^2 \xrightarrow{\phi} e^{-2\Phi}d\ell^2 \quad \underline{u} \xrightarrow{\phi} e^{\Phi}\underline{u}\right\}, \tag{281}$$

where $\Phi \in \mathcal{C}^\infty(\mathbb{R}\times\mathcal{S})$, $d\ell^2 = a_{ij}dx^i dx^j$ is the spatial metric on $\mathcal{S}$, and $\underline{u} = \frac{1}{\Omega}\partial_t$ the Carrollian time arrow.

This group is actually the zero-k contraction of CIsom $\left(\mathcal{H}, ds^2\right)$, the group of conformal isometries of the original finite-k relativistic metric ds^2 on the boundary $\mathcal{H}$ of the corresponding AdS bulk:

$$\text{CIsom}\left(\mathcal{H}, ds^2\right) = \left\{\phi \in \text{Diff}(\mathcal{H}), \quad ds^2 \xrightarrow{\phi} e^{-2\Phi}ds^2\right\} \tag{282}$$

with $\Phi \in \mathcal{C}^\infty(\mathcal{H})$. Indeed, consider the Lie algebra of conformal symmetries of ds^2, denoted $\mathfrak{cisom}\left(\mathcal{H}, ds^2\right)$ and spanned by vector fields $\underline{X} = X^0\partial_0 + X^i\partial_i$ such that

$$\mathcal{L}_{\underline{X}}ds^2 = -2\lambda ds^2 \tag{283}$$

for some function λ on $\mathcal{H}$.

In order to perform the zero-k contraction we write the generators as $\underline{X} = kX^t\partial_0 + X^i\partial_i$ (here $x^0 = kt$, thus $X^0 = kX^t$) and the metric ds^2 in the Randers-Papapetrou form (148). At zero k, (283) splits into:

$$\mathcal{L}_{\underline{X}}u = \lambda u, \quad \mathcal{L}_{\underline{X}}d\ell^2 = -2\lambda d\ell^2. \tag{284}$$

These are the equations the field $\underline{X}$ must satisfy for belonging to $\mathfrak{ccarr}\left(\mathbb{R}\times\mathcal{S}, d\ell^2, \underline{u}\right)$, the Lie algebra of the corresponding conformal Carroll group. This confirms that

$$\text{CIsom}\left(\mathcal{H}, ds^2\right) \underset{k\to 0}{\longrightarrow} \text{CCarr}\left(\mathbb{R}\times\mathcal{S}, d\ell^2, \underline{u}\right). \tag{285}$$

At last, if $\mathcal{S}$ is chosen to be the two-sphere and $d\ell^2$ the round metric, it can be shown that the corresponding conformal Carroll group is precisely the BMS(4) group, which describes the asymptotic symmetries of an asymptotically flat $3+1$-dimensional metric [82, 118].

Carrollian Covariance

In order to take the $k \to 0$ limit of the fluid equations of motion, we need to compute the relativistic Christoffel symbols. This chapter will allow us later on to elegantly check the Carroll covariance of the resulting Carrollian fluid equations.

The Randers-Papapetrou metric (148) has components (in the coframe $\left\{dx^0 = kdt, dx^i\right\}$):

$$g_{\mu\nu} \to \begin{pmatrix} -\Omega^2 & k\Omega b_j \\ k\Omega b_i & a_{ij} - k^2 b_i b_j \end{pmatrix}, \quad g^{\mu\nu} \to \frac{1}{\Omega^2}\begin{pmatrix} -1 + k^2 b^2 & k\Omega b^j \\ k\Omega b^i & \Omega^2 a^{ij} \end{pmatrix}, \tag{286}$$

where $b^k = a^{kj}b_j$. The metric determinant is:

$$\sqrt{-g} = \Omega\sqrt{a}. \tag{287}$$

Here, Ω, a_{ij} and b_i depend on time t and space $\boldsymbol{x}$.

The Christoffel symbols are computed exactly:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{k}\partial_t \ln\Omega + k\left(b^i\partial_i\Omega + \frac{1}{2}\left(\partial_t b^2 - b_i b_j \partial_t a^{ij}\right)\right), && (288)\\
\Gamma^0_{0i} &= \left(1-\frac{1}{2}k^2b^2\right)\partial_i \ln\Omega + \frac{1}{2}k^2 b^j\left(\partial_i b_j - \partial_j b_i - b_i\partial_j\ln\Omega\right)\\
&\quad + \frac{1}{2\Omega}b^j\partial_t\left(a_{ij}-k^2b_ib_j\right), && (289)\\
\Gamma^0_{ij} &= -\frac{k}{2\Omega}\left(\partial_i b_j + \partial_j b_i + k^2 b^n\left(b_i\left(\partial_j b_n - \partial_n b_j\right) + b_j\left(\partial_i b_n - \partial_n b_i\right)\right)\right)\\
&\quad + \frac{kb_n}{\Omega}\gamma^n_{ij} + \frac{1-k^2b^2}{2\Omega^2}\left(\frac{1}{k}\partial_t a_{ij} - kb_j\left(\partial_t b_i + \partial_i\Omega\right) - cb_i\left(\partial_t b_j + \partial_j\Omega\right)\right), && (290)\\
\Gamma^i_{00} &= \Omega a^{ij}\left(\partial_t b_j + \partial_j\Omega\right), && (291)\\
\Gamma^i_{j0} &= \frac{1}{2k}a^{in}\left(\partial_t\left(a_{nj}-k^2b_nb_j\right) + k^2\Omega\left(\partial_j b_n - \partial_n b_j\right) - k^2\left(b_n\partial_j\Omega + b_j\partial_n\Omega\right)\right), && (292)\\
\Gamma^i_{jn} &= \frac{k^2}{2}\left(\frac{b^i}{\Omega}\left(b_j\left(\partial_t b_n + \partial_n\Omega\right) + b_n\left(\partial_t b_j + \partial_j\Omega\right)\right) - a^{il}\left(b_j\left(\partial_n b_l - \partial_l b_n\right) + b_n\left(\partial_j b_l - \partial_l b_j\right)\right)\right)\\
&\quad + \gamma^i_{jn} - \frac{b^i}{2\Omega}\partial_t a_{jn}, && (293)
\end{aligned}$$

where γ^i_{jn} are the d-dimensional Christoffel symbols:

$$\gamma^i_{jn} = \frac{a^{il}}{2}\left(\partial_j a_{ln} + \partial_n a_{lj} - \partial_l a_{jn}\right), \tag{294}$$

which intervene in the definition of the Levi-Civita-Carroll connection (cf (248))

$$\hat\gamma^i_{jn} = \frac{a^{il}}{2}\left(\hat\partial_j a_{ln} + \hat\partial_n a_{lj} - \hat\partial_l a_{jn}\right) = \gamma^i_{jn} + c^i_{jn}. \tag{295}$$

Note also

$$\Gamma^\mu_{\mu 0} = \frac{1}{k}\partial_t \ln\left(\sqrt{a}\Omega\right), \quad \Gamma^\mu_{\mu i} = \partial_i \ln\left(\sqrt{a}\Omega\right). \tag{296}$$

With these data we will compute the divergence of the fluid energy-momentum tensor (as later reported in (323) and (324)).

In order to check the covariance of the fluid equations under Carrollian diffeomorphisms we can use several simple covariant blocks:

$$\begin{aligned}
\frac{1}{\Omega'}\partial'_t a'_{ij} &= \frac{1}{\Omega}\partial_t a_{nl} J^{-1n}{}_i J^{-1l}{}_j, && (297)\\
\frac{1}{\Omega'}\partial'_t \ln\sqrt{a'} &= \frac{1}{\Omega}\partial_t \ln\sqrt{a}, && (298)\\
\partial'_t b'_i + \partial'_i\Omega' &= \frac{1}{J}J^{-1j}{}_i\left(\partial_t b_j + \partial_j\Omega\right), && (299)\\
\hat\partial'_i &= J^{-1j}{}_i\hat\partial_j, && (300)
\end{aligned}$$

Using that the action of the Levi-Civita-Carroll covariant derivative on a scalar Φ, a vector V^i and a tensor S_{jn} is

$$\begin{aligned}
\hat\partial_i\Phi &= \partial_i\Phi + \frac{b_i}{\Omega}\partial_t\Phi, && (301)\\
\hat\nabla_i V^j &= \partial_i V^j + \frac{b_i}{\Omega}\partial_t V^j + \hat\gamma^j_{il}V^l\\
&= \nabla_i V^j + \frac{b_i}{\Omega}\partial_t V^j + c^j_{il}V^l, && (302)\\
\hat\nabla_i V^i &= \frac{1}{\sqrt{a}}\hat\partial_i\left(\sqrt{a}V^i\right) && (303)\\
\hat\nabla_i S_{jn} &= \partial_i S_{jn} + \frac{b_i}{\Omega}\partial_t S_{jn} - \hat\gamma^l_{ij}S_{ln} - \hat\gamma^l_{in}S_{jl}\\
&= \nabla_i S_{jn} + \frac{b_i}{\Omega}\partial_t S_{jn} - c^l_{ij}S_{ln} - c^l_{in}S_{jl}, && (304)
\end{aligned}$$

we can show that these transform as genuine tensors, namely:

$$\hat{\partial}'_i \Phi' = J^{-1j}{}_i \hat{\partial}_j \Phi, \tag{305}$$

$$\hat{\nabla}'_i V'^j = J^{-1n}{}_i J^j_l \hat{\nabla}_n V^l, \tag{306}$$

$$\hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i, \tag{307}$$

$$\hat{\nabla}'_i S'_{jh} = J^{-1m}{}_i J^{-1n}{}_j J^{-1l}{}_h \hat{\nabla}_m S_{nl}. \tag{308}$$

Further elementary transformation rules are as follows:

$$\frac{1}{\Omega'}\partial'_t \Phi' = \frac{1}{\Omega}\partial_t \Phi, \quad \frac{1}{\Omega'}\partial'_t V'^i = J^i_j \frac{1}{\Omega}\partial_t V^j, \quad \frac{1}{\Omega'}\partial'_t S'^{ij} = J^i_n J^j_l \frac{1}{\Omega}\partial_t S^{nl}, \tag{309}$$

as well as

$$\nabla'_i V'^i + \frac{b'_i}{\Omega'\sqrt{a'}}\partial'_t\left(\sqrt{a'}V'^i\right) = \hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i = \nabla_i V^i + \frac{b_i}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}V^i\right), \tag{310}$$

and

$$\begin{aligned}\nabla'_n S'^{ni} + \tfrac{b'_n}{\Omega'\sqrt{a'}}\left(\partial'_t\left(\sqrt{a'}S'^{ni}\right) - \sqrt{a'}S'^n{}_j \partial'_t a'^{ij}\right) - \tfrac{b'^i}{2\Omega'}S'^{nl}\partial'_t a'_{nl} = \hat{\nabla}'_n S'^{ni} = \\ = J^i_j \hat{\nabla}_n S^{nj} = J^i_j\left(\nabla_n S^{kj} + \tfrac{b_n}{\Omega\sqrt{a}}\left(\partial_t\left(S^{nj}\sqrt{a}\right) - \sqrt{a}S^n{}_l \partial_t a^{jl}\right) - \tfrac{b^j}{2\Omega}S^{nl}\partial_t a_{nl}\right).\end{aligned} \tag{311}$$

All these transformation rules play a key role in showing that the fluid equations are Carroll covariant once the limit $k \to 0$ has been implemented.

3.2 Equations of Motion Limit

We have seen that the geometry in the $k \to 0$ limit degenerates to what is called a Carrollian geometrical structure. Nothing wrong is undergoing here, simply the geometry at hand is not the usual pseudo-Riemannian one, and one has to accordingly be cautious with the limit of the various geometrical tensors. In some respects, at the level of the geometry this limit is a dimensional reduction, similar to a Kaluza-Klein reduction.

The geometrical setup being settled, we may now wonder what happens to the fluid conservation of the energy-momentum tensor in this limit. The resulting equations will dictate the equations of motion of the so-called Carrollian fluid, living on the null infinity $\mathscr{I}$ of asymptotically flat spacetimes. In a more general fashion, Carrollian fluids are a completely disentangled concept from holography, for they are self-consistent.

3.2.1 For Arbitrary Fluid and Dimension

Preliminary Remarks

As Carrollian particles, Carrollian fluids have no motion. From a relativistic perspective this is an observer-dependent statement, since boosts can turn on velocity. In the limit of vanishing velocity of light, however, these boosts are no longer permitted. Hence, being at rest becomes a genuinely intrinsic feature.

The fluid velocity must be set to zero faster than k[42] in order to avoid blow-ups in the energy-momentum conservation. The appropriate scaling, ensuring a non-trivial kinematic contribution is

$$v^i = k^2 \Omega \beta^i + O\left(k^4\right), \tag{312}$$

where $v^i = {}^{u^i}\!/\!_\gamma$. This leaves the Carrollian fluid with a kinematic variable $\underline{\beta} = \beta^i \partial_i$ of inverse-velocity dimension. We keep this dynamical degree of freedom in our general construction, even if for three-dimensional holographic Carrollian fluids it will turn out to vanish.

In order to reach covariant Carrollian fluid equations by expanding the relativistic fluid equations at small k, we need to define β^i in such a way that it transforms as components of a genuine Carrollian vector under (242) already at finite k. This is achieved by setting ($\beta^2 = \beta^j \beta_j$ and $\beta \cdot b = \beta^j b_j$)

$$v^i = \frac{k^2 \Omega \beta^i}{1 + k^2 \beta \cdot b} \Leftrightarrow \beta^i = \frac{v^i}{k^2 \Omega\left(1 - \frac{v \cdot b}{\Omega}\right)}, \tag{313}$$

[42] We would like to insist again on the fact that k is the speed of light (usually spelled c) for the hydrodynamic theory.

from which one checks that[43]

$$\beta'^i = J^i_j \beta^j. \tag{314}$$

The full fluid congruence reads then:

$$\begin{cases} u^0 = \gamma k = \dfrac{k}{\Omega}\dfrac{1+k^2\beta\cdot b}{\sqrt{1-k^2\beta^2}} = \dfrac{k}{\Omega} + O\left(k^3\right), & u_0 = -\dfrac{k\Omega}{\sqrt{1-k^2\beta^2}} = -k\Omega + O\left(k^3\right), \\ u^i = \gamma v^i = \dfrac{k^2\beta^i}{\sqrt{1-k^2\beta^2}} = k^2\beta^i + O\left(k^4\right), & u_i = \dfrac{k^2\left(b_i+\beta_i\right)}{\sqrt{1-k^2\beta^2}} = k^2\left(b_i+\beta_i\right) + O\left(k^4\right), \end{cases} \tag{315}$$

where the Lorentz factor has been obtained by imposing the usual normalization $\|u\|^2 = -k^2$:

$$\gamma = \frac{1+k^2\beta\cdot b}{\Omega\sqrt{1-k^2\beta^2}} = \frac{1}{\Omega}\left(1+\frac{k^2}{2}\beta\cdot(\beta+2b) + O\left(k^4\right)\right). \tag{316}$$

In the relativistic regime, i.e. before taking the zero-k limit, in the Randers-Papapetrou background (148) the perfect part of the energy-momentum tensor reads then:

$$\begin{aligned} T_{\text{perf}}{}^0{}_0 &= -\varepsilon - k^2(\varepsilon+p)\beta^l\left(b_l+\beta_l\right) + O\left(k^4\right), & (317)\\ k\Omega T_{\text{perf}}{}^0{}_i &= k^2(\varepsilon+p)\left(b_i+\beta_i\right) + O\left(k^4\right), & (318)\\ \frac{k}{\Omega} T_{\text{perf}}{}^j{}_0 &= -k^2(\varepsilon+p)\beta^j + O\left(k^4\right), & (319)\\ T_{\text{perf}}{}^j{}_i &= p\delta^j_i + k^2(\varepsilon+p)\beta^j\left(b_i+\beta_i\right) + O\left(k^4\right). & (320) \end{aligned}$$

Notice, on the one hand, that for vanishing β^i, these expressions are exact at finite k: most of the terms of order k^2 vanish as do all non-displayed higher-order contributions in k^2; on the other hand, for vanishing k, one recovers the perfect energy-momentum of a fluid at rest due to the simultaneous vanishing of v^i as a consequence of (312).

The eventual absence of motion, macroscopic or microscopic, and the shrinking of the light-cone raise many fundamental questions regarding the origin of pressure, temperature, thermalization, entropy etc. One may wonder in particular what causes viscosity and thermal conduction, what replaces the temperature derivative expansion of q_i and so on. Even the propagation of a signal such as sound, if possible, should be reconsidered. We have no definite answers to all these questions though. Our approach will be kinematical, aiming at writing the fundamental equations, covariant under Carrollian diffeomorphisms (242), starting from the relativistic equations (22). Alternative paths may exist, allowing to build some Carrollian dynamics without using the zero-k limit of a relativistic fluid, as for instance [97, 130].

The Structure of the Equations

The relativistic equations (22) should now be presented as

$$\nabla_\mu T^\mu{}_0 = 0, \quad \nabla_\mu T^{\mu i} = 0. \tag{321}$$

Under Carrollian diffeomorphisms (242), the divergence of the energy-momentum tensor transforms as:

$$\nabla'_\mu T'^\mu{}_0 = \frac{1}{J}\nabla_\mu T^\mu{}_0, \quad \nabla'_\mu T'^{\mu i} = J^i_l \nabla_\mu T^{\mu l}. \tag{322}$$

The two sets of equations (321) have separately a d-dimensional covariant transformation. This is part of the agenda for the Carrollian dynamics. Equations (321) are relativistic. Using the general energy-momentum tensor (21), we will show explicitly that we generally find:

$$\begin{aligned} \frac{k}{\Omega}\nabla_\mu T^\mu{}_0 &= \frac{1}{k^2}\mathcal{F} + \mathcal{E} + O\left(k^2\right), & (323)\\ \nabla_\mu T^{\mu i} &= \frac{1}{k^2}\mathcal{H}^i + \mathcal{G}^i + O\left(k^2\right). & (324) \end{aligned}$$

[43] This is easily proven by observing that $\beta_i + b_i = -\frac{\Omega u_i}{k u_0}$.

For zero β^i, these expressions are exact[44] with extra terms of order k^2 only, and requiring they vanish leads to the $d+1$ fully relativistic fluid equations. With $\beta^i \neq 0$, (323) and (324) are genuinely infinite series. Thanks to the validity of (314) at finite k, Carrollian diffeomorphisms do not mix the different orders of these series, making each term Carrollian-covariant. Here, we are interested in the zero-k limit, and in this case (323) and (324) split into $2+2d$ distinct equations:

- energy conservation $\mathcal{E}=0$;
- momentum conservation $\mathcal{G}^i=0$;
- constraint equations $\mathcal{F}=0$ and $\mathcal{H}^i=0$.

All of these are covariant under Carrollian diffeomorphisms.

The Carrollian fluid, obtained as Carrollian limit of a relativistic fluid in the appropriate Randers-Papapetrou background, is described in terms of β^i (d components), and the two variables p and ε.[45] The latter are related through an equation of state and the energy-conservation equation $\mathcal{E}=0$.

As we will see soon, the other $2d+1$ equations are setting consistency constraints among the $2d$ components of the heat currents Q_i and π_i (see below), the $d(d+1)$ components of the viscous stress tensors Σ_{ij} and Ξ_{ij}, the inverse-velocity components β^i and the geometric environment. Geometry is therefore expected to interfere more actively in the dynamics of Carrollian fluids than it did for Galilean hydrodynamics. Some of the aforementioned constraints are possibly rooted to more fundamental microscopic/geometric properties, yet to be unravelled.

Dissipative Tensors

In view of the subsequent steps of our analysis, an important question arises at this stage, which concerns the behaviour of q_i and τ_{ij} with respect to the velocity of light. Answering this question requires a microscopic understanding of the fluid i.e. a many-body (quantum-field-theory and statistical-mechanics) determination of the transport coefficients. In the absence of this knowledge, we may consider a large-k or small-k expansion of these quantities, in powers of k^2.

In the same spirit, we could also work out similar expansions for each of the functions entering the metric (148), as it possibly carries deep relativistic dynamics. The advantage of such an exhaustive analysis would be to set-up general conditions on a relativistic fluid and its spacetime environment for a large-k or a small-k regime to make sense. As a drawback, this approach would blur the universality of the equations we want to set.

We will therefore adopt a more pragmatic attitude and assume that Ω, b_i and a_{ij} are k-independent. Regarding the viscous stress tensor τ_{ij}, we will assume the following behaviours:[46]

$$\tau^{ij} = -\frac{\Sigma^{ij}}{k^2} - \Xi^{ij}. \tag{325}$$

This choice is inspired by flat-spacetime holography, where all the examples so far studied have this structure. This examples include all Petrov D asymptotically flat solutions and the Robinson-Trautman case. Similarly, for the heat current, we will adopt

$$q^i = Q^i + k^2 \pi^i. \tag{326}$$

The position of the spatial indices are designed to be covariant under Carrollian diffeomorphisms. One should notice that, in writing the energy-momentum tensor (21), we have not made any assumption regarding the hydrodynamic frame, which is therefore left generic, as we already discussed intensively.

Using now the velocity field in (312) and (315), the transversality conditions (24) in the Randers-Papapetrou background lead to

$$q^0 = \frac{k}{\Omega}\left(b_i + \beta_i\right) q^i, \quad q_0 = -k\Omega\beta_i q^i, \quad q_i = \left(a_{ij} + k^2 b_i \beta_j\right) q^j. \tag{327}$$

[44] This result is true for the particular structure of the dissipative tensors present in the next section, see (325) and (326).

[45] The proper energy density cannot be split in mass density and energy per mass, because the limit at hand is ultra-relativistic. Observe also that b is not a fluid variable but a Carrollian-frame parameter. The fluid kinematical variable is β.

[46] The viscous stress tensor diverges as $k \to 0$. This is not a problem nor a contradiction, for what matters in the limit are the equations of motion.

Similarly, the components of the viscous stress tensor are obtained from τ^{ij}:

$$\tau^{00} = \frac{k^2}{\Omega^2}(b_n+\beta_n)(b_l+\beta_l)\tau^{nl}, \tag{328}$$

$$\tau^{0i} = \frac{k}{\Omega}(b_n+\beta_n)\tau^{in}, \tag{329}$$

$$\tau_{00} = k^2\Omega^2\beta_n\beta_l\tau^{nl}, \tag{330}$$

$$\tau_{0i} = -k\Omega\beta_j\left(a_{in}+k^2b_i\beta_n\right)\tau^{jn}, \tag{331}$$

$$\tau_{ij} = \left(a_{in}+k^2b_i\beta_n\right)\left(a_{jl}+k^2b_j\beta_l\right)\tau^{nl}. \tag{332}$$

Under Carrollian diffeomorphisms (242) we obtain the following transformation rules

$$q'^i = q^j J^i_j, \quad \tau'^{ij} = \tau^{nl}J^i_nJ^j_l. \tag{333}$$

As remarked, this suggests to use q^i as components for the Carrollian d-dimensional heat current decomposed as in (326), and τ^{ij} for the Carrollian d-dimensional viscous stress tensors Σ^{ij} and Ξ^{ij} defined in (325).

We introduce as usual

$$Q_i = a_{ij}Q^j, \quad \Sigma^j{}_i = a_{il}\Sigma^{lj}, \quad \Sigma_{ij} = a_{jl}\Sigma^l{}_i, \tag{334}$$

$$\pi_i = a_{ij}\pi^j, \quad \Xi_i{}^j = a_{il}\Xi^{lj}, \quad \Xi_{ij} = a_{jl}\Xi_i{}^l. \tag{335}$$

Using the generic transformations (333) under Carrollian diffeomorphisms, we find that the above quantities transform as they should, for being eligible as d-dimensional tensors:

$$Q'_i = Q_jJ^{-1j}{}_i, \quad Q'^i = J^i_jQ^j, \tag{336}$$

$$\Sigma'_{ij} = J^{-1n}{}_iJ^{-1l}{}_j\Sigma_{nl}, \quad \Sigma'^j_i = J^{-1n}{}_i\Sigma^l{}_nJ^j_l, \quad \Sigma'^{ij} = \Sigma^{nl}J^i_nJ^j_l, \tag{337}$$

and similarly for π_i and Ξ_{jk}. We have eventually all the ingredients to simply insert everything into (323) and (324) and compute the four terms on the right-hand sides.

Scalar Equations

The computation of the spacetime divergence in (323) is straightforward and leads to the following:

$$\begin{aligned}\mathcal{E} = &-\left(\frac{1}{\Omega}\partial_t+\frac{d+1}{d}\theta\right)\left(\varepsilon+2\beta_iQ^i-\beta_i\beta_j\Sigma^{ij}\right)+\frac{1}{d}\theta\left(\Xi^i{}_i-\beta_i\beta_j\Sigma^{ij}+\varepsilon-dp\right)\\ &-\left(\hat{\nabla}_i+2\varphi_i\right)\left(Q^i-\beta_j\Sigma^{ij}\right)-\left(2Q^i\beta^j-\Xi^{ij}\right)\xi_{ij},\end{aligned} \tag{338}$$

$$\mathcal{F} = \Sigma^{ij}\xi_{ij}+\frac{1}{d}\Sigma^i{}_i\theta, \tag{339}$$

where we used the covariant derivative $\hat{\nabla}_i$ built using (248).

As already stated and readily seen by its equations, most of the fluid properties are of geometrical nature. In these equations we made use of all the various first oder derivatives: the acceleration (250), the vorticity (251) and the expansion and shear reported in (254).

With all our construction, we can elegantly check that (using e.g. (309) and (310))

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{F}' = \mathcal{F}. \tag{340}$$

Equation $\mathcal{F}=0$ sets a geometrical constraint on the Carrollian stress tensor Σ, whereas $\mathcal{E}=0$ is the energy conservation. The latter can be recast as follows:

$$\left(\frac{1}{\Omega}\partial_t+\theta\right)e_{\mathrm{e}} = -\left(\hat{\nabla}_i+2\varphi_i\right)\Pi^i-\Pi^{ij}\left(\xi_{ij}+\frac{1}{d}\theta a_{ij}\right), \tag{341}$$

written in terms of three Carrollian tensors, which capture the Carrollian energy exchanges:

$$e_{\mathrm{e}} = \varepsilon+2\beta_iQ^i-\beta_i\beta_j\Sigma^{ij}, \quad \Pi^i = Q^i-\beta_j\Sigma^{ij}, \quad \Pi^{ij} = Q^i\beta^j+\beta^iQ^j+pa^{ij}-\Xi^{ij}. \tag{342}$$

The first is a scalar e_{e}, which can be interpreted as an effective Carrollian energy density (observe the absence of kinetic energy, expected from the vanishing velocity). Its time variation, including the dilution/contraction effects due to the expansion, is driven by the gradient of a Carrollian energy flux, which is the vector Π^i, and by the coupling of the shear to a Carrollian flux tensor Π^{ij}.

Vector Equations

The vectorial part of the divergence is obtained from (324) and has two pieces. The first reads

$$\begin{aligned}\mathcal{G}_j &= \left(\hat{\nabla}_i+\varphi_i\right)\Pi^i{}_j+\varphi_j e_{\mathsf{e}}+2\Pi^i\varpi_{ij}+\left(\frac{1}{\Omega}\partial_t+\theta\right)\left(\pi_j+\beta_j\left(e_{\mathsf{e}}-2\beta_i\Pi^i-\beta_i\beta_n\Sigma^{in}\right)\right)\\ &\quad+\left(\frac{1}{\Omega}\partial_t+\theta\right)\left(\beta^n\left(\Pi_{nj}-\frac{1}{2}\beta_n\Pi_j-\frac{1}{2}\beta_n\beta^i\Sigma_{ij}\right)\right),\end{aligned} \tag{343}$$

while the second

$$\mathcal{H}_j=-\left(\hat{\nabla}_i+\varphi_i\right)\Sigma^i{}_j+\left(\frac{1}{\Omega}\partial_t+\theta\right)\Pi_j. \tag{344}$$

Equation $\mathcal{G}_j=0$ involves ε, p and their temporal and/or spatial derivatives, β, the heat current Q, and Ξ, expressed in terms of the effective energy density e_{e}, the Carrollian energy flux and flux tensor Π, as well as π and Σ. It is a momentum conservation. Notice also the coupling of the energy flux to the inertial vorticity.

Equation $\mathcal{H}_j=0$ depends neither on ε nor on p. This is an equation for the Carrollian energy flux Π and the viscous stress tensor Σ, of geometrical nature as it involves the metric a, the Carrollian connection b and the inertial acceleration φ. Under Carrollian diffeomorphisms (242) we obtain (see (311)):

$$\mathcal{G}'^i=J^i_j\mathcal{G}^j,\quad \mathcal{H}'^i=J^i_j\mathcal{H}^j. \tag{345}$$

One should observe at this point that Π and the energy flux associated with a Carrollian fluid defined in (342) are merely a repackaging of part of the dynamical data. Equation $\mathcal{F}=0$, as well as the vector equations need indeed more informations. There is pressure, energy density and velocity, on the one hand, and on the other hand, we find the two heat currents and the two viscous stress tensors. The zero-k limit produces a decoupling in the equations. This is the reason why $\mathcal{H}_j=0$ appears as an equation for the dissipative pieces of data only, while the non-dissipative ones mix with the heat currents inside $\mathcal{G}_j=0$.

First-order Carrollian Hydrodynamics

In order to acquire a better perspective on Carrollian fluid dynamics, we can study the first-order derivative expansion of its viscous tensors and heat currents. The first-derivative relativistic kinematical tensors as acceleration (27), expansion (28), shear (29), and vorticity (30), for a fluid with velocity behaving as (312) when $k\to 0$ read (the only independent components are the spatial ones):

$$a_i=\frac{k^2}{\Omega}\left(\partial_t\left(b_i+\beta_i\right)+\partial_i\Omega\right)+O\left(k^4\right)=k^2\left(\varphi_i+\gamma_i\right)+O\left(k^4\right), \tag{346}$$

$$\Theta=\frac{1}{\Omega}\partial_t\ln\sqrt{a}+O\left(k^2\right)=\theta+O\left(k^2\right), \tag{347}$$

$$\sigma_{ij}=\frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij}-\frac{1}{d}a_{ij}\partial_t\ln\sqrt{a}\right)+O\left(k^2\right)=\xi_{ij}+O\left(k^2\right), \tag{348}$$

$$\omega_{ij}=k^2\left(\partial_{[i}b_{j]}+\frac{1}{\Omega}b_{[i}\partial_{j]}\Omega+\frac{1}{\Omega}b_{[i}\partial_t b_{j]}+w_{ij}\right)+O\left(k^4\right)=k^2\left(\varpi_{ij}+w_{ij}\right)+O\left(k^4\right). \tag{349}$$

We find the corresponding Carrollian expansion θ and shear ξ_{ij}. These quantities are purely geometric and originate from the time dependence of the d-dimensional spatial metric. Similarly, the relativistic acceleration and vorticity allow to define the already introduced Carrollian, inertial acceleration φ_i and vorticity ϖ_{ij}, as well as the kinematical acceleration γ_i and kinematical vorticity w_{ij} defined as:

$$\gamma_i = \frac{1}{\Omega}\partial_t\beta_i, \tag{350}$$

$$w_{ij} = \hat{\partial}_{[i}\beta_{j]}+\beta_{[i}\varphi_{j]}+\beta_{[i}\gamma_{j]}. \tag{351}$$

Starting from the first-order relativistic viscous tensor (25) and heat current (26), in order to comply with the behaviours (325) and (326), we must assume that (up to possible higher orders in k^2)

$$\eta=\tilde{\eta}+\frac{\eta^C}{k^2},\quad \zeta=\tilde{\zeta}+\frac{\zeta^C}{k^2},\quad \kappa=k^2\tilde{\kappa}+\kappa^C. \tag{352}$$

Hence, putting these equations together, we find

$$\Sigma_{(1)ij} = 2\eta^C\xi_{ij} + \zeta^C\theta a_{ij}, \tag{353}$$

$$\begin{aligned} Q_{(1)i} &= -\frac{\kappa^C}{\Omega}\left(\partial_t(b_iT) + \beta_i\partial_t T + \partial_i(\Omega T)\right) \\ &= -\kappa^C\left(\hat{\partial}_i T + T\left(\varphi_i + \gamma_i\right)\right), \end{aligned} \tag{354}$$

and similarly for $\Xi_{(1)ij}$ and $\pi_{(1)i}$. These quantities will include respectively terms like $2\tilde{\eta}\xi_{ij} + \tilde{\zeta}\theta a_{ij}$ and $-\tilde{\kappa}\left(\hat{\partial}_i T + T\left(\varphi_i + \gamma_i\right)\right)$, plus extra terms coupled to η^C, ζ^C and κ^C, and originating from higher-order contributions in the k^2-expansion of the relativistic shear, acceleration and expansion. Notice that these are absent for vanishing β^i because in this case (346), (347), (348) and (349) are exact.

All the above expressions are covariant under Carrollian diffeomorphisms. The friction phenomena are geometric and due to time evolution of the background metric a_{ij}. The heat conduction depends also on the temperature, its microscopic understanding in Carrollian physic yet unknown. In the two-dimensional case one should take into account the Hall viscosity (44) in the relativistic viscous tensor at first order. Assuming again $\zeta_{\mathsf{H}} = \tilde{\zeta}_{\mathsf{H}} + \frac{\zeta^C_{\mathsf{H}}}{k^2}$, the extra term to be added to $\Sigma_{(1)ij}$ in (353) reads:

$$\zeta^C_{\mathsf{H}}\sqrt{a}\epsilon_{k(i}\xi_{j)l}a^{kl}, \tag{355}$$

and similarly for $\Xi_{(1)ij}$ with transport coefficients $\tilde{\zeta}_{\mathsf{H}}$ and ζ^C_{H} as already explained. The final first-order Carrollian equations are obtained by substituting $\Sigma_{(1)ij}$ and $Q_{(1)i}$ given in (353) and (354), and similarly for $\Xi_{(1)ij}$, and $\pi_{(1)i}$, inside the general expressions for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}_i$ and $\mathcal{H}_i$ derived above.

Conformal Carrollian Fluids

Carrollian fluids are ultra-relativistic and are thus compatible with conformal symmetry. For conformal relativistic fluids the energy-momentum tensor (21) is traceless and this requires

$$\varepsilon = dp, \quad \tau^\mu{}_\mu = 0. \tag{356}$$

In the Carrollian limit, the latter reads:

$$\Xi^i{}_i = \beta_i\beta_j\Sigma^{ij}, \quad \Sigma^i{}_i = 0. \tag{357}$$

In particular, we find $e_{\mathsf{e}} = \Pi^i{}_i$.

The dynamics of conformal fluids is covariant under Weyl transformations. Those act on the fluid variables as

$$\varepsilon \to \mathcal{B}^{d+1}\varepsilon, \quad \pi_i \to \mathcal{B}^d\pi_i, \quad Q_i \to \mathcal{B}^d Q_i, \quad \Xi_{ij} \to \mathcal{B}^{d-1}\Xi_{ij}, \quad \Sigma_{ij} \to \mathcal{B}^{d-1}\Sigma_{ij}, \tag{358}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The elements of the Carrollian geometry behave as (264). Moreover

$$\beta_i \to \frac{1}{\mathcal{B}}\beta_i, \quad \varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}, \quad w_{ij} \to \frac{1}{\mathcal{B}}w_{ij}, \quad \xi_{ij} \to \frac{1}{\mathcal{B}}\xi_{ij}. \tag{359}$$

The Carrollian inertial and kinematical accelerations, and the Carrollian expansion (347) transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln\mathcal{B}, \quad \gamma_i \to \gamma_i - \frac{\beta_i}{\Omega}\partial_t \ln\mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{d}{\Omega}\partial_t\mathcal{B}. \tag{360}$$

The first and the latter enable to define Weyl-Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$, as discussed in (269) and (270). With these derivatives, Carrollian expressions (338), (339), (343) and (344) read for a conformal fluid:

$$\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t e_{\mathsf{e}} - \hat{\mathscr{D}}_i\Pi^i - \Pi^{ij}\xi_{ij}, \tag{361}$$

$$\mathcal{F} = \Sigma^{ij}\xi_{ij}, \tag{362}$$

$$\begin{aligned} \mathcal{G}_j &= \hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)\left(\pi_i + \beta_i\left(e_{\mathsf{e}} - 2\beta_n\Pi^n - \beta_n\beta_l\Sigma^{nl}\right)\right) \\ &\quad + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)\left(\beta^n\left(\Pi_{ni} - \frac{1}{2}\beta_n\Pi_i - \frac{1}{2}\beta_n\beta^l\Sigma_{li}\right)\right), \end{aligned} \tag{363}$$

$$\mathcal{H}_j = -\hat{\mathscr{D}}_i\Sigma^i{}_j + \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi_j + \Pi_i\xi^i{}_j. \tag{364}$$

These equations are Weyl-covariant of weights $d+2$, $d+2$, $d+1$ and $d+1$.

The case of conformal Carrollian perfect fluids is remarkably simple. $\mathcal{F} = \mathcal{H}^i = 0$ are indeed automatically satisfied and

$$\mathcal{E} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t\varepsilon, \quad \mathcal{G}_j = \frac{1}{d}\hat{\mathcal{D}}_j\varepsilon + \frac{d+1}{d}\left(\frac{1}{\Omega}\hat{\mathcal{D}}_t\delta^i_j + \xi^i{}_j\right)\varepsilon\beta_i. \tag{365}$$

For these fluids the energy density is covariantly constant with respect to the Weyl-Carroll time derivative.

Conformal fluids play a particular role in this work, since they are eventually the holographic fluids we will be interested on. This was already true for the relativistic AdS situation, and will continue to hold in the Carrollian limit.

3.2.2 Conformal Carrollian Fluid in Three Dimensions

We will specialize here to three dimensions and conformal Carrollian fluids. These are the boundary data configurations to resum four-dimensional asymptotically flat bulk solution of Einstein equations, as we will scrutinize in the next section.

An important result holds for three-dimensional holographic fluids: they are always found to be with $\beta_i = 0$.[47] From this perspective these fluids are altogether even more geometrical, for they do not have any dynamical velocity. Nonetheless, they have non-trivial hydrodynamics based on the already spelled data that we recall here for the sake of clarity:

- the energy density ε and the pressure p, related here through a conformal equation of state $\varepsilon = 2p$;
- the heat currents $Q = Q_i dx^i$ and $\pi = \pi_i dx^i$;
- the viscous stress tensors $\Sigma = \Sigma_{ij}\, dx^i dx^j$ and $\Xi = \Xi_{ij} dx^i dx^j$.

They obey

$$\Sigma_{ij} = \Sigma_{ji}, \quad \Sigma^i{}_i = 0, \quad \Xi_{ij} = \Xi_{ji}, \quad \Xi^i{}_i = 0. \tag{366}$$

All these objects are Weyl-covariant with conformal weights 3 for the pressure and energy density, 2 for the heat currents, and 1 for the viscous stress tensors. They are well-defined in all examples we know from holography.

The equations for a Carrollian fluid are in dimension three as follows:

- a set of two scalar equations, both weight-4 Weyl-covariant:

$$-\frac{1}{\Omega}\hat{\mathcal{D}}_t\varepsilon - \hat{\mathcal{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0, \tag{367}$$

$$\Sigma^{ij}\xi_{ij} = 0; \tag{368}$$

- two vector equations, Weyl-covariant of weight 3:

$$\hat{\mathcal{D}}_j p + 2Q^i\varpi_{ij} + \frac{1}{\Omega}\hat{\mathcal{D}}_t\pi_j - \hat{\mathcal{D}}_i\Xi^i{}_j + \pi_i\xi^i{}_j = 0, \tag{369}$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t Q_j - \hat{\mathcal{D}}_i\Sigma^i{}_j + Q_i\xi^i{}_j = 0. \tag{370}$$

As already discussed in arbitrary dimension, (367) is the energy conservation, whereas (368) sets a geometrical constraint on the Carrollian viscous stress tensor Σ_{ij}. Equations (369) and (370) are dynamical equations involving the pressure $p = \varepsilon/2$, the heat currents Q_i and π_i, and the viscous stress tensors Σ_{ij} and Ξ_{ij}. They are reminiscent of a momentum conservation, although somewhat degenerate due to the absence of fluid velocity.

These equations are the main result here, and show the fate of the equations of motion in the $k \to 0$ limit for every three-dimensional conformal relativistic fluid.

[47] This is true for every algebraically special solution. Here indeed we are able to resum only a subset of bulk solutions. A possible attempt to include the complementary set of solutions could start by releasing this assumption.

3.2.3 Conformal Carrollian Fluid in Two Dimensions

Part of our scheme was to reconstruct AdS_3 solutions starting from a two-dimensional relativistic conformal fluid with conformal anomaly. These are very peculiar fluids, and two dimensions is also a particular setup, so we review here in details the main differences. The 2-dimensional Carrollian geometry $\mathbb{R} \times \mathcal{S}$ is obtained as the vanishing-k limit of the two-dimensional pseudo-Riemannian geometry $\mathcal{M}$ equipped with metric (130). In this limit, the line $\mathcal{S}$ inherits a metric[48]

$$d\ell^2 = a dx^2, \tag{371}$$

and $t \in \mathbb{R}$ is the Carrollian time.

The Carrollian frame is described by the form $b = b_x(t,x)\, dx$. In two dimensions the Carrollian derivative is written

$$\hat{\partial}_x = \partial_x + \frac{b_x}{\Omega}\partial_t, \tag{372}$$

and the Levi-Civita-Carroll connection becomes

$$\hat{\gamma}^x_{xx} = \hat{\partial}_x \ln \sqrt{a}. \tag{373}$$

The action of Weyl transformations on the elements of the Carrollian geometry and β_x is

$$a \to \frac{a}{\mathcal{B}^2}, \quad b_x \to \frac{b_x}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \quad \beta_x \to \frac{\beta_x}{\mathcal{B}}, \tag{374}$$

where $\mathcal{B} = \mathcal{B}(t, \boldsymbol{x})$ is an arbitrary function. Contrary to the three-dimensional scenario, here we can resum all bulk solutions, and the boundary fluid is as general as possible, including β.

As usual, we introduce the Carrollian acceleration φ_x and the Carrollian expansion θ,

$$\varphi_x = \frac{1}{\Omega}\left(\partial_t b_x + \partial_x \Omega\right) = \partial_t \frac{b_x}{\Omega} + \hat{\partial}_x \ln \Omega, \tag{375}$$

$$\theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \tag{376}$$

which transform as connections:

$$\varphi_x \to \varphi_x - \hat{\partial}_x \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{1}{\Omega}\partial_t \mathcal{B}. \tag{377}$$

In particular, these can be combined in

$$\alpha_x = \varphi_x - \theta b_x, \tag{378}$$

transforming under Weyl rescaling as

$$\alpha_x \to \alpha_x - \partial_x \ln \mathcal{B}. \tag{379}$$

The spatial Weyl-Carrol derivative is

$$\hat{\mathcal{D}}_x \Phi = \hat{\partial}_x \Phi + w \varphi_x \Phi, \tag{380}$$

for a weight-w scalar function Φ, and

$$\hat{\mathcal{D}}_x V^x = \hat{\nabla}_x V^x + (w-1)\varphi_x V^x, \tag{381}$$

for a vector with weight-w component V^x.

The temporal Weyl-Carroll derivative on a weight-w function Φ is here

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega}\partial_t \Phi + w\theta\Phi, \tag{382}$$

which is a scalar of weight $w+1$. Accordingly, the action of the Weyl-Carroll time derivative on a weight-w vector is

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t V^x = \frac{1}{\Omega}\partial_t V^x + w\theta V^x. \tag{383}$$

This is the component of a genuine Carrollian vector of weight $w+1$, and Leibinz rule allows to generalize this action to any tensor.

[48] This metric lowers all x indices. Here again, since the space is one-dimensional, we report its coordinate x not bold.

Here, the only non-vanishing piece of this derivative curvature is the one-form resulting from the commutation of $\hat{\mathcal{D}}_x$ and $\frac{1}{\Omega}\hat{\mathcal{D}}_t$, which has weight 1:

$$\mathcal{R}_x = \frac{1}{\Omega}\left(\partial_t \alpha_x - \partial_x(\theta\Omega)\right) = \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{384}$$

As stressed, the original relativistic fluid is not at rest, but has a velocity parametrized with $\beta = \beta_x dx$ This variable allows to define further kinematical objects.

- The acceleration $\gamma = \gamma_x dx$

$$\gamma_x = \frac{1}{\Omega}\partial_t\beta_x, \tag{385}$$

  which is not Weyl-covariant as opposed to the weight-0 object

$$\delta_x = \gamma_x - \theta\beta_x = \frac{\sqrt{a}}{\Omega}\partial_t\frac{\beta_x}{\sqrt{a}}. \tag{386}$$

- The weight-1 one-form (dubbed suracceleration)

$$\mathcal{A}_x = \frac{1}{\Omega}\hat{\mathcal{D}}_t\frac{1}{\Omega}\hat{\mathcal{D}}_t\beta_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right). \tag{387}$$

The latter can be combined with the curvature (384), which has equal weight,

$$s_x = \mathcal{A}_x + \mathcal{R}_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right) + \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{388}$$

This appears as a conformal Carrollian total, i.e. kinematical plus geometric, suracceleration, and enables us to define a weight-2 conformal Carrollian scalar:

$$s = \frac{s_x}{\sqrt{a}}. \tag{389}$$

The latter originates from the Weyl curvature F of the pseudo-Riemannian ascendent manifold $\mathcal{M}$:

$$s = -\lim_{k\to 0} kF. \tag{390}$$

Notice that the ordinary scalar curvature of $\mathcal{M}$ given in (98) is not Weyl-covariant (see (99)) and can be expressed in terms of Carrollian non-Weyl-covariant scalars of $\mathbb{R}\times\mathcal{S}$:

$$R = \frac{2}{k^2}\left(\theta^2 + \frac{1}{\Omega}\partial_t\theta\right) - 2\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x. \tag{391}$$

Besides the inverse velocity, acceleration and suracceleration, other physical data describe a Carrollian fluid.

- The energy density ε and the pressure p, related here through $\varepsilon = p$. The Carrollian energy and pressure are the zero-k limits of the corresponding relativistic quantities, and have weight 2.
- The heat current $\pi = \pi_x dx$ of conformal weight 1, inherited from the relativistic heat current as follows:[49]

$$q^x = k^2\pi^x + O\left(k^4\right). \tag{392}$$

  This translates the expected (see (136)) small-k behaviour of χ:

$$\chi = \chi_\pi k + O\left(k^3\right), \tag{393}$$

  leading to

$$\pi^x = \frac{\chi_\pi}{\sqrt{a}}. \tag{394}$$

[49] In arbitrary dimensions we generally admitted $q^x = Q^x + k^2\pi^x + O\left(k^4\right)$ (see (326)), which amounts assuming $\chi = \frac{\chi_Q}{k} + \chi_\pi k + O\left(k^3\right)$. This is actually more natural because vanishing χ_Q is not a hydrodynamic-frame-invariant feature in the presence of friction. Keeping $\chi_Q \neq 0$, however, is not viable from holography in two boundary dimensions because it would create a $1/k^2$ divergence inside the derivative expansion. Since the Carrollian limit affects anyway the hydrodynamic-frame invariance, our choice is consistent from every respect. Ultimately these behaviours should be justified within a microscopic quantum/statistical approach, missing at present.

- The weight-0 viscous stress tensors $\Sigma = \Sigma_{xx}dx^2$ and $\Xi = \Xi_{xx}dx^2$, obtained from the relativistic viscous stress tensor $\frac{\tau}{k^2} \star u \star u$ as
$$\tau^{xx} = -\frac{\Sigma^{xx}}{k^2} - \Xi^{xx} + O\left(k^2\right). \tag{395}$$
For this to hold, following (137), we expect
$$\tau = \frac{\tau_\Sigma}{k^2} + \tau_\Xi + O\left(k^2\right), \tag{396}$$
and find (in the Carrollian geometry, indices are lowered with $a_{xx} = a$):
$$\Sigma^x{}_x = -\tau_\Sigma, \quad \Xi^x{}_x = -\tau_\Xi - \beta^2\tau_\Sigma. \tag{397}$$
As we will see later, this is in agreement with the form of τ for the relativistic systems at hand (see Eqs. (391) and (196)).
- Finally, we assume that the components of the external force density behave as follows, providing further Carrollian power and tension:
$$\begin{cases} \frac{k}{\Omega}f_0 = \frac{f}{k^2} + e + O\left(k^2\right), \\ f^x = \frac{h^x}{k^2} + g^x + O\left(k^2\right). \end{cases} \tag{398}$$
This is again a posteriori justified by the success of the bulk reconstruction.

Eventually we are ready to present the equations of motion in this case:

$$-\left(\frac{1}{\Omega}\partial_t + 2\theta\right)\left(\varepsilon - \beta^2\Sigma^x{}_x\right) + \left(\hat{\nabla}^x + 2\varphi^x\right)\left(\beta_x\Sigma^x{}_x\right) + \theta\left(\Xi^x{}_x - \beta^2\Sigma^x{}_x\right) = e, \tag{399}$$
$$\theta\Sigma^x{}_x = f, \tag{400}$$
$$\left(\hat{\nabla}_x + \varphi_x\right)\left(\varepsilon - \Xi^x{}_x\right) + \varphi_x\left(\varepsilon - \beta^2\Sigma^x{}_x\right) + \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\pi_x + \beta_x\left(2\varepsilon - \Xi^x{}_x\right)\right) = g_x, \tag{401}$$
$$-\left(\hat{\nabla}_x + \varphi_x\right)\Sigma^x{}_x - \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\beta_x\Sigma^x{}_x\right) = h_x. \tag{402}$$

Generically, the above equations are not invariant under Carrollian local boosts, acting as
$$\beta'_x = \beta_x + B_x \tag{403}$$
(vanishing-k limit of (139)).

This should not come as a surprise. Such an invariance is exclusive to the relativistic case for obvious physical reasons, and is also known to be absent from Galilean fluid equations, which are not invariant under local Galilean boosts. Nevertheless, as we will shortly see, in specific situations a residual invariance persists.

We have finally obtained the two-dimensional equations of motion. This is the last required result for the boundary theory in order to address the problem of bulk reconstruction and limit $k \to 0$ of the derivative expansion. We will do this after discussing the fate of the relativistic energy-momentum tensor itself in the limit, and the dual Galilean limit $k \to \infty$. We will return to holography in Section 4, where all the results derived here will find good use.

3.3 The Fate of the Energy-Momentum Tensor

The ultra-relativistic limit breaks the spacetime metric into three independent data: the scalar density Ω, the connection b_i and the spatial metric a_{ij}. We saw that these geometric fields are nicely interpreted as constituents of the Carrollian geometry.

Consider an action defined on such a geometry, covariant under (242), we are facing a problem in defining the energy-momentum tensor. Indeed, in general-covariant theories it is obtained as the variation of the action with respect to the metric. This requires the existence of a regular metric (a pseudo-Riemannian manifold), but in the Carrollian case, as we mentioned, there is no spacetime non-degenerate metric. Therefore, we must introduce new objects that we will refer to as Carrollian momenta [129], and obtain as the variation of the action with respect to the three geometric fields mentioned above.

Notice that we are working here again in general boundary dimension $d+1$, with d the dimension of the spatial base (spanned by vectors with indices $i, j, \ldots$). We define the Carrollian equivalent of the energy-momentum tensor as:[50]

$$\mathcal{O} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta\Omega}, \quad \mathcal{B}^i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i} \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}. \tag{404}$$

Here $\Omega\sqrt{a}$ is the Carrollian counterpart of the relativistic $\sqrt{-g}$ and the variations are taken with respect to the 3 fields that replace the metric in the Carrollian setting.

From now on, we call (404) the Carrollian momenta. They transform under Carrollian diffeomorphisms as

$$\mathcal{O}' = J\mathcal{O} - \mathcal{B}^i j_i, \quad \mathcal{B}^{i\prime} = J^i_j \mathcal{B}^j, \quad \text{and} \quad \mathcal{A}^{ij\prime} = J^i_k J^j_l \mathcal{A}^{kl}. \tag{405}$$

The spatial vector $\mathcal{B}^i$ and matrix $\mathcal{A}^{ij}$ are indeed Carrollian tensors. However, $\mathcal{O}$ is not a scalar and, as we will see and use, it is wiser to introduce the scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i\mathcal{B}^i$. These objects replaces the energy-momentum tensor $\frac{\delta S}{\delta g_{\mu\nu}}$ in a Carrollian theory.

Given such a theory, the action is then invariant under Carrollian diffeomorphisms, generated by the spacetime vector $\underline{\xi}$

$$\delta_{\underline{\xi}} S = 0, \quad \underline{\xi} = \xi^t(t, \boldsymbol{x})\partial_t + \xi^i(\boldsymbol{x})\partial_i. \tag{406}$$

Notice that ξ^i only depends on $\boldsymbol{x}$, this is the infinitesimal translation of (242).

Under such an infinitesimal coordinate transformation we have

$$\delta_{\underline{\xi}} S = \int d^{d+1}x \left(\frac{\delta S}{\delta\Omega}\delta_{\underline{\xi}}\Omega + \frac{\delta S}{\delta b_i}\delta_{\underline{\xi}} b_i + \frac{\delta S}{\delta a_{ij}}\delta_{\underline{\xi}} a_{ij}\right) + \text{b.t.}. \tag{407}$$

We need to compute $\delta_{\underline{\xi}}\Omega$, $\delta_{\underline{\xi}} b_i$ and $\delta_{\underline{\xi}} a_{ij}$. In order to do so we compute the infinitesimal version of (244). If $x'^\mu = x^\mu - \xi^\mu$, then

$$\begin{aligned}
\delta_{\underline{\xi}}\Omega &= \underline{\xi}(\Omega) + \Omega\partial_t\xi^t, & (408)\\
\delta_{\underline{\xi}} b_i &= \underline{\xi}(b_i) - \Omega\partial_i\xi^t + b_j\partial_i\xi^j, & (409)\\
\delta_{\underline{\xi}} a_{ij} &= \underline{\xi}(a_{ij}) + \partial_i\xi^k a_{kj} + \partial_j\xi^k a_{ik}, & (410)
\end{aligned}$$

where $\underline{\xi}(f) \equiv \xi^t\partial_t f + \xi^i\partial_i f$.

We would like to write these transformations in terms of manifestly Carroll-covariant objects, so we define $X = \Omega\xi^t - b_i\xi^i$. By noticing that the components of a spacetime vector transform as

$$\xi^{t\prime} = J\xi^t + j_i\xi^i, \quad \xi^{i\prime} = J^i_k\xi^k, \tag{411}$$

it is straightforward to show that X is the right combination to get a scalar.

We thus rewrite (408), (409) and (410) in terms of X, ξ^i and the Carrollian geometrical tensors (250), (251), (254) introduced above[51]

$$\begin{aligned}
\delta_{\underline{\xi}}\Omega &= \partial_t X + \Omega\varphi_j\xi^j, & (412)\\
\delta_{\underline{\xi}} b_i &= -\hat{\partial}_i X + \varphi_i X - 2\varpi_{ij}\xi^j + \frac{b_i}{\Omega}\left(\partial_t X + \Omega\varphi_j\xi^j\right), & (413)\\
\delta_{\underline{\xi}} a_{ij} &= \hat{\nabla}_i\xi_j + \hat{\nabla}_j\xi_i + \frac{X}{\Omega}\partial_t a_{ij}. & (414)
\end{aligned}$$

This rewriting hints toward Carrollian covariance, as it replaces ξ^t with X. Therefore, we obtain $\delta_{\underline{\xi}} S = \delta_X S + \delta_{\xi^i} S$ with

$$\begin{aligned}
\delta_X S &= \int d^{d+1}x\Omega\sqrt{a}\left(\mathcal{O}\partial_t X - \mathcal{B}^i\hat{\partial}_i X + \mathcal{B}^i\varphi_i X + \mathcal{B}^i\frac{b_i}{\Omega}\partial_t X + \mathcal{A}^{ij}\frac{X}{\Omega}\partial_t a_{ij}\right), & (415)\\
\delta_{\xi^i} S &= \int d^{d+1}x\Omega\sqrt{a}\left(\mathcal{O}\Omega\varphi_j\xi^j - 2\mathcal{B}^i\varpi_{ij}\xi^j + \mathcal{B}^i b_i\varphi_j\xi^j + 2\mathcal{A}^{ij}\hat{\nabla}_i\xi_j\right). & (416)
\end{aligned}$$

[50] We call $\mathcal{B}^i$ the Carrollian momentum associated with b_i. It is always expressed with a suffix index, which therefore avoids confusion with the Weyl rescaling function $\mathcal{B}$.

[51] We recall that $\hat{\nabla}$ is the Carroll-covariant derivative introduced previously, with Christoffel symbols (248).

Finally, demanding $\delta_X S$ and $\delta_{\xi^i} S$ be zero separately and manipulating them, we obtain two conservation equations which are manifestly Carroll-covariant:[52]

$$\begin{aligned} \left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} &= 0, & (417)\\ 2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i\varpi_{ij} - \mathcal{E}\varphi_j &= 0, & (418) \end{aligned}$$

where we used the scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i\mathcal{B}^i$ introduced previously.

Let us briefly summarize. By strict comparison with the relativistic situation, we have defined the momenta of our Carrollian theory to be the variation of the action under the geometrical set of data that characterizes the background. Exploiting the underlying Carrollian symmetry we reached a set of two equations which encode the conservation properties of the momenta. As expected, these equations are fully Carroll-covariant.

Weyl Covariance

If the action is invariant under the Weyl transformations of the geometrical objects (264), then

$$\delta_\lambda S = \int d^{d+1}x\Omega\sqrt{a}\left(\mathcal{O}\delta_\lambda\Omega + \mathcal{B}^i\delta_\lambda b_i + \mathcal{A}^{ij}\delta_\lambda a_{ij}\right) = \int d^{d+1}x\Omega\sqrt{a}\lambda\left(\mathcal{O}\Omega + \mathcal{B}^i b_i + 2\mathcal{A}^{ij}a_{ij}\right) \quad (419)$$

has to vanish for every $\lambda(t, \boldsymbol{x})$. Therefore

$$\delta_\lambda S = 0 \quad \Rightarrow \quad \mathcal{E} = -2\mathcal{A}^i_i. \quad (420)$$

We will refer to this condition as the conformal state equation, it is the equivalent of the tracelessness of the energy-momentum tensor in the relativistic case. From (264) again, we deduce the following transformations of the Carrollian momenta

$$\mathcal{O} \to \mathcal{B}^{d+2}\mathcal{O}, \quad \mathcal{B}^i \to \mathcal{B}^{d+2}\mathcal{B}^i \quad \text{and} \quad \mathcal{A}^{ij} \to \mathcal{B}^{d+3}\mathcal{A}^{ij}. \quad (421)$$

This implies also $\mathcal{E} \to \mathcal{B}^{d+1}\mathcal{E}$.

We would like to write the conservation equations in a manifestly Weyl-covariant form. To do so we define $A^i_i = -\frac{d}{2}\mathcal{P}$. Then we decompose $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$ with Ξ^{ij} traceless, such that the constraint (420) becomes $\mathcal{E} = d\mathcal{P}$. This enable us to write (417) and (418) as

$$\begin{aligned} \left(\frac{1}{\Omega}\partial_t + \frac{d+1}{d}\theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \Xi^{ij}\xi_{ij} &= 0, & (422)\\ \left(\hat{\nabla}_i + \varphi_i\right)\Xi^i_j - \frac{1}{d}\left(\hat{\partial}_j + (d+1)\varphi_j\right)\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} &= 0. & (423) \end{aligned}$$

As already discussed, Carrollian derivatives are not Weyl covariant under Weyl rescaling. We therefore rewrite (422) and (423) using the Weyl-Carroll derivatives introduced in (269) and (270):

$$\begin{aligned} \frac{1}{\Omega}\hat{\mathscr{D}}_t\mathcal{E} - \hat{\mathscr{D}}_i\mathcal{B}^i - \Xi^{ij}\xi_{ij} &= 0, & (424)\\ -\frac{1}{d}\hat{\mathscr{D}}_j\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} + \hat{\mathscr{D}}_i\Xi^i_j &= 0. & (425) \end{aligned}$$

Not only these equations are now very compact, they are also manifestly Weyl-Carroll covariant.

Flat Case

So far we have worked on general Carrollian geometry, i.e. we did not impose any particular value of Ω, b_i and a_{ij}. We now restrict our attention to the flat Carrollian background.[53] At the relativistic level, the Poincaré group is defined as the set of coordinate transformations that leave the Minkowski metric invariant. By strict analogy, the Carroll group is defined as the set of transformations that preserve the Carrollian flatness, [81].

[52] A useful result is $\mathcal{B}^i\hat{\partial}_i X = -X\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{B}^i$, valid up to total derivatives and for any scalar X and vector $\mathcal{B}^i$.

[53] We refer here to flat Carrollian geometry as the geometry for which the Carroll group is an isometry.

Therefore, the Carroll group corresponds to the transformations satisfying

$$\partial_t \to \partial_t, \quad \delta_{ij} dx^i dx^j \to \delta_{ij} dx^i dx^j, \quad b_{0i} \to R_i^j \left(b_{0j} + \beta_j\right), \tag{426}$$

with b_{0i} constant. The resulting change of coordinates is

$$t' = t + \beta_i x^i + t_0, \quad x'^i = R^i_j x^j + x^i_0, \tag{427}$$

where $t_0 \in \mathbb{R}$, $\{x^i_0, \beta_i\} \in \mathbb{R}^d$ and $R^i_j \in O(d)$. This group is known in the literature as the Carroll group.[54]
Recasting (417) and (418) for $a_{ij}(t, \boldsymbol{x}) = \delta_{ij}$, $\Omega(t, \boldsymbol{x}) = 1$ and $b_i(t, \boldsymbol{x}) = b_{0i}$, we obtain

$$\partial_t \mathcal{O} - \partial_i \mathcal{B}^i = 0, \tag{428}$$

$$2\partial_i \mathcal{A}^i_j + 2b_{0i}\partial_t \mathcal{A}^i_j = 0. \tag{429}$$

The momenta appearing in these two equations can be packaged in a spacetime energy-momentum tensor

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} \\ -\mathcal{B}^j & -2\mathcal{A}^{ij} \end{pmatrix}. \tag{430}$$

The usual conservation of this tensor $\partial_\mu T^{\mu\nu} = 0$ is ensured by the conservation equations of the momenta, namely (428) and (429).

This tensor is not symmetric, but this should not come as a surprise: it is not defined throughout the variation of the action with respect to the spacetime metric (symmetric by construction), instead it is defined using the Carrollian metric fields. Finally notice that this spacetime lifting procedure was possible here solely due to the flatness of the Carrollian geometry. In general backgrounds, this is not possible, and the very concept of spacetime energy-momentum tensor is ambiguous – whereas the Carrollian momenta are by construction well suited.

As a conclusive remark notice that the Carroll group contains spacetime translations, so if a theory is invariant under this group, there will be a set of $d+1$ Nœther currents associated with spacetime translations. Packaging them in a $d+1$-dimensional kind of Nœther energy-momentum tensor, enables us to compare it with (430). Before discussing the definition of charges in our framework, we would like to insist on the relevance of these momenta: as we saw holography is implemented on a metric sourcing an energy-momentum tensor in AdS. In flat holography one may expect something very similar to take place, namely the Carrollian geometrical objects sourcing the Carrollian momenta defined here. Defining the latter properly constitutes certainly a step toward a flat holographic dictionary.

Emergence of Carrollian Physics

In the previous sections, we have intrinsically defined the Carrollian momenta starting from the metric fields of a Carrollian geometry. The Carrollian geometry was inspired by the ultra-relativistic contraction of the relativistic metric. Consider the relativistic decomposition of the energy-momentum tensor (21), and impose the already discussed scaling of dissipative tensors (325) and (326)

$$\tau^{ij} = -\frac{\Sigma^{ij}}{k^2} - \Xi^{ij} \quad \text{and} \quad q^i = -\mathcal{B}^i + k^2\pi^i, \tag{431}$$

where we identify the leading order of the heat current in the limit as the Carrollian spin-1 momentum $\mathcal{B}^i$.

The $k \to 0$ limit of the equations of motion reported in (338), (339), (343) and (344), calling again $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$, read here

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} = 0, \tag{432}$$

$$2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i\varpi_{ij} - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j = 0, \tag{433}$$

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j + \left(\hat{\nabla}_i + \varphi_i\right)\Sigma^i_j = 0, \tag{434}$$

$$\Sigma^{ij}\xi_{ij} + \frac{\theta}{d}\Sigma^i_i = 0. \tag{435}$$

[54]The Carroll group was already shown to be the symmetry group of flat zero signature geometries in the precursory work [89].

Notice that these equations reduce to the Carrollian equations (417) and (418) when the dissipative terms have no k-dependence, $\Sigma^{ij} = 0 = \pi^i$, together with the additional constraint $\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j = 0$.

This result undoubtedly shows the nature of the ultra-relativistic limit: it is a Carrollian limit, as we have already argued. Conversely, this analysis gives credit to our intrinsic Carrollian construction of the previous sections. We conclude with an aside important remark: we have taken the ultra-relativistic limit of the conservation equations because it would have been inconsistent to compute directly the limit of the energy-momentum tensor itself. Indeed we would have lost information on the fields which survive and the conservation equations they satisfy. This confirms that we have to give up the concept of spacetime energy-momentum tensor on general Carrollian backgrounds.

3.3.1 Intrinsic Carrollian Charges

This section is dedicated to the definition of charges in the Carrollian framework. Charges are conserved quantities associated with a symmetry of the theory. Relativistically, the latter can be generated by a Killing vector field. By projecting the energy-momentum tensor on the Killing vector, we obtain a conserved current.

We will show here how to implement this procedure in the Carrollian case. In order to do so, we firstly derive charges starting from a conserved Carrollian current. Secondly, we define Carrollian Killing and conformal Killing vectors. Thirdly, we build conserved charges associated with conformal Killing vectors.

Conserved Carrollian Current and its Charges

We show here a way to define a conserved charge starting from a conserved current. In this derivation we never impose the current to be associated with a Killing vector, therefore our construction is very general.

Whenever we have a scalar $\mathcal{J}$ and a vector $\mathcal{J}^i$ satisfying

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{J} + \left(\hat{\nabla}_i + \varphi_i\right)\mathcal{J}^i = 0, \tag{436}$$

we can build the conserved charge

$$\mathcal{Q} = \int_{\Sigma_t} d^d x \sqrt{a}\left(\mathcal{J} + b_i \mathcal{J}^i\right), \tag{437}$$

where Σ_t is a constant-time slice.

A way to derive this formula is to start from the relativistic counterpart: consider a conserved current J^μ, the charge is then

$$Q = \int_{\Sigma_t} d^d x \sqrt{\sigma} n_\mu J^\mu, \tag{438}$$

with n_μ the unit vector normal to Σ_t and $\sigma_{\mu\nu}$ the induced metric on Σ_t.

In order to perform the zero-k limit, we decompose J^μ in an already Carroll-covariant basis

$$\underline{J} = \mathcal{J}\left(\frac{k}{\Omega}\partial_0\right) + \mathcal{J}^i\left(\partial_i + \frac{k b_i}{\Omega}\partial_0\right). \tag{439}$$

Then, using Randers-Papapetrou parametrization (148), we obtain

$$\sqrt{\sigma} = \sqrt{a} + O\left(k^2\right), \quad n_0 = k\Omega + O\left(k^3\right), \quad J^0 = \frac{k}{\Omega}\left(\mathcal{J} + b_i \mathcal{J}^i\right). \tag{440}$$

Therefore, we find $Q \underset{k\to 0}{\to} k^2 \mathcal{Q}$, showing the relevance of the proposed Carrollian charge (437).

Carrollian Killing Vectors and their Currents

A Killing vector is a vector field that preserves the metric. Analogously, we define the Carrollian Killing vector $\underline{\xi}$ to be the vector satisfying[55]

$$\delta_{\underline{\xi}}\Omega = 0, \qquad \delta_{\underline{\xi}} a_{ij} = 0, \tag{441}$$

[55] This is the translation in our language of $\mathcal{L}_{\underline{X}} g = 0$ and $\mathcal{L}_{\underline{X}} \xi = 0$ of (III.6) in [81], see also [88]. Notice that the variation of b_i is left arbitrary. This is what we define to be Carrollian Killing vectors, other definitions may be use instead.

where $\delta_{\underline{\xi}}$ is the Lie derivative. This gives rise to two Killing equations on $\underline{\xi}$, which are exactly (412) and (414),[56]

$$\begin{aligned} \partial_t X + \Omega \varphi_j \xi^j &= 0, && (442)\\ \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega}\partial_t a_{ij} &= 0, && (443) \end{aligned}$$

where we recall $X = \Omega\xi^t - b_i\xi^i$. Notice that these equations do not actually depend on b_i.

The generalization to conformal Carrollian Killing vectors is straightforward. We call $\underline{\xi}$ a conformal Carrollian Killing vector if

$$\delta_{\underline{\xi}}\Omega = \lambda\Omega \quad \text{and} \quad \delta_{\underline{\xi}} a_{ij} = 2\lambda a_{ij}. \tag{444}$$

It obeys the following conformal Killing equations:

$$\begin{aligned} \partial_t X + \Omega \varphi_j \xi^j &= \lambda\Omega, && (445)\\ \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega}\partial_t a_{ij} &= 2\lambda a_{ij}. && (446) \end{aligned}$$

In particular from the last equation we obtain $\lambda = \frac{1}{d}\left(\hat{\nabla}_i\xi^i + \frac{X}{\Omega}\partial_t \ln\sqrt{a}\right)$. This general construction is very useful, as we will shortly confirm.

The associated conserved current can now be obtained projecting the Carrollian momenta on a Carrollian Killing vector, exactly like in the relativistic case. Indeed consider the following Carrollian current:

$$\mathcal{J} = \xi_i \mathcal{B}^i, \quad \mathcal{J}^i = \xi_j \Sigma^{ij}. \tag{447}$$

It is conserved provided $\underline{\xi}$ satisfies (443), and the Carrollian conservation equations (434) and (435) are verified. The corresponding conserved charge is then

$$\mathcal{Q}_{\underline{\xi}} = \int_{\Sigma_t} d^d x \sqrt{a}\, \xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right), \tag{448}$$

This charge is also conserved when $\underline{\xi}$ satisfies (446), if we further impose the condition $\Sigma^i_i = 0$.

A Particular Set of Charges

It can be shown that the equations describing the dynamics of asymptotically flat spacetimes in 3 and 4 dimensions can be related to Carrollian conservation laws for $\mathcal{B}^i = 0$.[57] For this reason we focus here on this particular case and build other conserved currents associated with conformal Killing vectors.

The Carrollian conservation equations obtained from the ultra-relativistic limit (432) and (433), for $\mathcal{B}^i = 0$, become

$$\begin{aligned} \left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} &= 0, && (449)\\ 2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j &= 0. && (450) \end{aligned}$$

We could have also reported the two equations on Σ^{ij}, (434) and (435), but they are immaterial here.

Consider a Killing vector $\underline{\xi}$, the following charge, up to boundary terms, is conserved

$$\mathcal{C}_{\underline{\xi}} = \int_{\Sigma_t} d^d x \sqrt{a}\left(X\mathcal{E} - \xi^i \pi_i + 2b_i \xi^j \mathcal{A}^i_j\right), \tag{451}$$

assuming only (449) and (450). This charge is also conserved when $\underline{\xi}$ is a conformal Killing vector, if we further impose the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$.

The corresponding conserved current reads[58]

$$\mathcal{J} = X\mathcal{E} - \xi^i \pi_i, \quad \mathcal{J}^i = 2\xi^j \mathcal{A}^i_j. \tag{452}$$

[56] On top of these equations, a Carrollian Killing vector has a time independent spatial part, i.e. $\partial_t \xi^i = 0$.
[57] We will discuss this in the examples of next section and in linearized four-dimensional gravity shortly.
[58] Its conservation (436) is ensured thanks to the Killing equations together with (449) and (450).

In conclusion, it is interesting to investigate the flat case $a_{ij}(t,\boldsymbol{x}) = \delta_{ij}$, $\Omega(t,\boldsymbol{x}) = 1$ and $b_i(t,\boldsymbol{x}) = b_{0i}$. Here, (449) and (450) can be written as $\partial_\mu T^{\mu\nu} = 0$ with[59]

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} + \pi^i \\ 0 & -2\mathcal{A}^{ij} \end{pmatrix}, \tag{453}$$

and we notice that the charge, up to a divergenceless term, takes the usual form

$$\mathcal{C}^{\text{Flat}}_{\underline{\xi}} = \int_{\Sigma_t} d^d x \left(\xi^t \mathcal{O} - \xi^i b_{0i}\mathcal{O} - \xi^i \pi_i + 2b_{0i}\xi^j \mathcal{A}^i_j\right) = -\int_{\Sigma_t} d^d x T^{0\mu}\xi_\mu + \tilde{\mathcal{C}}_{\xi^i}, \tag{454}$$

with $\tilde{\mathcal{C}}_{\xi^i} = -\int_{\Sigma_t} d^d x \xi^i b_{0i}\mathcal{O}$ separately conserved.

For $\underline{\xi}$ and $\underline{\eta}$ Killing vectors, we define the brackets

$$\{\mathcal{Q}_{\underline{\xi}}, \mathcal{Q}_{\underline{\eta}}\} \equiv \int_{\Sigma_t} d^d x \delta_{\underline{\eta}} \left[\sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right)\right], \tag{455}$$

$$\{\mathcal{C}_{\underline{\xi}}, \mathcal{C}_{\underline{\eta}}\} \equiv \int_{\Sigma_t} d^d x \delta_{\underline{\eta}} \left[\sqrt{a}\left(X\mathcal{E} - \xi^i \pi_i + 2b_i \xi^j \mathcal{A}^i_j\right)\right]. \tag{456}$$

Here $\delta_{\underline{\eta}}$ is the Lie derivative acting on the metric fields and the momenta, but not on ξ^t and ξ^i.

A lengthly computation shows that the charges $\mathcal{Q}_{\underline{\xi}}$ and $\mathcal{C}_{\underline{\xi}}$ equipped with these brackets form two representations of the Carrollian Killing algebra:

$$\{\mathcal{Q}_{\underline{\xi}}, \mathcal{Q}_{\underline{\eta}}\} = \mathcal{Q}_{[\underline{\xi},\underline{\eta}]} \quad \text{and} \quad \{\mathcal{C}_{\underline{\xi}}, \mathcal{C}_{\underline{\eta}}\} = \mathcal{C}_{[\underline{\xi},\underline{\eta}]}. \tag{457}$$

We can extend these results to the conformal Killing algebra when imposing the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$ for the charge $\mathcal{C}_{\underline{\xi}}$ and the condition $\Sigma^i_i = 0$ for the charge $\mathcal{Q}_{\underline{\xi}}$.

This last important result concludes our wondering on the fate of the energy-momentum in the Carrollian limit. Raising this question allowed us to find Carrollian counterparts of $T_{\mu\nu}$ and to further introduce well-defined Carrollian charges. The results detailed here will be applied in concrete examples in the next chapter of this work, and in the very next paragraph to four-dimensional gravity. For the moment being, we would like to stress again how unnatural would have been to take naively the limit $k \to 0$ at the level of the energy-momentum tensor directly. Indeed doing so we would have missed important dynamical contributions coming from the non-trivial geometrical structure.

Application to Four-dimensional Linearized Gravity

We choose to report the example of four-dimensional linearized gravity to corroborate our Carrollian findings: we compute the various charges just defined and show how naturally the bulk dynamics matches with the Carrollian expectations in the boundary. Specifically, we prove that the boundary equations of motion, which are the linearized Einstein equations after gauge fixing, can be interpreted as a Carrollian conservation, and that the asymptotic charges are also charges associated with conformal Carrollian Killing vectors.

The bulk metric is $g_{MN} = \eta_{MN} + h_{MN}$ with[60]

$$\begin{aligned} \eta &= -dt^2 - 2dtdr + r^2\gamma_{ij}dx^i dx^j, \\ h_{tt} &= \frac{2}{r}m_B + O\left(r^{-2}\right), \\ h_{tj} &= \frac{1}{2}\nabla^i C_{ij} + \frac{1}{r}N_j + O\left(r^{-2}\right), \\ h_{ij} &= rC_{ij} + O(1), \\ h_{rM} &= 0. \end{aligned} \tag{458}$$

The perturbation h_{MN} is traceless, so $\gamma^{ij}C_{ij} = 0$, where γ^{ij} is the metric of the two-sphere and ∇_i the associated covariant derivative. We recognize the mass aspect m_B, the angular momentum aspect N_i and the gravitational

[59] We recall that for $\mathcal{B}^i = 0$, $\mathcal{E} = \Omega\mathcal{O}$. Thus in the flat case $\mathcal{E} = \mathcal{O}$.

[60] We recall our conventions: (M, N) are four-dimensional bulk indices that we split in $M = \{r, \mu\} = \{r, t, x^i\}$, with x^i the two-dimensional indices $(i = 1, 2)$ of the spatial co-dimension two sections.

wave aspect C_{ij}, all depending on t and x^i (see [118, 202]). In this gauge, the linearized Einstein equations become:[61]

$$\partial_t m_B = \frac{1}{4}\partial_t \nabla^i\nabla^j C_{ij}, \tag{459}$$

$$\partial_t N_i = \frac{2}{3}\partial_i m_B - \frac{1}{6}\left[(\Delta-1)\nabla^j C_{ji} - \nabla_i\nabla^k\nabla^j C_{jk}\right]. \tag{460}$$

We first consider the case

$$\nabla^i\nabla^j C_{ij} = 0. \tag{461}$$

Then (459) and (460) admit a Carrollian interpretation and are recovered from (417) and (418) with the following metric data

$$\Omega = 1, \quad b_i = 0, \quad a_{ij} = \gamma_{ij}, \tag{462}$$

and Carrollian momenta

$$\Sigma^{ij} = \mathcal{B}^i = \Xi^i_i = 0, \tag{463}$$

$$\mathcal{E} = 4m_B, \quad \mathcal{A}^{ij} = -\frac{1}{2}\left(\frac{\mathcal{E}}{2}a^{ij} - \Xi^{ij}\right), \quad \pi^i = -3N^i, \quad \Xi^i_j = \frac{1}{2}(\Delta-4)\, C^i_j, \tag{464}$$

where $\mathcal{E} = -2\mathcal{A}^i_i$ and $\Xi^i_i = 0$–we are in the conformal case. We obtain the following conservation equations:

$$\partial_t \mathcal{E} = 0, \tag{465}$$

$$\partial_t \pi_i + \nabla_j\left(\frac{\mathcal{E}}{2}\gamma^j_i - \Xi^j_i\right) = 0. \tag{466}$$

This type of Carrollian conservation falls again into the general class previously described.

The asymptotic Killing vectors $\hat{\underline{\xi}} = \hat{\xi}^r\partial_r + \hat{\xi}^t\partial_t + \hat{\xi}^i\partial_i$ associated with the gauge (459) have the following leading order in r^{-1}

$$\hat{\xi}^r = -\lambda(\mathbf{x})r + O(1), \quad \hat{\xi}^t = \xi^t(t,\mathbf{x}) + O(r^{-1}) \quad \text{and} \quad \hat{\xi}^i = \xi^i(\mathbf{x}) + O(r^{-1}), \tag{467}$$

where $\underline{\xi} = \xi^t\partial_t + \xi^i\partial_i$ is a conformal Killing vector (i.e. satisfying (445) and (446)) of the Carrollian geometry given by $\{\Omega = 1, a_{ij} = \gamma_{ij}, b_i = 0\}$ and λ is the conformal factor. The solutions to the corresponding conformal Killing equations reproduce exactly the bms$_4$ algebra: $\xi^t = \frac{t}{2}\nabla_i\xi^i + \alpha(\mathbf{x})$, α being any function on S^2, ξ^i a conformal Killing of S^2 and $\lambda = \frac{1}{2}\nabla_i\xi^i$. We compute the corresponding surface charges. When $\nabla^i\nabla^j C_{ij} = 0$ they take the form

$$Q_{\hat{\underline{\xi}}}[g] = \int_{S^2} d^2x\sqrt{\gamma}\left(\xi^t\mathcal{E} - \xi^i\pi_i\right) = \mathcal{C}_{\underline{\xi}}, \tag{468}$$

with $\mathcal{E}$ and π_i given by (464). We recognize again the charges defined from purely Carrollian considerations, associated with the data (462–464). These charges are automatically conserved. Physically, this is due to the fact that part of the effect of gravitational radiation has suppressed by demanding $\nabla^i\nabla^j C_{ij} = 0$. We will find shortly that relaxing this condition has an effect on the charge conservation.

Integrating (465) and (466) we obtain

$$\mathcal{E} = \mathcal{E}_0(\mathbf{x}), \quad \pi_i = -\frac{1}{2}\partial_i\mathcal{E}_0 t + \int dt'\nabla_j\Xi^j_i + \pi_{0i}(\mathbf{x}). \tag{469}$$

The charges become

$$\begin{aligned}
\mathcal{C}_{\underline{\xi}} &= \int_{S^2} d^2x\sqrt{\gamma}\left(\left(\frac{\nabla_i\xi^i}{2}t + \alpha\right)\mathcal{E}_0 - \xi^i\left(-\frac{1}{2}\partial_i\mathcal{E}_0 t + \int dt'\nabla_j\Xi^j_i + \pi_{0i}\right)\right) \\
&= t\int_{S^2} d^2x\sqrt{\gamma}\left(\frac{1}{2}\nabla_i(\xi^i\mathcal{E}_0)\right) + \int_{S^2} d^2x\sqrt{\gamma}\left(\alpha\mathcal{E}_0 - \xi^i\left(\int dt'\nabla_j\Xi^j_i + \pi_{0i}\right)\right) \\
&= \int_{S^2} d^2x\sqrt{\gamma}\left(\alpha\mathcal{E}_0 - \xi^i\pi_{0i}\right) - \int dt'\int_{S^2} d^2x\sqrt{\gamma}\xi^i\nabla_j\Xi^j_i + \text{b.t.} \\
&= \int_{S^2} d^2x\sqrt{\gamma}\left(\alpha\mathcal{E}_0 - \xi^i\pi_{0i}\right) + \text{b.t.}.
\end{aligned} \tag{470}$$

[61] Solving empty linearized Einstein equations order by order in r^{-1} allows to express the various subleading coefficients in terms of m_B, C_{ij} and N_i. The only residual equations are then the ones that we present here.

The last step follows from the fact that ξ^i is a conformal Killing vector on S^2 and Ξ^i_j is traceless. We observe that $\mathcal{C}_{\underline{\xi}}$ is now manifestly conserved.

When $\nabla^i\nabla^j C_{ij} \neq 0$, on the gravity side the radiation affects the surface charges and spoils their conservation. Therefore, these charges do not match those we defined earlier. This situation can be further investigated and recast in Carrollian language. To this end, we define $\sigma = \nabla^i\nabla^j C_{ij}$ and rewrite (459) and (460)

$$\partial_t \mathcal{E} = 0, \tag{471}$$

$$\partial_t \pi_i + \nabla_j \left(\mathcal{P}\gamma^j_i - \Xi^j_i\right) = 0. \tag{472}$$

Here, the metric fields are

$$\Omega = 1, \quad b_i = 0, \quad a_{ij} = \gamma_{ij}, \tag{473}$$

together with the Carrollian momenta

$$\Sigma^{ij} = \mathcal{B}^i = 0, \tag{474}$$

$$\mathcal{E} = 4m_B - \sigma, \quad \mathcal{P} = \frac{\mathcal{E}}{2} + \sigma, \quad \pi^i = -3N^i, \quad \Xi^i_j = \frac{1}{2}\left(\Delta - 4\right) C^i_j. \tag{475}$$

Hence turning on σ can be interpreted as spoiling the conformal state equation: $\mathcal{E} = -2\left(\mathcal{A}^i_i + \sigma\right)$. It appears as a sort of conformal anomaly in the boundary theory. The surface charges become

$$Q_{\underline{\xi}}[g](t) = \int_{S^2} d^2x\sqrt{\gamma}\left(\xi^t(\mathcal{E}+\sigma) - \xi^i\pi_i\right), \tag{476}$$

and, as already stated, they are no longer conserved

$$\partial_t Q_{\underline{\xi}}[g] = \int_{S^2} d^2x\sqrt{\gamma}\left(\delta_{\underline{\xi}} + \lambda\right)\sigma, \tag{477}$$

where $\delta_{\underline{\xi}}$ is the usual Lie derivative and $\lambda = \frac{1}{2}\nabla_i\xi^i$ the conformal factor. These charges were obtained in [173].[62] For non linear gravity see [87, 203], where the charges are now non-integrable.

This example shows the value of the Carrollian charges introduced before and allows to familiarize with our findings. We will again discuss these charges for asymptotically flat full (as opposed to linearized) solutions of Einstein gravity, in relationship with our resummation in Section 4.

3.4 Dual Galilean Limit

At this point of this work the reader finds her/himself with a very concrete and fully developed method to start from the most general relativistic fluid in any dimension and take the Carrollian limit $k \to 0$. One spontaneous question arises: can we use this machinery to compute also the dual non-relativistic limit $k \to \infty$? The answer is yes, and this is the main result of this chapter. As advertised, we are pausing our discussion on holography, but we will go back to it soon after.

3.4.1 Geometrical Setup

The Galilean group is an infinite-k contraction of the Poincaré group. The latter acts locally in general $d+1$-dimensional pseudo-Riemannian manifolds $\mathcal{M}$. As much as we started from the Randers-Papapetrou parametrization of the relativistic metric before taking the $k \to 0$ limit in order to retrieve Carrollian diffeomorphisms, we parametrize here the relativistic metric using the so-called Zermelo gauge, to obtain in the $k \to \infty$ limit Galilean diffeomorphisms, defined as

$$t' = t'(t) \quad \text{and} \quad \boldsymbol{x}' = \boldsymbol{x}'(t, \boldsymbol{x}). \tag{478}$$

In fact, these diffeomorphisms maintain time absolute, as required in Galilean physics.[63]

We consequently choose the form of the metric on $\mathcal{M}$:

$$ds^2 = -\Omega^2 k^2 dt^2 + a_{ij}\left(dx^i - w^i dt\right)\left(dx^j - w^j dt\right). \tag{479}$$

[62] See the $n = 2$ case of Sec. 3. Their charges coincide with (476) with $\alpha = T$, $\xi^i = v^i$, $\mathcal{E}_0 = 4\mathcal{M}$ and $\pi^i_0 = -3\mathcal{N}^i$.

[63] It is precisely in this sense that we refer to Carrollian and Galilean diffeomorphisms as dual: in the former space is absolute while in the latter it is time to be absolute [81].

This is the natural choice because, under (478), Ω, a_{ij} and w^i transform as

$$a'_{ij} = a_{nl} J^{-1n}{}_i J^{-1l}{}_j, \qquad w'^n = \frac{1}{J}\left(J^n_i w^i + j^n\right), \qquad \Omega' = \frac{\Omega}{J}. \tag{480}$$

We thus see that a_{ij} behaves as a spatial metric while w^i is a connection, which will be identified with the non-inertiality of the frame at hand in the limit.

Every metric is compatible with the gauge (479), provided a_{ij}, w^i and Ω, are free to depend on $(t, \boldsymbol{x})$. The existence of a Galilean limit requires, however, Ω to depend on t only. Indeed, the proper time element for a physical observer is $d\tau = \sqrt{\frac{-ds^2}{k^2}}$. When k becomes infinite, $\lim\limits_{k\to\infty} d\tau = \Omega\, dt$ must coincide with the absolute Newtonian time, and this requires the absence of $\boldsymbol{x}$-dependence in Ω.

The spacetime Jacobian matrix associated with (478) reads

$$J^\mu_\nu(t,\boldsymbol{x}) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \to \begin{pmatrix} J(t) & 0 \\ J^i(t,\boldsymbol{x}) & J^i_j(t,\boldsymbol{x})\end{pmatrix} \quad \text{with} \quad J^i = \frac{j^i}{k}. \tag{481}$$

The metric form (479) is referred to as Zermelo, [150]. A relativistic particle moving in it is described by the components of its velocity $\underline{u}$, normalized as $\|u\|^2 = -k^2$:

$$u^\mu = \frac{dx^\mu}{d\tau} \Rightarrow u^0 = \gamma k,\ u^i = \gamma v^i, \tag{482}$$

where the Lorentz factor γ is defined as usual (although here, it depends also on the spacetime coordinates):[64]

$$\gamma(t,\boldsymbol{x},v) = \frac{dt}{d\tau} = \frac{1}{\Omega\sqrt{1-\left(\frac{v-w}{k\Omega}\right)^2}}. \tag{483}$$

Under a Galilean diffeomorphism the transformation of the components of u,

$$u'^0 = Ju^0, \quad u'^i = J^i_n u^n + J^i u^0, \quad u'_0 = \frac{1}{J}\left(u_0 - u_j J^{-1j}{}_n J^n\right), \quad u'_i = u_n J^{-1n}{}_i, \tag{484}$$

induces the following transformation on v^n

$$v'^n = \frac{1}{J}\left(J^n_i v^i + j^n\right), \tag{485}$$

which is the same as the transformation of w^i written in (480).

As announced, the role played by the latter become clear in the $k \to \infty$ limit. Indeed, in such a limit we are left with the positive-definite metric on the spatial base (called $\mathcal{S}$)

$$d\ell^2 = a_{ij}dx^i dx^j, \tag{486}$$

observed from a frame with non-inertial velocity $-\underline{w} = -w^i\partial_i$. Notice moreover that, since $J = J(t)$ and $\Omega = \Omega(t)$, Galilean transformations lead to $\Omega' = \Omega'(t')$, leaving invariant the absolute Newtonian time $\int dt\, \Omega(t) = \int dt'\, \Omega'(t')$. Observe also that $\dfrac{\underline{v} - \underline{w}}{\Omega}$ is a genuine vector of the spatial metric, being the latter a difference of connections. This vector expresses the velocity of a moving object with respect to the inertial frame, and as such it has to be covariant under Galilean diffeomorphisms.

We would like to conclude with a particular non-relativistic structure, which is invariant under the Galilean group. Consider the spatial metric to be the Euclidean space E_d with Cartesian coordinates ($a_{ij} = \delta_{ij}$), $\Omega = 1$, and the connection $\underline{w}$ constant. This system describes the non-relativistic motion of a free particle in Euclidean space, observed from an inertial frame. The Galilean group then acts as

$$\begin{cases} t' = t + t_0, \\ x'^n = R^n_i x^i + V^n t + x^n_0 \end{cases} \tag{487}$$

with all parameters being $(t, \boldsymbol{x})$-independent, and R^n_i the entries of an orthogonal matrix. It is only in this instance that the Galilean group acts globally as the group of isometries of the structure under analysis. In more general structures, the Galilean group acts only locally and it is no more a global symmetry. Before discussing the limit of the fluid equations of motion, it is useful to report the Christoffel symbols of the relativistic metric (479) in the large-k expansion and the inferred Levi-Civita connection in the limit.

[64] Expressions as v^2 stand for $a_{ij}v^i v^j$, not to be confused with $\|u\|^2 = g_{\mu\nu}u^\mu u^\nu$.

Christoffel Symbols

The Zermelo metric (479) has components (in the coframe $\{dx^0 = kdt, dx^i\}$):

$$g_{\mu\nu} \to \begin{pmatrix} -\Omega^2 + \frac{w^2}{k^2} & -\frac{w_n}{k} \\ -\frac{w_i}{k} & a_{in} \end{pmatrix}, \quad g^{\mu\nu} \to \frac{1}{\Omega^2} \begin{pmatrix} -1 & -\frac{w^j}{k} \\ -\frac{w^i}{k} & \Omega^2 a^{ij} - \frac{w^i w^j}{k^2} \end{pmatrix}, \tag{488}$$

The Christoffel symbols are easily computed. We are interested in their large-k behaviour for which one obtains the following:

$$\Gamma^0_{00} = \frac{1}{k}\partial_t \ln \Omega + + \frac{w^i}{2k^3\Omega^2}\left(\partial_i w^2 + w^j \partial_t a_{ij}\right) + O\left(1/k^5\right), \tag{489}$$

$$\Gamma^0_{0i} = -\frac{1}{2k^2\Omega^2}\left(w_j \partial_i w^j + w^j \partial_j w_i + w^j \partial_t a_{ij}\right) + O\left(1/k^4\right), \tag{490}$$

$$\Gamma^0_{ij} = \frac{1}{k\Omega^2}\left(\frac{1}{2}\left(\partial_i w_j + \partial_j w_i + \partial_t a_{ij}\right) - w_n \gamma^n_{ij}\right), \tag{491}$$

$$\Gamma^i_{00} = \frac{1}{k^2}\left(w^i \partial_t \ln \Omega - a^{in}\left(\partial_t w_n + \partial_n \frac{w^2}{2}\right)\right) + O\left(1/k^4\right), \tag{492}$$

$$\Gamma^i_{j0} = \frac{a^{in}}{2k}\left(\partial_n w_j - \partial_j w_n + \partial_t a_{jn}\right) + O\left(1/k^3\right), \tag{493}$$

$$\Gamma^i_{jn} = \gamma^i_{jn} + O\left(1/k^2\right), \tag{494}$$

where

$$\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right) \tag{495}$$

are the Christoffel symbols for the d-dimensional metric a_{ij}. Note also

$$\Gamma^\mu_{\mu 0} = \frac{1}{k}\partial_t \ln\left(\sqrt{a}\Omega\right), \quad \Gamma^\mu_{\mu i} = \partial_i \ln \sqrt{a}. \tag{496}$$

These data will be useful to compute the divergence of the fluid energy-momentum tensor.

3.4.2 Fluid Classical Limit

We will consider in the following the ordinary non-relativistic limit of fluid equations, formally reached at infinite k. The physical validity of this situation is based on two assumptions.

The first is kinematical: it assumes that the global velocity of the fluid with respect to the observer is small compared to k. This is easily implemented using the Zermelo form of the metric (479), where the control parameter for the validity of the classical limit is $\left|\frac{v-w}{k}\right|$. We find

$$\begin{cases} u^0 = \gamma k = \frac{k}{\Omega} + O\left(1/k\right), & u_0 = -k\Omega + O\left(1/k\right), \\ u^i = \gamma v^i = \frac{v^i}{\Omega} + O\left(1/k^2\right), & u_i = \frac{v_i - w_i}{\Omega} + O\left(1/k^2\right). \end{cases} \tag{497}$$

The second is microscopic. The internal particle motion should also be Galilean, in other words the energy density should be large compared to the pressure: $\varepsilon \gg p$. This sets restrictions on the equation of state, as not every equation of state is compatible with such a microscopic assumption.[65]

An important consequence of the microscopic assumption is the separation of mass and energy, now both independently conserved. It is customary to introduce the following:

- ϱ the usual mass per unit of volume (mass density);
- ϱ_0 the usual mass per unit of proper volume (rest-mass density);
- e the internal energy per unit of mass;

[65] For example, the conformal equation of state, $\varepsilon = dp$ is not compatible with the non-relativistic limit at hand.

- h the enthalpy per unit of mass.

These local thermodynamic quantities are related as

$$\begin{cases} \varepsilon = \left(e + k^2\right) \varrho_0, \\ h = e + \frac{p}{\varrho}, \\ \varrho_0 = \dfrac{\varrho}{\Omega\gamma} = \varrho\sqrt{1 - \left(\frac{v-w}{k\Omega}\right)^2} \approx \varrho - \frac{\varrho}{2}\left(\frac{v-w}{k\Omega}\right)^2, \end{cases} \tag{498}$$

where we have used (483) for the Lorentz factor γ, and expanded it for small $\left|\frac{v-w}{k}\right|$.

The Structure of the Equations

The fluid equations are the conservation of the energy-momentum tensor, in the background (479). It is computationally wise to split these equations as:

$$\nabla_\mu T^{\mu 0} = 0, \quad \nabla_\mu T^\mu{}_i = 0. \tag{499}$$

Indeed, applying a Galilean diffeomorphism (478), the time components up and space components down transform faithfully and irreducibly:

$$\nabla'_\mu T'^{\mu 0} = J\nabla_\mu T^{\mu 0}, \quad \nabla'_\mu T'^\mu{}_i = J^{-1l}{}_i \nabla_\mu T^\mu{}_l. \tag{500}$$

Hence, the two sets of equations (499) do not mix and have furthermore a d-dimensional covariant transformation, which is our goal for the Galilean fluid dynamics.

The expressions displayed so far are fully relativistic. The next step is to consider the large-k regime, where (499) can be expanded in powers of $1/k$. This expansion must be performed with care as the time equation needs an extra k factor with respect to the other d spatial equations because it describes the evolution of energy, which is a momentum multiplied by k. We find

$$k\nabla_\mu T^{\mu 0} = k^2 \frac{\mathcal{C}}{\Omega} + \frac{\mathcal{E}}{\Omega} + O\left(\frac{1}{k^2}\right), \tag{501}$$

$$\nabla_\mu T^\mu{}_i = \mathcal{M}_i + O\left(\frac{1}{k^2}\right). \tag{502}$$

At infinite k this leads to $d+2$ equations (rather than $d+1$, since in the Galilean limit, mass and energy are separately conserved) for ϱ, e, p and v^i:

- continuity equation (mass conservation) $\mathcal{C} = 0$;
- energy conservation $\mathcal{E} = 0$;
- momentum conservation $\mathcal{M}_i = 0$;

this system is completed with the equation of state $p = p(e, \varrho)$.

It is important to stress that Galilean diffeomorphisms (478) do not involve k, and consequently they do not mix the various terms in the expansions (501) and (502). All $d+2$ fluid equations reached this way on general backgrounds are guaranteed to be covariant under Galilean diffeomorphisms. Another important result that we should stress is that these fluids are described on the most general background. This is one of the novelties of our work.

Dissipative Tensors

As for the Carrollian counterparts, we need here to specify the behaviour of the dissipative tensors for the large-k limit. Regarding the viscous stress tensor τ_{ij}, we will assume

$$\tau_{ij} = -\Sigma_{ij}, \tag{503}$$

which is standard and considered e.g. in [42], where it is named σ'_{ij}. Similarly, for the heat current, we will adopt

$$q_i = Q_i. \tag{504}$$

The position of the spatial indices is different here with respect to (326) and (325). This comes about because they are designed to be covariant under different classes of diffeomorphisms. Orthogonality conditions (24) allow to express every component of these tensors in terms of q_i and τ_{ij}.

We assume here the Zermelo form of the metric (479), and a fluid velocity field as in (482), (483). We find

$$q_0 = -\frac{v^i q_i}{k}, \quad q^0 = \frac{\left(v^i - w^i\right) q_i}{k\Omega^2}, \quad q^i = a^{ij} q_j + \frac{w^i \left(v^j - w^j\right) q_j}{k^2\Omega^2}. \tag{505}$$

Similarly, the components of the stress tensor are obtained from τ_{ij}. For example:

$$\tau_{00} = \frac{v^n v^l \tau_{nl}}{k^2}, \quad \tau_{0j} = -\frac{v^n \tau_{nj}}{k}, \quad \tau^0{}_j = -\frac{\left(v^n - w^n\right) \tau_{nj}}{k\Omega^2}, \quad \tau^{00} = \frac{\left(v^n - w^n\right)\left(v^l - w^l\right) \tau_{nl}}{k^2\Omega^4}, \dots \tag{506}$$

We now define

$$Q^i = a^{ij} Q_j, \tag{507}$$

and

$$\Sigma_i{}^j = \Sigma_{in} a^{nj}, \quad \Sigma^{ij} = a^{in} \Sigma_n{}^j. \tag{508}$$

Using the generic transformation rules of q_μ and $\tau_{\mu\nu}$ under spacetime diffeomorphisms, we find that Q and Σ transform as they should, namely as d-dimensional tensors under Galilean diffeomorphisms (478):

$$Q'_i = Q_n J^{-1n}{}_i, \qquad Q'^i = J^i_n Q^n, \tag{509}$$

$$\Sigma'_{ij} = J^{-1n}{}_i J^{-1l}{}_j \Sigma_{nl}, \qquad \Sigma'^j_i = J^{-1n}{}_i \Sigma_n{}^l J^j_l, \quad \Sigma'^{ij} = \Sigma^{nl} J^i_n J^j_l. \tag{510}$$

Continuity and Energy Conservation

Using (21) for the energy-momentum tensor $T^{\mu\nu}$ with $g^{\mu\nu}$ and u^μ given in (479) and (482) and the results for the dissipative tensors described above, we can perform the large-k expansion of the relativistic energy conservation equation (501).

At $O(k^2)$ we find

$$\mathcal{C} = \frac{\partial_t \sqrt{a}\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i, \tag{511}$$

where a stands for the determinant of the d-dimensional metric $a_{ij}(t, \boldsymbol{x})$, and ∇_i is the Levi-Civita covariant derivative associated with $a_{ij}(t, \boldsymbol{x})$ and Christoffel symbols given in (495).

The standard continuity equation $\mathcal{C} = 0$ is thus recovered. It is customary to decompose $\mathcal{C}$ as

$$\frac{\partial_t \sqrt{a}\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i = \frac{1}{\Omega}\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho\theta, \tag{512}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + v^i \nabla_i \tag{513}$$

is the material derivative, and

$$\theta = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \nabla_i v^i\right) \tag{514}$$

the effective Galilean fluid expansion. The latter combines the divergence of the fluid congruence with the logarithmic expansion of the volume form to produce a genuine scalar under Galilean diffeomorphisms, as shortly discussed. We will also show that the material derivative (513), in the form $\frac{1}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}$, is also an "invariant" when acting on a scalar function whereas when acting on arbitrary tensors it should be supplemented with the appropriate w-connection terms.

At the next $O(k^0)$ order, we obtain:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\varrho\left(e+\frac{1}{2}\left(\frac{\underline{v}-\underline{w}}{\Omega}\right)^2\right)\right)+\frac{1}{\Omega}\nabla_i\left(\varrho v^i\left(e+\frac{1}{2}\left(\frac{\underline{v}-\underline{w}}{\Omega}\right)^2\right)\right)\\
&\quad+\frac{1}{\Omega}\nabla_i\left(\left(v^j-w^j\right)\left(p\delta^i_j-\Sigma_j{}^i\right)\right)+\nabla_iQ^i+\frac{1}{\Omega}\Pi^{ij}\left(\nabla_iw_j+\frac{1}{2}\partial_ta_{ij}\right) \qquad (515)\\
&= \frac{\varrho}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(e+\frac{1}{2}\left(\frac{\underline{v}-\underline{w}}{\Omega}\right)^2\right)+\frac{1}{\Omega}\nabla_i\left(p\left(v^i-w^i\right)\right)+\nabla_iQ^i\\
&\quad-\frac{1}{\Omega}\nabla_i\left(\left(v^j-w^j\right)\Sigma_j{}^i\right)+\frac{1}{\Omega}\Pi^{ij}\left(\nabla_iw_j+\frac{1}{2}\partial_ta_{ij}\right), \qquad (516)
\end{aligned}$$

where the second expression is obtained from the first using the continuity equation $\mathcal{C}=0$.

Here we introduced

$$\Pi^{ij}=\varrho\frac{\left(v^i-w^i\right)\left(v^j-w^j\right)}{\Omega^2}+pa^{ij}-\Sigma^{ij}, \qquad (517)$$

the components of the Galilean spatial energy-momentum tensor, following [42]. They are expressed in terms of the fluid velocity, measured in an inertial-like frame, i.e. $\underline{v}-\underline{w}$, and we will show they transform under Galilean diffeomorphisms (478) as a genuine rank-two d-dimensional tensor on $\mathcal{S}$:

$$\Pi^{ij\prime}=J^i_kJ^j_l\Pi^{kl}. \qquad (518)$$

Equation $\mathcal{E}=0$ is the Galilean energy conservation equation for a viscous fluid in motion on arbitrary, time-dependent d-dimensional space $\mathcal{S}$, and observed from an arbitrary frame (moving at velocity $-\underline{w}(t,\boldsymbol{x})$ with respect to a local inertial frame). In a short while, we will recast this equation in a suitable form for recognizing the underlying phenomena. Notice that both friction and thermal conduction occur, driven by the viscous stress tensor Σ and the heat current Q. As opposed to the energy-conservation equation at hand, the continuity (mass-conservation) equation depends neither on the motion of the observer ($\underline{w}$) nor on the friction properties of the fluid. This is expected because energy is frame-dependent while mass it is not.

We proceed now to check that under Galilean diffeomorphisms (478):

$$\mathcal{C}'=\mathcal{C},\quad \mathcal{E}'=\mathcal{E}. \qquad (519)$$

In order to show this, it is convenient to recognize some well-behaved blocks in the expressions at hand, based on the quoted transformation rules. We first remind:

$$a'_{ij}=a_{nl}J^{-1n}{}_iJ^{-1l}{}_j,\quad v'^j=\frac{1}{J}\left(J^j_iv^i+j^j\right),\quad w'^j=\frac{1}{J}\left(J^j_iw^i+j^j\right),\quad \Omega'=\frac{\Omega}{J}.$$

Consequently

$$v'_n=\frac{J^{-1i}{}_n}{J}\left(v_i+a_{ij}J^{-1j}{}_lj^l\right),\quad w'_n=\frac{J^{-1i}{}_n}{J}\left(w_i+a_{ij}J^{-1j}{}_lj^l\right) \qquad (520)$$

with

$$\begin{aligned}
\partial'_t &= \frac{1}{J}\left(\partial_t-j^nJ^{-1i}{}_n\partial_i\right), \qquad (521)\\
\partial'_j &= J^{-1i}{}_j\partial_i. \qquad (522)
\end{aligned}$$

Consider now A^i and B^i, the components of fields transforming like v^i or w^i (gauge-like transformation) and V^i a field transforming like $\frac{v^i-w^i}{\Omega}$ i.e. like a genuine vector:

$$A'^j=\frac{1}{J}\left(J^j_iA^i+j^j\right),\quad B'^j=\frac{1}{J}\left(J^j_iB^i+j^j\right),\quad V'^j=J^j_iV^i. \qquad (523)$$

Consider also a scalar and a rank-two tensor

$$\Phi'=\Phi,\quad S'_{ij}=S_{nl}J^{-1n}{}_iJ^{-1l}{}_j. \qquad (524)$$

The basic transformation rules are as follows:

$$\begin{aligned}
\frac{A'^i - B'^i}{\Omega'} &= J^i_j \frac{A^j - B^j}{\Omega}, && (525)\\
\frac{1}{\sqrt{a'}}\partial'_t\left(\sqrt{a'}\Phi'\right) + \nabla'_i\left(\Phi' A'^i\right) &= \frac{1}{J}\left(\frac{1}{\sqrt{a}}\partial_t\left(\sqrt{a}\Phi\right) + \nabla_i\left(\Phi A^i\right)\right), && (526)\\
\nabla'_i V'^i &= \nabla_i V^i, && (527)\\
\nabla'_{(i}A'_{j)} + \frac{1}{2}\partial'_t a'_{ij} &= \frac{1}{J}\left(\nabla_{(n}A_{l)} + \frac{1}{2}\partial_t a_{nl}\right) J^{-1n}{}_i J^{-1l}{}_j, && (528)\\
\nabla'^{(i}A'^{j)} - \frac{1}{2}\partial'_t a'^{ij} &= \frac{1}{J}\left(\nabla^{(n}A^{l)} - \frac{1}{2}\partial_t a^{nl}\right) J^i_n J^j_l, && (529)\\
\nabla'_i S'^{ij} &= J^j_l \nabla_i S^{il}, && (530)\\
\frac{1}{\Omega'}\left(\partial'_t V'_i + A'^j\nabla'_j V'_i + V'_j \nabla'_i B'^j\right) &= \frac{J^{-1n}{}_i}{\Omega}\left(\partial_t V_n + A^j\nabla_j V_n + V_j\nabla_n B^j\right), && (531)\\
\Delta' A'_i + r'^m_i A'_m + a'_{ij}a'^{mn}\partial'_t\gamma'^j_{mn} &= \frac{J^{-1j}{}_i}{J}\left(\Delta A_j + r_j{}^m A_m + a_{jl}a^{mn}\partial_t\gamma^l_{mn}\right). && (532)
\end{aligned}$$

In the above expressions, ∇_i, Δ and r_{ij} are associated with the d-dimensional Levi–Civita connection γ^i_{jn} displayed in (495). The action of ∂_t spoils the transformation rules displayed in (523) and (524). This is both due to the transformation property of the partial time derivative (521), and to the time dependence of the Jacobian matrix J^i_j. A Galilean covariant time-derivative can be introduced, acting as follows on a vector:

$$\frac{1}{\Omega}\frac{\mathrm{D}V^i}{\mathrm{d}t} = \frac{1}{\Omega}\left[\left(\partial_t + v^j\nabla_j\right)V^i - V^j\nabla_j w^i\right] = \frac{1}{\Omega}\frac{\mathrm{d}V^i}{\mathrm{d}t} - \frac{1}{\Omega}V^j\nabla_j w^i, \tag{533}$$

and resulting in a genuine vector under Galilean diffeomorphisms. Here, the frame velocity w^i plays the role of a connection, and the Galilean covariant time-derivative generalizes the material derivative $^{\mathrm{d}}/_{\mathrm{d}t}$ introduced in (513). The latter is covariant only when acting on scalar functions f, hence we set $\frac{\mathrm{D}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$.

Expression (533) is easily extended to tensors of arbitrary rank using the Leibniz rule, as e.g. for one-forms:

$$\frac{1}{\Omega}\frac{\mathrm{D}V_i}{\mathrm{d}t} = \frac{1}{\Omega}\frac{\mathrm{d}V_i}{\mathrm{d}t} + \frac{1}{\Omega}V_j\nabla_i w^j. \tag{534}$$

Notice that the Galilean covariant time-derivative at hand is not metric compatible:

$$\frac{1}{\Omega}\frac{\mathrm{D}a_{ij}}{\mathrm{d}t} = \frac{1}{\Omega}\left(\partial_t a_{ij} + 2\nabla_{(i}w_{j)}\right). \tag{535}$$

This result is actually expected because a covariant time-derivative of the metric should be interpreted as an extrinsic curvature. Indeed, expression (535) divided by $2k$ is exactly identified with the spatial components K_{ij} of constant-t hypersurfaces extrinsic curvature in the Zermelo background (479).

Using all these expressions it is eventually possible to straightforwardly show (518) and (519)

$$\mathcal{C}' = \mathcal{C}, \quad \mathcal{E}' = \mathcal{E}. \tag{536}$$

as previously claimed.

The Galilean covariant time derivative will be used in the next section to manipulate the Euler equation. Furthermore, the transformation rules introduced here will serve to show the covariance of the latter under Galilean diffeomorphims.

Euler Equation

Following the same pattern we applied for the scalar equations, we can process the large-k behaviour of the relativistic momentum-conservation equations. Along with (502) we find:

$$\mathcal{M}_i = \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\varrho\frac{v_i - w_i}{\Omega}\right) + \frac{1}{\Omega}\nabla_j\left(\varrho w^j\left(\frac{v_i - w_i}{\Omega}\right)\right) + \frac{\varrho}{\Omega}\left(\frac{v^j - w^j}{\Omega}\right)\nabla_i w_j + \nabla_j \Pi_i{}^j \tag{537}$$

with $\Pi_i{}^j$ as in (517).

The equation $\mathcal{M}_i = 0$ is the ultimate generalization of the standard Euler equation. It is remarkably simple. The second and third terms in (537) contribute to inertial forces (Coriolis, centrifugal etc.), and are usually absent in Euclidean space with inertial frames. Together with the first term, they provide the components of a one-form on $\mathcal{S}$ transforming as $\frac{v^i - w^i}{\Omega}$.

This is also how $\mathcal{M}_i$ behave under Galilean diffeomorphisms (478):

$$\mathcal{M}'_i = J^{-1l}{}_i \mathcal{M}_l, \tag{538}$$

where to prove it one uses the results just depicted above. The Euler equation (537) can be casted in terms of the acceleration $\gamma = \gamma_i dx^i$ of the Galilean fluid. This is defined covariantly as

$$a_i = \gamma_i + O\left({}^1\!/\!{}_{k^2}\right) \tag{539}$$

with a_i the spatial components of the relativistic fluid acceleration as in (27). We find:

$$\Omega^2 \gamma_i = \Omega \frac{\mathsf{d}^{v_i}/_\Omega}{\mathsf{d}t} - \Omega \partial_t {}^{w_i}/_\Omega - \frac{1}{2}\partial_i w^2 - v^j \left(\partial_j w_i - \partial_i w_j\right) \tag{540}$$

with ${}^{\mathsf{d}}/_{\mathsf{d}t}$ defined in (513).

In this expression, γ_i appear as the components of the acceleration in the local inertial frame and $\frac{\mathsf{d}^{v_i}/_\Omega}{\Omega \mathsf{d}t}$ are the components of the effectively measured acceleration in the coordinate frame at hand. In the right-hand side, the second term is the dragging acceleration, the third accounts for the centrifugal acceleration, and the last is Coriolis contribution. We can alternatively write (540) as

$$\gamma_i = \frac{\mathsf{d}^{(v_i - w_i)}/_\Omega}{\Omega \mathsf{d}t} - \frac{1}{2}\partial_i \frac{w^2}{\Omega^2} + \frac{v^j}{\Omega}\nabla_i \frac{w_j}{\Omega} = \frac{\mathsf{D}^{(v_i - w_i)}/_\Omega}{\Omega \mathsf{d}t}, \tag{541}$$

where we used the Galilean covariant time-derivative (534) in the second equality. By construction, the γ_i transforms as a genuine d-dimensional form and $\gamma^i = a^{ij}\gamma_j$ as a vector under Galilean diffeomorphisms

$$\gamma'_i = J^{-1l}{}_i \gamma_l. \tag{542}$$

One can also check explicitly the covariance of (540) using (531). Using γ_i and the expression (517) for the Galilean energy-momentum tensor, we can recast $\mathcal{M}_i$ in (537) à la Euler:

$$\mathcal{M}_i = \varrho \gamma_i + \partial_i p - \nabla_j \Sigma_i{}^j. \tag{543}$$

This equation is eventually written here in a very clear and physically insightful form.

Energy and Entropy

The momentum equation $\mathcal{M}_i = 0$ together with continuity equation $\mathcal{C} = 0$ can also be used in order to provide a sharper expression for $\mathcal{E}$ given in (515):

$$\frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\varrho\left(e + \frac{v^2 - w^2}{2\Omega^2}\right)\right) = -\nabla_i \Pi^i - \frac{1}{2\Omega}\Pi^{ij}\partial_t a_{ij} + \varrho \frac{v_j - w_j}{\Omega^2}\partial_t \frac{w^j}{\Omega}. \tag{544}$$

In this equation, $\varrho\left(e + \frac{v^2 - w^2}{2\Omega^2}\right)$ is the total energy density of the fluid in the natural, non-inertial frame. The energy density has three contributions: $e\varrho$ as internal energy, the kinetic energy $\frac{\varrho v^2}{2\Omega^2}$, and the potential energy of inertial forces $\frac{-\varrho w^2}{2\Omega^2}$. Furthermore

$$\Pi^i = \varrho \frac{v^i}{\Omega}\left(h + \frac{v^2 - w^2}{2\Omega^2}\right) + Q^i - \frac{v^j}{\Omega}\Sigma_j{}^i \tag{545}$$

appears as the Galilean energy flux. It receives contributions from the enthalpy, the kinetic and inertial-potential energies, as well as from dissipative processes: thermal conduction and friction, with the corresponding heat current Q and viscous stress current $\frac{-v \cdot \Sigma}{\Omega}$.

The general energy conservation equation $\mathcal{E} = 0$ has now a simple interpretation: the time variation of energy in a local domain is due to the energy flux through the frontier plus the work due to the time dependence of a_{ij} and w^i.

Dissipative processes create entropy. One can readily determine the variation of the latter by recasting the energy variation in a manner slightly different than (544). For that we compute $\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i$ with (515), (541) and (543).

Using continuity and (514) we find

$$\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i = \frac{\varrho}{\Omega}\frac{\mathrm{d}e}{\mathrm{d}t} + p\theta + \nabla_i Q^i - \frac{1}{\Omega}\Sigma^{ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right). \tag{546}$$

In this expression, we can trade the energy per mass e with the entropy per mass s, obeying

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v = T\mathrm{d}s + \frac{p}{\varrho^2}\mathrm{d}\varrho, \tag{547}$$

where $v = {}^1\!/\!_\varrho$. Substituting this in (546) and using continuity, we finally obtain

$$\frac{\varrho T}{\Omega}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\Omega}\Sigma^{ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^i. \tag{548}$$

The entropy is not conserved as a consequence of friction and heat conduction, which encode dissipative processes. The latter are globally captured in a generalized dissipation function

$$\psi = \frac{1}{\Omega}\Sigma^{ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^i, \tag{549}$$

appearing both in energy and entropy equations (546), (548). Observe that ψ depends explicitly on Christoffel symbols as well as on the time variation of the metric. Hence time dependence and inertial forces contribute the dissipation phenomena.[66]

First-order Galilean Hydrodynamics and Incompressibility

The viscous stress tensor Σ and the heat current Q are constructed phenomenologically as velocity and temperature derivative expansions. Since these objects transform tensorially under Galilean diffeomorphisms (see (509), (510)), they must be expressed in terms of tensorial derivative quantities.

At first order, we have θ defined in (514), which is an invariant, and

$$\frac{1}{\Omega}\left(\nabla_{(n} v_{l)} + \frac{1}{2}\partial_t a_{nl}\right), \tag{550}$$

which is a rank-two symmetric tensor (see (528)).

We can therefore set

$$\Sigma_{(1)ij} = 2\eta^{\mathsf{G}}\xi_{ij} + \zeta^{\mathsf{G}} a_{ij}\theta, \tag{551}$$

$$Q_{(1)i} = -\kappa^{\mathsf{G}}\partial_i T. \tag{552}$$

The transport coefficients are as usual the shear viscosity η^{G}, coupled to the Galilean shear,

$$\xi_{ij} = \frac{1}{\Omega}\left(\nabla_{(i} v_{j)} + \frac{1}{2}\partial_t a_{ij}\right) - \frac{1}{d}a_{ij}\theta, \tag{553}$$

which receives also contributions from the derivative of the metric; the bulk viscosity ζ^{G}, coupled to the Galilean expansion, and the thermal conductivity κ^{G} coupled to the temperature gradient.

Using the definitions of relativistic expansion and shear (27), (29), we can find their behaviour at large k in the Zermelo background:

$$\sigma_{ij} = \xi_{ij} + O\left({}^1\!/\!_{k^2}\right), \tag{554}$$

$$\Theta = \theta + O\left({}^1\!/\!_{k^2}\right). \tag{555}$$

[66] The effect of inertial forces on dissipation has been recently studied by simulation of flows on curved static films without heat current (i.e. $d = 2$, $\Omega = 1$, $\underline{w} = 0$, $\partial_t a_{ij} = 0$, $Q^{\mathsf{G}} = 0$) [204]. One might consider performing similar simulations or experiments for probing the more general sources of dissipation present in (549).

For completeness we also display the leading behaviour of the vorticity (30), even though it plays no role in first-order hydrodynamics:

$$\omega_{ij} = \frac{1}{\Omega}\left(\partial_{[i}(v-w)_{j]}\right) + O\left({}^{1}\!/\!{}_{k^2}\right). \tag{556}$$

It is important to stress at this point that transport coefficients are determined as modes of microscopic correlation functions, and are therefore sensitive to the velocity of light. In writing (503), we have assumed the following large-k behaviour:

$$\eta = \eta^{\mathsf{G}} + O\left({}^{1}\!/\!{}_{k^2}\right), \quad \zeta = \zeta^{\mathsf{G}} + O\left({}^{1}\!/\!{}_{k^2}\right), \quad \kappa = \kappa^{\mathsf{G}} + O\left({}^{1}\!/\!{}_{k^2}\right). \tag{557}$$

The case $d=2$ is peculiar because $\Sigma_{(1)ij}$ admits an extra term:

$$\zeta_{\mathrm{H}}^{\mathsf{G}}\,\eta_{n(i}\,\xi_{j)l}\,a^{nl} = \frac{\zeta_{\mathrm{H}}^{\mathsf{G}}}{2\Omega}\left(\eta_{n(i}\,\nabla_{j)}v^n + \eta_{n(i}\,a_{j)l}\left(\nabla^n v^l - \frac{\partial_t\sqrt{a}a^{nl}}{\sqrt{a}} - a^{nl}\nabla_m v^m\right)\right) \tag{558}$$

with $\eta_{nl} = \sqrt{a}\,\epsilon_{nl}$. This is indeed (up to a global sign) the infinite-k limit of the relativistic Hall-viscosity contribution in three spacetime dimensions given in (44), assuming again $\zeta_{\mathrm{H}} = \zeta_{\mathrm{H}}^{\mathsf{G}} + O\left(\frac{1}{k^2}\right)$.

Going back to arbitrary dimension, we can now combine the first-derivative contribution (551) of the viscous stress tensor with expression (543) for $\mathcal{M}_i$ in order to obtain the momentum conservation equation $\mathcal{M}_i = 0$ of first-order Galilean hydrodynamics. We obtain

$$\varrho\gamma_i + \partial_i p - \frac{\eta^{\mathsf{G}}}{\Omega}\left(\Delta v_i + r_i{}^j v_j + a_{in}a^{jl}\partial_t\gamma^n_{jl}\right) - \left(\zeta^{\mathsf{G}} + \frac{d-2}{d}\eta^{\mathsf{G}}\right)\partial_i\theta = 0, \tag{559}$$

where $\Delta = \nabla^i\nabla_i$ is the Laplacian operator in d dimensions and r_{ij} the Ricci tensor of the d-dimensional Levi-Civita connection γ^n_{ij}.

Similarly, substituting (551), (552) and (553) in (548), we find the entropy equation in first-order hydrodynamics on general backgrounds:[67]

$$\frac{\varrho T}{\Omega}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2\eta^{\mathsf{G}}}{\Omega^2}\left(\left(\nabla^i v^j\right)\left(\nabla_i v_j\right) + \left(\nabla^i v^j\right)\partial_t a_{ij} - \frac{1}{4}\left(\partial_t a^{ij}\right)\left(\partial_t a_{ij}\right)\right) + \left(\zeta^{\mathsf{G}} - \frac{2\eta^{\mathsf{G}}}{d}\right)\theta^2 + \kappa^{\mathsf{G}}\Delta T, \tag{560}$$

where we assumed κ^{G} constant (otherwise the last term would read $\nabla^i(\kappa^{\mathsf{G}}\nabla_i T)$).

A special class of Galilean fluids deserves further analysis. These are the incompressible fluids for which $\varrho(t,\boldsymbol{x})$ obeys

$$\frac{\mathrm{d}\varrho(t,\boldsymbol{x})}{\mathrm{d}t} = 0 \tag{561}$$

with $\frac{\mathrm{d}}{\mathrm{d}t}$ the material derivative defined in (513). Using the expressions (511) and (512), we recast the incompressibility requirement as the vanishing of the effective fluid expansion:

$$\theta = 0. \tag{562}$$

In this case, the bulk viscosity drops from the stress tensor (551) and the Galilean shear (553) simplifies. The first-order hydrodynamics momentum equation for an incompressible fluid thus reads:

$$\varrho\frac{\mathrm{d}^{v_i}\!/\!_{\Omega}}{\Omega\,\mathrm{d}t} = \varrho\frac{\mathrm{d}^{w_i}\!/\!_{\Omega}}{\Omega\,\mathrm{d}t} + \frac{\varrho}{2}\partial_i\frac{w^2}{\Omega^2} - \varrho\frac{v^j}{\Omega}\nabla_i\frac{w_j}{\Omega} - \partial_i p + \frac{\eta^{\mathsf{G}}}{\Omega}\left(\Delta v_i + r_i{}^j v_j + a_{in}a^{jl}\partial_t\gamma^n_{jl}\right). \tag{563}$$

We immediately recognize in this expression the generalized covariant Navier-Stokes equation, valid for incompressible fluids on any space $\mathcal{S}$, observed from an arbitrary frame. To the best of our knowledge this equation is new. The first three terms in the right-hand side are contributions of frame inertial forces, the fourth is the pressure force, and next come the friction forces at first-order derivative.

Eventually, for Euclidean space with $\Omega = 1$ and $w = 0$ we recover the textbook form

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{\nabla^i p}{\varrho} + \frac{\eta^{\mathsf{G}}}{\varrho}\Delta v^i. \tag{564}$$

[67] Possible impositions on the metric and the velocity are necessary to guarantee positivity of this expression, not discussed here.

3.4.3 Examples of Galilean Fluids

We provide here two applications: the flat space in rotating frame, which is well known and has the virtue of giving confidence to our methods, and the inflating space, combining both time-dependence and non-flatness of the host $\mathcal{S}$.

Rotating Frame in Three Dimensions

We will present the hydrodynamic equations for a non-perfect fluid moving in Euclidean space E_3 with Cartesian coordinates, and observed from a uniformly rotating frame

$$a_{ij} = \delta_{ij}, \quad \Omega = 1, \quad \underline{w}(\boldsymbol{x}) = \underline{x} \times \underline{\omega}. \tag{565}$$

For this fluid, the continuity equation is simply

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho \nabla \cdot v = 0. \tag{566}$$

The Euler equation in first-order hydrodynamics (559) reads:

$$\frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = (\underline{\omega} \times \underline{x}) \times \underline{\omega} + 2\underline{v} \times \underline{\omega} - \frac{\underline{\nabla} p}{\varrho} + \frac{\eta^{\mathsf{G}}}{\varrho} \Delta \underline{v} + \frac{1}{\varrho}\left(\zeta^{\mathsf{G}} + \frac{\eta^{\mathsf{G}}}{3}\right) \underline{\nabla}(\nabla \cdot v), \tag{567}$$

and we recognize the various, already spelled contributions to the dynamics. This equation has been obtained and used in many instances, see e.g. [155, 205, 206].

We also find the energy conservation equation (544):

$$\partial_t \left(\varrho\left(e + \frac{v^2 - \omega^2 x^2 + (\omega \cdot x)^2}{2}\right)\right) = -\nabla \cdot \Pi, \tag{568}$$

with

$$\Pi_i = \varrho v_i \left(h + \frac{v^2 - \omega^2 x^2 + (\omega \cdot x)^2}{2}\right) - \kappa^{\mathsf{G}} \nabla_i T - (v \cdot \Sigma_{(1)})_i \tag{569}$$

and

$$\Sigma_{(1)ij} = \eta^{\mathsf{G}} \left(\partial_i v_j + \partial_j v_i\right) + \left(\zeta^{\mathsf{G}} - \frac{2}{3}\eta^{\mathsf{G}}\right) \delta_{ij} \partial_n v^n. \tag{570}$$

Alternatively, using (516), the energy equation reads:

$$\varrho \frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{v^2 - \omega^2 x^2 + (\omega \cdot x)^2}{2}\right) = -\nabla \cdot (pv) + \kappa^{\mathsf{G}} \Delta T + \nabla \cdot \left(v \cdot \Sigma_{(1)}\right). \tag{571}$$

The temporal variation of the total energy per mass is given by the divergences of the pressure, the thermal conduction and the viscous stress fluxes.

Inflating Space

The dynamics of a non-perfect fluid moving on an inflating space can be studied considering:

$$a_{ij}(t, \boldsymbol{x}) = \exp\left(\alpha(t)\right) \tilde{a}_{ij}(\boldsymbol{x}), \quad \Omega = 1, \quad \underline{w} = 0. \tag{572}$$

The space dimension d is arbitrary here, therefore:

$$\ln \sqrt{a} = d\,\frac{\alpha}{2} + \ln \sqrt{\tilde{a}}. \tag{573}$$

The fluid equations obtained from (511), (516) and (543) become (α' stands for the time derivation)

$$\partial_t \varrho + \frac{\alpha'}{2} d\varrho + \nabla \cdot \varrho v = 0, \tag{574}$$

$$\varrho \frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{v^2}{2}\right) + \frac{\alpha'}{2}\left(\varrho v^2 + dp - tr\Sigma\right) + \nabla \cdot \left(pv + Q - v \cdot \Sigma\right) = 0, \tag{575}$$

$$\varrho \frac{\mathrm{d}v^i}{\mathrm{d}t} + \alpha' \varrho v^i + \nabla^i p - \nabla_j \Sigma^{ij} = 0. \tag{576}$$

where $\alpha' = \frac{\mathrm{d}\alpha}{\mathrm{d}t}$ and $tr\Sigma = a^{ij}\Sigma_{ij}$.

The continuity equation (574) has an extra term proportional to ϱ. This reflects the change of density due to α'. For a static fluid one finds the familiar result $\varrho = \varrho_0 e^{-d\alpha/2}$: for a space expanding in time, the density is getting diluted. In Euler's equation (576), a similar term creates a force proportional to the velocity field. For positive α', time dependence acts effectively like a friction. A similar conclusion is drawn from the energy conservation equation (575).

This example concludes the chapter on Galilean hydrodynamics. We have seen how to obtain the most general Galilean fluid on completely arbitrary background. The equations are fully covariant under Galilean diffeomorphisms, and they reduce under suitable conditions to well-known situations.

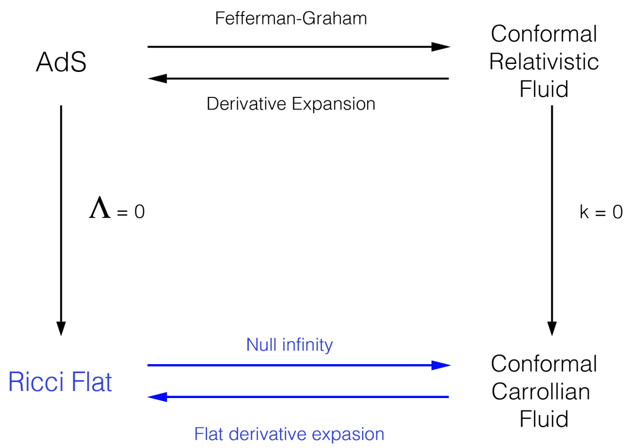
This concludes our description of the different limits of a relativistic fluid and its energy-momentum tensor. We have learned that we should be careful whenever the background is kept general, and work directly at the level of the equations of motion, which involve the divergence of the energy-momentum tensor itself. This is a posteriori expected for Galilean fluids, for we know that a spacetime energy-momentum is impossible to construct there. In parallel we show a similar result to hold for a Carrollian fluid, where the energy-momentum tensor gets replaced by the Carrollian momenta. The general equations of motion obtained in the Carrollian setting, fully covariant under Carrollian diffeomorphisms, will be fundamentals in the next section, where we show that they are the appropriate boundary dual of bulk Einstein equations for asymptotically flat spacetimes.

4 Flat Limit of Fluid-Gravity

We are now fully equipped to address the final missing part of our web of dualities. We have seen that the fluid/gravity duality in AdS relates a relativistic fluid on a conformal $d+1$-dimensional boundary with a $d+2$-dimensional solution of AdS Einstein equations through the derivative expansion, with $\Lambda = -\frac{d(d+1)}{2}k^2$. Therefore, the bulk flat limit, for which the AdS solution under consideration becomes Ricci flat, translates to $k \to 0$ in the boundary theory.[68]

We thus proceeded and considered the $k \to 0$ in the boundary theory. This had mainly two important implications. Firstly the boundary metric became degenerate, in a precise sense we carefully described. This is consistent with the fact that the null boundary $\mathscr{I}$ of an asymptotically flat spacetime is indeed a degenerate manifold. Secondly the conformal relativistic fluid became a conformal Carrollian fluid, and we thoroughly analyzed its equations of motion, obtained as the ultra-relativistic limit of the conservation of the energy-momentum tensor.

So the only missing step in the construction concerns the derivative expansion: what happens to it when we take $k \to 0$? The crucial result will be that it is finite. This is perhaps the most important result of this work. We know in fact that the FG gauge diverges in this limit. Therefore we unravel here a powerful tool and the final link (in blue) to complete the square:



[68] As we mentioned multiple times, the bulk Ricci flat limit is a straightforward result of general relativity. Every AdS solution admits an asymptotically flat counterpart, solution of Einstein equation with $\Lambda = 0$. What is not trivial is the gauge in which this limiting procedure is performed, and we are going to show that the derivative expansion is the right one.

We will discuss here the results for dimension four and three bulks. The latter will be the natural playground to compute conserved charges. In both situations we will present detailed examples to corroborate our findings. At present, the form of the derivative expansion is missing in bulk dimensions higher than four, which constitutes a natural direction of investigation.

4.1 The Four-dimensional Case

This chapter is fully devoted to the four-dimensional bulk picture, where the derivative expansion under the shearless condition allows to resum every algebraically special bulk solution.

4.1.1 Flat Derivative Expansion

Our starting point is the derivative expansion of an asymptotically locally AdS spacetime (158). The fundamental question is whether the latter admits a smooth zero-k limit. We have implicitly assumed that the Randers-Papapetrou data of the three-dimensional pseudo-Riemannian conformal boundary associated with the original Einstein spacetime, a_{ij}, b_i and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new null boundary $\mathcal{I}$.[69]

We can match the various three-dimensional Riemannian quantities with the corresponding Carrollian ones:

$$u = -k^2\left(\Omega dt - b\right) \tag{577}$$

and

$$\begin{aligned} \omega &= \frac{k^2}{2}\varpi_{ij}dx^i\wedge dx^j, \\ \gamma &= \star\varpi, \\ \Theta &= \theta, \\ a &= k^2\varphi_i dx^i, \\ A &= \alpha_i dx^i + \frac{\theta}{2}\Omega dt, \\ \sigma &= \xi_{ij}dx^i dx^j, \end{aligned} \tag{578}$$

where the left-hand-side quantities are Riemannian given in (150, 152–155), and the right-hand-side ones Carrollian as reported in (346–349), and we recall that for three-dimensional holographic fluid we impose $\beta_i = 0$.

In the list (578), we have dealt with the first derivatives, i.e. connexion-related quantities. We move now to second-derivative objects and collect the tensors relevant for the derivative expansion, following the same pattern (Riemannian vs. Carrollian):

$$\mathcal{R} = \frac{1}{k^2}\xi_{ij}\xi^{ij} + 2\hat{\mathcal{K}} + 2k^2\star\varpi^2, \tag{579}$$

$$\omega_\mu{}^\lambda\omega_{\lambda\nu}dx^\mu dx^\nu = k^4\varpi_i{}^l\varpi_{lj}dx^i dx^j, \tag{580}$$

$$\omega^{\mu\nu}\omega_{\mu\nu} = 2k^4\star\varpi^2, \tag{581}$$

$$\mathcal{D}_\nu\omega^\nu{}_\mu dx^\mu = k^2\hat{\mathcal{D}}_j\varpi^j{}_i dx^i - 2k^4\star\varpi^2\Omega dt + 2k^4\star\varpi^2 b_i dx^i. \tag{582}$$

Using (147) this leads to

$$S = S_\mu dx^\mu = -\frac{k^2}{2}\left(\Omega dt - b_i dx^i\right)^2\xi_{ij}\xi^{ij} + k^4 s - 5k^6\left(\Omega dt - b_i dx^i\right)^2\star\varpi^2 \tag{583}$$

with the Weyl-invariant tensor

$$s = 2\left(\Omega dt - b_i dx^i\right)dx^i\tilde{\eta}^j{}_i\hat{\mathcal{D}}_j\star\varpi + \star\varpi^2 d\ell^2 - \hat{\mathcal{K}}\left(\Omega dt - b_i dx^i\right)^2. \tag{584}$$

In the derivative expansion (non-resummed) (146), two explicit divergences appear at vanishing k. The first originates from the first term of S, which is the shear contribution to the Weyl-covariant scalar curvature $\mathcal{R}$ of the

[69] Indeed our ultimate goal is to set up a derivative expansion (in a closed resummed form under appropriate assumptions) for building up four-dimensional Ricci-flat spacetimes from a boundary Carrollian fluid, irrespective of its AdS origin. For this it is enough to assume a_{ij}, b_i and Ω k-independent, and use these data as fundamental blocks for the Ricci-flat reconstruction. It should be kept in mind, however, that for general Einstein spacetimes, these may depend on k with well-defined limit and subleading terms. Due to the absence of shear and to the particular structure of these solutions, the latter do not alter the Carrollian equations. This occurs for instance in Plebański-Demiański or in the Kerr-Taub-NUT family, which will be discussed as example.

three-dimensional AdS boundary , (579).[70] The second divergence comes from the Cotton tensor and is also due to the shear. It is not explicitly reported here but possible to recover taking the $k \to 0$ limit of the Cotton tensor with a shearfull congruence. It is fortunate – and expected – that counterterms coming from equal-order (non-explicitly written) σ^2 contributions, cancel out these singular terms. This is suggestive that already the non-resummed expansion (146) is well-behaved at zero-k, showing the success of the reconstruction of Ricci-flat spacetimes.

We will not take this rode, but rather confine our analysis to situations without shear, as we discussed already for Einstein spacetimes. Vanishing σ in the pseudo-Riemannian boundary implies indeed vanishing ξ_{ij} in the Carrollian (see (578)), and in this case, the divergent terms in S and C are absent. Of course, other divergences may occur from higher-order terms in the derivative expansion. To avoid dealing with these issues, we will focus on the resummed version of (146) i.e. (158), valid for algebraically special bulk geometries. This closed form is definitely smooth at zero k and reads:

$$\boxed{ds^2_{\text{res. flat}} = -2\left(\Omega dt - b\right)\left(dr + r\alpha + \frac{r\theta\Omega}{2}dt\right) + r^2 d\ell^2 + s + \frac{\left(\Omega dt - b\right)^2}{\rho^2}\left(8\pi G_N \varepsilon r + c\star\varpi\right).} \tag{585}$$

Here

$$\rho^2 = r^2 + \star\varpi^2, \tag{586}$$

$d\ell^2$, Ω, $b = b_i dx^i$, $\alpha = \alpha_i dx^i$, θ and $\star\varpi$ are the Carrollian geometric objects introduced earlier, while c and ε are the zero-k (finite) limits of the corresponding relativistic functions. Expression (585) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (158) was Einstein are fulfilled in the zero-k limit. These conditions are the set of conformal Carrollian hydrodynamic equations (367–370), and the integrability conditions, as they emerge from (161) and (163) at vanishing k. Making the latter explicit is the scope of next section. Notice eventually that the Ricci-flat line element (585) inherits Weyl invariance from its relativistic ancestor. The set of transformations (374), (377) and (379), supplemented with $\star\varpi \to \mathcal{B}\star\varpi$, $\varepsilon \to \mathcal{B}^3\varepsilon$ and $c \to \mathcal{B}^3 c$, can indeed be absorbed by setting $r \to \mathcal{B}r$, resulting thus in the invariance of (585). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this is rooted to null infinity $\mathscr{I}$. Before moving on we would like to stress again the fundamental result that the derivative expansion is finite in the $k \to 0$ limit, which is not at all an a priori guaranteed result but rather an important finding.

4.1.2 Conditions on the Flat Derivative Expansion

The Cotton tensor was a key tensor in the AdS boundary: it encodes the properties of the boundary global structure. In order to proceed with our resummability analysis, we need to describe the zero-k limit of this tensor (65) and of its conservation equation (66).

As already mentioned, at vanishing k divergences do generally appear for some components of the Cotton tensor. These divergences are no longer present in the absence of shear, which is precisely the assumption under which we are working. Every piece of the three-dimensional relativistic Cotton tensor appearing in (67) has thus a well-defined limit. We therefore introduce

$$\chi_i = \lim_{k\to 0} c_i, \quad \psi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_i - \chi_i\right), \tag{587}$$

$$X_{ij} = \lim_{k\to 0} c_{ij}, \quad \Psi_{ij} = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_{ij} - X_{ij}\right). \tag{588}$$

The time components c_0, c_{00} and $c_{0i} = c_{i0}$ vanish already at finite k (due to (69)), and χ_i, ψ_i, X_{ij} and Ψ_{ij} are thus genuine Carrollian tensors transforming covariantly under Carrollian diffeomorphisms. Actually, in the absence of shear the Cotton current and stress tensor are given exactly (i.e. for finite k) by $c_i = \chi_i + k^2\psi_i$ and $c_{ij} = X_{ij} + k^2\Psi_{ij}$.

The scalar c is Weyl-covariant of weight 3 (like the energy density). As expected, it is expressed in terms of geometric Carrollian objects built on third-derivatives of the 2-dimensional metric $d\ell^2$, b_i and Ω:

$$c = \left(\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right)\star\varpi. \tag{589}$$

[70] This divergence is traced back in the Gauss-Codazzi equation relating the intrinsic and extrinsic curvatures of an embedded surface, to the intrinsic curvature of the host. When the size of a fiber shrinks, the extrinsic-curvature contribution diverges.

Similarly, the forms χ_i and ψ_i, of weight 2, are (recall that $\tilde{\eta}_{ij} = \sqrt{a}\epsilon_{ij}$, the zero-$k$ limit of the spatial components of $\tilde{\eta}_{\mu\nu}$):

$$
\begin{aligned}
\chi_j &= \frac{1}{2}\tilde{\eta}^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j \hat{\mathscr{A}} - 2 \star\varpi \hat{\mathscr{R}}_j, && (590)\\
\psi_j &= 3\tilde{\eta}^l{}_j \hat{\mathscr{D}}_l \star\varpi^2. && (591)
\end{aligned}
$$

Finally, the weight-1 symmetric and traceless rank-two tensors read:

$$
\begin{aligned}
X_{ij} &= \frac{1}{2}\tilde{\eta}^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{R}}_i + \frac{1}{2}\tilde{\eta}^l{}_i \hat{\mathscr{D}}_j \hat{\mathscr{R}}_l, && (592)\\
\Psi_{ij} &= \hat{\mathscr{D}}_i \hat{\mathscr{D}}_j \star\varpi - \frac{1}{2}a_{ij}\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l \star\varpi - \tilde{\eta}_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t \star\varpi^2. && (593)
\end{aligned}
$$

Observe that c and the subleading terms ψ_i and Ψ_{ij} are present only when the vorticity is non-vanishing ($\star\varpi \neq 0$). All these are of gravito-magnetic nature.

The tensors c, χ_i, ψ_i, X_{ij} and Ψ_{ij} should be considered as the two-dimensional Carrollian resurgence of the three-dimensional Riemannian Cotton tensor. They should be referred to as Cotton descendants (there is no Cotton tensor in two dimensions anyway), and obey identities inherited at zero k from its conservation equation. These are similar to the hydrodynamic equations (367–370), satisfied by the different pieces of the energy-momentum tensor ε, Q_i, π_i, Σ_{ij} and Ξ_{ij}, and translating its conservation. In the case at hand, the absence of shear trivializes (368) and discards the last term in the other three equations:

$$
\begin{aligned}
\frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i &= 0, && (594)\\
\frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j &= 0, && (595)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j &= 0. && (596)
\end{aligned}
$$

One appreciates from these equations why it is important to keep the subleading corrections at vanishing k, both in the Cotton current c_μ and in the Cotton stress tensor $c_{\mu\nu}$. As for the energy-momentum tensor, ignoring them would simply lead to wrong Carrollian dynamics.

We are now ready to address the problem of integrability in Carrollian framework, for Ricci-flat spacetimes. In the relativistic case, where one describes relativistic hydrodynamics on the pseudo-Riemannian boundary of an asymptotically locally AdS spacetime the relevant equations are (161) and (163). These determine the friction components of the fluid energy-momentum tensor in terms of geometric data, captured by the Cotton tensor (current and stress components), via a sort of gravitational electric-magnetic duality, transverse to the fluid congruence. Equipped with those, the fluid equations (22) guarantee that the bulk is Einstein, i.e. that bulk Einstein equations are satisfied.

Correspondingly, using the results just detailed for the Cotton descendants, the zero-k limit of (161) sets up a duality relationship among the Carrollian-fluid heat current Q_i and the Carrollian-geometry third-derivative vector χ_i:

$$
Q_i = \frac{1}{8\pi G_N}\tilde{\eta}^j{}_i \chi_j = -\frac{1}{16\pi G_N}\left(\hat{\mathscr{D}}_i \hat{\mathscr{K}} - \tilde{\eta}^j{}_i \hat{\mathscr{D}}_j \hat{\mathscr{A}} + 4\star\varpi \tilde{\eta}^j{}_i \hat{\mathscr{R}}_j\right), \tag{597}
$$

while (163) allows to relate the Carrollian-fluid quantities Σ_{ij} and Ξ_{ij}, to the Carrollian-geometry ones X_{ij} and Ψ_{ij}:

$$
\Sigma_{ij} = \frac{1}{8\pi G_N}\tilde{\eta}^l{}_i X_{lj} = \frac{1}{16\pi G_N}\left(\tilde{\eta}^n{}_j \tilde{\eta}^l{}_i \hat{\mathscr{D}}_n \hat{\mathscr{R}}_l - \hat{\mathscr{D}}_j \hat{\mathscr{R}}_i\right), \tag{598}
$$

and

$$
\Xi_{ij} = \frac{1}{8\pi G_N}\tilde{\eta}^l{}_i \Psi_{lj} = \frac{1}{8\pi G_N}\left(\tilde{\eta}^l{}_i \hat{\mathscr{D}}_l \hat{\mathscr{D}}_j \star\varpi + \frac{1}{2}\tilde{\eta}_{ij}\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l \star\varpi - a_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t \star\varpi^2\right). \tag{599}
$$

One readily shows that (366) is satisfied as a consequence of the symmetry and tracelessness of X_{ij} and Ψ_{ij}.

We can finally recast the Carrollian hydrodynamic equations (367–370) for the fluid under consideration. Recalling that the shear is assumed to vanish,

$$
\xi_{ij} = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right) = 0, \tag{600}
$$

we see that (368) is trivialized. Furthermore, (370) is automatically satisfied with Q_j and $\Sigma^i{}_j$ given above, thanks also to (596). We are therefore left with two equations for the energy density ε and the heat current π_i:

- one scalar equation from (367):

$$-\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon+\frac{1}{16\pi G_N}\hat{\mathscr{D}}^i\left(\hat{\mathscr{D}}_i\hat{\mathscr{K}}-\tilde{\eta}^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}}+4\star\varpi\tilde{\eta}^j{}_i\hat{\mathscr{R}}_j\right)=0; \tag{601}$$

- one vector equation from (369):

$$\hat{\mathscr{D}}_j\varepsilon+4\star\varpi\tilde{\eta}^i{}_jQ_i+\frac{2}{\Omega}\hat{\mathscr{D}}_t\pi_j-2\hat{\mathscr{D}}_i\Xi^i{}_j=0 \tag{602}$$

  with Q_i and $\Xi^i{}_j$ given in (597) and (599).

These last two are Carrollian equations, describing time and space evolution of the fluid energy and heat current, as a consequence of transport phenomena like heat conduction and friction. These phenomena have been identified by duality to geometric quantities, and one recognizes distinct gravito-electric (like $\hat{\mathscr{K}}$) and gravito-magnetic contributions (like $\hat{\mathscr{A}}$). It should also be stressed that not all the terms are independent and one can reshuffle them using identities relating the Carrollian curvature elements. In the absence of shear, (263) holds and all information about $\hat{\mathscr{R}}_{ij}$ in (279) is stored in $\hat{\mathscr{K}}$ and $\hat{\mathscr{A}}$, while other geometrical data are supplied by $\hat{\mathscr{R}}_i$ in (278). All these obey

$$\begin{aligned}\tfrac{2}{\Omega}\hat{\mathscr{D}}_t\star\varpi+\hat{\mathscr{A}}&=0,\\ \tfrac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{K}}-a^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j&=0,\\ \tfrac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{A}}+\tilde{\eta}^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j&=0,\end{aligned} \tag{603}$$

which originate from three-dimensional Riemannian Bianchi identities and emerge along the k-to-zero limit.

Summarizing

As we did for the relativistic AdS counterpart, we now summarize our findings and recall the procedure one has to follow to, given a boundary conformal Carrollian fluid, obtain a bulk Ricci-flat solution.

Our analysis of the zero-k limit in the derivative expansion (158), valid assuming the absence of shear, has the following salient features.

- It reveals a degenerate null spacetime $\mathscr{I}$ endowed with a Carrollian geometry, encoded in a_{ij}, b_i and Ω, all functions of t and $\mathbf{x}$. This is inherited from the conformal three-dimensional pseudo-Riemannian boundary of the original Einstein space.
- The Carrollian null boundary is the host of a Carrollian fluid, obtained as the limit of a relativistic fluid, and described in terms of its energy density ε, and its friction tensors Q_i, π_i, Σ_{ij} and Ξ_{ij}.
- When the friction tensors Q_i, Σ_{ij} and Ξ_{ij} of the Carrollian fluid are given in terms of the geometric objects χ_i, X_{ij} and Ψ_{ij} using (597), (598) and (599), and when the energy density ε and the current π_i obey the hydrodynamic equations (601) and (602), the limiting resummed derivative expansion (585) is an exact Ricci-flat spacetime.
- The bulk spacetime is in general asymptotically locally flat. This property is encoded in the zero-k limit of the Cotton tensor, i.e. in the Cotton Carrollian descendants c, χ_i and X_{ij}.

As for the AdS scenario, the next question is the domain of validity of this resummation formula. There, we found it to cover all algebraically special solutions (see Appendix B). Also here, the bulk Ricci-flat spacetime obtained following the above procedure is algebraically special. We indeed observe that the bulk congruence ∂_r is null. Moreover, it is geodesic and shear-free.[71] According to the Goldberg-Sachs theorem, the bulk spacetime (585) is therefore of Petrov type II, III, D, N or O. The precise type is encoded in the Carrollian tensors $\varepsilon^\pm$, $Q_i^\pm$ and $\Sigma_{ij}^\pm$

$$\begin{aligned}\varepsilon^\pm&=\varepsilon\pm\tfrac{i}{8\pi G_N}c,\\ Q_i^\pm&=Q_i\pm\tfrac{i}{8\pi G_N}\chi_i,\\ \Sigma_{ij}^\pm&=\Sigma_{ij}\pm\tfrac{i}{8\pi G_N}X_{ij}.\end{aligned} \tag{604}$$

[71] This is proved in Appendix B.

Working again in holomorphic coordinates, we find the compact result

$$Q^{+} = \frac{i}{4\pi G_N}\chi_{\zeta}d\zeta, \quad (605)$$

$$\Sigma^{+} = \frac{i}{4\pi G_N}X_{\zeta\zeta}d\zeta^2, \quad (606)$$

and their complex-conjugates Q^- and Σ^-. These Carrollian geometric tensors encode the information on the canonical complex functions describing the Weyl-tensor decomposition in terms of principal null directions.

4.1.3 Examples

There is a plethora of examples that can be studied. We will analyze here the class of perfect conformal fluids and the dual stationary Kerr-Taub-NUT family, and the Carrollian Robinson-Trautman fluid dual to Robinson-Trautman. In each case, assuming the integrability conditions (597), (598) and (599) are fulfilled and the hydrodynamic equations (601) and (602) are obeyed, a Ricci-flat spacetime is reconstructed from the boundary $\mathscr{I}$. More examples exist like the Plebański-Demiański or the Weyl axisymmetric solutions, assuming extra symmetries (but not necessarily stationarity) for a viscous Carrollian fluid.

Stationary Perfect Fluids and Kerr-Taub-NUT

We would like to illustrate our findings and reconstruct from purely Carrollian fluid dynamics the family of Kerr-Taub-NUT stationary Ricci-flat black holes. We pick for that the following geometric data: $a_{ij}(\boldsymbol{x})$, $b_i(\boldsymbol{x})$ and $\Omega = 1$. Stationarity is implemented in these fluids by requiring that all the quantities involved are time independent.

Under this assumption, the Carrollian shear ξ_{ij} vanishes together with the Carrollian expansion θ, whereas constant Ω makes the Carrollian acceleration φ_i vanish as well. Consequently

$$\hat{\mathscr{A}} = 0, \quad \hat{\mathscr{R}}_i = 0, \quad (607)$$

and we are left with non-trivial curvature and vorticity:

$$\hat{\mathscr{K}} = \hat{K} = K, \quad \varpi_{ij} = \partial_{[i}b_{j]} = \tilde{\eta}_{ij} \star \varpi. \quad (608)$$

The Weyl-Carroll spatial covariant derivative $\hat{\mathscr{D}}_i$ reduces to the ordinary covariant derivative ∇_i, whereas the action of the Weyl-Carroll temporal covariant derivative $\hat{\mathscr{D}}_t$ vanishes.

We further assume that the Carrollian fluid is perfect: Q_i, π_i, Σ_{ij} and Ξ_{ij} vanish. This assumption is made according to the relativistic AdS pattern, where the asymptotically AdS Kerr-Taub-NUT spacetime is obtained starting from relativistic perfect fluids. Due to the duality relationships (597), (598) and (599) among the friction tensors of the Carrollian fluid and the geometric quantities χ_i, X_{ij} and Ψ_{ij}, the latter must also vanish. Using (590), (592) and (593), this sets the following simple geometric constraints:

$$\chi_i = 0 \Leftrightarrow \partial_i K = 0, \quad (609)$$

and

$$\Psi_{ij} = 0 \Leftrightarrow \left(\nabla_i\nabla_j - \frac{1}{2}a_{ij}\nabla_l\nabla^l\right) \star \varpi = 0, \quad (610)$$

whereas X_{ij} vanishes identically without bringing any further restriction. These are equations for the metric $a_{ij}(\boldsymbol{x})$ and the scalar vorticity $\star\varpi$, from which we can extract $b_i(\boldsymbol{x})$. Using (589), we also learn that

$$c = (\Delta + 2K) \star \varpi, \quad (611)$$

where $\Delta = \nabla_l\nabla^l$ is the ordinary Laplacian operator on $\mathscr{S}$. The last piece of the geometrical data, (591), it is non-vanishing and reads:

$$\psi_j = 3\tilde{\eta}^l{}_j\partial_l \star \varpi^2. \quad (612)$$

Finally, we must impose the fluid equations (601) and (602), leading to

$$\partial_t\varepsilon = 0, \quad \partial_i\varepsilon = 0. \quad (613)$$

The energy density ε of the Carrollian fluid is therefore a constant, which will be identified to the bulk mass parameter $M = 4\pi G_N \varepsilon$.

Every stationary Carrollian geometry encoded in $a_{ij}(\boldsymbol{x})$ and $b_i(\boldsymbol{x})$ with constant scalar curvature K hosts a conformal Carrollian perfect fluid with constant energy density, and is associated with the exact Ricci-flat spacetime with line element written using (585):

$$ds^2_{\text{perfect fluid}} = -2\,(dt-b)\,dr + \frac{2Mr + c\star\varpi - K\rho^2}{\rho^2}\,(dt-b)^2 + (dt-b)\,\frac{\psi}{3\star\varpi} + \rho^2 d\ell^2, \tag{614}$$

where $\rho^2 = r^2 + \star\varpi^2$. The vorticity $\star\varpi$ is determined by (610), solved on a constant-curvature background.

Using holomorphic coordinates, a constant-curvature metric on $\mathcal{S}$ reads:

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \tag{615}$$

with

$$P = 1 + \frac{K}{2}\zeta\bar{\zeta}, \quad K = 0, \pm 1, \tag{616}$$

corresponding to S^2 and E_2 or H_2 (sphere and Euclidean or hyperbolic planes). Using these expressions we can integrate (610). The general solution depends on three real, arbitrary parameters, n, a and ℓ:

$$\star\,\varpi = n + a - \frac{2a}{P} + \frac{\ell}{P}\,(1-|K|)\,\zeta\bar{\zeta}. \tag{617}$$

The parameter ℓ is relevant in the flat case exclusively. We can further integrate to obtain b:

$$b = \frac{i}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\,(1-|K|)\,\zeta\bar{\zeta}\right)\left(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}\right). \tag{618}$$

It is straightforward to determine the last pieces entering the bulk resumed metric (614):

$$c = 2Kn + 2\ell\,(1-|K|) \tag{619}$$

and

$$\frac{\psi}{3\star\varpi} = 2\tilde{\eta}^j{}_i \partial_j \star\varpi dx^i = 2i\frac{Ka + \ell\,(1-|K|)}{P^2}\left(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}\right). \tag{620}$$

In order to reach a more familiar form for the line element (614), it is convenient to trade the complex-conjugate coordinates ζ and $\bar{\zeta}$ for their modulus[72] and argument

$$\zeta = Ze^{i\Phi}, \tag{621}$$

and move from Eddington-Finkelstein to Boyer-Lindquist by setting

$$dt \to dt - \frac{r^2 + (n-a)^2}{\Delta_r} dr\,, \quad d\Phi \to d\Phi - \frac{Ka + \ell(1-|K|)}{\Delta_r} dr \tag{622}$$

with

$$\Delta_r = -2Mr + K\left(r^2 + a^2 - n^2\right) + 2\ell(n-a)(|K|-1). \tag{623}$$

We obtain finally:

$$\begin{aligned} ds^2_{\text{perfect fluid}} &= -\frac{\Delta_r}{\rho^2}\left(dt + \frac{2}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\,(1-|K|)\,Z^2\right)Z^2 d\Phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 \\ &+ \frac{2\rho^2}{P^2}dZ^2 + \frac{2Z^2}{\rho^2 P^2}\left((Ka + \ell\,(1-|K|))\,dt - \left(r^2 + (n-a)^2\right)d\Phi\right)^2 \end{aligned} \tag{624}$$

with

$$P = 1 + \frac{K}{2}Z^2, \quad \rho^2 = r^2 + \left(n + a - \frac{2a}{P} + \frac{\ell}{P}\,(1-|K|)\,Z^2\right)^2. \tag{625}$$

[72] The modulus and its range depend on the curvature. It is commonly expressed as: $Z = \sqrt{2}\tan\frac{\Theta}{2}$, $0 < \Theta < \pi$ for S^2; $Z = \frac{R}{\sqrt{2}}$, $0 < R < +\infty$ for E_2; $Z = \sqrt{2}\tanh\frac{\Psi}{2}$, $0 < \Psi < +\infty$ for H_2.

This bulk metric is Ricci-flat for any value of the parameters M, n, a and ℓ with $K = 0, \pm 1$. For vanishing n, a and ℓ, and with $M > 0$ and $K = 1$, one recovers the standard asymptotically flat Schwarzschild solution with spherical horizon. For $K = 0$ or -1, this is no longer Schwarzschild, but rather a metric belonging to the A class (see e.g. [49]). The parameter a switches on rotation, while n is the standard NUT charge. The parameter ℓ is also a rotational parameter available only in the flat-$\mathcal{S}$ case. Scanning over all these parameters, in combination with the mass and K, we recover the whole Kerr-Taub-NUT family of black holes, plus other, less familiar configurations, like the A-metric quoted above.

For the solutions at hand, the only potentially non-vanishing Carrollian boundary Cotton descendants are c and $\boldsymbol{\psi}$, displayed in (619) and (620). The first is non-vanishing for asymptotically locally flat spacetimes, and this requires non-zero n or ℓ. The second measures the bulk null congruence twist. In every case the metric (624) is Petrov type D.

We would like to make a comment regarding the isometries of the associated resummed Ricci-flat spacetimes with line element (624). For vanishing a and ℓ, there are four isometry generators and the field is in this case a stationary gravito-electric and/or gravito-magnetic monopole (mass and NUT parameters M, n). Constant-r hypersurfaces are homogeneous spaces in this case. The number of Killing fields is reduced to two (∂_t and ∂_Φ) whenever any of the rotational parameters a or ℓ is non-zero. These parameters make the gravitational field dipolar.

The bulk isometries are generally inherited from the boundary symmetries, i.e. the symmetries of the Carrollian geometry and the Carrollian fluid. The time-like Killing field ∂_t is clearly rooted to the stationarity of the boundary data. The space-like ones have legs on ∂_Φ and ∂_Z, and are associated to further boundary symmetries. From a Riemannian viewpoint, the metric (615) with (616) on the two-dimensional boundary surface $\mathcal{S}$ admits three Killing vector fields:

$$\underline{X}_1 = i\left(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}\right), \tag{626}$$

$$\underline{X}_2 = i\left(\left(1-\frac{K}{2}\zeta^2\right)\partial_\zeta - \left(1-\frac{K}{2}\bar\zeta^2\right)\partial_{\bar\zeta}\right), \tag{627}$$

$$\underline{X}_3 = \left(1+\frac{K}{2}\zeta^2\right)\partial_\zeta + \left(1+\frac{K}{2}\bar\zeta^2\right)\partial_{\bar\zeta}, \tag{628}$$

closing in $\mathfrak{so}(3)$, $\mathfrak{e}_2$ and $\mathfrak{so}(2,1)$ algebras for $K = +1, 0$ and -1 respectively. The Carrollian structure is however richer because it is constructed on the set $\{a_{ij}, b_i, \Omega\}$. Hence, not all Riemannian isometries generated by a Killing field $\underline{X}$ of $\mathcal{S}$ are necessarily promoted to Carrollian symmetries. For the latter, it is natural to further require the Carrollian vorticity be invariant:

$$\mathcal{L}_{\underline{X}} \star\varpi = \underline{X}\left(\star\varpi\right) = 0. \tag{629}$$

Condition (629) is fulfilled for all fields $\underline{X}_A$ ($A = 1, 2, 3$) in (626), (627) and (628), only as long as $a = \ell = 0$, since $\star\varpi = n$. Otherwise $\star\varpi$ is non-constant and only $\underline{X}_1 = i\left(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}\right) = \partial_\Phi$ leaves it invariant.

Using the general results reported in section 3.3.1, we would like to conclude this example with the computation of the Carrollian charges. We will do it in the specific case $K = 1$ and with θ, ϕ spatial coordinates. The generic metric (624) boils down to

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt-b\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{\sin^2\theta}{\rho^2}\left(adt - \left(r^2 + (n-a)^2\right)d\phi\right)^2, \tag{630}$$

with

$$\Delta_r = -2Mr + r^2 + a^2 - n^2, \tag{631}$$

$$\rho^2 = r^2 + (n - a\cos\theta)^2, \tag{632}$$

$$b = \left(2n(\cos\theta - 1) + a\sin^2\theta\right)d\phi. \tag{633}$$

The boundary spatial line element is written in these coordinates $d\ell^2 = d\theta^2 + \sin^2\theta d\phi^2$. We can interpret these data in terms of the Carrollian momenta

$$\Xi^{ij} = \pi^i = \Sigma^{ij} = \mathcal{B}^i = 0 \quad \mathcal{E} = M \quad \mathcal{A}^{ij} = -\frac{M}{4}a^{ij}, \tag{634}$$

The conformal Carrollian Killing equations can be solved and the result is

$$\underline{\xi} = \left(T(\boldsymbol{x}) + \frac{1}{2}t\nabla_i\xi^i\right)\partial_t + \xi^i(\boldsymbol{x})\partial_i. \tag{635}$$

where T is any smooth function on S^2 and ξ^i a Killing vector of the sphere. This is precisely the bms$_4$ generator [203]. The charges (448) are identically zero in this case. Conversely, the charges (451) are non-trivial

$$\mathcal{C}_{\underline{\xi}} = M \int_{S^2} d\theta d\phi \sin\theta \left(T - \frac{3}{2}\xi^i b_i \right). \tag{636}$$

They explicitly depend on the Kerr-Taub-NUT parameters thanks to the presence of the metric field b_i, and they are manifestly conserved.

Ricci-Flat Robinson-Trautman

The boundary geometry in this case is defined by $\Omega = 1$, $b_i = 0$ and $d\ell^2 = \dfrac{2}{P(t,\zeta,\bar\zeta)^2} d\zeta d\bar\zeta$, which is shearfree. It is straightforward to check that the general formulas (589–593) give $c = 0$ together with

$$\chi = \frac{i}{2}\left(\partial_\zeta K d\zeta - \partial_{\bar\zeta} K d\bar\zeta\right), \quad X = \frac{i}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right) d\zeta^2 - \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right) d\bar\zeta^2\right), \tag{637}$$

while $\psi_i = 0 = \Psi_{ij}$. These expressions satisfy (594–596), and the duality relations (597), (598) and (599) lead to the friction components of the energy-momentum tensor Q_i, Σ_{ij} and Ξ_{ij}:

$$Q = -\tfrac{1}{16\pi G_N}\left(\partial_\zeta K d\zeta + \partial_{\bar\zeta} K d\bar\zeta\right), \quad \pi = 0, \tag{638}$$

$$\Sigma = -\tfrac{1}{8\pi G_N P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta\ln P\right) d\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right) d\bar\zeta^2\right), \quad \Xi = 0. \tag{639}$$

We have completed our boundary procedure: we prescribed the boundary geometrical data, we built the Cotton descendant using them and we obtain the dissipative tensors. The next step is to compute the bulk resummed line element and impose the Carrollian fluid equations. Notice that the dissipative tensors match by construction the direct $k \to 0$ limit of the relativistic ancestors (177) and (178). In this particular case no k-expansion is needed since they are already k-independent.

Our goal is to present here the resummation of the derivative expansion (585) into a Ricci-flat spacetime dual to the fluid at hand. With the data written above (585) reads

$$ds^2_{\text{RT}} = -2dt\left(dr + Hdt\right) + 2\frac{r^2}{P^2} d\zeta d\bar\zeta, \tag{640}$$

where

$$2H = -2r\partial_t \ln P + K - \frac{2M(t)}{r}, \tag{641}$$

with $K = 2P^2\partial_{\bar\zeta}\partial_\zeta \ln P$ the Gaussian curvature of the line element $d\ell^2$.

Assuming now $\pi_i = 0$,[73] the general hydrodynamic equations (601) and (602) require $\varepsilon = \varepsilon(t)$ and

$$\Delta\Delta \ln P + 12M\partial_u \ln P - 4\partial_u M = 0, \tag{642}$$

with $\varepsilon(t) = \frac{M(t)}{4\pi G_N}$. This equation is indeed the Einstein Ricci-flat bulk equation (called Robinson-Trautman equation) for the metric (640), which shows that Carrollian fluids equations are the bulk Einstein equations.

We would like to underline this result, which is at the core of our findings: from purely Carrollian boundary consideration on $\mathscr{I}$ we reconstructed a highly non-trivial bulk solution, on shell only if the boundary conformal Carrollian fluid is on shell.

The solutions obtained here are algebraically special spacetimes of all types, as opposed to the Kerr-Taub-NUT family studied earlier (Schwarzschild solution is common to these two families). Furthermore they never have twist ($\psi = \Psi = 0$) and are generically asymptotically locally but not globally flat due to χ and X.

The specific Petrov type of Robinson-Trautman solutions is determined by analyzing the tensors (604), or (605) and (606) in holomorphic coordinates:

$$\varepsilon^+ = \frac{M(t)}{4\pi G_N}, \quad Q^+ = -\frac{1}{8\pi G_N}\partial_\zeta K d\zeta, \quad \Sigma^+ = -\frac{1}{4\pi G_N P^2}\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right) d\zeta^2. \tag{643}$$

We find the following classification [36, 60]:

[73] Since π_i is not related to the geometry by duality as the other friction and heat tensors, it can a priori assume any value. It is part of the Carrollian Robinson-Trautman fluid definition to set it to zero. It is an open intriguing question to see its effects if kept arbitrary.

II generic;

III with $\varepsilon^+ = 0$ and $\nabla_i Q^{+i} = 0$;

N with $\varepsilon^+ = 0$ and $Q_i^+ = 0$;

D with $2Q_i^+ Q_j^+ = 3\varepsilon^+ \Sigma_{ij}^+$ and vanishing traceless part of $\nabla_{(i} Q_{j)}^+$.

We would like at this point to compute the conformal Carrollian Killing vectors and their associated charges as discussed on general grounds in section 3.3.1. To make contact with the general objects defined there, we identify

$$\Xi^{ij} = \pi^i = \Sigma_i^i = 0, \tag{644}$$

$$\mathcal{E} = 4M, \quad \mathcal{B}^i = \nabla^i K, \quad \mathcal{A}^{ij} = -Ma^{ij}, \quad \Sigma^{ij} = \nabla^i \nabla^j \theta - \frac{1}{2} a^{ij} \nabla^k \nabla_k \theta, \tag{645}$$

where we called for brevity $d\ell^2 = a_{ij} dx^i dx^j = \dfrac{2}{P(t,\zeta,\bar\zeta)^2} d\zeta d\bar\zeta$. Weyl covariance is ensured by the conformal state equation $\mathcal{E} = -2\mathcal{A}_i^i$, together with $\Sigma_i^i = 0$. We then introduce a conformal Carrollian Killing vector $\underline{\xi}$, with (445) and (446) here given by

$$\begin{aligned} \partial_t \xi^t &= \lambda, & (646)\\ \nabla_i \xi_j + \nabla_j \xi_i + \xi^t \partial_t a_{ij} &= 2\lambda a_{ij}. & (647) \end{aligned}$$

The solution is the following vector[74]

$$\underline{\xi} = \left(\sqrt{a}\right)^{\frac{1}{2}} \left(\alpha(\boldsymbol{x}) + \frac{1}{2} \int dt \left(\sqrt{a}\right)^{-\frac{1}{2}} \nabla_i \xi^i \right) \partial_t + \xi^i(\boldsymbol{x}) \partial_i, \tag{648}$$

where ξ^i is a spatial conformal Killing vector, i.e. it satisfies

$$\nabla_i \xi_j + \nabla_j \xi_i = \nabla_k \xi^k a_{ij}. \tag{649}$$

The associated charges (448) become

$$\mathcal{Q}_{\underline{\xi}} = \int_{S^2} d^2x \sqrt{a} \xi_j \mathcal{B}^j = \int_{S^2} d\zeta d\bar\zeta P^{-2} \left(\xi^\zeta \partial_\zeta K + \xi^{\bar\zeta} \partial_{\bar\zeta} K \right). \tag{650}$$

They are conserved by construction.

Even though the second family of charges (451) were defined only for $\mathcal{B}^i = 0$, we can nevertheless study what their expression is for the solution at hand. We find

$$\mathcal{C}_{\underline{\xi}} = \int_{S^2} d^2x \sqrt{a} \xi^t \mathcal{E} = \int_{S^2} d\zeta d\bar\zeta P^{-3} \left(\alpha(\zeta,\bar\zeta) + \frac{1}{2} \int dt P \nabla_i \xi^i \right) 4M. \tag{651}$$

As expected, they are indeed not generically conserved:

$$\partial_t \mathcal{C}_{\underline{\xi}} = -\int_{S^2} d^2x \sqrt{a} \partial_i \xi^t \mathcal{B}^i. \tag{652}$$

These charges are not conserved, they potentially translate the fact that gravitational radiation is reaching $\mathscr{I}$, due to the non-trivial temporal dynamics of the solution at hand.

4.2 The Three-dimensional Case

In this section we discuss the reconstruction of asymptotically flat three-dimensional spacetimes starting from two-dimensional conformal Carrollian fluids living on null infinity [95].

[74] This vector follows in the class of conformal Killing vectors for Carroll structures described in [88].

4.2.1 Flat Derivative Expansion

Our starting point is the finite derivative expansion of an asymptotically AdS_3 spacetime, (195). The fundamental question is whether the latter admits a smooth zero-k limit.

We have implicitly assumed that the Randers-Papapetrou data of the two-dimensional pseudo-Riemannian conformal boundary associated with the original Einstein spacetime, a, b and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new null boundary $\mathscr{I}$. We can furthermore match the various two-dimensional Riemannian quantities with the corresponding one-dimensional Carrollian ones:

$$u = -k^2 \left(\Omega dt - (b_x + \beta_x)\, dx\right) + O\left(k^4\right), \quad \star u = k\sqrt{a}dx + O\left(k^3\right) \tag{653}$$

and

$$\begin{aligned} \Theta &= \theta + O\left(k^2\right), \\ a &= k^2 \left(\varphi_x + \gamma_x\right) dx + O\left(k^4\right), \\ A &= \theta\Omega dt + \left(\alpha_x + \delta_x\right) dx + O\left(k^2\right), \end{aligned} \tag{654}$$

where the left-hand-side quantities are Riemannian, and the right-hand-side ones Carrollian (see (375, 376, 378, 385, 386)).

The closed form (195) is smooth at zero k. In this limit the metric reads:[75]

$$\boxed{\begin{aligned} ds^2_{\text{flat}} = &-2\left(\Omega dt - b - \beta\right)\left(dr + r\left(\varphi + \gamma + \theta\left(\Omega dt - b - \beta\right)\right)\right) \\ &+ r^2 d\ell^2 + 8\pi G_N \left(\Omega dt - b - \beta\right)\left(\varepsilon\left(\Omega dt - b - \beta\right) - \pi\right). \end{aligned}} \tag{655}$$

Here $d\ell^2$, Ω, $b = b_x dx$, $\varphi = \varphi_x dx$ and θ are the Carrollian geometric objects introduced earlier. The bulk Ricci-flat spacetime is now dual to a Carrollian fluid with kinematics captured in $\beta = \beta_x dx$ and $\gamma = \gamma_x dx$, energy density ε (zero-k limit of the corresponding relativistic function), and heat current $\pi = \pi_x dx$ (as defined in (392), (393) and (394)).

For the fluid under consideration, there is also a pair of Carrollian stress tensors originating from the anomaly (196). Using expressions (391) and (396), we can determine τ_Σ and τ_Ξ, and (397) provide in turn the Carrollian stress:

$$\Sigma^x{}_x = -\frac{1}{4\pi G_N}\left(\theta^2 + \frac{\partial_t \theta}{\Omega}\right), \quad \Xi^x{}_x = \frac{1}{4\pi G_N}\left(\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x - \beta^2\left(\theta^2 + \frac{\partial_t\theta}{\Omega}\right)\right). \tag{656}$$

This is the Carrollian emanation of the relativistic conformal anomaly.

Expression (655) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (195) was Einstein are fulfilled in the zero-k limit. These are the set of Carrollian hydrodynamic equations (399–402), with Carrollian power and force densities e, f, g_x, h_x obtained using their definition (398) and the expressions of f_μ displayed in (200). Equations (400) and (402) are automatically satisfied, whereas (399) and (401) lead to[76]

$$\begin{cases} \dfrac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \dfrac{1}{4\pi G_N}\left(\dfrac{2s_x}{\Omega}\hat{\mathscr{D}}_t\beta^x + \dfrac{\beta_x}{\Omega}\hat{\mathscr{D}}_t s^x + \hat{\mathscr{D}}^x s_x\right) = 0, \\ \hat{\mathscr{D}}_x\varepsilon - \dfrac{\beta_x}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \dfrac{1}{\Omega}\hat{\mathscr{D}}_t\left(\pi_x + 2\varepsilon\beta_x\right) = 0 \end{cases} \tag{657}$$

with s_x given in (388). The unknown functions, which bear the fluid configuration, are $\varepsilon(t,x)$, $\pi_x(t,x)$ and $\beta_x(t,x)$. These cannot be all determined by the two equations at hand. Hence, there is some redundancy, originating from the relativistic fluid frame invariance – responsible e.g. for the arbitrariness of $\xi(x^+,x^-)$ in the description of AdS spacetimes using the light-cone boundary frame.

Equations (657) are Weyl-Carroll covariant. The Ricci-flat line element (655) inherits Weyl invariance from its relativistic ancestor. The set of transformations (374), (377) and (379), supplemented with $\varepsilon \to \mathcal{B}^2\varepsilon$ and $\pi_x \to \mathcal{B}\pi_x$, can indeed be absorbed by setting $r \to \mathcal{B}r$, resulting thus in the invariance of (655). Exactly like in four bulk dimensions, Weyl invariance is rooted in the location of the null boundary $\mathscr{I}$.

We would like to close this chapter with a specific but general enough situation to encompass all Barnich-Troessaert Ricci-flat three-dimensional spacetimes [119]. The Carrollian geometric data are $b_x = 0$, $\Omega = 1$ and

[75] We remind that for three-dimensional bulks, contrarily to the four-dimensional case, we allow β to be arbitrary.

[76] We remind that Weyl-Carroll covariant derivatives are defined in (380–383). Here ε, β^x, π_x and s^x have weights 2, 1, 1 and 3. For example $\hat{\mathscr{D}}_x s^x = \hat{\nabla}_x s^x + 2\varphi_x s^x = \frac{1}{\sqrt{a}}\hat{\partial}_x(\sqrt{a}s^x) + 2\varphi_x s^x$.

$a = \exp 2\Phi(t,x)$, and the kinematic variable of the Carrollian dual fluid β_x is left free. Consequently (655) reads:

$$\begin{aligned} ds^2_{\text{flat}} &= -2\left(dt-\beta_x dx\right)\left(dr + r\left(\partial_t\Phi dt + \left(\partial_t - \partial_t\Phi\right)\beta_x dx\right)\right) \\ &\quad + r^2 e^{2\Phi}dx^2 + 8\pi G_N \left(dt-\beta_x dx\right)\left(\varepsilon dt - \left(\pi_x + \varepsilon\beta_x\right)dx\right), \end{aligned} \tag{658}$$

where $\varepsilon(t,x)$ and $\pi(t,x)$ (x is not bold because one-dimensional) obey (657) in the form

$$\begin{cases} \left(\partial_t + 2\partial_t\Phi\right)\varepsilon + \dfrac{1}{4\pi G_N}\left(2s_x\left(\partial_t+\partial_t\Phi\right)\beta^x + \beta_x\left(\partial_t + 3\partial_t\Phi\right)s^x + \left(\partial_x + \partial_x\Phi\right)s^x\right) = 0, \\ \partial_x\varepsilon + \left(\partial_t + \partial_t\Phi\right)\pi_x + 2\varepsilon\partial_t\beta_x + \beta_x\partial_t\varepsilon = 0. \end{cases} \tag{659}$$

Here, s_x takes the simple form

$$s_x = \partial_t^2\beta_x - \partial_t\left(\beta_x\partial_t\Phi\right) - \partial_t\partial_x\Phi. \tag{660}$$

For vanishing β_x, the results (658) and (659) coincide precisely with those obtained in [119] by demanding Ricci-flatness in the BMS gauge. Here, they are reached from purely Carrollian-fluid considerations, and for generic $\beta_x(t,x)$, the metric (658) lays outside the BMS gauge.

4.2.2 Charges Analysis

The absence of anomaly in the Carrollian framework is equivalent to setting $\Sigma^x{}_x = \Xi^x{}_x = 0$, whereas the Weyl-Carroll flatness requires $s = 0$. This amounts to take $\Omega = a = 1$ and $b_x = 0$,[77] and with those data $s = 0$ reads

$$\partial_t^2\beta_x = 0. \tag{661}$$

In the Carrollian spacetime at hand, the fluid equations of motion (657) are

$$\begin{cases} \partial_t\varepsilon = 0, \\ \partial_x\varepsilon + \partial_t(\pi_x + 2\varepsilon\beta_x) = 0. \end{cases} \tag{662}$$

They can be integrated in terms of four arbitrary functions of x: $\varepsilon(x)$, $\varpi(x)$, $\lambda(x)$ and $\mu(x)$. We find

$$\pi_x(t,x) = -2\varepsilon(x)\beta_x(t,x) + \varpi(x) - t\partial_x\varepsilon, \tag{663}$$

$$\beta_x(t,x) = \frac{\lambda(x)}{2\varepsilon(x)} - \frac{t\partial_x\mu}{2\mu(x)} \tag{664}$$

(this parameterization of β_x will be appreciated later). The Ricci-flat (even locally flat) reconstructed spacetime from these Carrollian fluid data is obtained from the general expression (655):

$$ds^2_{\text{flat}} = -2\left(dt - \beta_x dx\right)\left(dr + r\partial_t\beta_x dx\right) + r^2dx^2 + 8\pi G_N\left(\varepsilon(dt-\beta_x dx)^2 - \pi_x dx(dt - \beta_x dx)\right), \tag{665}$$

where β_x and π_x are meant to be as in (663) and (664).

On the one hand, the arbitrary functions $\varepsilon(x)$ and $\varpi(x)$ are reminiscent of the functions $L_\pm(x^\pm)$ (or $\varepsilon(t,x)$ and $\chi(t,x)$) present in the AdS solutions. A vanishing-k limit was indeed used in [102] to obtain $\varepsilon(x)$ and $\varpi(x)$ from $L_\pm(x^\pm)$. On the other hand, $\lambda(x)$ and $\mu(x)$ remind $\xi^\pm(x^\pm)$, and are indeed a manifestation of a residual hydrodynamic frame invariance, which survives the Carrollian limit. Considering the Carrollian hydrodynamic-frame transformations (403)

$$\beta_x' = \beta_x + B_x, \tag{666}$$

in the present framework ($\Sigma^x{}_x = \Xi^x{}_x = 0$), and using (140–143, 392–394), we obtain the transformations:

$$\varepsilon' = \varepsilon, \quad \pi_x' = \pi_x - 2\varepsilon B_x, \tag{667}$$

which leave the Carrollian fluid equations (662) invariant. The new velocity field β_x' is compatible with the Weyl-Carroll flatness (661) provided the transformation function B_x is linear in time, hence parameterized in terms of two arbitrary functions of x. This is how $\lambda(x)$ and $\mu(x)$ emerge.

[77] Actually the absence of anomaly requires rather $\Omega = \Omega(t)$, $a = a(x)$ and $b_x = b_x(x)$, which can be reabsorbed trivially with Carrollian diffeomorphisms.

Observe also that the residual Carrollian hydrodynamic frame invariance enables us to define here a Carrollian Landau-Lifshitz hydrodynamic frame. Indeed, combining (663) and (664) we obtain

$$\pi_x(t,x) = -\lambda(x) + \varpi(x) + t\varepsilon(x)\partial_x \ln \frac{\mu(x)}{\varepsilon(x)}. \tag{668}$$

Adjusting the velocity field β_x such that

$$\lambda(x) = \varpi(x) \quad \text{and} \quad \frac{\mu(x)}{\varepsilon(x)} = \frac{1}{\varepsilon_0} \tag{669}$$

with ε_0 a constant, makes the Carrollian fluid perfect: $\pi_x = 0$.

In complete analogy with the AdS analysis, we will first compute the charges for vanishing velocity $\beta_x = 0$ (which is given by $\lambda(x) = 0$ and $\mu(x) = 1$) in terms of $\varepsilon(x)$ and $\varpi(x)$, and next perform the similar computation for perfect fluids with velocity β_x parameterized with two arbitrary functions $\lambda(x)$ and $\mu(x)$. Here empty Minkowski bulk is realized with $\mu = 1$, $\lambda = 0$, $\varpi = 0$ and $\varepsilon_0 = -\frac{1}{8\pi G_N}$.

As for the AdS case, the class (665) is not in the BMS gauge, unless β_x is constant, which can then be reabsorbed by a global Carrollian boost (constant B_x).[78] We will first discuss this situation, where the asymptotic Killings are the canonical generators of bms$_3$. Outside the BMS, we will perform the determination of the asymptotic isometry for metrics reconstructed from perfect fluids, and proceed with the surface charges and their algebra. Our conclusion is here that asymptotically flat fluid/gravity correspondence is sensitive to the residual hydrodynamic-frame invariance, as we will now prove. Eventually we will compute charges for a dissipative static fluid which is not hosted by a Weyl-Carroll flat boundary.

Dissipative Static Fluid

The metric (665) for vanishing β_x takes the simple form (from now on we denote with a prime the spatial derivative $\epsilon' = \partial_x \epsilon$)

$$ds^2_{\text{flat}} = -2dtdr + r^2dx^2 + 8\pi G_N \left(\varepsilon dt - (\varpi - t\varepsilon')\, dx\right) dt, \tag{670}$$

compatible with BMS gauge with asymptotic Killing vectors

$$\underline{\zeta} = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^x\partial_x, \tag{671}$$

where

$$\zeta^r = -rY' + H'' + tY''' + \frac{4\pi G}{r}\left(\varpi - t\varepsilon'\right)\left(H' + tY''\right), \tag{672}$$

$$\zeta^t = H + tY', \tag{673}$$

$$\zeta^x = Y - \frac{1}{r}\left(H' + tY''\right). \tag{674}$$

Here H and Y are functions of x only. Vectors (672–674) are the vanishing-k limit of (214–216), reached using $x^\pm = x \pm kt$, and setting $Y^\pm(x^\pm) = Y(x) \pm k\left(H(x) + tY'(x)\right)$.

This family of vectors produces the following variation on the metric fields:

$$-\mathcal{L}_{\underline{\zeta}} g_{MN} = \delta_{\underline{\zeta}} g_{MN} = \frac{\partial g_{MN}}{\partial \varepsilon}\delta_{\underline{\zeta}}\varepsilon + \frac{\partial g_{MN}}{\partial \varepsilon'}\delta_{\underline{\zeta}}\varepsilon' + \frac{\partial g_{MN}}{\partial \varpi}\delta_{\underline{\zeta}}\varpi, \tag{675}$$

with

$$\delta_{\underline{\zeta}}\varepsilon = -2\varepsilon Y' - Y\varepsilon' + \frac{Y'''}{4\pi G_N}, \tag{676}$$

$$\delta_{\underline{\zeta}}\varpi = -\frac{H'''}{4\pi G_N} + \frac{1}{H}\left(\varepsilon H^2\right)' - \frac{1}{Y}\left(\varpi Y^2\right)'. \tag{677}$$

[78] The functions $\lambda(x)$ and $\mu(x)$ entering (665) via (663) and (664) can be reabsorbed in any case by performing the coordinate transformation $dx \to \frac{dx}{\sqrt{\mu(x)}}$, $dt \to \frac{1}{\sqrt{\mu(x)}}\left(dt + \beta_x dx\right)$ and $r \to r\sqrt{\mu(x)}$. This leads to the same form as the one reached by setting $\mu = 1$ and $\lambda = 0$, i.e (670).

Their algebra closes for the same modified Lie bracket (219) with $\underline{\zeta}_a = \underline{\zeta}(H_a, Y_a)$ and

$$Y_3 = Y_1 Y_2' - Y_2 Y_1' \quad H_3 = Y_1 H_2' + H_1 Y_2' - Y_2 H_1' - H_2 Y_1'. \tag{678}$$

We can compute the charges of g in (670), using Minkowski as reference background $\bar{g}$. They read:

$$Q_{H,Y}[g - \bar{g}, \bar{g}] = \frac{1}{2}\int_0^{2\pi} dx \left[H\left(\varepsilon + \frac{1}{8\pi G_N}\right) - Y\varpi \right]. \tag{679}$$

With a basis of functions e^{imx} for H and Y, we find the standard collection of charges

$$P_m = \frac{1}{2}\int_0^{2\pi} dx\, e^{imx}\left(\varepsilon + \frac{1}{8\pi G_N}\right), \quad J_m = -\frac{1}{2}\int_0^{2\pi} dx\, e^{imx}\varpi, \tag{680}$$

which coincide with the computation performed e.g. in [102]. Using

$$\{Q_{H_1,Y_1}, Q_{H_2,Y_2}\} = \delta_{\underline{\zeta}_1} Q_{H_2,Y_2} = -\delta_{\underline{\zeta}_2} Q_{H_1,Y_1}, \tag{681}$$

we obtain the following surface-charge algebra:

$$i\{J_m, P_n\} = (m-n)P_{m+n} + \frac{c}{12} m\left(m^2 - 1\right)\delta_{m+n,0}\,, \quad i\{J_m, J_n\} = (m-n)J_{m+n}\,, \quad \{P_m, P_n\} = 0 \tag{682}$$

with $c = \frac{3}{G_N}$. This is the bms$_3$ algebra, and this analysis demonstrates that a non-perfect Carrollian fluid, even with $\beta_x = 0$, is sufficient to generate all Barnich-Troessaert flat three-dimensional spacetimes. This goes along with the analogue conclusion reached in AdS for Bañados spacetimes.

Perfect Fluid with Velocity

Consider now the resummed metric (665) assuming (669). We obtain

$$ds^2_{\text{flat}} = -2\left(dt - \beta_x dx\right)\left(dr - \frac{r\mu'}{2\mu}dx\right) + r^2 dx^2 + 8\pi G_N \varepsilon_0 \mu \left(dt - \beta_x dx\right)^2 \tag{683}$$

with β_x given by

$$\beta_x = \frac{1}{2\mu}\left(\frac{\lambda}{\varepsilon_0} - t\mu'\right). \tag{684}$$

Unless β_x is constant, the metric (683) is not in BMS gauge. The BMS subset is entirely captured by $\mu = 1$, $\lambda = 0$ with resulting solutions plain Minkowski ($\varepsilon_0 = -\frac{1}{8\pi G_N}$) and the non-spinning zero-modes of Barnich-Troessaert family:

$$ds^2_{\text{flat}} = -2dtdr + r^2 dx^2 + 8\pi G_N \varepsilon_0 dt^2. \tag{685}$$

The asymptotic isometries of (683) are now generated by[79]

$$\underline{\eta} = \eta^r \partial_r + \eta^t \partial_t + \eta^x \partial_x, \tag{686}$$

expressed in terms of two arbitrary functions $h(x)$ and $\rho(x)$

$$\eta^r = -r\rho', \quad \eta^t = h + t\rho', \quad \eta^x = \rho. \tag{687}$$

The algebra of asymptotic Killing vectors closes for the ordinary Lie bracket

$$\left[\underline{\eta}_1, \underline{\eta}_2\right] = \underline{\eta}_3 \tag{688}$$

with $\underline{\eta}_a = \underline{\eta}(h_a, \rho_a)$ and

$$\rho_3 = \rho_1' \rho_2 - \rho_2 \rho_1', \quad h_3 = \rho_1 h_2' + h_1 \rho_2' - \rho_2 h_1' - h_2 \rho_1'. \tag{689}$$

[79] Again the fields (686) and (687) are alternatively obtained by an appropriate zero-k limit of (228) and (229).

It respects the form of the metric

$$-\mathcal{L}_{\underline{\eta}} g_{MN} = \delta_{\underline{\eta}} g_{MN} = \frac{\partial g_{MN}}{\partial \mu}\delta_{\underline{\eta}}\mu + \frac{\partial g_{MN}}{\partial \mu'}\delta_{\underline{\eta}}\mu' + \frac{\partial g_{MN}}{\partial \lambda}\delta_{\underline{\eta}}\lambda \tag{690}$$

with

$$\delta_{\underline{\eta}}\lambda = -2\lambda\rho' - \rho\lambda' + \varepsilon_0\left(2\mu h' + h\mu'\right), \tag{691}$$

$$\delta_{\underline{\eta}}\mu = -2\mu\rho' - \rho\mu'. \tag{692}$$

The charges of g in (683) are computed as usual with Minkowski as reference background $\bar{g}$. They read:

$$Q_{h,\rho}[g-\bar{g},\bar{g}] = \frac{1}{2}\int_0^{2\pi} dx\left[h\left(\varepsilon_0\mu + \frac{1}{8\pi G_N}\right) - \rho\lambda\right]. \tag{693}$$

With a basis of unimodular exponentials for h and ρ, we now find

$$M_m = \frac{1}{2}\int_0^{2\pi} dx\, e^{imx}\left(\varepsilon_0\mu + \frac{1}{8\pi G_N}\right), \quad I_m = -\frac{1}{2}\int_0^{2\pi} dx\, e^{imx}\lambda, \tag{694}$$

and

$$\{Q_{h_1,\rho_1}, Q_{h_2,\rho_2}\} = \delta_{\underline{\eta}_1} Q_{h_2,\rho_2} = -\delta_{\underline{\eta}_2} Q_{h_1,\rho_1} \tag{695}$$

provide the surface-charge algebra:

$$i\{I_m, M_n\} = (m-n)M_{m+n} - \frac{m}{4G_N}\delta_{m+n,0}, \quad i\{I_m, I_n\} = (m-n)I_{m+n}, \quad \{M_m, M_n\} = 0. \tag{696}$$

As for the AdS case, the central extension of this algebra can be reabsorbed in a modes redefinition. Indeed, translating the modes

$$\tilde{M}_m = M_m - \frac{1}{8G_N}\delta_{m,0}, \tag{697}$$

we obtain

$$i\left\{I_m, \tilde{M}_n\right\} = (m-n)\tilde{M}_{m+n}, \quad i\{I_m, I_n\} = (m-n)I_{m+n}, \quad \left\{\tilde{M}_m, \tilde{M}_n\right\} = 0. \tag{698}$$

This algebra (that could have been obtained from (239) in the zero-k limit) has no explicit central charge. Therefore, our computation shows that holographic locally flat spacetimes based on perfect Carrollian fluids have asymptotic charges different from spacetimes based on dissipative static fluids.

Carrollian Charges

We would at this point to compute the charges defined in Section 3.3.1. We first of all need to deduce the boundary Carrollian data and momenta. To do this, we consider the restrictive case in which we set $\beta_x = 0$. This case is different from the two treated above: it is a dissipative static fluid where the boundary host is so far general – not Weyl-Carroll flat. The bulk line element (655) reads

$$ds^2_{\text{flat}} = -2\left(\Omega dt - b\right)\left(dr + r\left(\varphi + \theta\left(\Omega dt - b\right)\right)\right) + r^2 d\ell^2 + 8\pi G_N\left(\Omega dt - b\right)\left(\varepsilon\left(\Omega dt - b\right) - \pi\right). \tag{699}$$

From this metric we can extract the corresponding Carrollian geometry on null infinity $\mathscr{I} = \{r \to \infty\}$. The following procedure is general but we will use the specific case of three-dimensional asymptotically flat spacetimes as an illustration. Consider the conformal extension of (699)

$$d\tilde{s}^2_{\text{flat}} = r^{-2} ds^2_{\text{flat}}, \tag{700}$$

the factor r^{-2} is present to regularize the metric on $\mathscr{I}$. We perform the change of variable $\omega = r^{-1}$ in the conformal metric, it becomes[80]

$$d\tilde{s}^2_{\text{flat}} = -2\left(\Omega dt - b\right)\left(-d\omega + \omega\left(\varphi + \theta\left(\Omega dt - b\right)\right)\right) + d\ell^2 + 8\pi G_N \omega^2\left(\Omega dt - b\right)\left(\varepsilon\left(\Omega dt - b\right) - \pi\right). \tag{701}$$

[80] The null asymptote is thus $\mathscr{I} = \{\omega \to 0\}$.

We can deduce the Carrollian geometry on $\mathscr{I}$

$$\tilde{g}^{-1}\left(.,d\omega\right)_{|\mathscr{I}}=\frac{1}{\Omega}\partial_t,\quad d\tilde{s}^2_{\text{flat }|\mathscr{I}}=d\ell^2=adx^2\quad\text{and}\quad\tilde{g}\left(.,\partial_\omega\right)_{|\mathscr{I}}=\Omega dt-b. \tag{702}$$

We now move to the dynamics. Using $\hat{\mathscr{D}}_x s^x=0$ with $s_x=\frac{1}{\Omega}\partial_t\varphi_x-\theta\varphi_x-\hat{\partial}_x\theta$ (see (388) with $\beta_x=0$), Einstein equations reduce to

$$\begin{aligned}\left(\frac{1}{\Omega}\partial_t+2\theta\right)\mathcal{E} &= 0, &(703)\\ \left(\hat{\partial}_x+2\varphi_x\right)\mathcal{E}+\left(\frac{1}{\Omega}\partial_t+\theta\right)\pi_x &= 0. &(704)\end{aligned}$$

We interpret them as the Carrollian conservation equations (432–435) for $\Sigma^{xx}=\mathcal{B}^x=0$ and $\mathcal{E}=\mathcal{P}$ (conformal case). Furthermore Ξ^{xx} is automatically zero due to its tracelessness.

We would like at this point to obtain the surface charges. We thus compute the asymptotic Killing vectors of ds^2_{flat} whose leading orders in r^{-1} are

$$\hat{\xi^r}=-r\lambda(t,x)+O(1),\quad\hat{\xi}^t=\xi^t(t,x)+O(r^{-1})\quad\text{and}\quad\hat{\xi}^x=\xi^x(x)+O(r^{-1}). \tag{705}$$

Here $\lambda=\hat{\nabla}_x\xi^x+\frac{X}{\Omega}\partial_t\ln\sqrt{a}$ and $\underline{\xi}=\xi^t\partial_t+\xi^x\partial_x$ is a conformal Killing vector (i.e. satisfying (445) and (446)) of the corresponding Carrollian geometry $\{\Omega,a,b_x\}$. We calculate the associated surface charge through covariant phase-space formalism (see for instance [61]) and obtain that they are integrable:

$$Q_{\underline{\hat{\xi}}}[g]=\int_0^{2\pi}dx\sqrt{a}\left(\left(\Omega\xi^t-2b_x\xi^x\right)\mathcal{E}-\xi^x\pi_x\right). \tag{706}$$

It is readily seen that these charges have exactly the same expression as the conserved charges defined in (451) out of purely Carrollian considerations

$$Q_{\underline{\hat{\xi}}}[g]=\mathcal{C}_{\underline{\hat{\xi}}}. \tag{707}$$

Notice eventually that if we restrict our attention to the case $\Omega=1$, $a=1$ and $b_x=0$, we recover the usual Bondi gauge for asymptotically flat spacetimes, and the charges become exactly the ones derived in (679) for the dissipative static fluid on Weyl-Carroll flat boundary geometry.

To recap and conclude, we have analyzed different charges associated to different gauges and parametrizations. The heat current is of paramount importance in our construction. Disregard it a priori is not wise, as the charges computation clearly indicates. Using hydrodynamics to build bulk metrics was a very powerful tool. An important question, which is part of the projects we are addressing, is to analyze the solution space of the metric (655) in full generality, find the most general asymptotic Killing vectors that preserve it and their charges, at the light of recent works on the most general boundary conditions [207].

5 Weyl Symmetry

We are now fully equipped to tackle the last part of this work, which is devoted to the understanding of Weyl symmetry in holography. Although the previous part of this manuscript was focused on the flat limit of holography, this chapter will only touch upon the AdS construction. The flat limit of the ideas reported here is an open chapter, to some extent yet to be written.

As we mentioned multiple times so far, Weyl symmetry is a key ingredient in the fluid derivative expansion. Indeed, the derivative expansion builds a bulk metric based on boundary Weyl covariance. Nevertheless, the FG gauge is not form invariant under this symmetry. We recall here that Fefferman and Graham in their seminal works [77,78] found a bulk gauge (FG gauge) preserving the structure of time-like hypersurfaces in AdS$_{d+2}$ spacetimes.[81] This is useful to discuss the time-like conformal boundary; which we saw that in suitable coordinates is located at $z = 0$, z being the holographic coordinate such that $z = const$ hypersurfaces are time-like. The FG gauge induces on the boundary a metric, while the bulk Levi-Civita connection gives at first order the boundary Levi-Civita connection.

Although everything is consistent, we already insisted that there exists some leftover freedom in choosing the boundary metric. This comes about because the induced metric on the $z = 0$ hypersurface is defined, because of certain bulk diffeomorphisms, up to a rescaling by a non-trivial function of the boundary coordinates. We therefore often refer to the boundary as possessing a conformal class of metrics and say that the boundary enjoys Weyl symmetry. The latter is however ignored in physical applications, for we usually fix the boundary metric and thus break this symmetry.

The main observation is that the Levi-Civita connection is not Weyl-covariant, the metricity condition being the source of this non-covariance. This problem can be sidestepped by introducing the notion of a Weyl connection and more generally of Weyl geometry [175,176]. We will show that Weyl connections play a role in the holographic correspondence, on the field theory side of the duality. Indeed, we will prove that, by slightly generalizing the FG ansatz to what we call the Weyl-Fefferman-Graham gauge (WFG), the Weyl diffeomorphism responsible for the rescaling of the boundary metric becomes a geometric symmetry. The consequences of this modification are: the bulk geometry induces on the boundary a metric and a Weyl connection, instead of its Levi-Civita counterpart. In the dual quantum field theory, these objects act as backgrounds and sources for current operators.

It is a familiar aspect of the FG formalism that the on-shell bulk action diverges as one approaches the boundary. Traditionally, this is dealt with by including local counterterms which are functionals of the induced geometry, in a solution-independent way [10, 11, 13, 177]. There remains one physical subtlety, which is the appearance of a simple pole in $d + 1 - 2k$, with k integer. This effect is more appropriately thought of as an anomaly in the Weyl Ward identity, a basic feature of renormalization theory [184]. This anomaly can be traced back to the fact that holographic renormalization breaks Weyl covariance by fixing a $z = \epsilon$ hypersurface to regulate the theory. No Weyl-covariant renormalization procedures exist, which indicates that a Weyl anomaly is present and contributes in any even-dimensional boundary theory.[82] We will unravel a different packaging of the Weyl anomaly, through the use of the WFG gauge – the Weyl anomaly will in fact become an integral over Weyl-covariant geometrical tensors. Inspired by [185], we will present a simple cohomological interpretation of the Weyl anomaly, based on the difference of two Weyl-related bulk top forms.

We will furthermore advocate a different interpretation of the boundary sources and Ward identity, corroborated by the WFG extension. Specifically, we interpret the boundary theory as defined on a background metric and a background Weyl connection, given by the leading order of the bulk dual. We are now really sourcing two different currents, which can and indeed do both participate in the boundary Ward identity. We will in particular show that the holographic dictionary furnishes directly this boundary Ward identity relating the trace of the energy-momentum with the divergence of the Weyl current. This will be elegantly verified directly from the boundary action, without invoking holography.

[81] The boundary in our conventions has dimension $d + 1$. This rather unusual choice has been made to emphasize the spatial d-dimensional subspace. With respect to previous chapters, we refer to the boundary tensors here with a subscript $^{(k)}$, to underline the holographic order at which they appear.

[82] It is not an easy task to prove that Weyl anomalies arise only in even dimensions. To do so one has to prove that non-trivial cocycles of the Weyl group arise from local functionals that are Weyl invariant in and only in even integer dimension, [185].

5.1 Weyl Invariance and Holography

The Fefferman-Graham theorem says that the metric of a locally asymptotically AdS$_{d+2}$ geometry can be always put in the form (144), i.e.

$$ds^2 = L^2 \frac{dz^2}{z^2} + h_{\mu\nu}(z;x)dx^\mu dx^\nu \tag{708}$$

Using the expansion for $h_{\mu\nu}$, (145), and regarding the boundary dimension $d+1$ as variable,[83] $g^{(0)}_{\mu\nu}(x)$ has an interpretation as an induced boundary metric:

$$\frac{z^2}{L^2} ds^2 \underset{z\to 0}{\longrightarrow} g^{(0)}_{\mu\nu}(x)dx^\mu dx^\nu = ds^2_{bdy}. \tag{709}$$

It is this object that sources the stress energy tensor in the dual field theory, with $T^{(0)}_{\mu\nu}(x)$ its vev, as discussed. All of the other terms in the series are determined in terms of $g^{(0)}_{\mu\nu}(x), T^{(0)}_{\mu\nu}(x)$ by the bulk classical equations of motion.

Equation (709) defines the induced boundary metric up to a Weyl transformation. We see indeed that there is an ambiguity in the construction of this metric which amounts in defining the latter up to a scalar function of the boundary coordinates. Although it is often stated, this ambiguity is usually disregarded.

The following bulk diffeomorphism (which we refer to as the Weyl diffeomorphism)

$$z \to z' = z/\mathcal{B}(x), \qquad x^\mu \to x'^\mu = x^\mu \tag{710}$$

plays an important role. It has the effect of inducing a Weyl transformation of the boundary metric: using (709) with now holographic coordinate z' we obtain

$$ds^2_{\text{bdy}} = \frac{g^{(0)}_{\mu\nu}(x)}{\mathcal{B}(x)^2} dx^\mu dx^\nu. \tag{711}$$

However, this diffeomorphism does not leave the bulk metric in the Fefferman-Graham gauge, but rather transforms it to

$$ds^2 = L^2 \left(\frac{dz'}{z'} + \partial_\mu \ln \mathcal{B}(x)\, dx^\mu \right)^2 + h_{\mu\nu}(z'\mathcal{B}(x);x)dx^\mu dx^\nu \tag{712}$$

where

$$h_{\mu\nu}(z'\mathcal{B}(x);x) = \frac{L^2}{z'^2}\left[\frac{g^{(0)}_{\mu\nu}(x)}{\mathcal{B}(x)^2} + \frac{z'^2}{L^2} g^{(2)}_{\mu\nu}(x) + \frac{z'^4}{L^4}\mathcal{B}(x)^2 g^{(4)}_{\mu\nu}(x) + ...\right] \tag{713}$$

$$+\frac{z'^{d-1}}{L^{d-1}}\left[\mathcal{B}(x)^{d-1} T^{(0)}_{\mu\nu}(x) + \frac{z'^2}{L^2}\mathcal{B}(x)^{d+1} T^{(2)}_{\mu\nu}(x) + ...\right]. \tag{714}$$

Thus, this diffeomorphism takes us out of FG gauge (as it is one of the diffs that was fixed in going to that gauge), and acts on the boundary tensors $g^{(k)}_{\mu\nu}(x)$ and $T^{(k)}_{\mu\nu}(x)$ by a local Weyl rescaling with specific k-dependent weights.

The standard way to deal with the fact that we have been taken out of FG gauge is to employ an additional diffeomorphism acting on the $x^\mu \to x^\mu + \xi^\mu(z;x)$ which becomes trivial at the conformal boundary in such a way that $g^{(0)}_{\mu\nu}(x)$ is left unchanged, but the cross term in (712) is cancelled. However, this diffeomorphism unfortunately has a complicated effect on all of the subleading terms in the metric – they no longer transform linearly as in (713), but instead transform non-linearly under the combined transformations and this obscures the geometric significance of the sub-leading terms. There is nothing inconsistent here: indeed, in FG gauge, the subleading terms are given on-shell by expressions involving the Levi-Civita curvature of the induced metric, which themselves transform non-linearly under Weyl transformations.

We will instead consider here a revised ansatz, which we refer to as Weyl-Fefferman-Graham (WFG) gauge, defined as[84]

$$ds^2 = L^2 \left(\frac{dz}{z} - a_\mu(z;x)dx^\mu \right)^2 + h_{\mu\nu}(z;x)dx^\mu dx^\nu. \tag{715}$$

[83] This avoids the necessary introduction of logarithms that occur when $d+1$ is an even integer. In fact, using dimensional regularization we will allow $(d+1) \in \mathbb{C}$, the analytic continuation of the number of spacetime dimensions.

[84] It is also possible to generalize the ansatz by the inclusion of a scalar function in front of the first term, essentially a radial lapse function. We will discuss this further in the following. Notice furthermore that the flat limit of this ansatz is still divergent, as for the FG one.

The constant-z hypersurface Σ at $z = 0$ remains the conformal boundary with induced metric $g^{(0)}$, as

$$\frac{z^2}{L^2} ds^2 \underset{z\to 0}{\longrightarrow} g^{(0)}_{\mu\nu}(x) dx^\mu dx^\nu. \tag{716}$$

Thus the presence of a_μ in the ansatz does not modify the induced metric at $z = 0$. However, the metric is no longer diagonal in the z, x^μ coordinates, and so we must take greater care in interpreting how we approach the conformal boundary.

It is natural, given the metric ansatz (715), to introduce the 1-form

$$e \equiv \Omega(z;x)^{-1} \left(\frac{dz}{z} - a_\mu(z;x) dx^\mu \right) \tag{717}$$

This form defines a distribution $C_e \subset TM$ defined as

$$C_e = ker(e) = span \left\{ \underline{X} \in \Gamma(TM) \middle| i_{\underline{X}} e = 0 \right\}. \tag{718}$$

Note that there is an ambiguity in multiplying e (or equivalently the $\underline{X}$'s) by a function on M, and we have represented this ambiguity by introducing the function Ω.

We remark that if a_μ were zero, then C_e is the span of the vectors ∂_μ and can be thought of as related to constant-z hypersurfaces. More generally, it is convenient to introduce a basis for C_e as the set of vectors

$$\underline{D}_\mu \equiv \partial_\mu + a_\mu(z;x) z \partial_z. \tag{719}$$

This implies that we can regard a_μ as providing a lift[85] from $T\Sigma$ (with basis $\{\partial_\mu\}$) to C_e, that is, it can be thought of as an Ehresmann connection. By the Frobenius theorem, C_e is an integrable distribution if

$$\left[\underline{D}_\mu, \underline{D}_\nu\right] \in C_e. \tag{720}$$

To understand this condition, it is convenient to introduce a vector dual to e,

$$\underline{e} \equiv \Omega(z;x) z \partial_z \tag{721}$$

which has been normalized to $e(\underline{e}) = 1$, and we regard $\{\underline{e}, \underline{D}_\mu\}$ as a basis for $T_{(z;x)}M$. We then compute

$$\left[\underline{D}_\mu, \underline{D}_\nu\right] = \Omega(z;x)^{-1} f_{\mu\nu} \underline{e}, \qquad f_{\mu\nu} \equiv D_\mu a_\nu - D_\nu a_\mu \tag{722}$$

So we find that integrability is the condition $f_{\mu\nu} = 0$, and thus by Frobenius, the distribution C_e would define under that circumstance a foliation of M by co-dimension one hypersurfaces.

By taking $\underline{e}$ in the form (721), we have fixed some of the diffeomorphism invariance;[86] the diffeomorphisms that preserve the form of $\underline{e}$ are given by $z' = z'(z;x), x'^\mu = x'^\mu(x)$. Given the interpretation of holography in terms of renormalization, we expect that these diffeomorphisms correspond to generic local (in x) coarse grainings. These residual diffeomorphisms act on the form e as

$$\frac{\partial x'^\nu(x)}{\partial x^\mu} a'_\nu(z';x') = \frac{\partial \ln z'(z;x)}{\partial \ln z} a_\mu(z;x) + \frac{\partial \ln z'(z;x)}{\partial x^\mu}, \qquad \Omega'(z';x') = \frac{\partial \ln z'(z;x)}{\partial \ln z} \Omega(z;x). \tag{724}$$

The first equation is consistent with the interpretation of a as an Ehresmann connection. The second equation implies that the inherent ambiguity in the definition of the distribution C_e represented by $\Omega(z;x)$ can be thought of as the (local) reparametrization invariance of z. We can for example use this reparametrization invariance to set

[85] Here, we are regarding Σ as an isolated hypersurface in M. We can thus regard M as a fibre bundle $\pi : M \to \Sigma$. An Ehresmann connection provides a splitting of the tangent bundle $TM = H \oplus V$, and the $\underline{D}_\mu$ vectors form a basis of H, identified with C_e, at the point (z, x^μ).

[86] Indeed, the vector field $\underline{e}$ could more generally be of the form

$$\underline{e} \to \underline{e}' = \underline{e} + \theta^\mu(z;x) \underline{D}_\mu \tag{723}$$

which satisfies $e(\underline{e}) = 1$ for any θ^μ. (In the language of footnote 85 (see page 95), the $\underline{e}$ of (721) is special in that $\underline{e} \in V$). In the general case, we have $\left[\underline{D}_\mu, \underline{D}_\nu\right] = f_{\mu\nu} \underline{e}' - f_{\mu\nu} \theta^\lambda \underline{D}_\lambda$ and thus integrability remains the condition $f_{\mu\nu} = 0$. The second diffeomorphism, discussed earlier, that returns the metric to the FG ansatz after a boundary Weyl transformation corresponds on the contrary to setting $a_\mu \to 0$ at the expense of keeping $\theta^\mu \neq 0$.

$\Omega(z;x) \to L^{-1}$ if we wish. The residual diffeomorphisms that preserve this choice (or, more generally preserve any specific $\Omega(z;x)$) are of the form $z' = z/\mathcal{B}(x), x'^{\mu} = x'^{\mu}(x)$, which are the Weyl diffeomorphisms. These give

$$\frac{\partial x'^{\nu}(x)}{\partial x^{\mu}} a'_{\nu}(z';x') = a_{\mu}(z;x) - \partial_{\mu} \ln \mathcal{B}(x), \tag{725}$$

and so we are to interpret the $a_{\mu}(z;x)$ as a connection for the Weyl diffeomorphisms (710). Given this result, it will not come as a surprise that there will be an induced Weyl connection on the conformal boundary. To recap, using $\Omega = L^{-1}$, we have the following setup

$$\{\underline{e}, \underline{D}_{\mu}\} = \Big\{L^{-1} z\partial_z, \partial_{\mu} + a_{\mu} z\partial_z\Big\}, \qquad [\underline{D}_{\mu}, \underline{D}_{\nu}] = L f_{\mu\nu} \underline{e}. \tag{726}$$

To proceed further, we Fourier analyze $a_{\mu}(z;x)$ and $h_{\mu\nu}(z;x)$ in the sense that we will expand them in eigenfunctions of $\underline{e}$. Such eigenfunctions are of course just the monomials in $z \in \mathbb{R}^{+}$. For $h_{\mu\nu}(z;x)$ we obtain then the same expansion as before, (145), and for $a_{\mu}(z;x)$ we write

$$a_{\mu}(z;x) = \left[a^{(0)}_{\mu}(x) + \frac{z^2}{L^2} a^{(2)}_{\mu}(x) + ...\right] + \frac{z^{d-1}}{L^{d-1}} \left[p^{(0)}_{\mu}(x) + \frac{z^2}{L^2} p^{(2)}_{\mu}(x) + ...\right], \tag{727}$$

which is of the same form as the expansion of a massless gauge field in Fefferman-Graham. Given these expressions, we observe that $a^{(0)}_{\mu}$ is not part of the boundary metric, although as we will show, it is part of the induced boundary connection and thus should be regarded as part of the boundary geometry.

More precisely, what we will show is that for the WFG ansatz, the induced connection is not the Levi-Civita connection of the induced metric, but instead a Weyl connection. Given the expansions (145,727), we see that the Weyl diffeomorphism (710) acts as

$$g^{(k)}_{\mu\nu}(x) \to g^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{k-2}, \qquad T^{(k)}_{\mu\nu}(x) \to T^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{d-1+k} \tag{728}$$

$$a^{(k)}_{\mu}(x) \to a^{(k)}_{\mu}(x)\mathcal{B}(x)^{k} - \delta_{k,0}\partial_{\mu} \ln \mathcal{B}(x), \qquad p^{(k)}_{\mu}(x) \to p^{(k)}_{\mu}(x)\mathcal{B}(x)^{d-1+k} \tag{729}$$

and so in particular

$$g^{(0)}_{\mu\nu}(x) \to g^{(0)}_{\mu\nu}(x)/\mathcal{B}(x)^2, \qquad a^{(0)}_{\mu}(x) \to a^{(0)}_{\mu}(x) - \partial_{\mu} \ln \mathcal{B}(x) \tag{730}$$

and thus we may anticipate that $a^{(0)}_{\mu}$ will play the role of a boundary Weyl connection. All of the other subleading functions in the expansions (145,727) are interpreted to have, à la (728–729), definite Weyl weights, that is they are Weyl tensors. It is then natural to expect that they will be determined in terms of the Weyl curvature, which we discussed in the last section.

We introduced the concept of the distribution C_e precisely in order to properly discuss the notion of an induced connection, as C_e is a sub-bundle of TM. That is, given a connection ∇ on TM (which we will take to be the Levi-Civita connection), we can apply it to vectors in C_e, which will be of the general form

$$\nabla_{\underline{D}_{\mu}} \underline{D}_{\nu} = \Gamma^{\lambda}_{\mu\nu} \underline{D}_{\lambda} + \Gamma^{e}_{\mu\nu} \underline{e} \tag{731}$$

The coefficients of the induced connection on C_e are by definition the $\Gamma^{\lambda}_{\mu\nu}$ appearing in (731). Notice that these connection coefficients should not be confused with the usual Christoffel symbols, which are associated with coordinate bases. By direct computation, we find

$$\Gamma^{\lambda}_{\mu\nu} = \gamma^{\lambda}_{\mu\nu} \equiv \tfrac{1}{2} h^{\lambda\rho}\Big(D_{\mu} h_{\rho\nu} + D_{\nu} h_{\mu\rho} - D_{\rho} h_{\nu\mu}\Big) \tag{732}$$

and furthermore if we evaluate this expression at $z = 0$, we find

$$\gamma^{(0)\lambda}{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho}_{(0)}\Big((\partial_{\mu} - 2a^{(0)}_{\mu})g^{(0)}_{\nu\rho} + (\partial_{\nu} - 2a^{(0)}_{\nu})g^{(0)}_{\mu\rho} - (\partial_{\rho} - 2a^{(0)}_{\rho})g^{(0)}_{\mu\nu}\Big) \tag{733}$$

This result can be compared to the result (782) reported in Appendix A,[87] from which we conclude that the induced connection on the boundary is in fact a Weyl connection, with the role of the geometric data g_{ab} and A_a in (782) being played here by $g^{(0)}_{\mu\nu}$ and $a^{(0)}_{\mu}$. In comparing, we make use of the fact that here the intrinsic rotation coefficients are $C_{\mu\nu}{}^{\lambda} = 0$, as in (722). We will use the notation $\nabla^{(0)}$ for the corresponding Weyl connection (whose Weyl-Christoffel

[87] We report in this Appendix an account on the definition of the Weyl connection.

symbols are given by (733)), and the curvature as $R^{(0)\lambda}{}_{\mu\rho\nu}$. A tensor with components $t_{\mu_1\ldots\mu_n}(x)$ that has Weyl weight w_t transforms as $t_{\mu_1\ldots\mu_n}(x) \mapsto \mathcal{B}(x)^{w_t} t_{\mu_1\ldots\mu_n}(x)$, while $\mathcal{D}_\nu t_{\mu_1\ldots\mu_n}(x) \equiv \nabla^{(0)}_\nu t_{\mu_1\ldots\mu_n}(x) + w_t a^{(0)}_\nu t_{\mu_1\ldots\mu_n}(x)$ transforms covariantly with the same weight. As noted above, all of the component fields aside form $a^{(0)}_\mu$ transform covariantly with respect to arbitrary Weyl transformations, and the Weyl weights of the various component fields are given above in (728). In the next section, we will briefly study some aspects of the holographic dictionary, and we will find that every equation is covariant with respect to arbitrary Weyl transformations – it is a bona fide (background) symmetry of the dual field theory. In particular, we will find that the appearance of $a^{(0)}_\mu(x)$, since it transforms non-linearly under Weyl transformations, is through Weyl-covariant derivatives of other fields, or through expressions involving the Weyl-invariant field strength $f^{(0)}_{\mu\nu}$.

Before moving on, we would like to stress again the main result of this section: the usual bulk Levi-Civita connection built using the bulk metric in the enhanced WFG gauge induces on the boundary a Weyl connection and therefore a boundary Weyl-covariant geometry.

5.2 The Holographic Dictionary and the Weyl Anomaly

In this section, we will explore some details of the holographic dictionary corresponding to the WFG ansatz. The Levi-Civita connection in the bulk has the form

$$\nabla_{\underline{D}_\mu} \underline{D}_\nu = \gamma^\lambda_{\mu\nu} \underline{D}_\lambda - h_{\nu\lambda}\psi^\lambda{}_\mu \underline{e} \tag{734}$$

$$\nabla_{\underline{D}_\mu} \underline{e} = \psi^\lambda{}_\mu \underline{D}_\lambda \tag{735}$$

$$\nabla_{\underline{e}} \underline{D}_\mu = \psi^\lambda{}_\mu \underline{D}_\lambda + L\varphi_\mu \underline{e} \tag{736}$$

$$\nabla_{\underline{e}} \underline{e} = -Lh^{\lambda\rho}\varphi_\rho \underline{D}_\lambda \tag{737}$$

where

$$\psi^\mu{}_\nu = \rho^\mu{}_\nu + \frac{L}{2} h^{\mu\lambda} f_{\lambda\nu}, \qquad \rho^\mu{}_\nu = \tfrac{1}{2} h^{\mu\lambda} \underline{e}(h_{\lambda\nu}), \qquad \varphi_\mu = \underline{e}(a_\mu), \qquad f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu \tag{738}$$

and we note that φ_μ is proportional to the rotation coefficient $C_{e\mu}{}^e$, i.e., $[\underline{e}, \underline{D}_\mu] = L\varphi_\mu \underline{e}$. In addition, we will use the notation[88] $\theta = tr\rho = \underline{e}(\ln\sqrt{-h})$ and $\zeta^\mu{}_\nu = \rho^\mu{}_\nu - \frac{1}{d+1}\theta\delta^\mu{}_\nu$.

As we have detailed above, the WFG metric ansatz has two bulk fields $h_{\mu\nu}$ and a_μ, and $g^{(0)}_{\mu\nu}(x)$ and $a^{(0)}_\mu(x)$ appear as sources (and/or backgrounds), while $T^{(0)}_{\mu\nu}(x)$ and $p^{(0)}_\mu(x)$ appear as the corresponding vevs. The corresponding operators in the dual field theory are Weyl-covariant currents $\hat{T}_{\mu\nu}(x)$ and $\hat{J}_\mu(x)$, each of Weyl weight $d-1$. We will discuss these operators more fully in Section 5.3.

As usual, one finds that the bulk equations of motion determine the subleading component fields in terms of $g^{(0)}_{\mu\nu}(x), a^{(0)}_\mu(x), T^{(0)}_{\mu\nu}(x)$ and $p^{(0)}_\mu(x)$. Here we will assume that we have a vacuum solution that is asymptotically locally anti-de Sitter. For example, the ee-component of the vacuum Einstein equations is

$$0 = G_{ee} + \Lambda g_{ee} = -\tfrac{1}{2} tr(\rho\rho) - \frac{3L^2}{8} tr(ff) - \tfrac{1}{2}\overline{R} + \tfrac{1}{2}\theta^2 \tag{739}$$

where $\Lambda = -\frac{(d+1)d}{2L^2}$ is the cosmological constant of AdS$_{d+2}$ and we define for the sake of brevity

$$\overline{R}^\lambda{}_{\mu\rho\nu} = D_\rho \gamma^\lambda_{\nu\mu} - D_\nu \gamma^\lambda_{\rho\mu} + \gamma^\delta_{\nu\mu}\gamma^\lambda_{\rho\delta} - \gamma^\delta_{\rho\mu}\gamma^\lambda_{\nu\delta} \tag{740}$$

with $\overline{R} = h^{\mu\nu}\overline{R}^\rho{}_{\mu\rho\nu}$ the corresponding Ricci scalar. Expanding (739) we find

$$0 = \left[\Lambda + \frac{d(d+1)}{2L^2}\right] - \frac{1}{2}\frac{z^2}{L^2}\left[2dL^{-2}X^{(1)} + R^{(0)}\right] + \ldots - d\frac{z^{d+1}}{L^{d+1}}\left[\frac{d+1}{2}L^{-2}Y^{(1)} + \mathcal{D}\cdot p_{(0)}\right] + \ldots \tag{741}$$

where $R^{(0)}$ is the boundary Weyl-Ricci scalar and

$$X^{(1)} = g^{\mu\nu}_{(0)} g^{(2)}_{\mu\nu}, \qquad Y^{(1)} = g^{\mu\nu}_{(0)} T^{(0)}_{\mu\nu}. \tag{742}$$

[88] The notation used here can be interpreted in terms of expansion (θ), shear (ζ), vorticity (f) and acceleration (φ) of the radial congruence $\underline{e}$.

In (741), the order one equation is trivially satisfied while the z^2 contribution gives $X^{(1)}$ entirely in terms of the Weyl-Ricci scalar curvature:

$$X^{(1)} = -\frac{L^2}{2d}R^{(0)}. \tag{743}$$

As in the FG story, we must be careful with the $O(z^{d+1})$ terms here because of divergences in the evaluation of the on-shell action – those divergences are responsible for the Weyl anomaly in the dual field theory, the structure of which we will discuss in detail below. Nevertheless, we may read off the 'left-hand-side' of the Weyl Ward identity from this,

$$Y^{(1)} + \frac{2L^2}{d+1}\mathcal{D}\cdot p_{(0)}. \tag{744}$$

We will see later that this is the expected form given the interpretation of $T^{(0)}_{\mu\nu}$ and $p^{(0)}_\mu$ as vevs of currents in the dual field theory. We will also study the form of the anomalous right-hand-side later.

Similarly, one finds that the leading $O(z^2)$ term in $G_{e\mu}$ is proportional to

$$g^{\lambda\nu}_{(0)}\nabla^{(0)}_\nu\left(G^{(0)}_{\lambda\mu} + f^{(0)}_{\lambda\mu}\right) = 0, \tag{745}$$

the vanishing of which is the twice-contracted Bianchi identity of the Weyl connection, as discussed in the Appendix A (see eq. (800)).

The leading non-trivial terms in the $\mu\nu$-components of the Einstein equations determine

$$g^{(2)}_{\mu\nu} = -\frac{L^2}{d-1}\left(Ric^{(0)}_{(\mu\nu)} - \frac{1}{2d}R^{(0)}g^{(0)}_{\mu\nu}\right) = -\frac{L^2}{d-1}L^{(0)}_{(\mu\nu)}, \tag{746}$$

where $L^{(0)}$ is the Weyl-Schouten tensor. Its trace (742) correctly reproduces (743). We take each of these results as representative of the fact that the subleading terms in the expansion of the metric are determined by the Weyl curvature, analogous to what happens in the usual FG gauge in which they are determined by the Levi-Civita curvature of the induced metric. As we mentioned previously, the difference is that now all of the subleading terms in the bulk fields are Weyl-covariant. One expects that the same is true for a_μ as well, along with the transversality of such solutions. For example, the $O(z^4)$ term in the $e\mu$-component of the bulk Einstein equation involves $a^{(2)}_\mu$ in the form $Max(a^{(2)})_\mu$ where Max refers to the Weyl-Maxwell differential operator

$$Max(a^{(2)})_\mu \equiv \mathcal{D}\cdot\mathcal{D}a^{(2)}_\mu - \mathcal{D}_\mu(\mathcal{D}\cdot a^{(2)}) + (Ric^{(0)}_{\nu\mu} + 4f^{(0)}_{\nu\mu})g^{\nu\lambda}_{(0)}a^{(2)}_\lambda. \tag{747}$$

The appearance of the Maxwell operator here is the analogue of the appearance of the tranverse tensor $\Pi^{\mu\nu}$ in the bulk solutions for a massless gauge field, when the boundary is Minkowski space-time. Note that both the Weyl-Ricci tensor and $f^{(0)}_{\mu\nu}$ appear in the Laplacian because $a^{(2)}$ is a vector field that has non-zero Weyl charge (weight).

The holographic dictionary for WFG will be taken to be the obvious generalization of the usual relationship, i.e.,

$$Z_{\text{bulk}}[g; g^{(0)}, a^{(0)}] = exp(-S_{\text{o.s.}}[h, a; g^{(0)}, a^{(0)}]) = Z_{\text{FT}}[g^{(0)}, a^{(0)}] \tag{748}$$

where on the left we have the on-shell action of the bulk classical theory whose metric is given by h, a with asymptotic configurations $g^{(0)}, a^{(0)}$, while the right-hand-side is the generating functional of correlation functions of operators sourced by $g^{(0)}, a^{(0)}$. Although this is expressed in terms of the "bare" sources, it is implicit that a regularization scheme for the left-hand-side is employed and that the boundary counter-terms are introduced to absorb power divergences that arise in the evaluation of the on-shell action. Here, we will organize the discussion by taking the space-time dimension $d+1$ to be formally complex; the on-shell action is convergent for sufficiently small $d+1$, and as we move $d+1$ up along the real axis, we encounter additional divergences as $d+1$ approaches an even integer. It is well-known in the context of Fefferman-Graham that as a byproduct this divergence induces the Weyl anomaly of the dual field theory, and is associated with the appearance of logarithms in the field expansions when $d+1$ is precisely an even integer. Here we will review this bit of physics, as the existence of the Weyl connection, as we will see, organizes the Weyl anomaly in a much more symmetric fashion than is usually described.

It is taken for granted that Z_{bulk} is diffeomorphism invariant. Under the holographic map this implies, among other things, that the dual field theory can be regulated in a diffeomorphism-invariant fashion. However, the bulk calculation is classical, and thus, in principle, is a functional of the bulk metric g as well as the boundary values. We therefore suppose that

$$\frac{Z_{\text{bulk}}\left[g';g'_{(0)},a'_{(0)},...\middle|z',x'\right]}{Z_{\text{bulk}}\left[g;g_{(0)},a_{(0)},...\middle|z,x\right]}=1, \tag{749}$$

where the notation refers to the fact that we are computing the partition function in different coordinate systems. Here of course we are particularly interested in the Weyl diffeomorphism $(z',x')=(z/\mathcal{B}(x),x)$ which relates the boundary values $g'_{(0)}=g_{(0)}/\mathcal{B}^2$, $a'_{(0)}=a_{(0)}-d\ln\mathcal{B}$. Z_{bulk} is given in the classical limit by evaluating the (renormalized) on-shell action, $Z_{\text{bulk}}=e^{-S_{\text{o.s.}}[g;g_{(0)},a_{(0)},...|z,x]}$. We then ask, is it also true that this cleanly induces a Weyl transformation on the boundary? That is, is it true that

$$\frac{Z_{\text{bdy}}[x;g'_{(0)},a'_{(0)},...]}{Z_{\text{bdy}}[x;g_{(0)},a_{(0)},...]}\stackrel{?}{=}1, \tag{750}$$

where Z_{bdy} is the generating functional in the given background. As is well-established, what happens is that there is an anomaly

$$\frac{Z_{\text{bulk}}\left[g';g'_{(0)},a'_{(0)},...\middle|z',x\right]}{Z_{\text{bulk}}\left[g;g_{(0)},a_{(0)},...\middle|z,x\right]}=e^{\mathcal{A}_k}\frac{Z_{\text{bdy}}[x;g'_{(0)},a'_{(0)},...]}{Z_{\text{bdy}}[x;g_{(0)},a_{(0)},...]} \tag{751}$$

in dimension $d+1=2k$. Recall that we are employing the specific Weyl diffeomorphism, which is inducing a Weyl transformation on the boundary, but no boundary diffeomorphism. If we take the log of these expressions, the result is that

$$0=S_{\text{bulk}}[g';g'_{(0)},...|z',x]-S_{\text{bulk}}[g;g_{(0)},...|z,x]=S_{\text{bdy}}[x;g'_{(0)},a'_{(0)},...]-S_{\text{bdy}}[x;g_{(0)},a_{(0)},...]+\mathcal{A}_k. \tag{752}$$

That is, when we compare the evaluation of the bulk on-shell action in different coordinate systems, the result appears as the difference of boundary actions in Weyl-equivalent backgrounds, up to an anomalous term, which is not the difference of two such actions. The only source for such a term is a pole at $d+1=2k$ (i.e. $\frac{1}{d+1-2k}$) in the evaluation of the bulk action, which arises because the on-shell action is not a boundary term, but contains pieces that must be integrated over z. The bulk action is given by ($vol_S=\sqrt{-h}d^{d+1}x$)

$$S_{\text{bulk}}[g;g_{(0)},...|z,x]=\frac{1}{16\pi G_N}\int_M e\wedge vol_S(R-2\Lambda). \tag{753}$$

On shell, it evaluates to

$$S_{\text{bulk}}[g;g_{(0)},...|z,x]=-\frac{d+1}{8\pi G_N L^2}\int_M e\wedge vol_S=-\frac{d+1}{8\pi G_N L}\int_M\frac{dz}{z}\wedge d^{d+1}x\sqrt{-h}, \tag{754}$$

where we remind that $d+1$ is the boundary dimension. We then expand $\sqrt{-h}$ in powers of z:

$$S_{\text{bulk}}[g;g_{(0)},...|z,x]=-\frac{d+1}{8\pi G_N L}\int_M dz\wedge d^{d+1}x\Big(\frac{L}{z}\Big)^{d+2}\sqrt{-g^{(0)}}\Big(1+\frac{z^2}{L^2}\frac{X^{(1)}}{2}+\dots\Big). \tag{755}$$

Consider now the difference of Weyl-transformed bulk actions as in (752) and define $vol_\Sigma=\sqrt{-g^{(0)}}d^{d+1}x$. The idea is to start with $S_{\text{bulk}}[g';g'_{(0)},...|z',x]$, use the explicit Weyl transformation of the different quantities in the expansion (see (728)) and then change the name of the integration variable from z' to z.[89] We will demonstrate this for the first pole, which occurs at $d+1=2$. We then obtain

$$\begin{aligned}0\;=\;&\frac{d+1}{8\pi G_N}\int_M d\left(\frac{\mathcal{B}^{-(d+1)}}{d+1}\left(\frac{L}{z}\right)^{d+1}\right)\wedge vol_\Sigma-\frac{d+1}{8\pi G_N}\int_M d\left(\frac{1}{d+1}\left(\frac{L}{z}\right)^{d+1}\right)\wedge vol_\Sigma\\&+\frac{d+1}{16\pi G_N}\int_M d\left(\frac{\mathcal{B}^{-(d-1)}}{d-1}\left(\frac{L}{z}\right)^{d-1}\right)\wedge\mathcal{G}_\Sigma-\frac{d+1}{16\pi G_N}\int_M d\left(\frac{1}{d-1}\left(\frac{L}{z}\right)^{d-1}\right)\wedge\mathcal{G}_\Sigma+\dots,\end{aligned} \tag{756}$$

[89] To evaluate these expressions, a regulator is required. The last step of renaming the integration variable has a corresponding effect on the cutoff and thus is not innocuous in the renormalization procedure. Such a regulator is not Weyl-covariant, which is consistent with the fact that an anomaly arises. Most of the details of the renormalization occur in expressions that are the difference of two Weyl-equivalent actions, whereas the anomaly is not and has been cleanly extracted.

with $\mathcal{G}_\Sigma = X^{(1)} vol_\Sigma$ (Weyl weight $-(d-1)$). We observe that the offending term in $d \to 1^-$ (that is, boundary dimension 2^-) is

$$\frac{d+1}{16\pi G_N}\int_M d\left(\frac{\mathcal{B}^{-(d-1)}}{d-1}\left(\frac{L}{z}\right)^{d-1}\right)\wedge\mathcal{G}_\Sigma - \frac{d+1}{16\pi G_N}\int_M d\left(\frac{1}{d-1}\left(\frac{L}{z}\right)^{d-1}\right)\wedge\mathcal{G}_\Sigma = -\frac{1}{8\pi G_N L}\int_\Sigma \ln\mathcal{B}\,\mathcal{G}_\Sigma. \tag{757}$$

The equality in this equation is obtained expanding $\mathcal{B}$ around 1 and eventually imposing $d=1$. For concreteness we expand this final result using the holographic value of $X^{(1)}$, (743). Then, we read from (752):

$$\mathcal{A}_1 = \frac{1}{8\pi G_N L}\int_\Sigma \ln\mathcal{B}\,\mathcal{G}_\Sigma = -\frac{L}{16\pi G_N}\int_\Sigma \ln\mathcal{B}\,R^{(0)} vol_\Sigma. \tag{758}$$

This numerical coefficient is the correct one that leads to the central charge $c = \frac{3L}{2G_N}$. We will shortly comment on the implications, but notice already that $R^{(0)}$ is not the Levi-Civita curvature, as usually found, but rather the Weyl curvature. As such, it is a Weyl-covariant scalar.

The Weyl anomaly in $d=1$ then is best expressed cohomologically as the difference:

$$(e\wedge\mathcal{G}_\Sigma)' - (e\wedge\mathcal{G}_\Sigma) = d(\ln\mathcal{B}\,\mathcal{A}_1\,vol_\Sigma), \tag{759}$$

with $\mathcal{A}_1$ proportional to $X^{(1)}$. Each term on the left is expected to be closed (because they are top forms in the bulk!) but the difference is in general exact, with its strength determining the Weyl anomaly of the boundary theory.

Some comments are in order here. Firstly, we have obtained a very powerful new result: the Weyl anomaly $\mathcal{A}_1$ is now dictated in two boundary dimensions by the Weyl-covariant scalar curvature $R^{(0)}$. This is not the case if we start with the FG gauge in the bulk, for which the Levi-Civita scalar curvature appears. The Weyl covariance of all the subleading terms in the WFG gauge implies that the anomaly in every even boundary dimension will have Weyl-covariant curvature coefficients in our framework. Secondly, we expect the cohomological derivation of the anomaly to be a general feature, not restricted to the two-dimensional case. In fact, recalling that the metric determinant is expanded as

$$\sqrt{-h(z;x)} \quad = \quad \left(\frac{L}{z}\right)^{d+1}\sqrt{-g^{(0)}(x)}\left[1+\frac{1}{2}\frac{z^2}{L^2}X^{(1)}+\frac{1}{2}\frac{z^4}{L^4}X^{(2)}+...+\frac{1}{2}\frac{z^{d+1}}{L^{d+1}}Y^{(1)}+...\right], \tag{760}$$

we deduce that a similar derivation as for the two-dimensional case holds in any even dimension, with $\mathcal{G}_\Sigma$ generally replaced by

$$\mathcal{G}^{(k)}_\Sigma = X^{(k)} vol_\Sigma. \tag{761}$$

We therefore claim that in any even boundary dimension $d+1=2k$,

$$\left(e\wedge\mathcal{G}^{(k)}_\Sigma\right)' - \left(e\wedge\mathcal{G}^{(k)}_\Sigma\right) = d(\ln\mathcal{B}\,\mathcal{A}_k\,vol_\Sigma), \tag{762}$$

the $\mathcal{A}_k$ term on the right-hand side being proportional to $X_{(k)}$. Looking for a universal form of $X_{(k)}$ as a function of the Weyl curvature tensors of the boundary is an appealing future direction of investigation.

5.3 Field Theory Aspects

In this section, we will make some preliminary remarks about the dual field theory. The holographic analysis implies that we should now consider a field theory coupled to a background metric and Weyl connection, with action $S[g^{(0)}, a^{(0)}; \Phi]$ where Φ denotes some collection of dynamical fields to which we will assign some definite Weyl weights. As we will explain, this is perfectly natural from the field theory perspective as well, but constitutes a new organization of such field theories (which in the usual formulation are coupled only to a background metric). The quantum theory possesses a partition function $Z[g^{(0)}, a^{(0)}]$ that depends on the background, both through explicit dependence in the action and in the definition of the functional integral measure. A background Ward identity is generated by changing integration variables $\Phi(x) \to \mathcal{B}(x)^{w_\Phi}\Phi(x)$ giving

$$Z[g^{(0)}, a^{(0)}] = e^{\mathcal{A}[\mathcal{B}]} Z[\mathcal{B}(x)^{-2} g^{(0)}, a^{(0)} - d\ln\mathcal{B}(x)] \tag{763}$$

with $\mathcal{A}$ a possible anomalous contribution. Thus the Weyl Ward identity is a relationship between different theories, that is, field theories in different backgrounds and so, more properly, we refer to the above equation as a background

Ward identity. Strictly speaking, this argument applies to free theories, whereby (if Φ is a scalar) $w_\Phi = \frac{1}{2}(d-1)$ is the engineering dimension. An example of an action in this context is

$$S[g^{(0)}, a^{(0)}; \Phi] = -\tfrac{1}{2} \int d^{d+1}x \sqrt{-g^{(0)}}\; g^{\mu\nu}_{(0)} \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi \tag{764}$$

where $\mathcal{D}_\mu \Phi = \partial_\mu \Phi + w_\Phi a^{(0)}_\mu \Phi$ is Weyl-covariant.[90] Notice that the stress tensor of this theory has the form

$$\mathrm{T}^{\mu\nu}_{g^{(0)},a^{(0)}}(x) = \frac{2}{\sqrt{-g^{(0)}}} \frac{\delta S[g^{(0)}, a^{(0)}; \Phi]}{\delta g^{(0)}_{\mu\nu}(x)} = \mathcal{D}^\mu \Phi(x) \mathcal{D}^\nu \Phi(x) - \frac{1}{2} g^{(0)\mu\nu}(x) g^{(0)\alpha\beta}(x) \mathcal{D}_\alpha \Phi(x) \mathcal{D}_\beta \Phi(x) \tag{765}$$

Here we have used pedantic notation to emphasize that the definition of the operator depends on the background fields. This operator is Weyl-covariant, by which we mean

$$\mathrm{T}^{\mu\nu}_{\mathcal{B}(x)^{-2} g^{(0)}, a^{(0)} - d \ln \mathcal{B}(x)}(x) = \mathcal{B}(x)^{d+1} \mathrm{T}^{\mu\nu}_{g^{(0)},a^{(0)}}(x) \tag{766}$$

That is, if we compare correlation functions of the stress tensor in two Weyl-related backgrounds, there will be a relative factor of $\mathcal{B}(x)^{d+1}$ for each instance of the stress tensor; for brevity, we refer to this as the stress tensor (with two upper indices) having Weyl weight $w_T = d+1$. Similarly, we have the Weyl current

$$\mathrm{J}^{\mu}_{g^{(0)},a^{(0)}}(x) = \frac{1}{\sqrt{-g^{(0)}}} \frac{\delta S[g^{(0)}, a^{(0)}; \Phi]}{\delta a^{(0)}_{\mu}(x)} = w_\Phi \Phi(x) \mathcal{D}^\mu \Phi(x) \tag{767}$$

This operator is also Weyl-covariant in the same sense as the stress tensor and is of weight $d+1$. Thus $\hat{\mathrm{T}}^{\mu\nu}$ and $\hat{\mathrm{J}}^{\mu}$ have the properties of the operators sourced in the holographic WFG theory. In a holographic theory, we would not have the free field discussion given here, but we can still discuss sourcing these operators (in a given background).

Earlier, we saw that the classical Weyl Ward identity involved a linear combination of the trace of the stress tensor and the divergence of the Weyl current. This is in fact easily established in general terms. Here we will use classical language, but the argument easily extends to the quantum case by making use of (763). Indeed, suppose that the classical action satisfies

$$S[g^{(0)}, a^{(0)}; \mathcal{B}^{w_\Phi} \Phi] = S[g^{(0)}/\mathcal{B}^2, a^{(0)} - d \ln \mathcal{B}; \Phi] \tag{768}$$

As mentioned above, this is what we mean by Weyl being a background symmetry. By expanding both sides for small $\ln \mathcal{B}$ and going on-shell, we find

$$0 = \int d^{d+1}x \frac{\delta S}{\delta a^{(0)}_\mu(x)} \partial_\mu \ln \mathcal{B}(x) + \int d^{d+1}x \frac{\delta S}{\delta g^{(0)}_{\mu\nu}(x)} \Big(-2 \ln \mathcal{B}(x) g^{(0)}_{\mu\nu}(x) \Big) \tag{769}$$

We recognize that this may be written as

$$0 = \int d^{d+1}x \sqrt{-g^{(0)}}\; \mathrm{J}^\mu(x) \partial_\mu \ln \mathcal{B}(x) + \int d^{d+1}x \sqrt{-g^{(0)}}\; \mathrm{T}^{\mu\nu}(x) \Big(-\ln \mathcal{B}(x) g^{(0)}_{\mu\nu}(x) \Big) \tag{770}$$

and, by integrating by parts, we have

$$0 = -\int d^{d+1}x \sqrt{-g^{(0)}} \Big(\mathcal{D}_\mu \mathrm{J}^\mu(x) + \mathrm{T}^{\mu\nu}(x) g^{(0)}_{\mu\nu}(x) \Big) \ln \mathcal{B}(x) \tag{771}$$

This result serves to identify the relative normalization of $T^{(0)}_{\mu\nu}$ and $p^{(0)}_\mu$ and their relation with the currents defined here. Incidentally, the Weyl-covariant derivative appears in (771) precisely because the current J^μ (with raised index) has Weyl weight $d+1$.

We remark that typical discussions of related topics are rife with "improvements" to operators such as the stress tensor [188, 189], including mixing with a so-called 'virial current'. The operators that we have defined here have the advantage of transforming linearly, and in particular do not mix with each other, under Weyl transformations. Note also that the Weyl current in the free theory is in fact a total derivative. Thus, at least in the absence of edges or boundaries [92, 202], one might suppose that this operator is in a sense trivial.

This last comment concludes our wondering regarding the appearance of Weyl symmetry in holography, at least in the familiar AdS setting. Future directions and perspectives will be detailed in the conclusions, hereafter.

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[90] An independently Weyl invariant action term is $\int d^{d+1}x \sqrt{-g^{(0)}}\; R^{(0)} \Phi^2$. It is well-known that using the Levi-Civita connection, only a specific linear combination of the kinetic term and such a curvature term is Weyl invariant, at least up to a total derivative.

6 Conclusions

Since our road was long and not always straightforward to pave, we are going to recap our results and further comment on them. We will then outline future avenues of research, to be explored in either the short or the long term.

Our Findings

The focus of this thesis has been on the flat limit of the fluid/gravity dictionary. In an abuse of language, we constantly referred to the latter as "flat holography". To some degree, what we have investigated is really an asymptotically flat holographic picture. However, as highlighted throughout the analysis, we miss at present a fully understood microscopic duality.

We began our manuscript with a review of the way one obtains the AdS fluid/gravity duality starting from AdS/CFT. In principle, one could have just postulated that a bulk solution of Einstein equations with negative cosmological constant is dual to a relativistic conformal fluid living on its boundary together with an energy-momentum tensor given by the subleading order of the bulk-to-boundary expansion of the metric, in a suitable (FG) gauge.

From the boundary viewpoint, we need a $d+1$-dimensional metric and an energy-momentum tensor $T_{\mu\nu}$, and we are in business. In other words, given $T_{\mu\nu}$, this means that we can expand it geometrically along a boundary normalized time-like congruence, interpreted as the fluid velocity, and the hydrodynamic equations of motion will be encoded in the divergence of $T_{\mu\nu}$ being zero. We thus expanded and characterized the most general energy-momentum tensor in arbitrary dimension and payed particular attention to the geometrical setting. Indeed we kept the latter as general as possible, which as a byproduct returns many geometrical tensors we dealt with.

After this, we restricted our attention to three-dimensional fluids, eventually dual to four-dimensional gravitational solutions. In three dimensions, given a congruence, one can introduce a transverse duality operation, which we called $\tilde{\eta}$, with remarkable properties. Specifically, it intervenes in the integrability conditions we will shortly comment. At this point we introduced an important tool in fluid/gravity: Weyl symmetry. We used it to organize the boundary geometry and hydrodynamics, both in three and two dimensions. The boundary Weyl tensor being zero in three dimensions, we defined its three-dimensional analogue, called Cotton tensor and described its main features. It shares all the properties of the energy-momentum tensor (if conformal), hinting already a relationship between the two.

We specified afterward to two dimensions, where the boundary hydrodynamics is easier to handle. In fact, it has been wrongly claimed to be trivial. Here, we discussed in detail that both the presence of a heat current and an anomalous trace makes a two-dimensional fluid interesting. The heat current is fully determined by a scalar, aligned on the only available transverse direction. In two dimensions, it is possible to concretely analyze the issue related to the hydrodynamic frame choice. We discussed this issue multiple times, so we recap it here. The velocity of a relativistic fluid is not a physical observable: since heat, friction and kinetic motion are all just energy exchanges, relativistically we are free to align a fluid congruence along any particular direction. This leaves us with some freedom in choosing the latter. We yet do not know at this point whether holography on the other hand is sensitive or not to a particular choice of the fluid velocity. Indeed, it is a priori possible (and a posteriori confirmed at least in two dimensions) that our holographic setup breaks this hydrodynamic frame covariance. There is even a more questionable discussion related to hydrodynamics itself. Namely, it is possible that the latter has some global issues for which it is not straightforward to move from a frame to another. We therefore decided to avoid gauge fixing and worked with the most general fluid, where a heat current is present. This choice means that we are not in the Landau-Lifshitz frame. It does not exclude, however, the possibility of being in the so-called Eckart frame. Indeed, in the latter one requires additional currents to be perfect. To show this we should in principle extend our setup to, for instance, charged fluids dual to Einstein-Maxwell spacetimes. This is part of our agenda and will discuss in the outlook.

The AdS gravity side was at this point studied, and the reconstruction of bulk solutions discussed. We used in fluid/gravity the so-called derivative expansion, better suits than the Fefferman-Graham one. In fact, the former is explicitly based on Weyl covariance and, most importantly for our goal, admits a smooth vanishing cosmological constant limit. We immediately specialized to four-dimensional bulks, and wrote the most general r-expansion of the bulk line element compatible with Weyl covariance, which solves the r-evolution parts of Einstein equations. We decided to parametrize the boundary three-dimensional metric à la Randers-Papapetrou and adapt on it the fluid congruence as $\underline{u} = \frac{1}{\Omega}\partial_t$. Subsequently we noticed that imposing the shear of such congruence to vanish creates severe simplifications of the expanded bulk line element, and suggests a resummation of the latter in a closed form. We moreover imposed boundary integrability conditions. These are relationships among the dissipative part of the energy-momentum tensor and the Cotton tensor, suggested by the expansion of the bulk Weyl tensor and encoded in a sort of electro-magnetic duality for gravity. Every fluid describes order by order in the derivative expansion a dual bulk solution. If we require the boundary fluid to be shearless and to satisfy integrability conditions, then this fluid is dual to an Einstein solution with line element written in closed form. It is exploring this path that we found that the shearless condition and the fact that we imposed integrability restrict the achievable class of solutions in the bulk to be the algebraically special ones, due to the Goldberg-Sachs theorem. This theorem states that a solution of Einstein equations is of Petrov class algebraically special if it admits a null geodesic and shearless congruence. The resummed derivative expansion is written based on a null geodesic congruence, which is found to be shearless due to the shearless condition on the boundary fluid velocity. It is moreover a solution of Einstein equations thanks to the integrability conditions. Therefore, all the Goldberg-Sachs theorem hypothesis are verified, hence the thesis.

After corroborating all these findings in the Robinson-Trautman example, we focused our attention on the two-dimensional situation. Here the heat current has a compatible Weyl weight to intervene in the bulk line element. Einstein solutions are labeled by their asymptotic charges, so we proceeded and computed them for different sub-classes. These are the dissipative static fluid case and the perfect fluid with arbitrary velocity. The former is in a generic fluid frame while the latter is by construction in what is known as the Landau-Lifshitz frame. We proved that the asymptotic charges have different algebras in the two cases, which ultimately shows that we cannot choose in holography the boundary fluid velocity at will, at least within our framework and in two dimensions. This concluded the discussion of the fluid/gravity dictionary in AdS. That is, the upper part of our square of dualities.

On top of being interesting per se, Section 2 prepared the ground for the flat limit. This comes about thanks to the important realization that the bulk cosmological constant is proportional to the boundary speed of light. Therefore, sending $\Lambda \to 0$ is equivalent to send $k \to 0$. This limit on the boundary theory is at first puzzling, and we devoted all Section 3 to it and its consequences. Inspired by the ultra-relativistic contraction of the Poincaré group made by Levy-Leblond [79], we called this limit a Carrollian limit. We firstly analyzed the boundary metric, singular in the limit. This is neither bothering nor surprising, because we saw that this is exactly the feature that occurs in passing from the boundary of an AdS spacetime to an asymptotically flat one. In the latter the boundary is a degenerate manifold $\mathscr{I}$, with the AdS time-like congruence replaced by the correspondent null congruence. This limit unraveled as well a privileged class of diffeomorphisms, which we called Carrollian diffeomorphisms. We determined the geometric structure associated with these diffeomorphisms and the fate of the Weyl connection, now Weyl-Carroll, to comply with metricity and Carrollian transformations. These break the $d+1$-dimensional manifold in a spatial d-dimensional base and a one-dimensional null direction. Carrollian covariance is an important result and a useful consistency check.

After exploring the geometry, we focused on the hydrodynamic equations of motion limit. This was addressed in arbitrary dimension and fluid. We saw how the conservation of the energy-momentum tensor splits in two scalar equations and two Carroll vector equations in the limit $k \to 0$. We checked Carroll covariance and then specialized to conformal fluids, where Weyl-Carroll covariance is at play, and allows to drastically simplify the form of the equations. We eventually reported the specific result of this limiting procedure in three and two dimensions.

We observed that there is no more a notion of energy-momentum tensor in a fully general Carrollian limit. We therefore introduced the Carrollian counterparts of T, which we called Carrollian momenta, and showed that Carrollian covariance implies their conservation. These conservation equations match nicely the limit of the divergence of T, as expected. We then defined Carrollian intrinsic conserved charges and found that they are equal to the usual surface charges in the case of linearized asymptotically flat four-dimensional gravity, further supporting the accuracy of our findings.

We then applied our method to the dual Galilean limit, where now k is sent to infinity. This is certainly different, but in a very precise way – Galilean vectors and scalars are dual to Carrollian ones. The first step was to choose the correct relativistic parametrization of the boundary metric to obtain Galilean diffeomorphisms, which is the Zermelo one. The next step then was to organize the theory with respect to Galilean diffeomorphisms, for which time is absolute. With all this at our disposal we found the most general continuity, energy and Euler equations. We eventually probed these equations in some examples.

With all the AdS fluid/gravity dictionary ready and the boundary $k \to 0$ limit mastered, we focused on the final part of the road, where we take the flat limit of the resummed line element, Section 4. Here the limit $k \to 0$ of the derivative expansion turned out to be neither trivial nor divergent. This result was not at all a priori guaranteed. It constituted an important traffic circle in our road.

We firstly considered the limit in the four-dimensional case. We wrote the line element only as a function of Carrollian data living on the null asymptote, and, for the same reason as its AdS precursor, we proved that this gives rise to all algebraically special asymptotically flat solutions. There is no notion of Cotton tensor on a degenerate null manifold, so we carefully extracted the $k \to 0$ limit of the integrability conditions, characterizing the Cotton descendants as functions of the Carrollian tensors introduced previously. This limit also enhanced the number of dissipative tensors, with now a couple of heat currents and stress tensors at play. All of this has then been tested in some examples. We presented the reconstruction of the Kerr-Taub-NUT family, where everything is time independent but there is boundary fluid vorticity. Then, we discussed the Ricci-flat Robinson-Trautman solution. Armed with the expertise acquired in the AdS case, we showed how this time-dependent solution can be obtained starting by Carrollian conservation equations, which eventually are encoded in the Robinson-Trautman bulk Einstein equation.

We then moved to the bulk three-dimensional case. Here again the flat limit of the derivative expansion is finite and, in contrast with the four-dimensional case, allows to gather in general all bulk solutions.[91] This has been checked computing the asymptotic charges and showing that they infer the general Virasoro algebra, with the central charge. As for its AdS ancestor, the case with perfect fluid is not arbitrary enough, which questions the role played by the heat current in holography. We concluded this section describing how the intrinsic Carrollian charges match with the surface charges also in this case.

This concluded our road toward flat holography. Many questions have been raised on the way and will be listed shortly. At this point we noticed how important Weyl covariance was in our construction. We therefore dedicated Section 5 to a through analysis of Weyl symmetry, in the context of AdS/CFT. In fact, Weyl symmetry was already implemented to certain extent in fluid/gravity but sidestepped at the microscopic level, in the FG formulation of holography. We confined our attention to AdS, for we worked with a slight generalization of the Fefferman-Graham gauge, that we named Weyl-Fefferman-Graham. The latter is indeed form invariant under Weyl diffeomorphism, which is the bulk transformation that induces a boundary Weyl rescaling of the metric and shift of the Weyl connection. The WFG gauge allowed to obtain a clear derivation and geometrical interpretation of the Weyl anomaly. Furthermore, the subleading terms of Einstein equations returned the boundary Ward identity that we elegantly showed in the boundary field theory, defined on a background given by both the metric and the Weyl connection.

Outlooks

This work raised many interesting questions. It touched upon only partially or even sometimes marginally to some of them, which represent appealing directions of investigation. While part of these questions are surely addressable in the near future, other are far from being understood. We organize the arguments starting from what we consider the closest to be achieved all the way to the more conceptual long-term questions.

The first question concerns the expansion of all this work to higher dimensions. Considering what happens in moving from three-dimensional bulks to four-dimensional ones, we suspect the presence of some conditions relating geometry and hydrodynamics even in higher dimensions. The role of the Cotton tensor is special to three-dimensional boundaries. As a result, it will have to be replaced with other conformal tensors, like the Weyl or Bach tensors, now non vanishing. Some steps in this direction have been made in [199], which we had the opportunity to closely follow. Useful results could be the characterization of conformal tensors and the formalism developed in [208].

Extensions of this work could be performed by including a charged fluid in the hydrodynamic boundary theory. The latter would then be dual to Einstein-Maxwell solutions in the bulk, [209]. In this scenario integrability conditions would relate then the geometry with the energy-momentum tensor and the electro-magnetic current in the boundary – the dynamics being encoded in the divergence of the energy-momentum tensor equal to the current. We addressed this problem in [210], but many questions are still open, in particular in relation with the next point: the hydrodynamic frame invariance.

The charged case could indeed shed light on the role played by the fluid congruence, both intrinsically and from holography. Concerning the former, it could be that only in the presence of an extra current the question of changing fluid frame is fully treatable. Regarding the latter, we know already that we cannot neglect the heat current, key ingredient in our construction. In the charged situation, however, we will have an extra dissipative current which can intervene together with the heat current and perhaps these two will turn out to be interchangeable, as the hydrodynamic frame invariance would suggest.

Relating to the previous point, another direction could be to study what happens to the derivative expansion when we change boundary hydrodynamic frame, even in the charge-less case. We know the line element is sensitive to this modification, but perhaps by analyzing the explicit form of it we can obtain an improvement of the line element such that it is form invariant under boundary frame redefinition. In fact, our line element in three dimensions can be enhanced at will as long as it leads to reasonable boundary conditions. This would also make contact with the order

[91] At least all known bulk solutions, compatible with particular choices of boundary conditions.

by order attempts made in [24, 25], where the fluid was forced to be in the Landau-Lifshitz gauge. The Robinson-Trautman example is also a situation where moving to this frame has been implemented [56, 57], so having an covariant framework could shed more light on this class of solutions also.

As we mentioned in the main text, integrability conditions infer the right boundary structure to reach particular classes of bulk solution. Indeed, every fluid can be settle to be dual to a solution of Einstein equations order by order. Nonetheless, only under integrability and shearless conditions we reach a bulk solution written in closed form. The reason why integrability conditions are needed, or rather what are their consequences, is still under investigation. We know they arise in the boundary expansion of the bulk Weyl tensor [30, 50, 51] as a sort of gravitational electro-magnetic duality. We should persist in this direction introducing for instance the analogue of the energy-momentum charges for the Cotton tensor instead. These would be magnetic charges. Therefore, integrability conditions could be thought as a relationship between the electric and magnetic spectra of the boundary theory. This has been investigated in the BMS construction [147, 211–213] and is related to the whole soft physics program [202], which is the study of infrared physics. Our fluid perspective could potentially lead to interesting results in this direction.

In three-dimensional bulk our results on the different boundary configurations and the correspondent asymptotic charges is intriguing. The most general fluid configuration has not yet been analyzed, and we suspect its bulk line element could be the explicit realization of the boundary condition of [207, 214, 215]. This result would indicate the power of our approach. Furthermore, it would allow to understand where the different contributions come from and how they can be tuned. Eventually a through analysis of the solution phase space is required.

Another appealing argument to explore is the boundary microscopic structure of the field theory introduced in Section 5. The Ward identity discussed there indicates that the energy momentum is not traceless. It would be relevant to study this equation with contact terms [216], and to probe the theory described there in first order formalism [217].

Eventually we arrive to the most long-term question: the microscopic realization of flat holography. Although parts of the road are yet to be paved, some parts of it have already been done, [120–122]. We believe the subject is moving toward the right direction but it is still unraveling. Indeed we know that the boundary theory possesses BMS symmetry [116, 118],[92] where the null-like direction plays a privileged role. Attempts have been done in constructing BMS field theories [110], putative dual of asymptotically flat solutions. The situation on the topic is still confusing in many aspects. Firstly, there is no definition of a boundary stress tensor because, as discussed, this is replaced by the Carrollian momenta. However we yet do not know how to extract the latter given an explicit bulk solution in full generality. Secondly, the bulk action diverges as in AdS, but here we are currently missing a full renormalization scheme. Lastly, there is no high energy construction underlining the duality so far. That is, we do not have any string theoretical realization to rely on. One could argue that there could be another quantum gravity theory that gives in the IR limit asymptotically flat solutions of general relativity such that they are dual to a BMS boundary field theory. Whatever the answer will turn out to be, we need some guidelines in this dark road, which could come from a bottom-up approach based on constructing a field theory based on the boundary symmetries, or from a top-down one, with a limit of some high energy theory which unravels a way to relate the bulk partition function with the boundary one – whatever theory the latter will then describe.

In conclusion, this work sets the stage to raise multiple questions in different domains and in the links among them. While parts of the open questions are already well-posed, others are more long-term questions addressing the core of holography.

[92] To obtain the asymptotic symmetries part of the diffeomorphisms are usually locked and ansatz are made on the asymptotic behavior of the metric fields. It is possible that new different conditions lead to different asymptotic symmetries. The state-of-art on the topic is BMS, but new results have been developed recently [218].

A Weyl Connections and Weyl Manifolds

We recall here the definition of a Weyl connection and its geometrical curvature tensors, further informations can be found in [174–176].

Given a manifold M with metric g and connection ∇ (on the tangent bundle TM), we define the metricity ∇g and torsion T via

$$\begin{aligned} \nabla_{\underline{X}} g(\underline{Y},\underline{Z}) &= \nabla_{\underline{X}}(g(\underline{Y},\underline{Z})) - g(\nabla_{\underline{X}}\underline{Y},\underline{Z}) - g(\underline{Y},\nabla_{\underline{X}}\underline{Z}), \quad (772)\\ T(\underline{X},\underline{Y}) &= \nabla_{\underline{X}}\underline{Y} - \nabla_{\underline{Y}}\underline{X} - [\underline{X},\underline{Y}]\,. \quad (773)\end{aligned}$$

Here for brevity we adapt an index-free notation where $\underline{X}, \ldots$ are arbitrary vector fields and $[\underline{X},\underline{Y}]$ denotes the Lie bracket.

Suppose we have a basis $\{\underline{e}_a\}$ of vector fields, and define the connection coefficients via

$$\nabla_{\underline{e}_a}\underline{e}_b = \Gamma^c_{ab}\underline{e}_c. \quad (774)$$

It is a familiar theorem that requiring both the metricity and torsion of the connection to vanish leads to a uniquely determined set of connection coefficients, those of the Levi-Civita connection. Indeed, further defining the rotation coefficients

$$[\underline{e}_a,\underline{e}_b] = C_{ab}{}^c\underline{e}_c, \quad (775)$$

we find the general result

$$\mathring{\Gamma}^d_{ac} = \tfrac{1}{2}g^{db}\Big(\underline{e}_a(g_{bc}) + \underline{e}_c(g_{ab}) - \underline{e}_b(g_{ca})\Big) - \tfrac{1}{2}g^{db}\Big(C_{ab}{}^f g_{fc} + C_{ca}{}^f g_{fb} - C_{bc}{}^f g_{fa}\Big), \quad (776)$$

where $g_{ab} \equiv g(\underline{e}_a,\underline{e}_b)$ and we use the circle notation to refer to the Levi-Civita quantities.[93] This reduces with the choice of coordinate basis $\underline{e}_a = \partial_a$ to the familiar Christoffel symbols.

The vanishing of metricity and torsion are certainly invariant under diffeomorphisms. Therefore, all the geometrical objects built using the Levi-Civita connection transform nicely under diffeomorphisms. We note though that metricity is not invariant under Weyl transformations[94] $g \to g/\mathcal{B}^2$, instead transforming as

$$\nabla g \to (\nabla g - 2d\ln\mathcal{B}\otimes g)/\mathcal{B}^2. \quad (779)$$

Consequently, if we wish to consider geometric theories in which Weyl transformations play a role, it is inconvenient to choose the usual Levi-Civita connection. Instead, one attains a connection that is covariant with respect to both Weyl transformations and diffeomorphisms by introducing a Weyl connection A which transforms non-linearly under a Weyl transformation

$$g \to g/\mathcal{B}^2, \qquad A \to A - d\ln\mathcal{B}. \quad (780)$$

By design then, the Weyl metricity is covariant[95]

$$(\nabla g - 2A\otimes g) \to (\nabla g - 2A\otimes g)/\mathcal{B}^2, \quad (781)$$

and it makes sense to set it to zero if one wishes. Fortunately, there is a theorem which states that there is a unique connection (also generally referred to as a Weyl connection) that has zero torsion and Weyl metricity. In this case, the connection coefficients are given by the formula

$$\begin{aligned}\Gamma^d_{ac} = {}& \tfrac{1}{2}g^{db}\Big(\underline{e}_a(g_{bc}) + \underline{e}_c(g_{ab}) - \underline{e}_b(g_{ca})\Big) - \tfrac{1}{2}g^{db}\Big(C_{ab}{}^f g_{fc} + C_{ca}{}^f g_{fb} - C_{bc}{}^f g_{fa}\Big)\\ & -\Big(A_a\delta^d{}_c + A_c\delta^d{}_a - g^{db}A_b g_{ca}\Big). \quad (782)\end{aligned}$$

[93] This is again a notation we exploit only where the Weyl connection is relevant, to avoid heavy notation.

[94] The Weyl transformation should not be confused with a conformal transformation, which is a diffeomorphism. They do look similar in their actions on the components of the metric,

$$\begin{aligned} Weyl: &\quad g_{ab}(x) \mapsto g_{ab}(x)/\mathcal{B}(x)^2, \quad (777)\\ conformal: &\quad g_{ab}(x) \mapsto g'_{ab}(x') = g_{ab}(x)/\omega(x)^2. \quad (778)\end{aligned}$$

Here though, $\mathcal{B}(x)$ is an arbitrary function, while $\omega(x)$ is a specific function, associated with a special diffeomorphism that is a conformal isometry.

[95] To be more specific, what we mean by this notation is

$$(\nabla g - 2A\otimes g)(\underline{X},\underline{Y},\underline{Z}) = \nabla_{\underline{X}}g(\underline{Y},\underline{Z}) - 2A(\underline{X})g(\underline{Y},\underline{Z})$$

The notation $A(\underline{X})$ used here and throughout the paper refers to the contraction of a 1-form with a vector, $A(\underline{X}) \equiv i_{\underline{X}}A \equiv A_a X^a$.

We note that these connection coefficients are in fact invariant under Weyl transformations. Consequently, the curvature of the Weyl connection has components[96]

$$R^a{}_{bcd} = \underline{e}_c(\Gamma^a_{db}) - \underline{e}_d(\Gamma^a_{cb}) + \Gamma^f_{db}\Gamma^a_{cf} - \Gamma^f_{cb}\Gamma^a_{df} - C_{cd}{}^f\Gamma^a_{fb} \tag{784}$$

that are themselves Weyl invariant. This Weyl-Riemann tensor possesses less symmetries than its Levi-Civita counterpart, and indeed the degrees of freedom contained within are in one-to-one correspondence with the Levi-Civita Riemann tensor, plus a 2-form F, which is the field strength $F = dA$. To see this, we can write the Weyl curvature components in terms of the Levi-Civita curvature components,

$$\begin{aligned} R^a{}_{bcd} &= \mathring{R}^a{}_{bcd} + \mathring{\nabla}_d A_b \delta^a{}_c - \mathring{\nabla}_c A_b \delta^a{}_d + (\mathring{\nabla}_d A_c - \mathring{\nabla}_c A_d)\delta^a{}_b + \mathring{\nabla}_c A^a g_{bd} - \mathring{\nabla}_d A^a g_{bc} && (785) \\ &\quad + A_b(A_d\delta^a{}_c - A_c\delta^a{}_d) + A^a(g_{bd}A_c - g_{bc}A_d) + A^2(g_{bc}\delta^a{}_d - g_{bd}\delta^a{}_c). && (786) \end{aligned}$$

The corresponding Weyl-Ricci tensor, which we define as $Ric_{ab} = R^c{}_{acb}$, is given by

$$Ric_{ab} = \mathring{R}ic_{ab} - \tfrac{d+1}{2}F_{ab} + (d-1)\Big(\mathring{\nabla}_{(a}A_{b)} + A_aA_b\Big) + \Big(\mathring{\nabla}\cdot A - (d-1)A^2\Big)g_{ab} \tag{787}$$

in space-time dimension $d+1$. We then read off that the Weyl-Ricci tensor has an antisymmetric part

$$Ric_{[ab]} = -\tfrac{d+1}{2}F_{ab}, \tag{788}$$

while the symmetric part differs from the Levi-Civita Ricci tensor,

$$Ric_{(ab)} = \mathring{R}ic_{ab} + (d-1)\Big(\mathring{\nabla}_{(a}A_{b)} + A_aA_b\Big) + \Big(\mathring{\nabla}\cdot A - (d-1)A^2\Big)g_{ab}. \tag{789}$$

The corresponding Weyl-Ricci scalar is the trace,

$$R = \mathring{R} + 2d\mathring{\nabla}\cdot A - d(d-1)A^2. \tag{790}$$

Under a Weyl transformation, $R \to R\mathcal{B}^2$, so we see that the Levi-Civita Ricci scalar must transform very non-trivially under Weyl,

$$\mathring{R} \to \mathcal{B}^2\Big(\mathring{R} + 2d\mathring{\nabla}^2\ln\mathcal{B} - 2d(d-1)A\cdot d\ln\mathcal{B} + d(d-1)(d\ln\mathcal{B})^2\Big) \tag{791}$$

in order to cancel the transformation of the non-Weyl-invariant expression involving the Weyl connection A. We thus see the important role played by the Weyl connection. Organize the theory with respect to the latter is a more natural prescription, whenever this theory includes Weyl transformations.

Given a Weyl connection, we can organize tensors in such a way that they have a specific Weyl weight and we use the notation

$$\mathcal{D}_{\underline{X}}t = \nabla_{\underline{X}}t + w_t A(\underline{X})\, t. \tag{792}$$

whereby

$$t \to \mathcal{B}^{w_t}t, \qquad \mathcal{D}t \to \mathcal{B}^{w_t}\mathcal{D}t. \tag{793}$$

For the specific case of a scalar field ϕ, we would then write $\mathcal{D}_a\phi = \underline{e}_a(\phi) + w_\phi A_a\phi$. The condition that Weyl metricity vanishes is translated in this notation as $\mathcal{D}g = 0$.

Finally we remark that the Bianchi identity for the Weyl-Riemann tensor is

$$\nabla_a R^e{}_{bcd} + \nabla_c R^e{}_{bda} + \nabla_d R^e{}_{bac} = 0 \tag{794}$$

Contracting the e, c indices, we get the once-contracted Bianchi identity

$$\nabla_a Ric_{bd} - \nabla_d Ric_{ba} + \nabla_c R^c{}_{bda} = 0. \tag{795}$$

which given that the Weyl-Riemann and Weyl-Ricci tensors are Weyl invariant, can also be written as

$$\mathcal{D}_a Ric_{bd} - \mathcal{D}_d Ric_{ba} + \mathcal{D}_c R^c{}_{bda} = 0. \tag{796}$$

[96] Here we are using the convention

$$R^a{}_{bcd}\underline{e}_a \equiv R(\underline{e}_b, \underline{e}_c, \underline{e}_d) \equiv \nabla_{\underline{e}_c}\nabla_{\underline{e}_d}\underline{e}_b - \nabla_{\underline{e}_d}\nabla_{\underline{e}_c}\underline{e}_b - \nabla_{[\underline{e}_c,\underline{e}_d]}\underline{e}_b \tag{783}$$

If we multiply by g^{ab}, we find

$$g^{ab}\mathcal{D}_a Ric_{bd} - \mathcal{D}_d R + \mathcal{D}_c(g^{ab}R^c{}_{bda}) = 0. \tag{797}$$

This can be simplified further by noting that

$$g^{ab}R^c{}_{bda} = g^{cb}\Big(Ric_{bd} + 2F_{bd}\Big) \tag{798}$$

and hence the twice contracted Bianchi identity can be simplified to

$$g^{ab}\mathcal{D}_a(G_{bc} + F_{bc}) = 0 \tag{799}$$

where $G_{ab} = Ric_{ab} - \frac{1}{2}Rg_{ab}$ is the Weyl-Einstein tensor. Since G and F have Weyl weight zero, this can also be written as

$$g^{ab}\nabla_a(G_{bc} + F_{bc}) = 0 \tag{800}$$

This is the analogue of the familiar result in Riemannian geometry, $\mathring{\nabla}^a\mathring{G}_{ac} = 0$.

B Petrov Classification and Goldberg-Sachs Theorem

We hereby recall the Petrov classification of the Weyl tensor (we work here in four dimension), which is relevant to the Goldberg-Sachs theorem. The latter will be enunciated and proved to hold for our four-dimensional bulk resummed metric (158) and congruence ∂_r. This Appendix uses results outlined in [37,44,48,210].

The petrov classification allows to study the eigenvalues of the Weyl tensor, here spelled C_{abcd}.[97] To do so, we introduce the complex null tetrad $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}$ and write the metric as

$$ds^2 = -2\boldsymbol{kl} + 2\boldsymbol{m}\overline{\boldsymbol{m}}. \tag{801}$$

The eigenvalue problem for the Weyl tensor boils down to solve

$$\frac{1}{2}C_{abcd}X^{cd} = \lambda X_{ab}, \tag{802}$$

with X_{ab} an eigen-bi-vector, i.e. a skew symmetric tensor. It is possible to prove that classify the eigenvalues of this equation is equivalent to the characterization of the Weyl tensor in terms of its principal null directions, in particular one obtains

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0 \;\Leftrightarrow\; \boldsymbol{\Psi}_0 = C_{abcd}k^a m^b k^c m^d = 0. \tag{803}$$

After all the symmetries being used, we need only ten real independent components of the Weyl tensor, which can be stored in five complex functions obtained with contractions of C with the various basis forms

$$\boldsymbol{\Psi}_0 = C_{abcd}k^a m^b k^c m^d \tag{804}$$
$$\boldsymbol{\Psi}_1 = C_{abcd}k^a l^b k^c m^d \tag{805}$$
$$\boldsymbol{\Psi}_2 = C_{abcd}k^a m^b \overline{m}^c l^d \tag{806}$$
$$\boldsymbol{\Psi}_3 = C_{abcd}k^a l^b \overline{m}^c l^d \tag{807}$$
$$\boldsymbol{\Psi}_4 = C_{abcd}\overline{m}^a l^b \overline{m}^c l^d. \tag{808}$$

Then equation (803), after applying the most general null rotation controlled by the complex parameter E, becomes

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0' - 4E\boldsymbol{\Psi}_1' + 6E^2\boldsymbol{\Psi}_2' - 4E^3\boldsymbol{\Psi}_3' + E^4\boldsymbol{\Psi}_4' = 0. \tag{809}$$

The multiplicity of the solution of this equation will then be also the multiplicity of the principal null directions, which determine the Petrov class of the spacetime under consideration. The possibilities are

[97] Throughout this Appendix we use indices a, b, c, d as bulk four-dimensional ones, splittable in $a = (r, \mu)$ with μ boundary three-dimensional indices.

Petrov type	Roots E	Multiplicity
I	$\frac{\sqrt{\lambda_2+2\lambda_1}\pm\sqrt{\lambda_1+2\lambda_2}}{\sqrt{\lambda_1-\lambda_2}}$	$(1,1,1,1)$
D	$0,\infty$	$(2,2)$
II	$0,\pm i\sqrt{\frac{3\lambda}{2}}$	$(2,1,1)$
III	$0,\infty$	$(3,1)$
N	0	(4)

A Weyl tensor is said algebraically special (and consequently the spacetime will be said of algebraically special Petrov class) if it admits at least one multiple principal null direction. That is, if it is of Petrov class D, II, III and N.[98]

One can moreover obtain the multiplicity of the principal null directions by directly inspecting the vanishing $\boldsymbol{\Psi}$

$$\boldsymbol{\Psi}_0 = 0,\ \boldsymbol{\Psi}_1 \neq 0, \dots \ \Leftrightarrow \text{Multiplicity } 1 \ \Leftrightarrow \text{Petrov } I \tag{810}$$

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = 0,\ \boldsymbol{\Psi}_2 \neq 0, \dots \ \Leftrightarrow \text{Multiplicity } 2 \ \Leftrightarrow \text{Petrov } D, II \tag{811}$$

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 = 0,\ \boldsymbol{\Psi}_3 \neq 0, \dots \ \Leftrightarrow \text{Multiplicity } 3 \ \Leftrightarrow \text{Petrov } III \tag{812}$$

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 = \boldsymbol{\Psi}_3 = 0,\ \boldsymbol{\Psi}_4 \neq 0, \dots \ \Leftrightarrow \text{Multiplicity } 4 \ \Leftrightarrow \text{Petrov } N. \tag{813}$$

For the reader non familiar with this classification, we would like to remark that all famous black hole solutions fall into Petrov type D. It is not an easy task to find and characterize a non algebraically special solutions.

Without further ado, let us now report the Goldberg-Sachs theorem (1961):

Goldberg-Sachs theorem: *a vacuum spacetime is algebraically special if and only if it admits a shear-free congruence of null geodesics.*

That is, if there exists a shearless, null and geodesic vector field, then the spacetime has Petrov class D, II, III and N (and of course O).

We now prove that the resummed line element (158) has a vector field $\underline{u} = \partial_r$ which is indeed null, geodesic and shearfree. That $\underline{u} = \partial_r$ is null is evident due to the fact that (158) has $g_{rr} = 0$. At the same time, using again $g_{rr} = 0$, it is straightforward to show that it is also geodesic, for its (bulk) acceleration vanishes ($u^a = \delta^a_r$)

$$a^a = u^b \nabla_b u^a = u^b \Gamma^a_{bc} u^c = \Gamma^a_{rr} = g^{ab} \partial_r g_{rb} = \frac{1}{k^2} g^{ab} \partial_r u_b = 0. \tag{814}$$

In the second-last passage we used the result $g_{ra} = \frac{u_a}{k^2}$, the metric dual of $\underline{u} = \partial_r$. Finally, in the last passage we use that the explicit form of this tensor[99]

$$u = -k^2(\Omega dt - b_i dx^i), \tag{815}$$

is r-independent. Therefore we have a congruence $\underline{u}$ which is null and geodesic. It remains to show that it is shearless.

To demonstrate it, we bring (158) in the form (801) via the identifications

$$\boldsymbol{k} = \frac{1}{k^2} u \tag{816}$$

$$\boldsymbol{l} = -dr - \frac{r}{k^2} a - \frac{H}{k^2} u + \mathcal{D}_\nu \omega^\nu{}_\mu dx^\mu \tag{817}$$

$$\boldsymbol{m} = \frac{\rho}{P} d\zeta \tag{818}$$

$$\overline{\boldsymbol{m}} = \frac{\rho}{P} d\bar{\zeta} \tag{819}$$

where we used that by assumption the boundary spatial part of the metric can be written as (160) due to the boundary shearlessness assumption. In this expressions we recognize all the various objects introduced in the main text: the boundary vorticity ω, the boundary acceleration a, the fucntion ρ defined in (157), and we introduce the function H in $\boldsymbol{l}$ given by

$$2H = -\theta r + k^2 r^2 - \frac{\mathcal{R}}{2} + \frac{1}{\rho^2}(8\pi G_N \varepsilon r + c\gamma). \tag{820}$$

[98] Although some authors consider it has a Petrov type (called O), we exclude from our analysis this trivial case where the Weyl tensor itself vanishes identically.

[99] The boundary metric dual of this form differs from the bulk one. In the former it is a time-like congruence while in the latter it is null.

By doing this identification, we elevate the metric dual of the congruence $\underline{u}$ to be one of the basis tetrad (indeed from now on we call $u_a = k_a$). Notice that the basis forms satisfy by construction

$$\boldsymbol{k}^2 = 0, \qquad \boldsymbol{l}^2 = 0, \qquad \boldsymbol{l} \cdot \boldsymbol{k} = -1. \tag{821}$$

To define the shear of our congruence, we introduce the rank-2 projector

$$\Delta_{ab} = g_{ab} + k_a l_b + k_b l_a \tag{822}$$

which is the projector orthogonal to the 2-dimensional vector space spanned by $\boldsymbol{k}$ and $\boldsymbol{l}$. We then define the projected covariant derivative of $\boldsymbol{k}$ (here of course ∇ is the bulk Levi-Civita covariant derivative):

$$B_{ab} = \Delta_a{}^c \Delta_b{}^d \nabla_c k_d \tag{823}$$

The acceleration being zero, we decompose the latter in its symmetric trace-free part (its shear), its skew-symmetric part (its vorticity), and its trace (its expansion)

$$B_{ab} = \sigma_{ab} + \omega_{ab} + \frac{\Theta}{2}\Delta_{ab}, \tag{824}$$

whereby

$$\sigma_{ab} = \frac{1}{2}(B_{ab} + B_{ba} - \Theta\Delta_{ab}), \qquad \omega_{ab} = \frac{1}{2}(B_{ab} - B_{ba}), \qquad \Theta = \Delta^{ab} B_{ab}. \tag{825}$$

To explicitly evaluate σ, we first notice, given the already-spelled relationships, that

$$\nabla_a k_b + \nabla_b k_a = \partial_r g_{ab}, \tag{826}$$

which allows to write, using the properties of $\boldsymbol{k}$ and $\boldsymbol{l}$

$$B_{ab} + B_{ba} = \Delta_a{}^c \Delta_b{}^d \partial_r \Delta_{cd}. \tag{827}$$

Computing now the expansion

$$\Theta = \Delta^{ab} \nabla_a k_b = \nabla_a k^a = \Gamma^a_{ar} = \partial_r \ln\sqrt{g}, \tag{828}$$

we eventually obtain

$$\sigma_{ab} = \frac{1}{2}\Delta_a{}^c \Delta_b{}^d \left(\partial_r \Delta_{cd} - \Delta_{cd}\partial_r \ln\sqrt{g}\right). \tag{829}$$

We then explicitly compute Δ and the determinant of the bulk metric. Here the assumption of boundary shearless is crucial. We obtain

$$\Delta_{ab} dx^a dx^b = \frac{2\rho^2}{P^2} d\zeta d\bar{\zeta}, \qquad \sqrt{g} = \frac{\Omega\rho^2}{P^2}. \tag{830}$$

Plugging these results in (829) we get $\sigma_{ab} = 0$ as a fine cancellation between the two contributions.

This remarkable result, pinned on the boundary shearless requirement, shows that the bulk line element (158) admits a null, geodesic and shearless congruence. Consequently, Goldberg-Sachs theorem applies. That is, the resummed bulk is algebraically special (it cannot be of Petrov type I), as we claimed in the main text.

Notice eventually that the flat limit of the bulk metric enjoys the same properties. Indeed, in the same way as for the AdS case we rewrite the metric (585) in terms of a null tetrad $(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}})$:

$$ds^2_{\text{res. flat}} = -2\boldsymbol{k}\boldsymbol{l} + 2\boldsymbol{m}\overline{\boldsymbol{m}}\,, \quad \boldsymbol{k}\cdot\boldsymbol{l} = -1\,, \quad \boldsymbol{m}\cdot\overline{\boldsymbol{m}} = 1\,, \tag{831}$$

where $\boldsymbol{k} = -(\Omega dt - b)$ is the dual of ∂_r and

$$\boldsymbol{l} = -dr - r\alpha - \frac{r\theta\Omega}{2}dt + \frac{\psi}{6\star\varpi} + \frac{\Omega dt - b}{2\rho^2}\left(8\pi G_N \varepsilon r + c\star\varpi - \rho^2\hat{\mathcal{K}}\right), \tag{832}$$

(here $\psi = \psi_i dx^i$), along with

$$2\boldsymbol{m}\overline{\boldsymbol{m}} = \rho^2 d\ell^2\,. \tag{833}$$

Using the above results we find that ∂_r is shear-free due to (600). Thus, also (585) is algebraically special.

References

[1] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[2] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," *Phys. Rev.* **D9** (1974) 3292–3300.

[3] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220. [,167(1975)].

[4] G. 't Hooft, "Dimensional reduction in quantum gravity," *Conf. Proc.* **C930308** (1993) 284–296, arXiv:gr-qc/9310026 [gr-qc].

[5] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, arXiv:hep-th/9409089 [hep-th].

[6] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *AIP Conf. Proc.* **484** (1999) no. 1, 51.

[7] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[8] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[9] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, arXiv:hep-th/9905111 [hep-th].

[10] R. Emparan, C. V. Johnson, and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," *Phys. Rev.* **D60** (1999) 104001, arXiv:hep-th/9903238 [hep-th].

[11] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, arXiv:hep-th/9902121 [hep-th].

[12] S. de Haro, K. Skenderis, and S. N. Solodukhin, "Gravity in warped compactifications and the holographic stress tensor," *Class. Quant. Grav.* **18** (2001) 3171–3180, arXiv:hep-th/0011230 [hep-th]. [,495(2000)].

[13] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595–622, arXiv:hep-th/0002230 [hep-th].

[14] K. Skenderis, "Asymptotically Anti-de Sitter space-times and their stress energy tensor," *Int. J. Mod. Phys.* **A16** (2001) 740–749, arXiv:hep-th/0010138 [hep-th]. [,394(2000)].

[15] J. Polchinski, "Introduction to Gauge/Gravity Duality," in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010*, pp. 3–46. 2010. arXiv:1010.6134 [hep-th].

[16] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158. 2002. arXiv:hep-th/0201253 [hep-th].

[17] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[18] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[19] G. Policastro, D. T. Son, and A. O. Starinets, "From AdS / CFT correspondence to hydrodynamics," *JHEP* **09** (2002) 043, arXiv:hep-th/0205052 [hep-th].

[20] G. Policastro, D. T. Son, and A. O. Starinets, "From AdS / CFT correspondence to hydrodynamics. 2. Sound waves," *JHEP* **12** (2002) 054, arXiv:hep-th/0210220 [hep-th].

[21] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in ads/cft and the membrane paradigm,"*Phys.Rev.D* **79:025023,2009** (Sept., 2008) , 0809.3808.

[22] M. Haack and A. Yarom, "Nonlinear viscous hydrodynamics in various dimensions using ads/cft,"*JHEP* **0810:063,2008** (June, 2008) , 0806.4602.

[23] R. Loganayagam, "Entropy current in conformal hydrodynamics,"*JHEP* **0805:087,2008.** (Jan., 2008) , 0801.3701.

[24] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, "Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions,"*JHEP* **0812:116,2008** (Sept., 2008) , 0809.4272.

[25] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," *JHEP* **02** (2008) 045, arXiv:0712.2456 [hep-th].

[26] M. Rangamani, "Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence," *Class. Quant. Grav.* **26** (2009) 224003, arXiv:0905.4352 [hep-th].

[27] V. E. Hubeny, S. Minwalla, and M. Rangamani, "The fluid/gravity correspondence," 1107.5780.

[28] S. Diles, "The role of weyl symmetry in hydrodynamics," 1711.00099.

[29] C. R. LeBrun, "H-Space with a Cosmological Constant," *Proc. Roy. Soc. Lond.* **A380** (1982) no. 1778, 171–185.

[30] I. Bakas, "Energy-momentum/cotton tensor duality for ads4 black holes,"*JHEP* **0901:003,2009** (Sept., 2008) , 0809.4852.

[31] S. de Haro, "Dual gravitons in ads4/cft3 and the holographic cotton tensor,"*JHEP* **0901:042,2009** (Aug., 2008) , 0808.2054.

[32] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, and K. Siampos, "Vorticity in holographic fluids," 1206.4351.

[33] A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, and K. Siampos, "Holographic perfect fluidity, cotton energy-momentum duality and transport properties," 1309.2310.

[34] P. M. Petropoulos, "Gravitational duality, topologically massive gravity and holographic fluids," 1406.2328.

[35] A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Geroch group for einstein spaces and holographic integrability," 1512.04970.

[36] J. Gath, A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Petrov classification and holographic reconstruction of spacetime," 1506.04813.

[37] P. M. Petropoulos and K. Siampos, "Integrability, einstein spaces and holographic fluids," 1510.06456.

[38] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, "Relativistic viscous hydrodynamics, conformal invariance, and holography," *JHEP* **04** (2008) 100, arXiv:0712.2451 [hep-th].

[39] P. Romatschke, "New developments in relativistic viscous hydrodynamics,"*Int.J.Mod.Phys.E* **19:1-53,2010** (Feb., 2009) , 0902.3663.

[40] P. Kovtun, "Lectures on hydrodynamic fluctuations in relativistic theories,"*J. Phys. A: Math. Theor.* **45** (May, 2012) 473001, 1205.5040.

[41] J. Hernandez and P. Kovtun, "Relativistic magnetohydrodynamics,"*JHEP* **1705** (Mar., 2017) 001, 1703.08757.

[42] L. Landau and E. Lifshitz, *Fluid Mechanics*. No. vol. 6. Elsevier Science, 2013. https://books.google.fr/books?id=CeBbAwAAQBAJ.

[43] L. Rezzolla and O. Zanotti, "Relativistic hydrodynamics," *Oxford university press* (2013) . https://www.oxfordscholarship.com/view/10.1093/acprof:oso/9780198528906.001.0001/acprof-9780198528906.

[44] J. N. Goldberg and R. K. Sachs, "Republication of: A theorem on Petrov types," *GENERAL RELATIVITY AND GRAVITATION -LONDON THEN DORDRECHT* (2009) . https://link.springer.com/article/10.1007/s10714-008-0722-5.

[45] B. Coll, J. Llosa, and D. Soler, "Three-dimensional metrics as deformations of a constant curvature metric," *Gen.Rel.Grav.* **34:269-282,2002** (Gen.Rel.Grav.34:269-282,2002) , gr-qc/0104070.

[46] T. M. Adamo, C. N. Kozameh, and E. T. Newman, "Null Geodesic Congruences, Asymptotically Flat Space-Times and Their Physical Interpretation," *Living Rev. Rel.* **12** (2009) 6, arXiv:0906.2155 [gr-qc]. [Living Rev. Rel.15,1(2012)].

[47] T. M. Adamo and E. T. Newman, "Asymptotically Stationary and Static Space-times and Shear-free Null Geodesic Congruences," *Class. Quant. Grav.* **26** (2009) 155003, arXiv:0906.2409 [gr-qc].

[48] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009. https://books.google.fr/books?id=SiWXP8FjTFEC.

[49] J. Griffiths and J. Podolskỳ, *Exact Space-Times in Einstein's General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009. https://books.google.fr/books?id=SbZpFMOwhzMC.

[50] D. S. Mansi, A. C. Petkou, and G. Tagliabue, "Gravity in the 3+1-split formalism i: Holography as an initial value problem,"*Class.Quant.Grav.* **26:045008,2009** (Aug., 2008) , 0808.1212.

[51] D. S. Mansi, A. C. Petkou, and G. Tagliabue, "Gravity in the 3+1-split formalism ii: Self-duality and the emergence of the gravitational chern-simons in the boundary,"*Class.Quant.Grav.* **26:045009,2009** (Aug., 2008) , 0808.1213.

[52] I. Robinson and A. Trautman, "Spherical Gravitational Waves," *Phys. Rev. Lett.* **4** (1960) 431–432.

[53] I. Robinson and A. Trautman, "Some spherical gravitational waves in general relativity," *Proc. Roy. Soc. Lond.* **A265** (1962) 463–473.

[54] J. Bicak and J. Podolsky, "Global structure of Robinson-Trautman radiative space-times with a cosmological constant," *Phys. Rev.* **D55** (1997) 1985–1993, arXiv:gr-qc/9901018 [gr-qc].

[55] J. Podolsky and R. Svarc, "Algebraic classification of robinson-trautman spacetimes,"*Phys. Rev. D* **94,** (Aug., 2016) 064043, 1608.07118.

[56] G. B. de Freitas and H. S. Reall, "Algebraically special solutions in ads/cft," 1403.3537.

[57] I. Bakas and K. Skenderis, "Non-equilibrium dynamics and ads_4 robinson-trautman," 1404.4824.

[58] I. Bakas, K. Skenderis, and B. Withers, "Self-similar equilibration of strongly interacting systems from holography,"*Phys. Rev. D* **93,** (Dec., 2015) 101902, 1512.09151.

[59] K. Skenderis and B. Withers, "Robinson-trautman spacetimes and gauge/gravity duality," 1703.10865.

[60] L. Ciambelli, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "The Robinson-Trautman spacetime and its holographic fluid," *PoS* **CORFU2016** (2017) 076, arXiv:1707.02995 [hep-th].

[61] G. Compère and A. Fiorucci, "Advanced Lectures on General Relativity," *Lect. Notes Phys.* **952** (2019) pp., arXiv:1801.07064 [hep-th].

[62] M. Banados, "Global charges in Chern-Simons field theory and the (2+1) black hole," *Phys. Rev.* **D52** (1996) 5816–5825, arXiv:hep-th/9405171 [hep-th].

[63] G. Barnich and F. Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," *Nucl. Phys.* **B633** (2002) 3–82, arXiv:hep-th/0111246 [hep-th].

[64] G. Barnich and G. Compere, "Surface charge algebra in gauge theories and thermodynamic integrability,"*J.Math.Phys.* **49:042901,2008** (Aug., 2007) , 0708.2378.

[65] S. Fischetti, W. Kelly, and D. Marolf, "Conserved charges in asymptotically (locally) ads spacetimes," 1211.6347.

[66] G. Compère, "Surfacecharges,". http://www.ulb.ac.be/sciences/ptm/pmif/gcompere/package.html.

[67] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl. Phys.* **B311** (1988) 46.

[68] G. Compère, W. Song, and A. Strominger, "New Boundary Conditions for AdS3," *JHEP* **05** (2013) 152, arXiv:1303.2662 [hep-th].

[69] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, arXiv:hep-th/9204099 [hep-th].

[70] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the (2+1) black hole," *Phys. Rev.* **D48** (1993) 1506–1525, arXiv:gr-qc/9302012 [gr-qc]. [Erratum: Phys. Rev.D88,069902(2013)].

[71] M. Banados, "Three-dimensional quantum geometry and black holes," *AIP Conf. Proc.* **484** (1999) no. 1, 147–169, arXiv:hep-th/9901148 [hep-th].

[72] K. Skenderis and S. N. Solodukhin, "Quantum effective action from the ads/cft correspondence," *Phys.Lett.B* **472:316-322,2000** (Phys.Lett.B472:316-322,2000) , hep-th/9910023.

[73] C. Eling, Y. Oz, S. Theisen, and S. Yankielowicz, "Conformal anomalies in hydrodynamics," 1301.3170.

[74] R. Banerjee, "Exact results in two dimensional chiral hydrodynamics with gravitational anomalies," 1303.5593.

[75] R. Banerjee, S. Dey, and B. R. Majhi, "Entropy current in two dimensional anomalous hydrodynamics and a bound on the sum of the response parameters,"*Phys. Rev. D* **92,** (Dec., 2014) 044019, 1412.5878.

[76] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Flat holography and Carrollian fluids," arXiv:1802.06809 [hep-th].

[77] C. Fefferman and C. R. Graham, "Conformal invariants," *in Elie Cartan et les mathématiques d'aujourd'hui, Astérisque 1985, numéro hors série Soc. Math. France, Paris, 95* .

[78] C. Fefferman and C. R. Graham, "The ambient metric," 0710.0919.

[79] J.-M. Lévy-Leblond, "Une nouvelle limite non-relativiste du groupe de poincaré," *Annales de l'I.H.P. Physique théorique* **3** (1965) no. 1, 1–12. http://eudml.org/doc/75509.

[80] C. Duval, G. W. Gibbons, and P. A. Horvathy, "Conformal Carroll groups," *J. Phys.* **A47** (2014) no. 33, 335204, arXiv:1403.4213 [hep-th].

[81] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, "Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time," *Class. Quant. Grav.* **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[82] C. Duval, G. W. Gibbons, and P. A. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class. Quant. Grav.* **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[83] R. Sachs, "Asymptotic symmetries in gravitational theory," *Phys. Rev.* **128** (1962) 2851–2864.

[84] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[85] T. Mädler and J. Winicour, "Bondi-Sachs Formalism," *Scholarpedia* **11** (2016) 33528, arXiv:1609.01731 [gr-qc].

[86] F. Alessio and G. Esposito, "On the structure and applications of the bondi-metzner-sachs group,"*Int.J.Geom.Meth.Mod.Phys.* **15** (Sept., 2017) no.02, 1709.05134.

[87] G. Barnich and C. Troessaert, "Finite BMS transformations," *JHEP* **03** (2016) 167, arXiv:1601.04090 [gr-qc].

[88] L. Ciambelli, R. G. Leigh, C. Marteau, and P. M. Petropoulos, "Carroll Structures, Null Geometry and Conformal Isometries," arXiv:1905.02221 [hep-th].

[89] M. Henneaux, "Geometry of zero signature spacetime," *Bull.Soc.Math.Belg. 31 47-63* (1979) .

[90] X. Bekaert and K. Morand, "Connections and dynamical trajectories in generalised Newton-Cartan gravity I. An intrinsic view," *J. Math. Phys.* **57** (2016) no. 2, 022507, arXiv:1412.8212 [hep-th].

[91] F. Hopfmüller and L. Freidel, "Gravity Degrees of Freedom on a Null Surface," *Phys. Rev.* **D95** (2017) no. 10, 104006, arXiv:1611.03096 [gr-qc].

[92] W. Donnelly and L. Freidel, "Local subsystems in gauge theory and gravity," *JHEP* **09** (2016) 102, arXiv:1601.04744 [hep-th].

[93] X. Bekaert and K. Morand, "Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective," *J. Math. Phys.* **59** (2018) no. 7, 072503, arXiv:1505.03739 [hep-th].

[94] F. Hopfmüller and L. Freidel, "Null Conservation Laws for Gravity," *Phys. Rev.* **D97** (2018) no. 12, 124029, arXiv:1802.06135 [gr-qc].

[95] A. Campoleoni, L. Ciambelli, C. Marteau, P. M. Petropoulos, and K. Siampos, "Two-dimensional fluids and their holographic duals," arXiv:1812.04019 [hep-th].

[96] R. F. Penna, "bms_3 invariant fluid dynamics at null infinity," 1708.08470.

[97] R. F. Penna, "BMS_3 invariant fluid dynamics at null infinity," *Class. Quant. Grav.* **35** (2018) no. 4, 044002, arXiv:1708.08470 [hep-th].

[98] A. Bagchi and R. Gopakumar, "Galilean conformal algebras and ads/cft,"*JHEP* **0907:037,2009** (Feb., 2009) , 0902.1385.

[99] A. Bagchi, "The bms/gca correspondence,"*Phys.Rev.Lett.* **105:171601,2010** (June, 2010) , 1006.3354.

[100] A. Bagchi, "Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories," *Phys. Rev. Lett.* **105** (2010) 171601, arXiv:1006.3354 [hep-th].

[101] G. Barnich, A. Gomberoff, and H. A. González, "The flat limit of three dimensional asymptotically anti-de sitter spacetimes," 1204.3288.

[102] G. Barnich, A. Gomberoff, and H. A. Gonzalez, "The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes," *Phys. Rev.* **D86** (2012) 024020, arXiv:1204.3288 [gr-qc].

[103] A. Bagchi, S. Detournay, R. Fareghbal, and J. Simon, "Holography of 3d flat cosmological horizons,"*Phys. Rev. Lett.* **110,** (Aug., 2012) 141302, 1208.4372.

[104] A. Bagchi and R. Fareghbal, "BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries," *JHEP* **10** (2012) 092, arXiv:1203.5795 [hep-th].

[105] R. Fareghbal and A. Naseh, "Flat-space energy-momentum tensor from bms/gca correspondence,"*JHEP* **03(2014)005** (Dec., 2013) 03005, 1312.2109.

[106] A. Bagchi, R. Basu, D. Grumiller, and M. Riegler, "Entanglement entropy in galilean conformal field theories and flat holography,"*Phys. Rev. Lett.* **114,** (Oct., 2014) 111602, 1410.4089.

[107] R. Fareghbal, A. Naseh, and S. Rouhani, "Aspects of ultra-relativistic field theories via flat-space holography,"*Phys.Lett. B* **771** (Nov., 2015) 189–193, 1511.01774.

[108] A. Bagchi, R. Basu, A. Kakkar, and A. Mehra, "Flat Holography: Aspects of the dual field theory," *JHEP* **12** (2016) 147, arXiv:1609.06203 [hep-th].

[109] R. Fareghbal and I. Mohammadi, "Flat-space holography and correlators of robinson-trautman stress tensor," 1802.05445.

[110] A. Bagchi, A. Mehra, and P. Nandi, "Field Theories with Conformal Carrollian Symmetry," *JHEP* **05** (2019) 108, arXiv:1901.10147 [hep-th].

[111] J. Hartong, "Holographic reconstruction of 3d flat space-time," 1511.01387.

[112] J. Hartong, "Gauging the carroll algebra and ultra-relativistic gravity," 1505.05011.

[113] G. Arcioni and C. Dappiaggi, "Exploring the holographic principle in asymptotically flat space-times via the BMS group," *Nucl. Phys.* **B674** (2003) 553–592, arXiv:hep-th/0306142 [hep-th].

[114] G. Arcioni and C. Dappiaggi, "Holography in asymptotically flat space-times and the BMS group," *Class. Quant. Grav.* **21** (2004) 5655, arXiv:hep-th/0312186 [hep-th].

[115] G. Arcioni and C. Dappiaggi, "Holography and BMS field theory," *AIP Conf. Proc.* **751** (2005) 176–178, arXiv:hep-th/0409313 [hep-th]. [,176(2004)].

[116] C. Dappiaggi, V. Moretti, and N. Pinamonti, "Rigorous steps towards holography in asymptotically flat spacetimes," *Rev. Math. Phys.* **18** (2006) 349–416, arXiv:gr-qc/0506069 [gr-qc].

[117] J. de Boer and S. N. Solodukhin, "A Holographic reduction of Minkowski space-time," *Nucl. Phys.* **B665** (2003) 545–593, arXiv:hep-th/0303006 [hep-th].

[118] G. Barnich and C. Troessaert, "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited," *Phys. Rev. Lett.* **105** (2010) 111103, arXiv:0909.2617 [gr-qc].

[119] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* **05** (2010) 062, arXiv:1001.1541 [hep-th].

[120] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, "2D Stress Tensor for 4D Gravity," *Phys. Rev. Lett.* **119** (2017) no. 12, 121601, arXiv:1609.00282 [hep-th].

[121] S. Pasterski, S.-H. Shao, and A. Strominger, "Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere," *Phys. Rev.* **D96** (2017) no. 6, 065026, arXiv:1701.00049 [hep-th].

[122] D. Kapec and P. Mitra, "A d-Dimensional Stress Tensor for Mink$_{d+2}$ Gravity," arXiv:1711.04371 [hep-th].

[123] A. Bagchi, S. Chakrabortty, and P. Parekh, "Tensionless strings from worldsheet symmetries," 1507.04361.

[124] B. Cardona, J. Gomis, and J. M. Pons, "Dynamics of carroll strings," 1605.05483.

[125] E. Bergshoeff, J. Gomis, and G. Longhi, "Dynamics of carroll particles," 1405.2264.

[126] E. Bergshoeff, J. Gomis, and L. Parra, "The Symmetries of the Carroll Superparticle," *J. Phys.* **A49** (2016) no. 18, 185402, arXiv:1503.06083 [hep-th].

[127] R. Basu and U. N. Chowdhury, "Dynamical structure of carrollian electrodynamics," 1802.09366.

[128] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids," *Class. Quant. Grav.* **35** (2018) no. 16, 165001, arXiv:1802.05286 [hep-th].

[129] L. Ciambelli and C. Marteau, "Carrollian conservation laws and ricci-flat gravity," 1810.11037.

[130] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, "Perfect fluids,"*SciPost Phys.* **5,** (Oct., 2017) 003, 1710.04708.

[131] T. Damour, *Quelques proprietes mecaniques, electromagnet iques, thermodynamiques et quantiques des trous noir.* PhD thesis, Paris U., VI-VII, 1979. http://pagesperso.ihes.fr/~damour/Articles/.

[132] R. H. Price and K. S. Thorne, "Membrane Viewpoint on Black Holes: Properties and Evolution of the Stretched Horizon," *Phys. Rev.* **D33** (1986) 915–941.

[133] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, "From Navier-Stokes To Einstein," *JHEP* **07** (2012) 146, arXiv:1101.2451 [hep-th].

[134] G. Dvali, C. Gomez, and D. Lüst, "Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry," *Phys. Lett.* **B753** (2016) 173–177, arXiv:1509.02114 [hep-th].

[135] L. Donnay, G. Giribet, H. A. Gonzalez, and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," *Phys. Rev. Lett.* **116** (2016) no. 9, 091101, arXiv:1511.08687 [hep-th].

[136] R. F. Penna, "BMS invariance and the membrane paradigm," *JHEP* **03** (2016) 023, arXiv:1508.06577 [hep-th].

[137] M. Blau and M. O'Loughlin, "Horizon Shells and BMS-like Soldering Transformations," *JHEP* **03** (2016) 029, arXiv:1512.02858 [hep-th].

[138] A. Averin, G. Dvali, C. Gomez, and D. Lust, "Gravitational Black Hole Hair from Event Horizon Supertranslations," *JHEP* **06** (2016) 088, arXiv:1601.03725 [hep-th].

[139] C. Eling and Y. Oz, "On the Membrane Paradigm and Spontaneous Breaking of Horizon BMS Symmetries," *JHEP* **07** (2016) 065, arXiv:1605.00183 [hep-th].

[140] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Extended Symmetries at the Black Hole Horizon," *JHEP* **09** (2016) 100, arXiv:1607.05703 [hep-th].

[141] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft Hair on Black Holes," *Phys. Rev. Lett.* **116** (2016) no. 23, 231301, arXiv:1601.00921 [hep-th].

[142] S. W. Hawking, M. J. Perry, and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," *JHEP* **05** (2017) 161, arXiv:1611.09175 [hep-th].

[143] A. Strominger and A. Zhiboedov, "Superrotations and Black Hole Pair Creation," *Class. Quant. Grav.* **34** (2017) no. 6, 064002, arXiv:1610.00639 [hep-th].

[144] R. F. Penna, "Near-horizon BMS symmetries as fluid symmetries," *JHEP* **10** (2017) 049, arXiv:1703.07382 [hep-th].

[145] L. Donnay, G. Giribet, H. A. González, and A. Puhm, "Black hole memory effect," *Phys. Rev.* **D98** (2018) no. 12, 124016, arXiv:1809.07266 [hep-th].

[146] S. Haco, S. W. Hawking, M. J. Perry, and A. Strominger, "Black Hole Entropy and Soft Hair," *JHEP* **12** (2018) 098, arXiv:1810.01847 [hep-th].

[147] V. Chandrasekaran, E. E. Flanagan, and K. Prabhu, "Symmetries and charges of general relativity at null boundaries," *JHEP* **11** (2018) 125, arXiv:1807.11499 [hep-th].

[148] R. F. Penna, "Near-horizon Carroll symmetry and black hole Love numbers," arXiv:1812.05643 [hep-th].

[149] L. Donnay and C. Marteau, "Carrollian Physics at the Black Hole Horizon," arXiv:1903.09654 [hep-th].

[150] G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick, and M. C. Werner, "Stationary metrics and optical zermelo-randers-finsler geometry,"*Phys.Rev.D* **79:044022,2009** (Nov., 2008) , 0811.2877.

[151] M. Vinokur, *A new formulation of the conservation equations of fluid dynamics*. 1974. https://books.google.fr/books?id=3jY8AQAAIAAJ.

[152] L. M. Avis, "A space-time tensor formulation for continuum mechanics in general curvilinear, moving, and deforming coordinate systems,".

[153] R. Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Books on Mathematics. Dover Publications, 1990. https://books.google.fr/books?id=QcZIAwAAQBAJ.

[154] H. Luo and T. R. Bewley, "On the contravariant form of the navier-stokes equations in time-dependent curvilinear coordinate systems," 2004.

[155] M. Charron, A. Zadra, and C. Girard, "Four-dimensional tensor equations for a classical fluid in an external gravitational field,"*Quarterly Journal of the Royal Meteorological Society* **140** (04, 2014) .

[156] M. Geracie, K. Prabhu, and M. M. Roberts, "Fields and fluids on curved non-relativistic spacetimes,"*JHEP* **08** (Mar., 2015) 042, 1503.02680.

[157] C. Duval, G. Burdet, H. P. Kunzle, and M. Perrin, "Bargmann Structures and Newton-cartan Theory," *Phys. Rev.* **D31** (1985) 1841–1853.

[158] B. Julia and H. Nicolai, "Null Killing vector dimensional reduction and Galilean geometrodynamics," *Nucl. Phys.* **B439** (1995) 291–326, arXiv:hep-th/9412002 [hep-th].

[159] C. Duval and P. A. Horvathy, "Non-relativistic conformal symmetries and Newton-Cartan structures," *J. Phys.* **A42** (2009) 465206, arXiv:0904.0531 [math-ph].

[160] D. T. Son, "Newton-Cartan Geometry and the Quantum Hall Effect," arXiv:1306.0638 [cond-mat.mes-hall].

[161] K. Jensen, "Aspects of hot galilean field theory," 1411.7024.

[162] K. Jensen and A. Karch, "Revisiting non-relativistic limits," *JHEP* **04** (2015) 155, arXiv:1412.2738 [hep-th].

[163] G. Festuccia, D. Hansen, J. Hartong, and N. A. Obers, "Torsional newton-cartan geometry from the noether procedure,"*Phys. Rev. D* **94,** (July, 2016) 105023, 1607.01926.

[164] D. Hansen, J. Hartong, and N. A. Obers, "Non-relativistic expansion of the Einstein-Hilbert Lagrangian," in *15th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG15) Rome, Italy, July 1-7, 2018*. 2019. arXiv:1905.13723 [gr-qc].

[165] A. Ashtekar, "Geometry and Physics of Null Infinity," arXiv:1409.1800 [gr-qc].

[166] A. Ashtekar, M. Campiglia, and A. Laddha, "Null infinity, the BMS group and infrared issues," *Gen. Rel. Grav.* **50** (2018) no. 11, 140, arXiv:1808.07093 [gr-qc].

[167] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, "The Holographic fluid dual to vacuum Einstein gravity," *JHEP* **07** (2011) 050, arXiv:1103.3022 [hep-th].

[168] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, "The relativistic fluid dual to vacuum Einstein gravity," *JHEP* **03** (2012) 076, arXiv:1201.2678 [hep-th].

[169] M. M. Caldarelli, J. Camps, B. Goutéraux, and K. Skenderis, "AdS/Ricci-flat correspondence," *JHEP* **04** (2014) 071, arXiv:1312.7874 [hep-th].

[170] N. Pinzani-Fokeeva and M. Taylor, "Towards a general fluid/gravity correspondence," *Phys. Rev.* **D91** (2015) no. 4, 044001, arXiv:1401.5975 [hep-th].

[171] C. Eling, A. Meyer, and Y. Oz, "The relativistic rindler hydrodynamics," 1201.2705.

[172] B. Oblak, *BMS Particles in Three Dimensions*. PhD thesis, Brussels U., 2016. arXiv:1610.08526 [hep-th].

[173] A. Campoleoni, D. Francia, and C. Heissenberg, "Asymptotic charges at null infinity in any dimension," 1712.09591.

[174] L. Ciambelli and R. G. Leigh, "Weyl Connections and their Role in Holography," arXiv:1905.04339 [hep-th].

[175] G. B. Folland, "Weyl manifolds," *J. Diff. Geom.* **4** (1970) 145–153.

[176] G. Hall, "Weyl manifolds and connections," *J Math Phys* **33** (1992) 2633.

[177] K. Skenderis, "Lecture notes on holographic renormalization," *Class. Quant. Grav.* **19** (2002) 5849–5876, arXiv:hep-th/0209067 [hep-th].

[178] D. M. Capper and M. J. Duff, "Trace anomalies in dimensional regularization," *Nuovo Cim.* **A23** (1974) 173–183.

[179] S. Deser, M. J. Duff, and C. J. Isham, "Nonlocal Conformal Anomalies," *Nucl. Phys.* **B111** (1976) 45–55.

[180] M. J. Duff, "Observations on Conformal Anomalies," *Nucl. Phys.* **B125** (1977) 334–348.

[181] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," *Phys. Lett.* **B103** (1981) 207–210. [,598(1981)].

[182] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," *Rev. Mod. Phys.* **60** (1988) 917.

[183] M. J. Duff, "Twenty years of the Weyl anomaly," *Class. Quant. Grav.* **11** (1994) 1387–1404, arXiv:hep-th/9308075 [hep-th].

[184] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," *JHEP* **07** (1998) 023, arXiv:hep-th/9806087 [hep-th].

[185] P. O. Mazur and E. Mottola, "Weyl cohomology and the effective action for conformal anomalies," *Phys. Rev.* **D64** (2001) 104022, arXiv:hep-th/0106151 [hep-th].

[186] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories," *Nucl. Phys.* **B363** (1991) 486–526.

[187] T. Arakelian, D. R. Karakhanian, R. P. Manvelyan, and R. L. Mkrtchian, "Trace anomalies and cocycles of the Weyl group," *Phys. Lett.* **B353** (1995) 52–56.

[188] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997. http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997.

[189] C. G. Callan, Jr., S. R. Coleman, and R. Jackiw, "A New improved energy - momentum tensor," *Annals Phys.* **59** (1970) 42–73.

[190] A. Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger, "Weyl gauging and conformal invariance," *Nucl. Phys.* **B495** (1997) 433–450, arXiv:hep-th/9607110 [hep-th].

[191] S. Dengiz, E. Kilicarslan, and B. Tekin, "Weyl-gauging of Topologically Massive Gravity," *Phys. Rev.* **D86** (2012) 104014, arXiv:1209.1251 [hep-th].

[192] C. Coriano, L. Delle Rose, C. Marzo, and M. Serino, "The dilaton Wess–Zumino action in six dimensions from Weyl gauging: local anomalies and trace relations," *Class. Quant. Grav.* **31** (2014) 105009, arXiv:1311.1804 [hep-th].

[193] G. K. Karananas and A. Monin, "Weyl and Ricci gauging from the coset construction," *Phys. Rev.* **D93** (2016) no. 6, 064013, arXiv:1510.07589 [hep-th].

[194] K. Farnsworth, M. A. Luty, and V. Prilepina, "Weyl versus Conformal Invariance in Quantum Field Theory," *JHEP* **10** (2017) 170, arXiv:1702.07079 [hep-th].

[195] F. Wu, "Note on Weyl versus Conformal Invariance in Field Theory," *Eur. Phys. J.* **C77** (2017) no. 12, 886, arXiv:1704.05210 [hep-th].

[196] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, "Correlation functions in the CFT(d) / AdS(d+1) correspondence," *Nucl. Phys.* **B546** (1999) 96–118, arXiv:hep-th/9804058 [hep-th].

[197] V. Balasubramanian, P. Kraus, and A. E. Lawrence, "Bulk versus boundary dynamics in anti-de Sitter space-time," *Phys. Rev.* **D59** (1999) 046003, arXiv:hep-th/9805171 [hep-th].

[198] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality*. Cambridge University Press, 2015. https://books.google.fr/books?id=OGOGBwAAQBAJ.

[199] M. Humbert, "Holographic reconstruction in higher dimension," *Internship report, Ecole normale superieure and Ecole Polytechnique* (2017) .

[200] J. Foster and E. T. Newman, "Note on the Robinson-Trautman Solutions," *Journal of Mathematical Physics* **8** (02, 1967) 189–194.

[201] K. P. Tod, "Analogue of the past horizon in the robinson-trautman metrics," *Classical and Quantum Gravity - CLASS QUANTUM GRAVITY* **6** (08, 1989) 1159–1163.

[202] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," arXiv:1703.05448 [hep-th].

[203] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **12** (2011) 105, arXiv:1106.0213 [hep-th].

[204] J. D. Debus, M. Mendoza, S. Succi, and H. J. Herrmann, "Energy dissipation in flows through curved spaces," 1511.08031.

[205] J. Pedlosky, *Geophysical Fluid Dynamics*. Springer study edition. Springer New York, 1992. https://books.google.fr/books?id=FXs-uRSDBFYC.

[206] O. Regev, O. M. Umurhan, and P. A. Yecko, *Modern Fluid Dynamics for Physics and Astrophysics*. 2016.

[207] D. Grumiller, W. Merbis, and M. Riegler, "Most general flat space boundary conditions in three-dimensional Einstein gravity," *Class. Quant. Grav.* **34** (2017) no. 18, 184001, arXiv:1704.07419 [hep-th].

[208] O. Miskovic and R. Olea, "Topological regularization and self-duality in four-dimensional anti-de sitter gravity," 0902.2082.

[209] M. M. Caldarelli, O. J. C. Dias, and D. Klemm, "Dyonic ads black holes from magnetohydrodynamics," *JHEP* **0903:025,2009** (Dec., 2008) , 0812.0801.

[210] L. Ciambelli, "Resummation of Einstein-Maxwell spacetimes," *Internship report, Ecole normale superieure and Ecole Polytechnique* (2016) .

[211] H. Godazgar, M. Godazgar, and C. N. Pope, "Tower of subleading dual BMS charges," *JHEP* **03** (2019) 057, arXiv:1812.06935 [hep-th].

[212] H. Godazgar, M. Godazgar, and C. N. Pope, "New dual gravitational charges," *Phys. Rev.* **D99** (2019) no. 2, 024013, arXiv:1812.01641 [hep-th].

[213] H. Godazgar, M. Godazgar, and C. N. Pope, "Subleading BMS charges and fake news near null infinity," *JHEP* **01** (2019) 143, arXiv:1809.09076 [hep-th].

[214] D. Grumiller and M. Riegler, "Most general AdS_3 boundary conditions," *JHEP* **10** (2016) 023, arXiv:1608.01308 [hep-th].

[215] S. Detournay and M. Riegler, "Enhanced Asymptotic Symmetry Algebra of 2+1 Dimensional Flat Space," *Phys. Rev.* **D95** (2017) no. 4, 046008, arXiv:1612.00278 [hep-th].

[216] H. Osborn and A. C. Petkou, "Implications of conformal invariance in field theories for general dimensions," *Annals Phys.* **231** (1994) 311–362, arXiv:hep-th/9307010 [hep-th].

[217] R. G. Leigh and A. C. Petkou, "Gravitational duality transformations on (A)dS(4)," *JHEP* **11** (2007) 079, arXiv:0704.0531 [hep-th].

[218] G. Compère, A. Fiorucci, and R. Ruzziconi, "The Λ-BMS_4 group of dS_4 and new boundary conditions for AdS_4," arXiv:1905.00971 [gr-qc].

Annexes

As stated in the introduction, this thesis relies on part of the papers I wrote with my various co-authors. These papers are reported here as they appear on Arxiv.

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The Robinson–Trautman spacetime and its holographic fluid

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ABSTRACT

We discuss the holographic reconstruction of four-dimensional asymptotically anti-de Sitter Robinson–Trautman spacetime from boundary data. We use for that a resummed version of the derivative expansion. The latter involves a vector field, which is interpreted as the dual-holographic-fluid velocity field and is naturally defined in the Eckart frame. In this frame the analysis of the non-perfect holographic energy–momentum tensor is considerably simplified. The Robinson–Trautman fluid is at rest and its time evolution is a heat-diffusion kind of phenomenon: the Robinson–Trautman equation plays the rôle of heat equation, and the heat current is identified with the gradient of the extrinsic curvature of the two-dimensional boundary spatial section hosting the conformal fluid, interpreted as an out-of-equilibrium kinematical temperature. The hydrodynamic-frame-independent entropy current is conserved for vanishing chemical potential, and the evolution of the fluid resembles a Moutier thermodynamic path. We finally comment on the general transformation rules for moving to the Landau–Lifshitz frame, and on possible drawbacks of this option.

Contents

1 The Robinson–Trautman spacetime and holography

Robinson–Trautman solutions to Einstein's equations were found in 1960-1962 [1].[1] They are obtained assuming the existence of a null, geodesic and shearless congruence. In vacuum, under these assumptions, Goldberg–Sachs theorem states that the corresponding spacetime is algebraically special, *i.e.* Petrov type II, III, N, D or O. This feature remains valid when a cosmological constant or even certain other classes of energy sources are added.

Asymptotically anti-de Sitter Robinson–Trautman spacetimes have attracted some attention in the framework of holography. The three-dimensional boundary metric and the dual conformal field theory expectation value of the energy–momentum tensor were found in [3], where further properties of the boundary state were also discussed, in particular from a hydrodynamic perspective (see also [4]).

Conformal fluid dynamics was thoroughly studied within fluid/gravity correspondence [5–7]. This holographic correspondence sets a relationship between Einstein spaces (possibly with a gauge field) and boundary conformal fluids (potentially charged), incarnated in the derivative expansion. The derivative expansion is an alternative to the Fefferman–Graham expansion [8, 9]. Besides the usual boundary data as the metric and the energy–momentum tensor (for pure gravity), it requires an extra piece, namely a velocity field assumed to slowly vary in spacetime.

[1] See e.g. [2] for a modern and more general presentation.

In fact, the velocity field is redundant since it is not needed in the Fefferman–Graham approach, and it is arbitrary because for non-perfect relativistic fluids the distinction between energy and mass is immaterial. Its rôle is to organize the expansion, and its choice a matter of convenience, or better, of physical framework. Often the derivative expansions are asymptotic series, and non-hydrodynamic (*i.e.* non-perturbative) modes can appear, triggering an alarm regarding the validity of the hydrodynamic interpretation. From this viewpoint, some hydrodynamic-frame (velocity-field) choices might be better designated than others.

Fluid/gravity correspondence raises an important question: given a boundary metric, what are the conditions it should satisfy, and which energy–momentum tensor should it be accompanied with in order for an *exact* dual bulk Einstein space to exist? This question has been successfully investigated in [10–15]. It turns out to be relevant both for the integrability of Einstein's equations (*à la Geroch*, see [16–18]) and because it gives access to exact transport properties of the holographic fluid. To answer this question the Fefferman–Graham expansion is not very useful because it is not resummable (except for trivial cases [19]), as opposed to the derivative expansion, which is resummable when the velocity field is chosen *shearless*.

The resummation process at hand reveals two main features: (i) the bulk Einstein spacetime is Petrov algebraically special, and (ii) the boundary fluid velocity is in the Eckart frame. This last property is interesting because, often, the general analysis of transport properties in relativistic fluids is performed in the Landau–Lifshitz frame, hence setting the heat flow to zero. In the present framework, however, this choice is not natural, and can even be questionable. This happens in particular for Robinson–Trautman spacetimes, which are algebraically special and emerge while resumming appropriate boundary data, and hence fall in the class under investigation here. In the following, we will review how Robinson–Trautman is obtained exclusively from boundary considerations (Sec. 2), and what is the corresponding holographic-fluid interpretation, with some emphasis on the issue of entropy (Sec. 3). Two appendices provide further useful information on relativistic hydrodynamics.

2 Reconstruction from the boundary

Our aim here is to review the holographic construction of Robinson–Trautman Einstein spaces as performed in [13]. We only refer to boundary data, which are designed and combined in order for the derivative expansion to be resummable.

2.1 The general resummation formula

If $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the boundary metric and $\mathrm{T} = T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the boundary energy–momentum tensor, the resummed bulk metric[2] reads:

$$\mathrm{d}s^2_{\text{res.}} = 2\mathrm{u}(\mathrm{d}r + r\mathrm{A}) + r^2 k^2 \mathrm{d}s^2 + \frac{\Sigma}{k^2} + \frac{\mathrm{u}^2}{\rho^2}\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6}\right). \tag{2.1}$$

- Here, u is a shearless, normalized, time-like vector field. It has acceleration $\mathrm{a} = (\mathrm{u}\cdot\nabla)\,\mathrm{u}$, expansion $\Theta = \nabla\cdot\mathrm{u}$, and vorticity $\omega = \frac{1}{2}\omega_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}\,(\mathrm{du} + \mathrm{u}\wedge\mathrm{a})$.

- The guideline for setting up the derivative expansion is *Weyl covariance* [6, 7]: the bulk geometry is required to be insensitive to a conformal transformation of the boundary metric. Covariantization with respect to rescalings is achieved with the Weyl connection one-form:
$$\mathrm{A} = \mathrm{a} - \frac{\Theta}{2}\mathrm{u}. \tag{2.2}$$
Covariant derivatives ∇ are thus traded for Weyl-covariant ones $\mathscr{D} = \nabla + w\,\mathrm{A}$, w being the conformal weight of the tensor under consideration. In three spacetime dimensions, Weyl-covariant quantities are e.g.
$$\mathscr{D}_\nu \omega^\nu{}_\mu = \nabla_\nu \omega^\nu{}_\mu, \tag{2.3}$$
$$\mathscr{R} = R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu, \tag{2.4}$$
$$\mathscr{D}_\mu u_\nu = \nabla_\mu u_\nu + u_\mu a_\nu - \frac{\Theta}{2}h_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{2.5}$$
(for the last we have used (A.1)), while
$$\Sigma = \Sigma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -2\mathrm{u}\mathscr{D}_\nu\omega^\nu{}_\mu \mathrm{d}x^\mu - \omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \mathrm{u}^2\frac{\mathscr{R}}{2}, \tag{2.6}$$
is Weyl-invariant and stands for the Weyl-covariantized Schouten tensor.

- The radial coordinate is r, and ρ performs the resummation of the derivative expansion as it is defined by
$$\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu} = r^2 + \frac{q^2}{4k^4}. \tag{2.7}$$
Boundary Weyl transformations $\mathrm{d}s^2 \to \mathrm{d}s^2/\mathcal{B}^2$ correspond to bulk diffeomorphisms, which can be reabsorbed into a redefinition of the radial coordinate: $r \to \mathcal{B}\,r$.

- The boundary metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ has in general non-vanishing Cotton tensor

[2] We have traded here the usual advanced-time coordinate used in the quoted literature on fluid/gravity correspondence for the retarded time, spelled t (see (2.13)).

$\mathrm{C} = C_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, where

$$C_{\mu\nu} = \eta_{\mu\rho\sigma}\nabla^\rho \left(R_\nu{}^\sigma - \frac{R}{4}\delta_\nu{}^\sigma\right), \tag{2.8}$$

with $\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma}$. Whenever C is non-zero, the bulk is asymptotically *locally* anti-de Sitter. The Cotton tensor has conformal weight one (like the energy–momentum tensor) and is identically conserved:

$$\nabla \cdot \mathrm{C} = 0. \tag{2.9}$$

The bulk metric $\mathrm{d}s^2_{\text{res.}}$ given in expression (2.1) is an *exact* Einstein space with $\Lambda = -3k^2$ provided the boundary energy–momentum tensor is *exactly* conserved:

$$\nabla \cdot \mathrm{T} = 0. \tag{2.10}$$

This statement might raise questions, and calls for a few remarks. The energy–momentum tensor is not meant to be necessarily of perfect-fluid type. At the same time, the time-like congruence u, chosen independently, is interpreted as the fluid velocity. It is somehow puzzling that despite the apparent (and, as we already discussed, legitimate) arbitrariness of this choice, the statement regarding the exact Einstein nature of $\mathrm{d}s^2_{\text{res.}}$ could hold. There is a simple explanation for this.

Firstly, we have imposed (as part of our resummation ansatz) u to be a shearless congruence. This assumption, not only enables us to discard the large number of Weyl-covariant tensors available when the shear is non-vanishing, which would have probably spoiled any resummation attempt; but it also selects the algebraically special geometries, known to be related with integrability properties. Indeed, on the bulk (2.1), u is a manifestly null congruence, associated with the vector ∂_r . One can show (see [14]) that this bulk congruence is also *geodesic* and *shear-free*. According to the generalizations of the Goldberg–Sachs theorem, the anticipated Einstein bulk metric (2.1) is therefore algebraically special, *i.e.* of Petrov type II, III, D, N or O.

Secondly, the freedom in choosing u is only apparent because we have required it to be shearless. In $2+1$ dimensions, such a time-like vector field is essentially unique – unless there are symmetries, in which case all choices are anyway equivalent due to the symmetries. Indeed, given a generic three-dimensional metric (rather, a conformal class of metrics), there is a unique way to express it as a fibration over a conformally flat two-dimensional base:[3]

$$\mathrm{d}s^2 = -(\mathrm{d}t - \mathrm{b})^2 + \frac{2}{k^2 P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta}, \tag{2.11}$$

[3]See e.g. [20] and the discussion in the appendix of [14].

with P an arbitrary real function of $(t, \zeta, \bar{\zeta})$, and

$$\mathrm{b} = B(t, \zeta, \bar{\zeta})\, \mathrm{d}\zeta + \bar{B}(t, \zeta, \bar{\zeta})\, \mathrm{d}\bar{\zeta}. \tag{2.12}$$

In this metric,

$$\mathrm{u} = -\mathrm{d}t + \mathrm{b} \tag{2.13}$$

is precisely normalized and shear-free (see [14]). This defines our fluid congruence.

Thirdly, using the above resummation technique, it is possible to control *from the boundary* the Petrov type of the bulk, encoded in the Weyl tensor. The Weyl tensor and its dual can be used to form a pair of complex-conjugate tensors. Their five independent complex components are naturally packaged inside two complex-conjugate symmetric 3×3 matrices $\mathrm{Q}^{\pm}$ with zero trace (see e.g. [2]). The eigenvalue structure of $\mathrm{Q}^{\pm}$ (*i.e.* the degeneracy of the Weyl principal null directions) determines the Petrov type. Performing the Fefferman–Graham expansion of the complex Weyl tensors $\mathrm{Q}^{\pm}$ for a general Einstein space, one can show [21–24] that the leading-order ($1/r^3$) coefficients $\mathrm{S}^{\pm}$ are related to the combination

$$T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{i}{8\pi G k^2} C_{\mu\nu} \tag{2.14}$$

of the components of the boundary Cotton and energy–momentum tensors, by a constant similarity relation: $\mathrm{T}^{\pm} = -\mathrm{P}\,\mathrm{S}^{\pm}\mathrm{P}^{-1}$ with $\mathrm{P} = \mathrm{diag}(\pm i, -1, 1)$. The Segre type of $\mathrm{S}^{\pm}$ determines precisely the Petrov type of the four-dimensional bulk metric and establishes a one-to-one map between the bulk Petrov type and the boundary data. We will see more precisely how this operates in the case of Robinson–Trautman spacetime. Notice for the moment that due to conservation equations (2.9) and (2.10),

$$\nabla \cdot \mathrm{T}^{\pm} = 0. \tag{2.15}$$

It is clear from the above that the absence of shear for the boundary fluid congruence plays a crucial rôle in the resummability of the derivative expansion, leading ultimately to exact algebraically special Einstein spaces. Nonetheless, we cannot exclude that some exact Einstein type I spaces might be successfully reconstructed, or that none exact resummation involves a congruence with shear. In favour of the first option, one could argue that, the velocity of a relativistic fluid being arbitrary, one can always choose it shearless, without loss of generality. However, the way this congruence enters the resummation formula suggests, via the Goldberg–Sachs theorem, that we can only reach algebraically special Einstein spaces. We see thus the importance of this congruence from the holographic viewpoint, since it crucially enters and characterizes the resummation process. It is the reason why we proceed in the next section with the hydrodynamic analysis based on this congruence, which turns out to describe the holographic fluid in the Eckart frame.

2.2 The reconstruction of Robinson–Trautman

Consider the boundary metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{2}{k^2 P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta}. \tag{2.16}$$

The vector ∂_t is hypersurface-orthogonal, and the normal hypersurfaces are constant-t sections. The Gaussian curvature of the latter is $k^2 K$, where

$$K = \Delta \ln P \tag{2.17}$$

with $\Delta = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta$. The Cotton tensor, computed using[4] (2.8), reads:

$$\mathrm{C} = i \begin{pmatrix} \mathrm{d}t & \mathrm{d}\zeta & \mathrm{d}\bar{\zeta} \end{pmatrix} \begin{pmatrix} 0 & -\frac{k^2}{2}\partial_\zeta K & \frac{k^2}{2}\partial_{\bar{\zeta}} K \\ -\frac{k^2}{2}\partial_\zeta K & -\partial_t \left(\frac{\partial_\zeta^2 P}{P}\right) & 0 \\ \frac{k^2}{2}\partial_{\bar{\zeta}} K & 0 & \partial_t \left(\frac{\partial_{\bar{\zeta}}^2 P}{P}\right) \end{pmatrix} \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}\zeta \\ \mathrm{d}\bar{\zeta} \end{pmatrix}, \tag{2.18}$$

which is a real tensor.

We must now introduce the canonical *reference* tensors $\mathrm{T}^{\pm}$ and apply the following strategy (valid more generally *i.e.* beyond the choice (2.16) of boundary metric):

1. Determine the components of $\mathrm{T}^{\pm}$ in terms of third derivatives of the boundary metric (2.16), using Eq. (2.18) in (see (2.14))

$$\mathrm{Im}\mathrm{T}^{+} = \frac{\mathrm{C}}{8\pi G k^2}. \tag{2.19}$$

2. Use this information for expressing the actual energy–momentum tensor

$$\mathrm{T} = \mathrm{Re}\mathrm{T}^{+} \tag{2.20}$$

   in terms of third derivatives of the metric.

3. Reconstruct the bulk spacetime metric using (2.1).

4. Impose the conservation of T (2.10) and obtain a set of three *a priori* fourth-order partial-differential equations for the boundary metric, which

   (a) play the rôle of resummability conditions for the derivative expansion,

[4]Together with the choice of retarded time quoted in note 2, we reverse here the orientation with respect to the one adopted in [13]: $\eta_{t\zeta\bar{\zeta}} = \frac{i}{k^2 P^2}$. With these conventions, time flows as in [4], but is reversed with respect to Ref. [3]. Incidentally, we also rescale some observables for convenience, resulting e.g. in extra $1/k^2$ factors, as in Eq. (2.23).

(b) capture the boundary fluid dynamics.

Several remarks are in order here. First, the partial-differential equations obtained in step number 4 guarantee that Einstein's equations are fulfilled with the resummed derivative expansion (2.1). Second, in step number 1, Eq. (2.19) may impose restrictions among the components of the metric (its third derivatives in fact). These, with whatever external further condition we may impose via the form of $\mathrm{T}^{\pm}$, control the Petrov type of the bulk.

The power of the method displayed here is that we do not make any ansatz for the form of the energy–momentum tensor T. Rather we supply the reference tensors $\mathrm{T}^{\pm}$ with a canonical form, which in turn delivers C and T. The latter leads to equations for the boundary metric, which are also the holographic fluid equations of motion.

Notice that we have no control on the frame in which the fluid is described, as the velocity field is the shearless congruence read off directly from the boundary metric (2.16) (see (2.13)):

$$\mathrm{u} = -\mathrm{d}t, \tag{2.21}$$

which has no vorticity, no acceleration but is expanding at a rate

$$\Theta = -2\partial_t \ln P. \tag{2.22}$$

We should already stress that in this frame, which we will describe more precisely later, the holographic fluid exhibits a finite number of corrections with respect to a perfect fluid, as the energy–momentum tensor is basically third-order in derivatives of geometric quantities. This is not surprising and it is a rather general feature of exact Einstein bulk spaces to lead to holographic fluid configurations which do not trigger all transport coefficients. Still, the kinematic state is non-trivial, and the absence of certain series of corrections in the energy–momentum tensor is really the signature of vanishing of the corresponding transport coefficients (see [11] for the original detailed discussion).

There are two basic and distinct canonical forms for $\mathrm{T}^{\pm}$, which exhaust all possibilities.

Perfect-fluid form For perfect-fluid reference tensors, we need two complex-conjugate reference velocity fields $\mathrm{u}^{\pm}$. Consider the normalized congruence[5]

$$\mathrm{u}^{+} = \mathrm{u} + \frac{\alpha^{+}}{k^2 P^2}\mathrm{d}\zeta \tag{2.23}$$

with $\alpha^{+} = \alpha^{+}(t,\zeta,\bar{\zeta})$, and its complex-conjugate $\mathrm{u}^{-} = \mathrm{u} + \frac{\alpha^{-}}{k^2P^2}\mathrm{d}\bar{\zeta}$ with $\alpha^{-} = \alpha^{+*}$. The

[5]This is the most general one: adding an extra leg along the missing direction, and adjusting the overall scale for keeping the norm to -1 amounts to the combination of a Weyl transformation and a diffeomorphism.

perfect-fluid energy–momentum tensors based on these reference congruences read:

$$\mathrm{T}^{\pm}_{\mathrm{pf}} = \frac{M_{\pm}(t,\zeta,\bar{\zeta})k^2}{8\pi G}\left(3\left(\mathrm{u}^{\pm}\right)^2 + \mathrm{d}s^2\right) \tag{2.24}$$

with $M_- = M_+^*$.

Radiation-matter form Consider finally

$$\mathrm{T}^{+}_{\mathrm{rm}} = \frac{1}{4\pi G}\mathrm{d}\zeta\left(\beta \mathrm{d}t + \frac{\gamma}{k^2}\mathrm{d}\zeta\right). \tag{2.25}$$

In this expression β and γ are *a priori* functions of t, ζ and $\bar{\zeta}$. The tensor is the symmetrized direct product of a light-like by a time-like vector. Notice that for vanishing β, we obtain a *pure-radiation* tensor *i.e.* the square of a null vector.

We will consider a general reference tensor of the form

$$\mathrm{T}^{+} = \mathrm{T}^{+}_{\mathrm{pf}} + \mathrm{T}^{+}_{\mathrm{rm}}, \tag{2.26}$$

the two components being given in Eqs. (2.24) and (2.25). For this combination,

$$8\pi G i\, \mathrm{Im}\mathrm{T}^{+} = \begin{pmatrix}\mathrm{d}t & \mathrm{d}\zeta & \mathrm{d}\bar{\zeta}\end{pmatrix} \begin{pmatrix} k^2\left(M_+ - M_-\right) & -\frac{3M_+\alpha^+}{2P^2} + \frac{\beta}{2} & \frac{3M_-\alpha^-}{2P^2} - \frac{\bar{\beta}}{2} \\ -\frac{3M_+\alpha^+}{2P^2} + \frac{\beta}{2} & \frac{3M_+(\alpha^+)^2}{2P^4k^2} + \frac{\gamma}{k^2} & \frac{M_+ - M_-}{2P^2} \\ \frac{3M_-\alpha^-}{2P^2} - \frac{\bar{\beta}}{2} & \frac{M_+ - M_-}{2P^2} & -\frac{3M_-(\alpha^-)^2}{2P^4k^2} - \frac{\bar{\gamma}}{k^2} \end{pmatrix} \begin{pmatrix}\mathrm{d}t \\ \mathrm{d}\zeta \\ \mathrm{d}\bar{\zeta}\end{pmatrix}, \tag{2.27}$$

while

$$8\pi G\, \mathrm{Re}\mathrm{T}^{+} = \begin{pmatrix}\mathrm{d}t & \mathrm{d}\zeta & \mathrm{d}\bar{\zeta}\end{pmatrix} \begin{pmatrix} k^2\left(M_+ + M_-\right) & -\frac{3M_+\alpha^+}{2P^2} + \frac{\beta}{2} & -\frac{3M_-\alpha^-}{2P^2} + \frac{\bar{\beta}}{2} \\ -\frac{3M_+\alpha^+}{2P^2} + \frac{\beta}{2} & \frac{3M_+(\alpha^+)^2}{2P^4k^2} + \frac{\gamma}{k^2} & \frac{M_+ + M_-}{2P^2} \\ -\frac{3M_-\alpha^-}{2P^2} + \frac{\bar{\beta}}{2} & \frac{M_+ + M_-}{2P^2} & \frac{3M_-(\alpha^-)^2}{2P^4k^2} + \frac{\bar{\gamma}}{k^2} \end{pmatrix} \begin{pmatrix}\mathrm{d}t \\ \mathrm{d}\zeta \\ \mathrm{d}\bar{\zeta}\end{pmatrix}. \tag{2.28}$$

The reference tensor at hand depends on four complex arbitrary functions of t, ζ and $\bar{\zeta}$: M_+, α^+, β and γ. We can now require (2.19), using (2.18) and (2.27). The first observation is that this identification of the Cotton tensor demands

$$M_+(t,\zeta,\bar{\zeta}) = M_-(t,\zeta,\bar{\zeta}), \tag{2.29}$$

which we will name $M(t,\zeta,\bar{\zeta})$, a real function. Furthermore, it appears a pair of independent conditions plus their complex-conjugates. The first reads:

$$3M\frac{\alpha^+}{P^2} + \partial_\zeta K = \beta \quad \text{and} \quad \text{c.c.}\,, \tag{2.30}$$

while the second is

$$\frac{3}{2}M\frac{(\alpha^+)^2}{P^4} + \gamma = \partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) \quad \text{and} \quad \text{c.c.}\,. \tag{2.31}$$

Equations (2.30) and (2.31) are *algebraic* for the functions $\alpha^\pm(t,\zeta,\bar\zeta)$, $\beta(t,\zeta,\bar\zeta)$ and $\gamma(t,\zeta,\bar\zeta)$, as well as the complex conjugate functions $\bar\beta(t,\zeta,\bar\zeta)$ and $\bar\gamma(t,\zeta,\bar\zeta)$. Extracting these functions and inserting them back into (2.28), we determine using (2.20) the boundary energy–momentum tensor in terms of third derivatives of the metric, as already anticipated:

$$\mathrm{T} = \frac{1}{16\pi G}\begin{pmatrix}\mathrm{d}t & \mathrm{d}\zeta & \mathrm{d}\bar\zeta\end{pmatrix}\begin{pmatrix}4Mk^2 & \partial_\zeta K & \partial_{\bar\zeta} K\\ \partial_\zeta K & \frac{2}{k^2}\partial_t\left(\frac{\partial_{\bar\zeta}^2 P}{P}\right) & \frac{2M}{P^2}\\ \partial_{\bar\zeta} K & \frac{2M}{P^2} & \frac{2}{k^2}\partial_t\left(\frac{\partial_{\zeta}^2 P}{P}\right)\end{pmatrix}\begin{pmatrix}\mathrm{d}t\\ \mathrm{d}\zeta\\ \mathrm{d}\bar\zeta\end{pmatrix}. \tag{2.32}$$

We are now ready to proceed and write the bulk metric as obtained using the resummed version of the derivative expansion, Eq. (2.1). We find:

$$\mathrm{d}s^2_{\text{res.}} = -2\mathrm{d}t(\mathrm{d}r + H\mathrm{d}t) + 2\frac{r^2}{P^2}\,\mathrm{d}\zeta\mathrm{d}\bar\zeta \tag{2.33}$$

with

$$2H = k^2r^2 - 2r\partial_t \ln P + K - \frac{2M}{r}\,. \tag{2.34}$$

According to our reasoning about the resummation of the derivative expansion into an exact Einstein space, the metric (2.33) is expected to be Einstein provided the boundary energy–momentum tensor (2.32) is conserved, *i.e.* obeys (2.10). Let us impose therefore the conservation of T:

$$\nabla\cdot\mathrm{T} = 0 \quad \Longleftrightarrow \quad \begin{cases}\Delta K + 12M\partial_t \ln P = 4\partial_t M,\\ \partial_\zeta M = 0, \quad \partial_{\bar\zeta} M = 0.\end{cases} \tag{2.35}$$

Not only the first equation in (2.35) is the *Robinson–Trautman* equation, which precisely guarantees that (2.33) is Einstein, but it also appears here as the longitudinal component of the energy–momentum conservation, *i.e.* as the *heat* equation for the boundary fluid, at rest in the frame at hand. We will further elaborate on the properties of the holographic fluid in the next section.

We would like at this point to remark that no reference to any *a priori* bulk property has been made in our approach. The Robinson–Trautman equation has been obtained from purely boundary considerations, by imposing the conservation of the boundary energy–

momentum tensor, and we can similarly tune the boundary data in order to control the bulk Petrov type of the bulk Einstein space. Generically the latter is type II because we can prove [14] that the bulk congruence ∂_r is null, geodesic and shearless, and using thus the extensions of Goldberg–Sachs theorem, the reconstructed bulk space is algebraically special.[6] By tuning the functions that define the reference tensors $\mathrm{T}^{\pm}$, namely $M(t), \alpha^{\pm}(t,\zeta,\bar{\zeta})$, $\beta(t,\zeta,\bar{\zeta})$, $\bar{\beta}(t,\zeta,\bar{\zeta})$, $\gamma(t,\zeta,\bar{\zeta})$ and $\bar{\gamma}(t,\zeta,\bar{\zeta})$, we can scan other classes (see [13] for details):

- If $M=0$, $\alpha^{\pm}$ are immaterial and $\beta(t,\zeta,\bar{\zeta})$ and $\gamma(t,\zeta,\bar{\zeta})$ are fully determined by Eqs. (2.30) and (2.31):

$$\beta = \partial_\zeta K \quad \text{and} \quad \text{c.c.}\,, \tag{2.36}$$

$$\gamma = \partial_t \left(\frac{\partial^2_\zeta P}{P}\right) \quad \text{and} \quad \text{c.c.}\,. \tag{2.37}$$

  Furthermore, the Robinson–Trautman equation guarantees holomorphicity for β, function of (t,ζ) only. Hence, the bulk is generically Petrov type III. When $\beta=0$, it becomes type N, where now $K=K(t)$, following (2.36). The most general $P(t,\zeta,\bar{\zeta})$ such that its curvature is a function of time only was found in [26], and reads:

$$P(t,\zeta,\bar{\zeta}) = \frac{1+\frac{\epsilon}{2}h(t,\zeta)\,\bar{h}(t,\bar{\zeta})}{\sqrt{2f(t)\,\partial_\zeta h(t,\bar{\zeta})\,\partial_{\bar{\zeta}}\bar{h}(t,\bar{\zeta})}} \tag{2.38}$$

  with $\epsilon = 0, \pm 1$ and arbitrary functions $f(t)$ and $h(t,\zeta)$.

- If $\beta=\gamma=0$, $\alpha^{\pm}$ are read-off from (2.30):

$$\alpha^+ = -\frac{P^2}{3M}\partial_\zeta K \quad \text{and} \quad \text{c.c.}\,, \tag{2.39}$$

  and the geometry is subject to a further constraint[7] obtained by combining (2.31) and (2.39):

$$6M\,\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) = \left(\partial_\zeta K\right)^2 \quad \text{and} \quad \text{c.c.}\,. \tag{2.40}$$

  The bulk is still type II, but choosing holomorphic $\alpha^- = \alpha^-(t,\zeta)$, *i.e.* (using (2.39))

$$\partial_\zeta\left(P^2\partial_\zeta K\right) = 0 \quad \text{and} \quad \text{c.c.}\,, \tag{2.41}$$

  together with the constraint (2.40), makes it type D. There are two independent type D

[6] Notice that Robinson–Trautman spacetimes were originally designed to be algebraically special – see [25] for more information regarding the principal null directions of Robinson–Trautman.

[7] Notice a useful identity: $\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) = \frac{1}{P^2}\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)$.

solutions:

1. The Schwarzschild, reached with $P = 1 + \frac{\epsilon}{2}\zeta\bar{\zeta}$ and $K = \epsilon$, which is asymptotically anti-de Sitter.
2. The C-metric, which requires $P^2\partial_\zeta K = h(\bar{\zeta}) \neq 0$ and is asymptotically locally anti-de Sitter due to a non-vanishing boundary Cotton tensor.

Let us mention here that the time dependence of M remains arbitrary, and can be reabsorbed by performing an appropriate bulk diffeomorphism, inducing a conformal transformation plus a diffeomorphism on the boundary [2]. The Robinson–Trautman equation reads then:

$$\partial_{\bar{\zeta}}\partial_\zeta K = 3M\partial_t\left(\frac{1}{P^2}\right) \tag{2.42}$$

with constant M. We will adopt this convention for the rest of our presentation.

Before moving to the hydrodynamic analysis of the energy–momentum tensor, we would like to end the current section with some general comments regarding the bulk Einstein spaces under consideration.

With the exception of the Petrov-D solutions quoted above, Robinson–Trautman spacetimes are time-dependent and carry gravitational radiation. Once this radiation is emitted, the spacetime settles down generically to an anti-de Sitter Schwarzschild black hole. The general features of this evolution are captured by the Robinson–Trautman equation, which, following [27], is a parabolic equation describing a Calabi flow on a two-surface. As long as $M \neq 0$, these spacetimes exhibit a past singularity at $r = 0$, past-trapped two-surfaces and a future horizon, which is the anti-de Sitter Schwarzschild horizon at late times. Unfortunately, singularities are often developed on this horizon and no smooth extension is possible beyond, in the interior region.

Irregularities of the two-surface $\mathcal{S}$ time-dependent metric

$$\mathrm{d}\ell^2 = \frac{2}{k^2 P(t,\zeta,\bar{\zeta})^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \tag{2.43}$$

possibly present at early times, are washed out by the evolution, as usual with geometric flows. The flow at hand, governed by the Robinson–Trautman equation (2.42), has the following salient properties:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}}\frac{\mathrm{d}^2\zeta}{P^2} = 0, \tag{2.44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}}\frac{\mathrm{d}^2\zeta}{P^2}K = 0, \tag{2.45}$$

where $\mathrm{d}^2\zeta = -i\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$ (this assumes there are no boundary-like contributions – the proof will be given and commented in Sec. 3). Hence, the area of $\mathcal{S}$ and its average curvature (*i.e.*

the Euler number) are preserved along the flow, which, at late times, brings the metric into a symmetric geometry compatible with the original topology. From the spacetime perspective, this situation corresponds indeed to the evolution towards an anti-de Sitter Schwarzschild black hole with conformal boundary $\mathbb{R} \times S^2$, E_2 or H_2.[8]

Closing this chapter, one should observe that Robinson–Trautman spacetimes appear as laboratories for investigating time-dependent black-hole exact solutions surrounded by gravitational radiation. As opposed to the stationary paradigms, very little is known here, even at a very elementary level: location of past horizon, definition of thermodynamic quantities such as energy, temperature or entropy, interpretation of the evolution as out-of-equilibrium thermodynamics. This is surprising because understanding deviations from equilibrium in these systems is at least as important as counting their microscopic degrees of freedom, which has attracted more attention. Any further comment on bulk thermodynamics would be, at this stage, daring.

3 The Robinson–Trautman holographic fluid

Following the general plan presented in Sec. 2.1, we have reached Robinson–Trautman spacetimes in Sec. 2.2, using in the derivative expansion (2.1), the boundary metric (2.16), the boundary energy–momentum tensor (2.32) and the boundary fluid velocity field (2.21). The latter defines the hydrodynamic frame where the resummation of the derivative expansion is successfully performed – for reasons that we have already discussed. This frame turns out to be very natural for describing the fluid properties.

3.1 The hydrodynamic frame and the fluid transport data

In the case at hand, the energy density of the fluid reads:[9]

$$\varepsilon = T_{\mu\nu}u^\mu u^\nu = \frac{Mk^2}{4\pi G}, \tag{3.1}$$

and is constant, as is the pressure ($\varepsilon = 2p$). We can split the energy–momentum tensor (see App. A and e.g. [29, 30]) as

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \tau_{\mu\nu} + u_\mu q_\nu + u_\nu q_\mu, \tag{3.2}$$

[8]The Calabi flow is set for a metric on a compact Kähler space, here two-dimensional. For this reason it was quoted in [4] for spherical geometry only. Probably, E_2 or H_2 could also support this flow, assuming they were made compact by modding out some discrete isometry. This line has not attracted much attention, and at present Calabi-flow results do not cover all Robinson–Trautman geometries. The statements regarding late-time behaviour should therefore be taken with care as they have not been demonstrated for all possible initial conditions. In particular, the possibility of reaching the C-metric has been discussed in [28]. In that work it was shown that Robinson–Trautman spacetimes admitting a space-like isometry generically decay to the C-metric.

[9]As pointed out in App. B, the kinematical out-of-equilibrium quantities $\boldsymbol{\varepsilon}$, $\boldsymbol{p}$ and $\boldsymbol{\varrho}$ are chosen to coincide with the thermodynamic local-equilibrium ε, p and ϱ.

with a conformal-perfect-fluid part

$$\mathrm{T}^{(0)} = \frac{\varepsilon}{2}\left(3\mathrm{u}^2 + \mathrm{ds}^2\right) \tag{3.3}$$

and a non-perfect piece $\tau_{\mu\nu} + u_\mu q_\nu + u_\nu q_\mu$, where $\tau_{\mu\nu}$ and q_μ are the components of the *stress tensor* and the *heat current* respectively. These are fully transverse:

$$\tau_{\mu\nu} u^\mu = 0, \quad q_\mu u^\mu = 0 \tag{3.4}$$

with

$$q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{3.5}$$

The non-perfect piece $u_\mu q_\nu + u_\nu q_\mu$ is *non-transverse*. The latter is absent in the Landau–Lifshitz frame.

Here we are *not in the Landau–Lifshitz*, but rather *in the Eckart* frame (see App. B for a detailed discussion on this subject). To show this we should consider the more general charged Robinson–Trautman solution, which solves bulk Einstein–Maxwell equations and has a conserved current J on the boundary.[10] In these solutions, the electromagnetic field has three components: magnetic, electric and radiation. On the boundary, there is a conserved current, a chemical potential and a magnetic field [31]. The latter couples to the current as $\nabla_\mu T^{\mu\nu} = 4\pi G J_\mu F^{\mu\nu}$, and vanishes if and only if the bulk radiation component is absent. In this case of ideal magnetohydrodynamics,[11] is again governed by the plain Robinson–Trautman equation, and the conserved current has the perfect form ($j_\nu = 0$ in (A.14)):

$$J_\nu = \varrho u_\nu \tag{3.6}$$

with

$$\varrho = \frac{k^2 Q}{4\pi G} P(t, \zeta, \bar{\zeta})^2 \tag{3.7}$$

and Q an arbitrary constant. This demonstrates the statement regarding the Eckart frame, since the current is fully longitudinal and perfect.

In the Eckart frame, the heat current is non-vanishing and we find, using (3.5),

$$\mathrm{q} = -\frac{1}{16\pi G}\left(\partial_\zeta K\, \mathrm{d}\zeta + \partial_{\bar{\zeta}} K \mathrm{d}\bar{\zeta}\right). \tag{3.8}$$

[10] Conserved currents may also appear without extra degrees of freedom, in systems with symmetries generated by Killing vectors k. Indeed, in those situations $k_\nu T^{\mu\nu}$ are components of divergence-free vectors. Since Robinson–Trautman spacetimes have generically no isometries, we will not investigate this direction.

[11] Keeping the radiation component opens the field of general magnetohydrodynamics – see [32] for a related discussion, and [33] for a more general perspective.

The non-perfect stress tensor (we have used the identity of footnote 7) is given by

$$\tau = \frac{1}{8\pi G k^2 P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right). \tag{3.9}$$

It reflects the friction, which is of kinematic origin. Hence, it is not surprising that we can express it in terms of the orthogonally projected covariant derivatives (see App. A) of the fluid velocity:[12]

$$\begin{aligned}\tau_{\mu\nu} &= -\frac{1}{16\pi G k^2}\left(D_\mu D_\nu\Theta - \frac{1}{2}h_{\mu\nu}D^\lambda D_\lambda\Theta\right)\\ &= -\frac{1}{16\pi G k^2}\left(h_\mu{}^\rho h_\nu{}^\sigma\nabla_\rho h_\sigma{}^\lambda\nabla_\lambda\Theta - \frac{1}{2}h_{\mu\nu}\nabla_\rho h^{\rho\sigma}\nabla_\sigma\Theta\right).\end{aligned} \tag{3.10}$$

This is not possible for q though. Generically, the heat flow cannot be expressed as a pure u-derivative expansion, it also involves the gradient of scalars like the temperature or the curvature, and betrays thermal conduction or similar phenomena.

As already mentioned in Sec. 2.2, when dealing with exact algebraically special Einstein spaces, the holographic energy–momentum tensor receives at most third-order derivative corrections with respect to the perfect fluid. The reason is simple. The bulk algebraic structure sets an intimate relationship between the energy–momentum tensor and the Cotton tensor, which is a third derivative of the boundary metric. Since the shearless velocity field is determined by the geometry itself, the energy–momentum is necessarily expressed with third derivatives of the velocity field.

This property is very general. It was extensively discussed in a wide class of situations like the Plebański–Demiański family, where the energy–momentum tensor is either third-order in u-derivatives (in the presence of a bulk acceleration parameter) [14], or is perfect [11]. This latter case does not imply that the fluid is perfect: some of the would-be corrections vanish just because of kinematic reasons (as $-2\eta\sigma_{\mu\nu}$), some other because infinite series of transport coefficients are indeed zero for the holographic fluid at hand.

In the Robinson–Trautman case, the unique available transport coefficient is read-off in q (Eq. (3.8)) or in τ (Eq. (3.9)). This coefficient is of order $1/16\pi G$, and we will further comment on it in Sec. 3.2. As long as we remain within Robinson–Trautman solutions, this is the only information we can get, and it is exact. Of course, in order to have access to more transport coefficients (possibly infinite series of them), we can consider changing hydrodynamic frame. But even in that case, the new ones will all stem out of the former, and all will be of

[12] In our case, due to the absence of shear, vorticity and acceleration, the velocity derivatives are expressed only in terms of derivatives of the expansion, as for example:

$$\nabla_\lambda\nabla_\mu u_\nu = \frac{1}{2}\partial_\lambda\Theta h_{\mu\nu} + \frac{1}{4}\Theta^2\left(h_{\lambda\mu}u_\nu + h_{\lambda\nu}u_\mu\right).$$

the same order.

For example, it is possible to move from Eckart to Landau–Lifshitz frame. As explained thoroughly in App. B, this requires some care. At the first place, these frames are built assuming the existence of a conserved matter current. Moving from Eckart to Landau–Lifshitz trades the heat current of the conserved energy–momentum tensor in Eckart for the transverse part of the matter current in Landau–Lifshitz. This is conceivable for the charged Robinson–Trautman, but audacious for the neutral case. At a second stage, the actual transformation is performed perturbatively, order by order in a parameter, which is $\|\mathrm{q}\|$ (see App. B for detailed expressions), required to be small compared to the energy scale. These series are usually asymptotic.

This philosophy was originally pursued in [3] with success regarding the determination of transport coefficients. Still, it has some caveats. From the mathematical viewpoint, this amounts to trading an exact quantity like τ or q, for an infinite series, which in general lacks convergence. Physics-wise, moving to Landau–Lifshitz blurs the simple and clear picture, which emerges in the Eckart frame as we will see; moreover, doing so while ignoring the matter current j is inappropriate, in particular when computing the entropy current (see Sec. 3.3).[13]

3.2 Physics and evolution in the Eckart frame

In the Eckart frame, the pressure is constant and the fluid is at rest on a spatial section $\mathcal{S}$ equipped with a metric $\mathrm{d}\ell^2$ (Eq. (2.43)). The physical phenomena taking place in the fluid are related to thermal conduction, materialized in the heat current q, Eq. (3.8), and captured by the Robinson–Trautman equation (2.42) appearing as the time component of the energy–momentum conservation (2.35). This is a heat-flow equation, and one can elegantly derive it directly from the general heat-current-divergence equation displayed in (A.17). In the case under investigation, a_μ, $\sigma_{\mu\nu}$ and $g_{\mu\nu}\tau^{\mu\nu}$ vanish, whereas ε is constant, so (A.17) reads:

$$\mathrm{div}_{(2)}\mathrm{q} = -\frac{3\varepsilon}{2}\Theta. \tag{3.11}$$

We have introduced $\mathrm{div}_{(2)}\mathrm{q} = \nabla_{(2)i}\, q^i$, which is equal to $\nabla_\mu q^\mu$ because q is transverse with respect to the hypersurface-orthogonal vector $\mathrm{u} = \partial_t$, so exclusively defined inside the spatial section $\mathcal{S}$. Geometric quantities referring to this surface and to the corresponding metric $\mathrm{d}\ell^2$ will carry a subindex "(2)":

- antisymmetric tensor: $\eta_{(2)\zeta\bar{\zeta}} = -\frac{i}{k^2P^2}$, and volume form: $\Omega_{(2)} = -i\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{k^2P^2} = \frac{\mathrm{d}^2\zeta}{k^2P^2}$;
- Laplacian operator: $\triangle_{(2)}f = k^2\triangle f = 2k^2P^2\partial_\zeta\partial_{\bar{\zeta}}f$, and scalar curvature: $R_{(2)} = 2k^2K$;

[13] The same attitude was adopted later on by the authors of [4], who insist in moving to Landau–Lifshitz in their follow-ups [34, 35].

- Hodge–Poincaré duality: $\mathrm{q} = q_\zeta \mathrm{d}\zeta + q_{\bar\zeta}\mathrm{d}\bar\zeta \Leftrightarrow \underset{(2)}{\star}\mathrm{q} = i\left(q_\zeta \mathrm{d}\zeta - q_{\bar\zeta}\mathrm{d}\bar\zeta\right)$.

Substituting in Eq. (3.11) the heat current (3.8) expressed as

$$\mathrm{q} = -\frac{1}{16\pi G}\mathrm{d}_{(2)}K, \tag{3.12}$$

the expansion Θ given in (2.22), and the constant energy density (3.1), we find indeed the Robinson–Trautman equation (2.42):

$$\partial_{\bar\zeta}\partial_\zeta K = 3M\partial_t\left(\frac{1}{P^2}\right).$$

Equation (3.11) can be used in integral form, over a fixed domain $\mathcal{D} \subseteq \mathcal{S}$ with boundary $\partial\mathcal{D}$. Thanks to Green's theorem,[14] we find:

$$\int_{\mathcal{D}} \frac{\mathrm{d}^2\zeta}{k^2P^2}\varepsilon\,\Theta = -\frac{2}{3}\oint_{\partial\mathcal{D}} \underset{(2)}{\star}\mathrm{q}. \tag{3.13}$$

Using specifically (2.22) for Θ, (3.1) for ε and (3.12) for q, we finally obtain:

$$k^2\frac{\mathrm{d}A_{\mathcal{D}}}{\mathrm{d}t} = \frac{i}{6M}\oint_{\partial\mathcal{D}}\left(\partial_\zeta K \mathrm{d}\zeta - \partial_{\bar\zeta}K\mathrm{d}\bar\zeta\right), \tag{3.14}$$

where

$$A_{\mathcal{D}} = \int_{\mathcal{D}}\frac{\mathrm{d}^2\zeta}{k^2P^2} \tag{3.15}$$

is the area of the domain $\mathcal{D}$. Multiplying by ε, the total energy stored by the fluid inside $\mathcal{D}$,

$$E_{\mathcal{D}} = \frac{M}{4\pi G}\int_{\mathcal{D}}\frac{\mathrm{d}^2\zeta}{P^2} \tag{3.16}$$

obeys

$$\frac{\mathrm{d}E_{\mathcal{D}}}{\mathrm{d}t} = \frac{i}{24\pi G}\oint_{\partial\mathcal{D}}\left(\partial_\zeta K\mathrm{d}\zeta - \partial_{\bar\zeta}K\mathrm{d}\bar\zeta\right). \tag{3.17}$$

Assuming $\mathcal{S}$ be a compact surface without boundaries, from Eq. (3.14), we conclude that the total area of $\mathcal{S}$, $A = A_{\mathcal{S}}$ remains constant in time.[15] This demonstrates (2.44). Accordingly, the total energy $E = E_{\mathcal{S}} = \varepsilon A$ is also conserved. Along time, the spatial section $\mathcal{S}$ hosting the fluid evolves and the fluid energy, conserved in total, moves from one region to another. With reasonable initial conditions, the system stabilizes at large times in a configu-

[14] Reminder of Green's theorem: for any vector/one-form v

$$\int_{\mathcal{D}}\frac{\mathrm{d}^2\zeta}{k^2P^2}\mathrm{div}_{(2)}\mathrm{v} = \oint_{\partial\mathcal{D}}\underset{(2)}{\star}\mathrm{v}.$$

[15] Under appropriate assumptions for K asymptotics, $\mathcal{S}$ could even be non-compact, and its area infinite.

ration with spatially constant K (see discussion at the end of Sec. 2.2).

Summarizing, the Robinson–Trautman holographic fluid is at rest in the Eckart frame and is subject to thermal conduction, with energy exchanges operating according to the dynamics described above, and driven by the heat current (3.8).

In order to simplify our discussion and fit within the framework of the the Robinson–Trautman spacetime built in Sec. 2.2, we will consider from now on vanishing chemical potential. This choice is holographically achievable [31]. We could alternatively set the density to zero; all of our conclusions would hold in that case, but we find the former option more convenient. Following (B.6) and (B.7), we find for the conformal fluid at hand the temperature as related to the energy density by standard Stefan's law:

$$\varepsilon = \sigma T^3 = \frac{Mk^2}{4\pi G} \tag{3.18}$$

with $\sigma = \frac{8\pi^2 G^2}{27k^4}$. Hence the local-equilibrium thermodynamic temperature T is constant.

The heat current of the Robinson–Trautman fluid can be expressed, like for any fluid, as a derivative expansion in the temperature, and in geometric or kinematic tensors. In the present case, however, this current is known exactly, and contains a single term, that would appear at third order in the derivative expansion. The would-be first-order term, displayed in the generic expression (B.4), is absent here. In this expression, appears the local-thermodynamic-equilibrium temperature T, given in (3.18), which is constant. Since the acceleration is vanishing, the first order does not contribute indeed.

One may be puzzled at this stage, discussing thermal conduction without temperature gradients. This attitude is probably too naive. As explained in App. B, quantities like temperature or chemical potential lack a microscopic definition when out-of-equilibrium phenomena take place. Even though the hydrodynamic hypothesis of local thermodynamic equilibrium may be justified, the local-equilibrium temperature $T(x)$ (in fact constant here) or chemical potential $\mu(x)$ (absent in our case) do not exhaust all available information, and more is captured in the kinematical, out-of-equilibrium functions $\boldsymbol{T}(x)$ and $\boldsymbol{\mu}(x)$.

The origin of the transport phenomena witnessed here being in essence geometric, it is tempting, inspired by (B.4), to recast the exact expression of the current (3.8) as

$$q_\mu = -\kappa D_\mu \boldsymbol{T} \tag{3.19}$$

with

$$\kappa \boldsymbol{T}(t,\zeta,\bar{\zeta}) = \kappa T + \frac{1}{16\pi G}\left(K(t,\zeta,\bar{\zeta}) - \langle K \rangle\right). \tag{3.20}$$

The Gaussian curvature $K(t,\zeta,\bar{\zeta})$ contributes thus to a kind of kinematical, out-of-equilibrium temperature $\boldsymbol{T}(t,\zeta,\bar{\zeta})$. It is naturally accompanied with a heat conductivity, read off as its coefficient in (3.20):

$$\kappa = \frac{1}{16\pi G}. \tag{3.21}$$

The latter is of geometric origin, as the transport phenomenon it triggers. This result is in agreement with the general analysis performed in [36].

In expression (3.20), we have introduced T given in (3.18), and the average curvature[16] over $\mathcal{S}$:

$$\langle K \rangle = \frac{1}{A} \int_{\mathcal{S}} \frac{\mathrm{d}^2\zeta}{k^2 P^2} K. \tag{3.22}$$

This turns out to be constant, as advertised in (2.45). Indeed, one easily shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}} \frac{\mathrm{d}^2\zeta}{P^2} K = -\frac{i}{2} \oint_{\partial\mathcal{D}} \left(\partial_\zeta \Theta \, \mathrm{d}\zeta - \partial_{\bar\zeta} \Theta \, \mathrm{d}\bar\zeta \right), \tag{3.23}$$

which vanishes when $\mathcal{D} = \mathcal{S}$, under the already spelled assumptions.[17] For asymptotic time, $K(t, \zeta, \bar\zeta)$ is expected to converge towards a constant, which is therefore identified with $\langle K \rangle$. Hence

$$\lim_{t \to +\infty} \boldsymbol{T}(t, \zeta, \bar\zeta) = T. \tag{3.24}$$

At late times, the fluid reaches global equilibrium with the kinematical temperature equal to the thermodynamic-equilibrium temperature, as expected. At any time, the thermodynamic-equilibrium temperature is the average kinematical temperature: $\langle \boldsymbol{T}(t, \zeta, \bar\zeta) \rangle = T$.

The validity of holographic approach in the present framework requires a large black-hole mass, hence a large temperature T. This leaves room for initial conditions on $P(t, \zeta, \bar\zeta)$ that do not violate the positivity of $\boldsymbol{T}(t, \zeta, \bar\zeta)$. Actually, the latter may not be mandatory since $\boldsymbol{T}(t, \zeta, \bar\zeta)$ is an instrument for probing transport, and not a fundamental quantity defined *ab initio* – reason why we insist calling it "kinematical, out-of-equilibrium temperature" as opposed to "local-thermodynamic-equilibrium temperature" (see discussion in App. B).

3.3 The entropy current and its conservation

The last important aspect of the Robinson–Trautman fluid dynamics we would like to discuss is the entropy, the associated current and its divergence. For the conformal case in three dimensions, the standard entropy current is given in (B.19) in the Eckart frame, and

[16] Defined as a limit for a non-compact surface.

[17] The identity (3.23) does not require the Robinson–Trautman equation to be satisfied. It is thus valid for any dynamics and not necessarily for the Calabi flow. Actually it reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}} \frac{\mathrm{d}^2\zeta}{P^2} \triangle f = i \oint_{\partial\mathcal{D}} \left(\partial_\zeta \partial_t f \, \mathrm{d}\zeta - \partial_{\bar\zeta} \partial_t f \, \mathrm{d}\bar\zeta \right),$$

for any function $f(t, \zeta, \bar\zeta)$.

reproduced here for clarity:

$$S^\mu = \frac{1}{T}\big((3p - \mu\varrho)u^\mu + q^\mu\big). \tag{3.25}$$

We remind that in this expression the local-equilibrium thermodynamic quantities and relations are used, following the discussion of App. B, as determined in the Eckart frame. It applies to the more general charged Robinson–Trautman solution with density displayed in Eq. (3.7). Since we have chosen zero chemical potential, the second term drops,[18] and the entropy is constant:

$$s = \frac{3p}{T} = \frac{3\sigma T^2}{2} = \left(\frac{M}{4}\right)^{2/3}. \tag{3.26}$$

In this case, the entropy current reads:

$$\mathrm{S} = \left(\frac{M}{4}\right)^{2/3}\left(\partial_t - \frac{P^2}{6M}\left(\partial_{\bar\zeta}K\,\partial_\zeta + \partial_\zeta K\,\partial_{\bar\zeta}\right)\right). \tag{3.27}$$

Using the general expression for the entropy-current divergence (B.20), we obtain:

$$\nabla_\mu S^\mu = 0. \tag{3.28}$$

This is the consequence of the local-equilibrium temperature and pressure being constant, and of the vanishing chemical potential, shear and acceleration. Put differently, s and T being both constant, the current S is divergence-free as a consequence of a fine cancellation between the velocity expansion Θ and the divergence of the heat current, displayed in (3.11).

The conservation of the entropy current is surprising at first sight because we are seemingly out of equilibrium and evolution towards equilibrium usually produces entropy. However, the thermal-conduction irreversible phenomenon described by the Robinson–Trautman dynamics is of geometric nature. Hence, it can reasonably accommodate a conserved entropy current. Indeed, the fluid is at rest. The evolution preserves the area and the energy, and occurs at a constant average kinematical temperature, equal to the local-equilibrium temperature. At the same time the absence of acceleration and shear wash out the effects of the heat current and the stress friction (see (B.16)), and the process ultimately appears as an adiabatic, even isentropic, redistribution of energy due to the kinetics of the surface rather than to the motion of the fluid, till the final global-equilibrium state is reached. In thermodynamic language this is a special case of *isothermal* Carnot's path,[19] known as Moutier's [37], which produces no work and has zero thermodynamic efficiency.[20] Carnot's evolution is reversible and this does not contradict anything here, as the origin of irreversibility for the described phenomenon is purely geometrical.

[18] This term also drops for vanishing density.

[19] We use intentionally "path" rather than "cycle" as in the process under consideration the system does not come back to the original state because of the time-evolving geometry.

[20] The thermodynamic efficiency of a cycle is defined as $\boldsymbol{\eta} = 1 - T_{\min}/T_{\max}$.

The above conclusion is *frame-independent* as is the actual entropy current. The latter can be expressed alternatively as in Eq. (B.17):

$$\mathrm{S} = s_{\mathrm{LL}}\mathrm{u}_{\mathrm{LL}} - \frac{\mu_{\mathrm{LL}}}{T_{\mathrm{LL}}}\mathrm{j}_{\mathrm{LL}}, \tag{3.29}$$

where all observables are evaluated in the Landau–Lifshitz frame. Following App. B, these observables appear as series expansions around their Eckart-frame counterparts, in powers of the heat-current norm $\|\mathrm{q}\|$. The latter, displayed below in (3.34), is inevitably unbounded for Robinson–Trautman because of the singular future behaviour of K. The validity of the hydrodynamic-frame change is therefore limited. This problem has been avoided in our preceding analysis, performed directly and exactly in the original Eckart frame.

Although in Eckart's our choice has been $\mu \equiv \mu_{\mathrm{E}} = 0$, this is no longer true in Landau–Lifshitz's (see (B.25) and (B.32)):[21]

$$\delta\left(\frac{\mu}{T}\right) = \frac{\mathrm{q}\cdot\tau\cdot\mathrm{q}}{\varrho T \mathrm{q}^2} - \frac{1}{\varrho T(p+\varepsilon)}\left(\mathrm{q}^2 + \frac{\mathrm{q}\cdot\tau\cdot\tau\cdot\mathrm{q}}{\mathrm{q}^2} - \left(\frac{\mathrm{q}\cdot\tau\cdot\mathrm{q}}{\mathrm{q}^2}\right)^2\right) + \cdots, \tag{3.30}$$

where the dots stand for higher-order terms in $\|\mathrm{q}\|$. As a consequence, in this frame, the entropy current (3.29) receives two distinct non-vanishing contributions, $\mathrm{S} = \mathrm{S}_{\mathrm{LL1}} + \mathrm{S}_{\mathrm{LL2}}$:

$$\mathrm{S}_{\mathrm{LL1}} = s_{\mathrm{LL}}\mathrm{u}_{\mathrm{LL}} = s\mathrm{u} + \frac{s}{p+\varepsilon}\mathrm{q} - \frac{\mu\varrho\mathrm{q}^2}{T(p+\varepsilon)^2}\mathrm{u} - s\frac{\tau\cdot\mathrm{q}}{(p+\varepsilon)^2} + \cdots, \tag{3.31}$$

$$\mathrm{S}_{\mathrm{LL2}} = -\frac{\mu_{\mathrm{LL}}}{T_{\mathrm{LL}}}\mathrm{j}_{\mathrm{LL}} = \frac{\mu\varrho}{T(p+\varepsilon)}\mathrm{q} + \frac{\mu\varrho\mathrm{q}^2}{T(p+\varepsilon)^2}\mathrm{u} + s\frac{\tau\cdot\mathrm{q}}{(p+\varepsilon)^2} - \cdots, \tag{3.32}$$

and we have used the explicit perturbative transformation rules provided in (B.22)–(B.32) (quantities without indices are evaluated in the Eckart frame). These two expressions are general and valid for any fluid. They sum up to $s\mathrm{u} + \mathrm{q}/T$, expression of S in the Eckart frame.

In the Robinson–Trautman conformal holographic fluid, the heat current q is given in (3.8):

$$\mathrm{q} = -\frac{k^2 P^2}{16\pi G}\left(\partial_{\bar\zeta} K \,\partial_\zeta + \partial_\zeta K\,\partial_{\bar\zeta}\right), \tag{3.33}$$

and its norm squared is

$$\mathrm{q}^2 = \frac{k^2 P^2}{128\pi^2 G^2}\partial_\zeta K\,\partial_{\bar\zeta} K, \tag{3.34}$$

while

$$\tau\cdot\mathrm{q} = -\frac{1}{128\pi^2 G^2}\left(\partial_{\bar\zeta} K\,\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta + \partial_\zeta K\,\partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta\right). \tag{3.35}$$

For vanishing chemical potential, $\mu = 0$ (or for vanishing density, $\varrho = 0$), the above equa-

[21] We use the notation $\mathrm{q}\cdot\tau\cdot\mathrm{q} = \tau_{\mu\nu}\,q^\mu q^\nu$ and similarly for other terms and contractions.

tions read:

$$\mathrm{S}_{\mathrm{LL1}} = \mathrm{S} - s\frac{\tau\cdot\mathrm{q}}{(3p)^2} + \cdots, \tag{3.36}$$

$$\mathrm{S}_{\mathrm{LL2}} = s\frac{\tau\cdot\mathrm{q}}{(3p)^2} - \cdots \tag{3.37}$$

with $p = \varepsilon/2$ given in (3.1), s in (3.26), S in (3.27) and $\tau\cdot\mathrm{q}$ in (3.35). None of the two pieces of the entropy current displayed in the Landau–Lifshitz frame (3.36) and (3.37) is divergence-free, but the sum is:

$$\begin{aligned}\nabla\cdot\mathrm{S}_{\mathrm{LL1}} &= -\nabla\cdot\mathrm{S}_{\mathrm{LL2}} \\ &= -\frac{s}{(3p)^2}\nabla\cdot(\tau\cdot\mathrm{q}) \\ &= \frac{P^2}{18k^2(2M)^{4/3}}\left(\partial_\zeta\left(\partial_\zeta K\,\partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\right)+\text{c.c.}\right).\end{aligned} \tag{3.38}$$

In previous analyses of the Robinson–Trautman fluid, $\mathrm{S}_{\mathrm{LL1}} = s_{\mathrm{LL}}\mathrm{u}_{\mathrm{LL}}$ was used alone as an entropy current, leading to the conclusion that it is not conserved.[22] This amounts to setting $\mu_{\mathrm{LL}} = 0$ in (3.29), which in turn would require $\mu_{\mathrm{E}} \neq 0$. Since in these works no chemical potential was introduced in the original frame reached holographically, it seems to us that the choice made subsequently for the entropy current is unjustified. Deciding which is the best choice for this current is certainly a long debate that we will not pursue here. Our choice is the standard one, originally proposed by Landau and Lifshitz [38]. More importantly, it is frame-invariant provided one is careful in trading the heat current q for a transverse matter current j, when discussing the change of hydrodynamic frame. This is often disregarded in the literature.

4 Conclusions

We would like now to summarize our analysis, which is twofold.

The first side concerns the general reconstruction of exact bulk Einstein spacetimes, from boundary data obeying appropriate conditions. This reconstruction is a resummation of the hydrodynamic derivative expansion, for which we choose a shearless congruence. Given a boundary metric, such a congruence is basically unique and has a double virtue: (i) reducing the number of terms allowed by conformal invariance, hence making the resummation potentially tractable;[23] (ii) being promoted into a bulk null, geodesic and shearless congruence, whenever the resummation is successful. This last feature makes the bulk algebraically spe-

[22] The expressions for $\mathrm{S}_{\mathrm{LL1}}$ and $\nabla\cdot\mathrm{S}_{\mathrm{LL1}}$ of [4] differ from the ones displayed here, Eqs. (3.36) and (3.38), because of technical inaccuracies.

[23] In the presence of shear the plethora of compatible terms makes the exercise difficult.

cial by Goldberg–Sachs theorem, and naturally expressed it in Eddington–Finkelstein coordinates. Moreover, it crucially sets a relationship between the boundary energy–momentum tensor and the Cotton tensor, through the structure it imposes on the reference conserved tensors $\mathrm{T}^{\pm} = \mathrm{T} \pm \frac{i}{8\pi G k^2}\mathrm{C}$, which is of prime importance. This scheme allows for a direct boundary control of the bulk Petrov type, and recasts the conservation of T as a bulk integrability equation, interpreted on the boundary as a heat-flow equation.

The method at hand is general and enables us to reach all known algebraically special Einstein spacetimes (see e.g. [11, 14] for the Plebański–Demiański class). It is fair to quote, though, that the issue of Petrov general spacetimes is still open, together with the rôle that a boundary shearless congruence will play in this case, or, stated differently, the possibility of reconstructing such spacetimes with shearless fluid velocities. Leaving aside this question, we have followed the pattern for a general boundary class with a shearless congruence without vorticity and reached the entire Robinson–Trautman family. The Robinson–Trautman equation comes out here holographically as the boundary energy–momentum conservation equation, given the structure the latter acquires from its relationship with the Cotton tensor.

The last property brings us to the second part of the present work, more specifically dedicated to the physics of the holographic fluid. Three main features emerge for it: (i) the hydrodynamic frame associated with the congruence at hand is the Eckart frame; (ii) in this frame, the energy–momentum tensor receives only third-order derivative corrections; (iii) the energy–momentum conservation is non-trivial in the time direction, and appears as the heat equation for the fluid. These properties can be traced back to our original choice of shearless congruence, and to the consequences it has both for the bulk and for the boundary. They are all expected to be generic for exact and algebraically special Petrov Einstein spaces, and valid beyond the Robinson–Trautman paradigm.

Here, the fluid is at rest on a surface which evolves in time keeping its area constant. The fluid has constant pressure and constant energy density. The transport phenomena occurring can be assimilated with thermal conduction, which drives the system towards global equilibrium by continuously redistributing a conserved total energy on the moving surface, in a fashion reminiscent of Solaris' ocean dynamics [39]. This is achieved according to the Calabi flow, here revealed as a genuine heat flow. The interpretation of the Gaussian curvature of the surface as the time-dependent part of a kinematical out-of-equilibrium temperature, and the exact determination of the corresponding geometric heat conductivity are novelties of our work. They provide a natural thermal-like interpretation to the geometric flow.

The other important aspect unravelled here concerns the hydrodynamic frame. The holographic fluids dual to exact Einstein (more precisely Einstein–Maxwell in order to produce a boundary current) spacetimes emerge often in the Eckart frame. Then, not only is the conserved current perfect, but the corrections to the energy–momentum tensor with respect to the perfect fluid are restricted and canonically related to the third derivatives of the metric and the velocity. This makes the fluid dynamics clear and provides a rich information on series of vanishing transport coefficients. It is unfortunate that in the framework of holography one systematically tries to reach the Landau–Lifshitz frame, irrespective of the context. This leads sometimes to inconsistencies, as we pointed out e.g. regarding the entropy current.

The present analysis of the Robinson–Trautman boundary fluid, and other studies of exact-Einstein-space holography, suggest that the underlying fluid dynamics is quite peculiar. The system is time-dependent and evolves generically towards equilibrium by thermal conduction. This process is of geometric origin though, as it is driven by the evolution of the surface itself, and is associated to a very specific correction with respect to perfect fluidity. Furthermore energy and area are conserved, and the standard entropy current has no divergence. Entropy is thus conserved as a fine tuning inside the out-of equilibrium process at hand. There is nothing to be worried about this state of affairs, except that one might legitimately question the practical usefulness of these holographic systems and the interest in elaborating further on their transport properties. In contrast, the investigation of this distinctive conformal fluid dynamics, might shed light on black-hole out-of-equilibrium thermodynamics, which is still in a quite primitive state.

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A On vector-field congruences

Consider a D-dimensional Lorentzian metric $g_{\mu\nu}$ and an arbitrary time-like vector field $\mathrm{u} = u^\mu \partial_\mu$, normalized as $u_\mu u^\mu = -1$, later identified with the fluid velocity. Its integral curves define a congruence which is characterized by its acceleration, shear, expansion and vorticity:

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \frac{1}{D-1}\Theta\, h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{A.1}$$

with[24]

$$
\begin{aligned}
a_\mu &= u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, && \text{(A.2)}\\
\sigma_{\mu\nu} &= \frac{1}{2} h_\mu{}^\rho h_\nu{}^\sigma \left(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho\right) - \frac{1}{D-1} h_{\mu\nu} h^{\rho\sigma} \nabla_\rho u_\sigma && \text{(A.3)}\\
&= \nabla_{(\mu} u_{\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{D-1} \Theta\, h_{\mu\nu}, && \text{(A.4)}\\
\omega_{\mu\nu} &= \frac{1}{2} h_\mu{}^\rho h_\nu{}^\sigma \left(\nabla_\rho u_\sigma - \nabla_\sigma u_\rho\right) = \nabla_{[\mu} u_{\nu]} + u_{[\mu} a_{\nu]}. && \text{(A.5)}
\end{aligned}
$$

These tensors satisfy several simple identities:

$$
u^\mu a_\mu = 0, \quad u^\mu \sigma_{\mu\nu} = 0, \quad u^\mu \omega_{\mu\nu} = 0, \quad u^\mu \nabla_\nu u_\mu = 0, \quad h^\rho{}_\mu \nabla_\nu u_\rho = \nabla_\nu u_\mu, \tag{A.6}
$$

and we have introduced the longitudinal and transverse projectors:

$$
U^\mu{}_\nu = -u^\mu u_\nu, \quad h^\mu{}_\nu = u^\mu u_\nu + \delta^\mu_\nu, \tag{A.7}
$$

where $h_{\mu\nu}$ is also the induced metric on the local plane orthogonal to u. The projectors satisfy the usual identities:

$$
U^\mu{}_\rho U^\rho{}_\nu = U^\mu{}_\nu, \quad U^\mu{}_\rho h^\rho{}_\nu = 0, \quad h^\mu{}_\rho h^\rho{}_\nu = h^\mu{}_\nu, \quad U^\mu{}_\mu = 1, \quad h^\mu{}_\mu = D-1. \tag{A.8}
$$

It is customary to define the orthogonally projected covariant derivative acting on any tensor as

$$
D_\gamma T_{\alpha_1 \dots \alpha_p}{}^{\beta_1 \dots \beta_q} = h_\gamma{}^\lambda h_{\alpha_1}{}^{\mu_1} \dots h_{\alpha_p}{}^{\mu_p} h_{\nu_1}{}^{\beta_1} \dots h_{\nu_q}{}^{\beta_q} \nabla_\lambda T_{\mu_1 \dots \mu_p}{}^{\nu_1 \dots \nu_q}. \tag{A.9}
$$

Any tensor can be decomposed in longitudinal, transverse and mixed components. Consider for concreteness the energy–momentum tensor, which is rank-two and symmetric with components $T_{\mu\nu}$:

$$
T_{\mu\nu} = \boldsymbol{\varepsilon} u_\mu u_\nu + \boldsymbol{p} h_{\mu\nu} + \tau_{\mu\nu} + u_\mu q_\nu + u_\nu q_\mu. \tag{A.10}
$$

The non-longitudinal part is

$$
\boldsymbol{p} h_{\mu\nu} + \tau_{\mu\nu} + u_\mu q_\nu + u_\nu q_\mu. \tag{A.11}
$$

We have defined

$$
\boldsymbol{\varepsilon} = u^\mu u^\nu T_{\mu\nu}, \quad \tau_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma T_{\rho\sigma} - \boldsymbol{p} h_{\mu\nu}, \quad q_\mu = -h_\mu{}^\nu T_{\nu\sigma} u^\sigma \tag{A.12}
$$

[24]Our conventions for symmetrization and antisymmetrization are:

$$
A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right), \quad A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right).
$$

such that

$$h_\mu{}^\nu q_\nu = q_\mu, \quad h_\mu{}^\rho \tau_{\rho\nu} = \tau_{\mu\nu}, \quad u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \boldsymbol{\varepsilon} u_\nu. \tag{A.13}$$

The purely transverse piece $\boldsymbol{p} h_{\mu\nu} + \tau_{\mu\nu}$ is the stress tensor, while q^μ is the heat current.

Similarly, any current with components J^μ can be decomposed in longitudinal and transverse parts:

$$J^\mu = \boldsymbol{\varrho} u^\mu + j^\mu \tag{A.14}$$

with

$$h_\mu{}^\nu j_\nu = j_\mu, \quad u^\mu j_\mu = 0, \quad \boldsymbol{\varrho} = -u^\mu J_\mu. \tag{A.15}$$

Assuming the energy–momentum tensor $T_{\mu\nu}$ being conserved:

$$\nabla_\mu T^{\mu\nu} = 0, \tag{A.16}$$

we can carry on and describe the dynamics for the heat current, using (A.13), together with (A.1) and (A.10). We obtain, for its divergence:[25]

$$\nabla_\mu q^\mu = -\mathrm{u}(\boldsymbol{\varepsilon}) - \left(\boldsymbol{p} + \boldsymbol{\varepsilon} + \frac{g_{\mu\nu}\tau^{\mu\nu}}{D-1}\right)\Theta - a_\mu q^\mu - \sigma_{\mu\nu}\tau^{\mu\nu}, \tag{A.17}$$

where $\mathrm{u}(f) = u^\mu \nabla_\mu (f) = u^\mu \partial_\mu (f)$.

The current J is also supposed to to obey

$$\nabla_\mu J^\mu = 0, \tag{A.18}$$

from which we extract the dynamics of its transverse component j using (A.2):

$$\nabla_\mu j^\mu = -\mathrm{u}(\boldsymbol{\varrho}) - \boldsymbol{\varrho}\,\Theta. \tag{A.19}$$

B Hydrodynamics and out-of-equilibrium states

Hydrodynamic functions and hydrodynamic frames

We recall here some basic facts regarding fluid dynamics (see [29, 30] as well as the pillar of hydrodynamics manuals [38] – we also recommend [40]). Hydrodynamics is by essence out-of-equilibrium. Every concept should therefore be considered with care, as no universal methods exist, which would embrace all facets of these phenomena, especially in the relativistic regime for non-ideal fluids.

[25] Notice that q being transverse, $D_\mu q^\mu = \nabla_\mu q^\mu - a_\mu q^\mu$.

Fluids are described in terms of their energy–momentum tensor and one (or more) current(s), all conserved in the absence external forces. The dynamical quantities are thus (see (A.10) and (A.14)) $\boldsymbol{\varepsilon}(x)$, $\boldsymbol{p}(x)$, $\boldsymbol{\varrho}(x)$, $q^{\mu}(x)$, $\tau^{\mu\nu}(x)$ and $j^{\mu}(x)$, assumed to be functionals of some fundamental quantities, equal in number to the available equations (A.16) and (A.18): $u^{\mu}(x)$, $\boldsymbol{T}(x)$ and $\boldsymbol{\mu}(x)$. This functional dependence is captured by the constitutive equations, expressed usually as a derivative expansion. As a matter of principle, these hydrodynamic functionals obey microscopic equations like Boltzmann's equation, but it is in practice difficult to extract information directly from there. The derivative expansion is the alternative, perturbative phenomenological approach.

At strict equilibrium and for an ideal fluid, u is aligned with a time-like Killing vector, *i.e.* the fluid is at rest, and $\boldsymbol{T}$ and $\boldsymbol{\mu}$ are constants. So are $\boldsymbol{\varepsilon}$, $\boldsymbol{p}$ and $\boldsymbol{\varrho}$. All these quantities are then defined within equilibrium thermodynamics as the temperature T, chemical potential μ, energy density ε, pressure p and matter (or better, Nœther-charge) density ϱ. The constitutive relations are the equation of state $p = p(T, \mu)$ and the usual Gibbs–Duhem relation for the grand potential $-p = \varepsilon - Ts - \mu\varrho$ with $\varrho = \left(\partial p/\partial \mu\right)_T$ and $s = \left(\partial p/\partial T\right)_{\mu}$.

Once the fluid is set to motion, the equilibrium is abandoned and assumed to be achieved locally, for hydrodynamics to make sense. Thermodynamic functions become local (and supposed to be slowly varying) but even within this basic assumption, for non-perfect fluids, neither $\boldsymbol{\varepsilon}(x)$, $\boldsymbol{p}(x)$ and $\boldsymbol{\varrho}(x)$ appearing in the fluid equations, nor $\boldsymbol{T}(x)$ and $\boldsymbol{\mu}(x)$ entering the constitutive relations need *a priori* to be identified with the corresponding local-equilibrium thermodynamic quantities. Even the velocity congruence $\mathrm{u}(x)$ has no first-principle definition in relativistic hydrodynamics. One has in particular the freedom to redefine

$$\boldsymbol{T}(x) \to \boldsymbol{T}'(x), \quad \boldsymbol{\mu}(x) \to \boldsymbol{\mu}'(x), \quad \mathrm{u}(x) \to \mathrm{u}'(x), \tag{B.1}$$

provided we modify accordingly $\boldsymbol{\varepsilon}(x)$, $\boldsymbol{p}(x)$, $\boldsymbol{\varrho}(x)$, $q^{\mu}(x)$, $\tau^{\mu\nu}(x)$ and $j^{\mu}(x)$.

The above freedom can be used to fix some of the hydrodynamic functions. This is how the concept of hydrodynamic frame emerges. The Eckart frame (also called particle frame, [41, 42]) is reached by requiring the matter current J be perfect *i.e.* $\mathrm{j} = 0$, while in the Landau–Lifshitz frame the heat current q is set to zero [38]. In every frame, the remaining non-vanishing hydrodynamic functionals are set as derivative expansions with respect to $\boldsymbol{T}(x)$, $\boldsymbol{\mu}(x)$ and $u^{\mu}(x)$. The coefficients are phenomenological data, which can in principle be determined from the microscopic theory. The consequence of changing frame is to reshuffle the various coefficients (sometimes trading one for an infinite number of others), which ultimately carry the relevant information about the fluid, irrespective of the frame.

It is worth noting at this stage that the definition of the Eckart frame and, by the logic of frame transformation, the corresponding definition of the Landau–Lifshitz counterpart, refer *explicitly* to the conserved matter current J. The heat current q, as part of the conserved energy–momentum tensor T, and the non-perfect contribution j to the conserved matter current J are interchanged in the course of the transformation. Regularity (or invertibility) of the latter makes it dangerous to set *a priori* both these vectors to zero, irrespective of the fact that ultimately the matter density $\boldsymbol{\varrho}$ or the chemical potential $\boldsymbol{\mu}$ may vanish.

The choice of frame is important for several reasons. At the first place, because of the nature of derivative expansions: these are often asymptotic series and only the first terms can be trusted. Hence, depending on the regime, some frames may not provide accurate results. Secondly, the precise physical context can play a rôle. For instance, when dealing with fluids in a quasi-Newtonian regime, the Eckart frame is superior as it is the one in which one recovers classical Euler's equations for non-relativistic fluids. Following the classical irreversible thermodynamics theory in Eckart frame,[26] we find at first order – dropping the index "E":

$$\boldsymbol{\varepsilon}_{(1)} = \varepsilon, \quad \boldsymbol{p}_{(1)} = p, \quad \boldsymbol{\varrho}_{(1)} = \varrho, \tag{B.2}$$

$$\tau^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta h^{\mu\nu}\Theta, \tag{B.3}$$

$$q^{\mu}_{(1)} = -\kappa h^{\mu\nu}\left(\partial_\nu T + T\,a_\nu\right). \tag{B.4}$$

In $D = 3$ spacetime dimensions there is also a term $-\zeta_{\mathrm{H}}\eta^{\rho\lambda(\mu}u_\rho\sigma_\lambda{}^{\nu)}$ in $\tau^{\mu\nu}_{(1)}$ with ζ_{H} the Hall viscosity.

Formally, the choice of frame (Eckart, Landau–Lifshitz, . . .) does not exhaust all freedom and it is always implicitly assumed that, owing to this residual latitude, $\boldsymbol{\varepsilon}(x)$, $\boldsymbol{p}(x)$ and $\boldsymbol{\varrho}(x)$ are identified with the local-equilibrium thermodynamic energy density $\varepsilon(x)$, pressure $p(x)$ and charge density $\varrho(x)$, *i.e.* not only at the first order as Eqs. (B.2) may suggest. Nothing guarantees, however, that the kinematic out-of-equilibrium temperature $\boldsymbol{T}(x)$ and chemical potential $\boldsymbol{\mu}(x)$ could be identified with the equilibrium data $T(x)$ and $\mu(x)$, even at lowest order – *a fortiori* when higher (and possibly all) orders in the derivative expansion are concerned. The literature is very poor on this issue, probably because we are here reaching the limits of the hydrodynamic approach. Answering this question would require to enter the realm of non-equilibrium many-body systems.

Conformal fluids

The case of conformal fluids deserves some further comments. From microscopic first principles, the energy–momentum tensor is traceless and this should hold even in the limit of extinct interactions. In other words, from Eq. (A.10) and following the above identification of kinematical energy and pressure $\boldsymbol{\varepsilon}$, $\boldsymbol{p}$ with thermodynamic ones ε, p, one obtains:

$$\varepsilon(x) = (D-1)\,p(x), \quad g_{\mu\nu}\tau^{\mu\nu} = 0. \tag{B.5}$$

[26] See [40] for a comprehensive review about classical irreversible thermodynamics (CIT) and the Eckart frame.

Equilibrium thermodynamics for conformal fluids then sets the equilibrium temperature $T(x)$ following Stefan's law, modified in the presence of a chemical potential to comply with the Gibbs–Duhem equation (B.11):

$$p = T^D f\left(\frac{\mu}{T}\right). \tag{B.6}$$

Here $f\left(\mu/T\right)$ encodes the equation of state for the conformal fluid. It is determined by its microscopic properties, and satisfies

$$f(0) = \frac{\sigma}{D-1}, \tag{B.7}$$

where σ is a Stefan–Boltzmann-like constant in D dimensions. The matter density and entropy therefore read:

$$\varrho = \left(\frac{\partial p}{\partial \mu}\right)_T = T^{D-1} f'\left(\frac{\mu}{T}\right), \tag{B.8}$$

$$s = \left(\frac{\partial p}{\partial T}\right)_\mu = \frac{1}{T}(Dp - \mu\varrho). \tag{B.9}$$

Vanishing density requires thus $f = \sigma/D-1$ constant, and we recover Stefan's law in this case too. As already emphasized, the thermodynamic temperature and chemical potential may not be meaningful in a plain non-equilibrium regime.

Entropy current

The next object we would like to discuss is the entropy current. The canonical expression for it is [29, 38, 40, 43]

$$S^\mu = \frac{1}{T}\left(pu^\mu - T^{\mu\nu}u_\nu - \mu J^\mu\right). \tag{B.10}$$

Using the decompositions (A.10) and (A.14), the identifications of the kinematical $\boldsymbol{\varepsilon}(x)$, $\boldsymbol{p}(x)$ and $\boldsymbol{\varrho}(x)$ with the thermodynamic ones, as well as the already quoted equilibrium thermodynamic relation

$$Ts = p + \varepsilon - \mu\varrho, \tag{B.11}$$

one finds:

$$S^\mu = su^\mu + \frac{1}{T}q^\mu - \frac{\mu}{T}j^\mu. \tag{B.12}$$

This current allows writing the thermodynamic entropy as:

$$s = -S^\mu u_\mu. \tag{B.13}$$

We should stress that the entropy current has raised many questions and its canonical form (B.10) may not be appropriate to all physical situations. It is based on local-equilibrium thermodynamic functions, $s(x)$, $T(x)$ and $\mu(x)$, and depending on the set-up, these may be far from the kinematical $\boldsymbol{T}(x)$ and $\boldsymbol{\mu}(x)$, which lack first-principle microscopic definition anyway.

It can be shown that the entropy current is frame-independent [29]. This holds in particular for Eckart and Landau–Lifshitz frames:

$$S^{\mu}_{\mathrm{E}} = S^{\mu}_{\mathrm{LL}}. \tag{B.14}$$

The formal expression of the current changes though, from one frame to another. In the Eckart frame, (B.10) becomes

$$S^{\mu}_{\mathrm{E}} = su^{\mu} + \frac{1}{T}q^{\mu}, \tag{B.15}$$

and using Eq. (A.17)

$$\begin{aligned}\nabla_{\mu} S^{\mu}_{\mathrm{E}} &= -\frac{\mu\varrho}{T}\Theta - \mathrm{u}\left(\frac{\mu\varrho}{T}\right) + \mathrm{u}\left(\frac{p}{T}\right) + \varepsilon\mathrm{u}\left(\frac{1}{T}\right) + \mathrm{q}\left(\frac{1}{T}\right)\\ &\quad -\frac{1}{T}\left(\frac{g_{\mu\nu}\tau^{\mu\nu}}{D-1}\Theta + a_{\mu}q^{\mu} + \sigma_{\mu\nu}\tau^{\mu\nu}\right).\end{aligned} \tag{B.16}$$

Similarly, we find in the Landau–Lifshitz frame

$$S^{\mu}_{\mathrm{LL}} = su^{\mu} - \frac{\mu}{T}j^{\mu}, \tag{B.17}$$

which is precisely the current originally proposed by Landau and Lifshitz in [38]. Thanks to the usual tools ((A.14), (A.18) and (A.19)), the divergence turns out to be

$$\nabla_{\mu} S^{\mu}_{\mathrm{LL}} = \mathrm{u}\left(\frac{p+\varepsilon}{T}\right) + \frac{p+\varepsilon}{T}\Theta - \varrho\mathrm{u}\left(\frac{\mu}{T}\right) - \mathrm{j}\left(\frac{\mu}{T}\right). \tag{B.18}$$

In order to avoid cluttering indices, it is understood that whatever quantity appears in the right-hand side of Eqs. (B.15)–(B.18) is determined in the hydrodynamic frame declared in the left-hand side (and similarly for (B.19)–(B.21) below).

Positivity of $\nabla_{\mu} S^{\mu}$ sets bounds on the transport coefficients that appear in the derivative expansion. Notice *en passant* that this divergence is Weyl-covariant as it matches the Weyl-divergence of the entropy current.[27]

For a conformal fluid, the entropy current (B.12) reads:

$$S^{\mu}_{\mathrm{E}} = \frac{1}{T}\left((Dp-\mu\varrho)u^{\mu} + q^{\mu}\right), \quad \text{or} \quad S^{\mu}_{\mathrm{LL}} = \frac{1}{T}\left((Dp-\mu\varrho)u^{\mu} - \mu j^{\mu}\right), \tag{B.19}$$

[27] Indeed, we would write $\mathscr{D}_{\mu} S^{\mu} = \nabla_{\mu} S^{\mu} + (w_{\mathrm{S}} - D)A_{\mu}S^{\mu}$, but the conformal weight w_{S} of the entropy current equals D.

while its divergence (B.16) or (B.18) is now

$$\nabla_\mu S^\mu_{\mathrm{E}} = -\frac{\mu\varrho}{T}\Theta - \mathrm{u}\left(\frac{\mu\varrho}{T}\right) + Dp\,\mathrm{u}\left(\frac{1}{T}\right) + \mathrm{q}\left(\frac{1}{T}\right) - \frac{1}{T}\left(a_\mu q^\mu + \sigma_{\mu\nu}\tau^{\mu\nu} - \mathrm{u}(p)\right), \tag{B.20}$$

or

$$\nabla_\mu S^\mu_{\mathrm{LL}} = D\,\mathrm{u}\left(\frac{p}{T}\right) + D\frac{p}{T}\Theta - \varrho\mathrm{u}\left(\frac{\mu}{T}\right) - \mathrm{j}\left(\frac{\mu}{T}\right). \tag{B.21}$$

The various kinematical and thermodynamic quantities appearing in the equations, are determined in the corresponding frame; they are different for Eckart and Landau–Lifshitz, contrary to the entropy current and its divergence.

Eckart-to-Landau–Lifshitz transformation

We would like to conclude this appendix with some explicit transformation rules. Writing $\mathscr{Q}_{\mathrm{LL}} = \mathscr{Q}_{\mathrm{E}} + \delta\mathscr{Q}$ for any kinematical or thermodynamic quantity $\mathscr{Q}$, the displacements can be computed linearly, quadratically, and so on, based on the fundamental rule that the energy–momentum tensor T and the matter current J are frame-invariant. In order to avoid any confusion, we restore the index "E" for the Eckart frame, and provide the results with minimal details.

The variation in the velocity field is determined in terms of the heat current, non-zero in Eckart frame, vanishing in Landau–Lifshitz frame, by solving perturbatively the eigenvalue problem:

$$\delta\mathrm{u}^{(1)} = \frac{\mathrm{q}}{p_{\mathrm{E}} + \varepsilon_{\mathrm{E}}}. \tag{B.22}$$

All other transformation rules are determined from the latter, using the quoted invariances and Gibbs–Duhem equation.[28] The non-perfect matter-current component j is vanishing in Eckart and non-zero in Landau–Lifshitz, where its first-order value is

$$\delta\mathrm{j}^{(1)} = -\frac{\varrho_{\mathrm{E}}}{p_{\mathrm{E}} + \varepsilon_{\mathrm{E}}}\mathrm{q}, \tag{B.23}$$

while

$$\delta\varepsilon^{(1)} = \delta\varrho^{(1)} = \delta s^{(1)} = \delta p^{(1)} = 0. \tag{B.24}$$

Similarly, we find

$$\delta\left(\frac{\mu}{T}\right)^{(1)} = \frac{q\cdot\tau_{\mathrm{E}}\cdot q}{\varrho_{\mathrm{E}} T_{\mathrm{E}} \mathrm{q}^2}, \tag{B.25}$$

and using $\delta p = \varrho\delta\mu + s\delta T$ we can read off $\delta T^{(1)}$ and $\delta\mu^{(1)}$.

It should be noticed that the stress tensor τ_{E} is a correction with respect to the perfect fluid, of similar order than the heat current q. The first correction it receives is therefore of

[28] The kinematical $\boldsymbol{\varepsilon}_{\mathrm{LL}}(x)$, $\boldsymbol{p}_{\mathrm{LL}}(x)$ and $\boldsymbol{\varrho}_{\mathrm{LL}}(x)$ are still identified with the local-equilibrium thermodynamic energy density $\varepsilon_{\mathrm{LL}}(x)$, pressure $p_{\mathrm{LL}}(x)$ and charge density $\varrho_{\mathrm{LL}}(x)$.

second order:

$$\delta\tau^{(2)\mu\nu} = \frac{q\cdot\tau_{\mathrm{E}}\cdot q}{(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})\,\mathrm{q}^2}\left(q^\mu u^\nu + q^\nu u^\mu\right) + \frac{\mathrm{tr}\,\delta\tau^{(2)}}{D-1}h^{\mu\nu}. \tag{B.26}$$

In this expression, the trace of the correction, $\mathrm{tr}\,\delta\tau^{(2)} = g_{\mu\nu}\,\delta\tau^{(2)\mu\nu}$, is left undetermined. This trace also appears in the second-order correction of the pressure,

$$\delta p^{(2)} = \frac{\delta\varepsilon^{(2)}}{D-1} - \frac{\mathrm{tr}\,\delta\tau^{(2)}}{D-1}, \quad \delta\varepsilon^{(2)} = -\frac{\mathrm{q}^2}{p_{\mathrm{E}}+\varepsilon_{\mathrm{E}}}, \tag{B.27}$$

so that a freedom remains to reabsorb it or not in the latter (see discussion in [29]). The other second-order corrections from Eckart to Landau–Lifshitz frame read:

$$\delta\mathrm{u}^{(2)} = \frac{1}{2\,(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})^2}\left(\mathrm{q}^2\mathrm{u}_{\mathrm{E}} - 2\tau_{\mathrm{E}}\cdot\mathrm{q}_{\mathrm{E}}\right), \tag{B.28}$$

$$\delta\mathrm{j}^{(2)} = -\frac{\varrho_{\mathrm{E}}}{(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})^2}\left(\mathrm{q}^2\mathrm{u}_{\mathrm{E}} - \tau_{\mathrm{E}}\cdot\mathrm{q}_{\mathrm{E}}\right), \tag{B.29}$$

$$\delta s^{(2)} = \frac{\mathrm{q}^2 s_{\mathrm{E}}}{2\,(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})^2} - \frac{\mathrm{q}^2}{T_{\mathrm{E}}\,(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})}, \tag{B.30}$$

$$\delta\varrho^{(2)} = \frac{\mathrm{q}^2\varrho_{\mathrm{E}}}{2\,(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})^2}, \tag{B.31}$$

$$\delta\left(\frac{\mu}{T}\right)^{(2)} = -\frac{1}{\varrho_{\mathrm{E}}T_{\mathrm{E}}\,(p_{\mathrm{E}}+\varepsilon_{\mathrm{E}})}\left(\mathrm{q}^2 + \frac{\mathrm{q}\cdot\tau_{\mathrm{E}}\cdot\tau_{\mathrm{E}}\cdot\mathrm{q}}{\mathrm{q}^2} - \left(\frac{\mathrm{q}\cdot\tau_{\mathrm{E}}\cdot\mathrm{q}}{\mathrm{q}^2}\right)^2\right). \tag{B.32}$$

Finding the latter requires to analyse the eigenvalue problem of the energy–momentum tensor at third order. We can further combine (B.27) with (B.32) and $\delta p = \varrho\delta\mu + s\delta T$, and extract $\delta T^{(2)}$ and $\delta\mu^{(2)}$.

We can proceed similarly and obtain the above quantities at next order, or even further. Their expressions follow the pattern already visible in the first and second orders. It is readily seen that the expansions of all Landau–Lifshitz observables around their Eckart values are controlled by the parameter $\|\mathrm{q}\|/p_{\mathrm{E}}+\varepsilon_{\mathrm{E}}$, *i.e.* basically the norm of the heat current. The magnitude of this quantity sets validity bounds on the frame transformation at hand. For a more general discussion on related issues, see the already quoted Refs. [29, 40].

References

[1] I. Robinson and A. Trautman, *Spherical gravitational waves*, Phys. Rev. Lett. **4** (1960) 431; *Some spherical gravitational waves in general relativity*, Proc. Roy. Soc. **A265** (1962) 463.

[2] J. B. Griffiths and J. Podolský, *Exact space–times in Einstein's general relativity*, Cambridge Monographs on Mathematical Physics, 2009.

[3] G.B. de Freitas and H.S. Reall, *Algebraically special solutions in AdS/CFT,* JHEP **1406** (2014) 148, arXiv:1403.3537 [hep-th].

[4] I. Bakas and K. Skenderis, *Non-equilibrium dynamics and AdS_4 Robinson–Trautman*, JHEP **1408** (2014) 056, arXiv:1404.4824 [hep-th].

[5] V.E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, arXiv:1107.5780 [hep-th].

[6] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT,* JHEP **0810** (2008) 063, arXiv:0806.4602 [hep-th].

[7] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, *Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions*, JHEP **0812** (2008) 116, arXiv:0809.4272 [hep-th].

[8] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Elie Cartan et les mathématiques d'aujourd'hui,* Astérisque, 1985, numéro hors série Soc. Math. France, Paris, 95.

[9] C. Fefferman and C.R. Graham, *The ambient metric*, arXiv:0710.0919 [math.DG].

[10] M.M. Caldarelli, R.G. Leigh, A.C. Petkou, P.M. Petropoulos, V. Pozzoli and K. Siampos, *Vorticity in holographic fluids*, Proc. of Science **Corfu11** (2012) 076, arXiv:1206.4351 [hep--th].

[11] A. Mukhopadhyay, A.C. Petkou, P.M. Petropoulos, V. Pozzoli and K. Siampos, *Holographic perfect fluidity, Cotton energy–momentum duality and transport properties*, JHEP **1404** (2014) 136, arXiv:1309.2310 [hep-th].

[12] P.M. Petropoulos, *Gravitational duality, topologically massive gravity and holographic fluids*, Lect. Notes Phys. **892** (2015) 331, arXiv:1406.2328 [hep-th].

[13] J. Gath, A. Mukhopadhyay, A.C. Petkou, P.M. Petropoulos and K. Siampos, *Petrov Classification and holographic reconstruction of spacetime*, JHEP **1509** (2015) 005, arXiv:1506.04813 [hep-th].

[14] P.M. Petropoulos and K. Siampos, *Integrability, Einstein spaces and holographic fluids*, proceedings of the workshop *About various kinds of interactions* in honour of the 65th birthday of Professor Philippe Spindel, N. Boulanger and S. Detournay, eds., Mons 2017, arXiv:1510.06456 [hep-th].

[15] A.C. Petkou, P.M. Petropoulos and K. Siampos, *Geroch group for Einstein spaces and holographic integrability*, Proc. of Science **PLANCK 2015** (2015) 104, arXiv:1512.04970 [hep--th].

[16] R. Geroch, *A method for generating solutions of Einstein's equations*, J. Math Phys. **12** (1971) 918; *A method for generating new solutions of Einstein's equation, II*, J. Math. Phys. **13** (1972) 394.

[17] J. Ehlers, *Les théories relativistes de la gravitation*, CNRS, Paris, 1959, 275-284.

[18] R.G. Leigh, A.C. Petkou, P.M. Petropoulos and P.K. Tripathy, *The Geroch group in Einstein spaces*, Class. Quant. Grav. **31** (2014) 225006, arXiv:1403.6511 [hep-th].

[19] K. Skenderis and S. N. Solodukhin, *Quantum effective action from the AdS/CFT correspondence*, Phys. Lett. **B472** (2000) 316, hep-th/9910023.

[20] B. Coll, J. Llosa and D. Soler, *Three-dimensional metrics as deformations of a constant curvature metric*, Gen. Rel. Grav. **34** (2002) 269, gr-qc/0104070.

[21] D.S. Mansi, A.C. Petkou and G. Tagliabue, *Gravity in the* $3+1$*-split formalism I: holography as an initial value problem*, Class. Quant. Grav. **26** (2009) 045008, arXiv:0808.1212 [hep-th].

[22] D.S. Mansi, A.C. Petkou and G. Tagliabue, *Gravity in the* $3+1$*-split formalism II: self-duality and the emergence of the gravitational Chern–Simons in the boundary*, Class. Quant. Grav. **26** (2009) 045009, arXiv:0808.1213 [hep-th].

[23] S. de Haro, *Dual gravitons in* AdS_4/CFT_3 *and the holographic Cotton tensor*, JHEP **0901** (2009) 042, arXiv:0808.2054 [hep-th].

[24] O. Miskovic and R. Olea, *Topological regularization and self-duality in four-dimensional anti-de Sitter gravity*, Phys. Rev. **D79** (2009) 124020, arXiv:0902.2082 [hep-th].

[25] J. Podolský and R. Svarc, *Algebraic classification of Robinson–Trautman spacetimes*, Phys. Rev. **D94** (2016) 064043, arXiv:1608.07118 [hep-th].

[26] J. Foster and E.T. Newman, *Note on the Robinson–Trautman solutions*, J. Math. Phys. **8** (1967) 189.

[27] K.P. Tod, *Analogues of the past horizon in the Robinson–Trautman metrics*, Class. Quantum. Grav. **6** (1989) 1159.

[28] C. Hoenselaers and Z. Perjés, *Remarks on the Robinson–Trautman solutions*, Class. Quantum. Grav. **10** (1993) 375.

[29] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, J. Phys. **A45** (2012) 473001, arXiv:1205.5040 [hep-th].

[30] P. Romatschke, *New developments in relativistic viscous hydrodynamics*, Int. J. Mod. Phys. **E19** (2010) 1, arXiv:0902.3663 [hep-th].

[31] C. Marteau, *On charged Robinson–Trautman holograph*y, internship report, internship report, Ecole normale supérieure, Ecole polytechnique, 2017.

[32] M.M. Caldarelli, O.J.C. Dias and D. Klemm, *Dyonic AdS black holes from magnetohydrodynamics* JHEP **0903** (2009) 025, arXiv:0812.0801 [hep-th].

[33] J. Hernandez and P. Kovtun, *Relativistic magnetohydrodynamics*, JHEP **1705** (2017) 001, arXiv:1703.08757 [hep-th].

[34] I. Bakas, K. Skenderis and B. Withers, *Self-similar equilibration of strongly interacting systems from holography*, Phys. Rev. **D93** (2016) 101902, arXiv:1512.09151 [hep-th].

[35] K. Skenderis and B. Withers, *Robinson–Trautman spacetimes and gauge/gravity duality*, PoS CORFU **2016** (2017) 097, arXiv:1703.10865 [hep-th].

[36] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, Phys. Rev. **D79** (2009) 025023, arXiv:0809.3808 [hep-th].

[37] J. Moutier, *Sur les transformations non réversibles*, Bulletin de la Société Philomathique 3e série, **I** (1877) 39.

[38] L.D. Landau et E.M. Lifchitz, *Physique Théorique*, Vol. 6 *Mécanique des fluides*, MIR 1969.

[39] A.A. Tarkovski, *Solaris*, Mosfilm, 1972.

[40] L. Rezzolla and O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, Oxford UK, 2013.

[41] R.C. Tolman, *On the weight of heat and thermal equilibrium in general relativity*, Phys. Rev. **35** (1930) 904.

[42] C. Eckart, *The thermodynamics of irreversible processes – III Relativistic theory of the simple fluid*, Phys. Rev. **58** (1940) 919.

[43] W. Israël, *Thermodynamics of relativistic systems*, Physica **106A** (1981) 204.

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Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids

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Abstract

We provide the set of equations for non-relativistic fluid dynamics on arbitrary, possibly time-dependent spaces, in general coordinates. These equations are fully covariant under either local Galilean or local Carrollian transformations, and are obtained from standard relativistic hydrodynamics in the limit of infinite or vanishing velocity of light. All dissipative phenomena such as friction and heat conduction are included in our description. Part of our work consists in designing the appropriate coordinate frames for relativistic spacetimes, invariant under Galilean or Carrollian diffeomorphisms. The guide for the former is the dynamics of relativistic point particles, and leads to the Zermelo frame. For the latter, the relevant objects are relativistic instantonic space-filling branes in Randers–Papapetrou backgrounds. We apply our results for obtaining the general first-derivative-order Galilean fluid equations, in particular for incompressible fluids (Navier–Stokes equations) and further illustrate our findings with two applications: Galilean fluids in rotating frames or inflating surfaces and Carrollian conformal fluids on two-dimensional time-dependent geometries. The first is useful in atmospheric physics, while the dynamics emerging in the second is governed by the Robinson–Trautman equation, describing a Calabi flow on the surface, and known to appear when solving Einstein's equations for algebraically special Ricci-flat or Einstein spacetimes.

Contents

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1 Introduction

Ordinary non-relativistic fluid dynamics is described in terms of a basic set of equations: continuity, energy conservation and momentum conservation (Euler equation). In most textbooks (as e.g. [1]) the fluid is observed from either inertial, or stationary rotating frames, using Cartesian or spherical/cylindrical coordinates. Although these set-ups are satisfactory for most practical purposes, they do not exhaust all possible situations because the equations at hand are not covariant under Galilean diffeomorphisms *i.e.* general coordinate transformations such as $t' = t'(t)$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$. Most importantly, the geometry hosting the fluid is assumed to be three- or two-dimensional Euclidean space. This is a severe limitation, as we may want to study the fluid moving on a surface, which is neither flat nor static, and equipped with an arbitrary coordinate system.

Progress has been made over the last decades, sustained by the needs of the space programs or meteorology [2–7]. The most recent work [7] beautifully highlights the various contributions, and provides a covariant frame-independent formulation. Still, these authors do not address the issue of trading Euclidean space for an arbitrary curved and time-dependent geometry, and subsequent analyses have focused to the case of static surfaces (see *e.g.* [8]). Part of our work consists in filling this gap, and presenting the most general equations describing a non-relativistic viscous fluid moving on a space endowed with a spatial, time-dependent metric, and observed from an arbitrary frame. Each geometric object involved in this description has a well-defined transformation rule under Galilean diffeomorphisms, making the set of equations covariant.

In order to achieve the above program, we carefully analyze the infinite-light-velocity limit inside the relativistic fluid equations. Although standard (see §125 of [1] for the original presentation and [9] for a modern approach), this method has been only partially developed outside the realm of Minkowski spacetime (as *e.g.* in [10]). Hence, it has mostly led to non-relativistic fluids on plain Euclidean space in inertial frames. Choosing the form of a general spacetime metric such that it allows for a non-relativistic limit, enables us to reach our goal.

Considering the infinite-light-velocity limit in a relativistic framework suggests to study in parallel the alternative zero-light-velocity limit. This is actually ultra-relativistic, but we will keep on calling it non-relativistic as it decouples time and contracts the Poincaré group down to the Carroll group, as originally described in [11].

Carrollian physics has attracted some attention over the recent years [12, 13]. Although kinematically restricted – due to the vanishing velocity of light, the light-cone collapses to a line and no motion is allowed – the freedom of choosing a frame is as big as for Galilean physics though. In particular, the single particle has degenerate motion [14], but extended instantonic[1] objects do still exist and have non-trivial dynamics, making this framework rich and interesting. Following the pattern described above, we study the corresponding general set of equations for viscous fluids. The form of the spacetime metric appropriate for the limit at hand is of Randers–Papapetrou, slightly different from the one used in the former case, which is the Zermelo form.[2] The obtained equations are covariant under Carrollian coordinate transformations, $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. In order to avoid any confusion, we will refer to the standard non-relativistic fluids as *Galilean*, whereas the latter will be called *Carrollian*.

Our motivation for the present work is twofold. On the one hand, as already mentioned, stands the need for a fully covariant formulation of Galilean fluid dynamics, on general spaces and from arbitrary frames, which might have useful physical applications. On the other hand, viscous Carrollian fluids were never studied and turn out to emerge in the context of asymptotically flat holography [16], in replacement of the relativistic fluids present in the usual fluid/gravity holographic correspondence of asymptotically anti-de Sitter space-

[1] In ordinary relativistic spacetime, we would call these objects tachyonic as they extend in space *i.e.* outside the local light-cone. Since the latter is everywhere degenerate in Carrollian spacetimes, instantonic is more illustrative.

[2] See [15] for an interesting discussion on Zermelo vs. Randers–Papapetrou forms.

times [17–20]. Performing this analysis in parallel is useful as both Galilean and Carrollian groups, and Zermelo and Randers–Papapetrou frames turn out to have intimate duality relationships.

We will start our exposition by designing the appropriate forms for relativistic spacetimes, hosting naturally the action of – *i.e.* being stable under – the two diffeomorphism groups that we want to survive in the infinite-c or zero-c limits, Secs. 2.1, 2.2. Local Galilean and Carrollian transformations are elegantly implemented in ordinary particle or instantonic space-filling brane dynamics, respectively. They are subsequently uplifted into Zermelo and Randers–Papapetrou metrics for the spacetime. The next step consists in studying ordinary viscous relativistic fluids on these environments and consider the infinite-c or zero-c limits in their equations. This is performed in Secs. 3.2 and 3.3, following a concise overview on relativistic fluids, Sec. 3.1. We find generalized continuity, energy-conservation and Euler equations for the usual Galilean fluids, as well as a set of two scalar (one for the energy) and two vector equations for the Carrollian ones. We analyze the covariance properties of the equations in both cases, and show that these transform as expected. Some examples are collected in Sec. 4: the Galilean fluid from a rotating frame or on an inflating surface, and the dynamics of a two-dimensional Carrollian viscous fluid. Further technical details, are provided in the appendix, where we introduce a new time connection for the Galilean geometry, and both temporal and spatial connections for the Carrollian and conformal-Carrollian geometry, together with their associated curvature tensors, allowing for a more elegant presentation of the corresponding covariant equations.

2 Galilean and Carrollian Poincaré uplifts

We present here the relativistic uplifts of Newton–Cartan and Carrollian non-relativistic structures. In these Lorentzian-signature spacetimes, respectively of the Zermelo and Randers–Papapetrou form, the Galilean and Carrollian diffeomorphisms are naturally realized, and the dynamics of free objects smoothly matches the ordinary Galilean and Carrollian dynamics, when the velocity of light becomes infinite or vanishes, respectively.

2.1 From Galileo Galilei ...

Consider a free particle on an arbitrary d-dimensional space $\mathscr{S}$, endowed with a positive-definite metric

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i,j\ldots \in \{1,\ldots,d\}, \tag{2.1}$$

and observed from a frame with respect to which the locally inertial frame has velocity $\mathbf{w} = w^i\partial_i$. Its classical (as opposed to relativistic) dynamics is captured by the following

Lagrangian:

$$\mathcal{L}(\mathbf{v},\mathbf{x},t) = \frac{1}{2\Omega^2} a_{ij} \left(v^i - w^i\right)\left(v^j - w^j\right) \tag{2.2}$$

with action

$$S[\mathbf{x}] = \int_{\mathscr{C}} \mathrm{d}t\, \Omega \mathcal{L}(\mathbf{v},\mathbf{x},t). \tag{2.3}$$

In this expression:

- a_{ij} and w^i are general functions of $(t,\mathbf{x})$;[3]
- $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$ are the usual components of the velocity $\mathbf{v} = v^i \partial_i$;
- $\mathcal{L}(\mathbf{v},\mathbf{x},t)$ appears as a Lagrangian density, with Lagrangian[4] $L(\mathbf{v},\mathbf{x},t) = \Omega\mathcal{L}(\mathbf{v},\mathbf{x},t)$.

Furthermore

- the Lagrange generalized momenta are (indices are lowered and raised with a_{ij} and its inverse)

$$p_i = \frac{\partial L}{\partial v^i} = \frac{1}{\Omega}(v_i - w_i), \tag{2.4}$$

- $H(\mathbf{p},\mathbf{x},t) = p_i v^i - L(\mathbf{v},\mathbf{x},t)$ is the Hamiltonian with Hamiltonian density $\mathcal{H} = \frac{1}{\Omega} H$:

$$\mathcal{H} = \frac{1}{2}\left(\mathbf{p}^2 + \frac{\mathbf{p}\cdot\mathbf{w}}{\Omega}\right). \tag{2.5}$$

The existence of an absolute Newtonian time requires Ω be a function of t only, the absolute time being thus $\int \mathrm{d}t\,\Omega(t)$. One should stress that keeping general $\Omega(t,\mathbf{x})$ does not spoil the consistency of the system (2.2), (2.3), but invalidates the interpretation of (2.1) as the spatial metric. Even though in practical situations we can set $\Omega = 1$, its rôle is important when dealing with general Galilean diffeomorphisms (see (2.11)–(2.15)), in the framework underlying the above dynamical system: the Newton–Cartan structures [21].[5]

We can compute the energy density expressing the Hamiltonian (2.5) in terms of the velocity:

$$\mathcal{H} = \frac{1}{2\Omega^2} a_{ij}\left(v^i + w^i\right)\left(v^j - w^j\right) = \frac{1}{2\Omega^2}\left(\mathbf{v}^2 - \mathbf{w}^2\right). \tag{2.6}$$

As usual $-\mathbf{w}^2/2\Omega^2$ plays the rôle of the potential for inertial forces. Using the energy theorem ($\mathrm{d}H/\mathrm{d}t = -\partial L/\partial t$) one finds

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\frac{1}{2\Omega^2}\left(v^i - w^i\right)\left(v^j - w^j\right)\partial_t a_{ij} + \frac{v_i - w_i}{\Omega}\partial_t \frac{w^i}{\Omega}. \tag{2.7}$$

[3] Here $\mathbf{x}$ stands for $\{x^1, \ldots, x^d\}$.

[4] Euler–Lagrange equations are $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i}$.

[5] Some modern references on Newton–Cartan structure are *e.g.* [22–25].

The most canonical example of (2.2) is that of a massive particle moving in Euclidean space E_3 with Cartesian coordinates, and observed from a non-inertial frame:

$$a_{ij} = \delta_{ij}, \quad \Omega = 1, \quad \mathbf{w}(t, \mathbf{x}) = \mathbf{x} \times \boldsymbol{\omega}(t) - \mathbf{V}(t). \tag{2.8}$$

Here $\mathbf{V}(t)$ is the dragging velocity of the non-inertial frame, $\boldsymbol{\omega}(t)$ the angular velocity of its rotating axes, and $\mathbf{v} - \mathbf{w} = \mathbf{v} + \mathbf{V} + \boldsymbol{\omega} \times \mathbf{x}$ is the velocity as measured in the original inertial frame (Roberval's theorem).

The action (2.3) is invariant under general *Galilean diffeomorphisms i.e.* transformations

$$t' = t'(t) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}), \tag{2.9}$$

for which we define the following Jacobian functions:

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{2.10}$$

The metric components transform as a tensor of $\mathscr{S}$:

$$a'_{ij} = a_{kl} J^{-1k}_{\ \ \ i} J^{-1l}_{\ \ \ j}, \tag{2.11}$$

the particle and frame velocities as gauge connections:

$$v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right), \tag{2.12}$$

$$w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right), \tag{2.13}$$

and the generalized momenta (2.4) as one-form components:

$$p'_i = p_k J^{-1k}_{\ \ \ i}; \tag{2.14}$$

Ω is just rescaled:

$$\Omega' = \frac{\Omega}{J}. \tag{2.15}$$

Since $J = J(t)$ and $\Omega = \Omega(t)$, Galilean transformations lead to $\Omega' = \Omega'(t')$, leaving invariant the absolute Newtonian time $\int \mathrm{d}t\, \Omega(t) = \int \mathrm{d}t'\, \Omega'(t')$. Observe also that $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ is a genuine vector of $\mathscr{S}$, which ensures the form-invariance of $\mathcal{L}$ and thus the covariance of the equations of motion.

There is a particular Newton–Cartan structure, which is invariant under the Galilean group: $\mathscr{S}$ is the Euclidean space E_d with Cartesian coordinates ($a_{ij} = \delta_{ij}$) and $\Omega = 1$, and the connection $\mathbf{w}$ is constant *i.e.* independent of $(t, \mathbf{x})$. This system describes the non-relativistic motion of a free particle in Euclidean space, observed from an inertial frame. The Galilean group acts as

$$\begin{cases} t' = t + t_0, \\ x'^k = R^k_i x^i + V^k t + x^k_0 \end{cases} \tag{2.16}$$

with all parameters being $(t, \mathbf{x})$-independent, and R^k_i the entries of an orthogonal matrix. The action of these transformations leave the Lagrangian and the equations of motion at hand *invariant*. In more general Newton–Cartan structures, the Galilean group acts in the tangent space equipped with a local orthonormal frame and it is no more a global symmetry.

The Galilean group is an infinite-c contraction of the Poincaré group. The latter acts locally in general $d+1$-dimensional pseudo-Riemannian manifolds $\mathscr{M}$. In order to recover the above Newton–Cartan structure and its class of diffeomorphisms (2.9) in the infinite-c limit, there is a natural choice for the form of the metric on $\mathscr{M}$:

$$\mathrm{d}s^2 = -\Omega^2 c^2 \mathrm{d}t^2 + a_{ij}\left(\mathrm{d}x^i - w^i \mathrm{d}t\right)\left(\mathrm{d}x^j - w^j \mathrm{d}t\right). \tag{2.17}$$

The form (2.17) is required for the functions Ω, a_{ij} and w^i to transform as in (2.11), (2.13) and (2.15) under a Galilean diffeomorphism (2.9). Actually, every metric is compatible with the gauge (2.17), provided a_{ij}, w^i and Ω, are free to depend on $x = (ct, \mathbf{x}) = \{x^\mu, \mu = 0, 1, \ldots, d\}$. The existence of a Galilean limit requires, however, Ω to depend on t only. Indeed, the proper time element for a physical observer is $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2/c^2}$. When c becomes infinite, $\lim_{c\to\infty} \mathrm{d}\tau = \Omega \mathrm{d}t$ must coincide with the absolute Newtonian time, and this requires the absence of $\mathbf{x}$-dependence in Ω, as expected from our previous discussion on the dynamics of (2.3).

The spacetime Jacobian matrix associated with (2.9), reads (using (2.10)):

$$J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \to \begin{pmatrix} J(t) & 0 \\ J^i(x) & J^i_j(x) \end{pmatrix} \quad \text{with} \quad J^i = \frac{j^i}{c}. \tag{2.18}$$

The metric form (2.17) is refered to as *Zermelo* (see [15]). A relativistic particle moving in (2.17) is described by the components of its velocity u, normalized as $\|\mathrm{u}\|^2 = -c^2$:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \Rightarrow u^0 = \gamma c,\ u^i = \gamma v^i, \tag{2.19}$$

where the Lorentz factor γ is defined as usual (although here, it depends also on the spacetime coordinates):[6]

$$\gamma(t, \mathbf{x}, \mathbf{v}) = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\Omega\sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2}}. \tag{2.20}$$

[6]Expressions as $\mathbf{v}^2$ stand for $a_{ij}v^i v^j$, not to be confused with $\|\mathrm{u}\|^2 = g_{\mu\nu}u^\mu u^\nu$.

Under a Galilean diffeomorphism (2.18), the transformation of the components of u,

$$u'^0 = Ju^0, \quad u'^i = J^i_k u^k + J^i u^0, \quad u'_0 = \frac{1}{J}\left(u_0 - u_j J^{-1j}_{\ \ k} J^k\right), \quad u'_i = u_k J^{-1k}_{\ \ i}, \tag{2.21}$$

induces a transformation on v^i, which matches precisely (2.12).

The dynamics of the relativistic free particle is described using *e.g.* the length of the world-line $\mathscr{C}$ as an action:

$$S[x] = \int_{\mathscr{C}} \mathrm{d}\tau = \int_{\mathscr{C}} \sqrt{-\frac{\mathrm{d}s^2}{c^2}}. \tag{2.22}$$

This is easily computed in the Zermelo environment (2.17), and expanded for large c:

$$\begin{aligned} S[x] &= \int_{\mathscr{C}} \mathrm{d}t\,\Omega \sqrt{1 - \frac{1}{c^2\Omega^2} a_{ij} \left(v^i - w^i\right)\left(v^j - w^j\right)} \\ &= \int_{\mathscr{C}} \mathrm{d}t\,\Omega \left(1 - \frac{1}{2c^2\Omega^2} a_{ij} \left(v^i - w^i\right)\left(v^j - w^j\right) + \mathrm{O}\left(1/c^4\right)\right). \end{aligned} \tag{2.23}$$

Hence, the dynamics (2.22), disregarding the first term in (2.23), which is a Galilean invariant, coincides in the infinite-c limit with the dynamics of the non-relativistic action displayed in (2.3). This shows that (2.17) is the natural relativistic spacetime uplift of a Galilean space $\mathscr{S}$ endowed with a Newton–Cartan structure.

2.2 . . . to Lewis Carroll

The Poincaré group admits another contraction at vanishing c [11]. Although this limit may sound degenerate as particle motion is frozen, it exhibits both an interesting dynamics and a rich mathematical structure.

A Euclidean space E_d with Cartesian coordinates, accompanied with a real time line t can be equipped with a structure alternative to Newton–Cartan's, known as Carrollian. This structure is left invariant by the Carrollian group acting as

$$\begin{cases} t' = t + B_i x^i + t_0, \\ x'^k = R^k_i x^i + x^k_0 \end{cases} \tag{2.24}$$

with all parameters being $(t, \mathbf{x})$-independent, and R^k_i the entries of an orthogonal matrix.

Invariant equations of motion can be considered for extended objects *i.e.* fields rather than particles. Indeed, at zero velocity of light, a particle cannot move in time but time can define an $\mathbf{x}$-dependent field. The scalar field $t(\mathbf{x})$ describes a d-brane, in other words a space-filling object in E_d, extended inside a portion of space $\mathscr{V} \subset E_d$.[7] Its invariant action can be

[7] Our guide in this section is symmetry, and our goal the adequate Poincaré uplift. The precise physical system and the nature of its dynamics are of secondary importance. Other systems with Carrollian symmetry may exist. It is interesting, though, to maintain a dual formulation for the two sides (Galilean and Carrollian), as for objects

e.g.

$$S[t] = \int_{\mathscr{V}} \mathrm{d}^d x \mathcal{L}(\boldsymbol{\partial} t) \tag{2.25}$$

with Lagrangian density

$$\mathcal{L}(\boldsymbol{\partial} t) = \frac{1}{2}\delta^{ij}\left(\partial_i t - b_i\right)\left(\partial_j t - b_j\right), \tag{2.26}$$

where b_i are constant parameters with inverse-velocity dimension, playing the rôle of a constant gauge-field background, and transforming by shift and rotation under (2.24): $b'_i = \left(b_j + B_j\right) R^{-1j}{}_i$.

More general Carrollian structures equip Riemannian manifolds $\mathscr{S}$ with metric (2.1) and time $t \in \mathbb{R}$. The Carrollian transformations (2.24) are realized locally, in the tangent space, and are no longer symmetries. The structure is covariant under *Carrollian diffeomorphisms*

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{2.27}$$

with Jacobian functions

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i'}}{\partial x^j}. \tag{2.28}$$

The covariant action describing the Carrollian dynamics in the more general case at hand is[8]

$$S[t] = \int_{\mathscr{V} \subset \mathscr{S}} \mathrm{d}^d x \sqrt{a} \mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}), \tag{2.29}$$

where a stands for the determinant of the matrix a_{ij} and $\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x})$ is the Lagrangian density:

$$\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}) = \frac{1}{2} a^{ij}\left(\Omega \partial_i t - b_i\right)\left(\Omega \partial_j t - b_j\right). \tag{2.30}$$

Here the components of the metric, the scale factor Ω, and the components of the background gauge field $\boldsymbol{b} = b_i \mathrm{d}x^i$ depend all on $(t, \mathbf{x})$.

Under Carrollian diffeomorphisms, the metric transforms as in (2.11) *i.e.*

$$a'^{ij} = J^i_k J^j_l a^{kl}, \tag{2.31}$$

Ω is rescaled as in (2.15) – where everything now depends both on t and $\mathbf{x}$ – while the field gradients and the gauge connection obey respectively

$$\partial'_k t' = \left(J \partial_i t + j_i\right) J^{-1i}{}_{k'}, \tag{2.32}$$

with dimension-one and codimension-one world-volumes.

[8] Notice that actions (2.25), (2.29) and (2.37) are all Euclidean-signature (instantonic) because of vanishing c.

and

$$b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1}{}^i_k. \tag{2.33}$$

Here

$$\beta_i = \Omega \partial_i t - b_i \tag{2.34}$$

transform as components of a one-form on $\mathscr{S}$, making the density Lagrangian form-invariant.

We will now uplift the above structure into a $d+1$-dimensional pseudo-Riemannian manifold $\mathscr{M}$, where the full Poincaré group is realized in the tangent space. Following the pattern used in the Galilean framework, Sec. 2.1, we can recover the general Carrollian structure and its class of diffeomorphisms (2.27) in the zero-c limit, starting from a metric on $\mathscr{M}$ of the form:

$$\mathrm{d}s^2 = -c^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.35}$$

The form (2.35) is known as *Randers–Papapetrou*. It is universal, as every metric can be recast in this gauge. Here, it is required for the functions $\Omega(x)$, $a^{ij}(x)$ and $b_i(x)$ to transform as in (2.15), (2.31) and (2.33) under a Carrollian diffeomorphism (2.27) – again $x \equiv (x^0 = ct, \mathbf{x})$. The spacetime Jacobian matrix associated with transformations (2.27), reads (using (2.28)):

$$J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \to \begin{pmatrix} J(x) & J_j(x) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix} \quad \text{with} \quad J_i = c j_i. \tag{2.36}$$

The Carrollian dynamics captured in the action (2.29) is the zero-c limit of a relativistic instantonic d-brane in a spacetime $\mathscr{M}$ with Randers–Papapetrou metric (2.35). As already mentioned (footnote 1), in this context instantonic means that the world-volume does not extend in time; it is a kind of codimension-one snap shot materialized in a space-like d-dimensional hypersurface $\mathscr{V}$, coordinated with $y^i, i = 1, \ldots, d$. Under these assumptions, the Dirac–Born–Infeld action reads:

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d y \sqrt{h}, \tag{2.37}$$

where h is the determinant of the induced metric matrix

$$h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \tag{2.38}$$

with $g_{\mu\nu}$ the background metric components.

For the Randers–Papapetrou environment displayed in (2.35), we find:

$$h_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \left(a_{kl} - c^2 (\Omega \partial_k t - b_k)(\Omega \partial_l t - b_l)\right). \tag{2.39}$$

In this expression, $\partial_k t$ stands for ${}^{\partial t}/_{\partial x^k}$. Consequently, we implicitly assume that the functions $x^k = x^k(y^i)$ are invertible, which is equivalent to saying that one can choose a gauge where

$y^i = x^i$. This is what happens in practice. Indeed, one can readily compute the root of the determinant and its expansion in powers of c^2. Naming $\alpha_i^k = \frac{\partial x^k}{\partial y^i}$, we obtain:

$$\sqrt{h} = \det\alpha \sqrt{a} \left(1 - \frac{c^2}{2} a^{kl} \left(\Omega \partial_k t - b_k\right) \left(\Omega \partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{2.40}$$

Hence (2.37) becomes

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d x \sqrt{a} \left(1 - \frac{c^2}{2} a^{kl} \left(\Omega \partial_k t - b_k\right) \left(\Omega \partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{2.41}$$

Neglecting the first term, which is invariant under Carrollian diffeomorphisms (2.27), (2.28), in the zero-c limit, (2.41) describes the same dynamics as (2.29), (2.30). This result, in close analogy with the Galilean discussion in the previous section, shows that the form (2.35) is well-suited for the zero-c limit.

3 Fluid dynamics in the non-relativistic limits

The aim of the present chapter is to exhibit the general fluid equations in the Galilean and Carrollian structures. This is achieved starting from plain relativistic viscous-fluid dynamics in the appropriate background – Zermelo or Randers–Papapetrou – and analyzing the associated, infinite or vanishing light-velocity limit. By construction, the equations reached this way are covariant under the corresponding diffeomorphisms. We study here neutral fluids, moving freely *i.e.* subject only to pressure, friction forces and thermal conduction processes. We conclude with some comments on a duality relating the two limits under consideration.

3.1 Relativistic fluids

Free relativistic viscous fluids are described in terms of their energy–momentum tensor obeying the set of $d+1$ conservation equations

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.1}$$

The time component is the energy conservation, the other d spatial ones, momentum conservation, usually called *Euler* equations.

The energy–momentum tensor is made of a perfect-fluid piece and terms resulting from friction and thermal conduction. It reads:

$$T^{\mu\nu} = (\varepsilon + p)\frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2}, \tag{3.2}$$

and contains $d+2$ dynamical variables:

- energy per unit of proper volume (rest density) ε, and pressure p;
- d velocity-field components u^i (u^0 is determined by the normalization $\|\mathrm{u}\|^2 = -c^2$).

A local-equilibrium thermodynamic equation of state[9] $p = p(T)$ is therefore needed for completing the system. We also have the usual Gibbs–Duhem relation for the grand potential $-p = \varepsilon - Ts$ with $s = \partial p/\partial T$. The viscous stress tensor $\tau^{\mu\nu}$ and the heat current q^μ are purely transverse:

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \tfrac{1}{c^2} T_{\mu\nu} u^\mu u^\nu. \tag{3.3}$$

Hence, they are expressed in terms of u^i and their spatial components q_i and τ_{ij}.

The quantities q_i and τ_{ij} capture the physical properties of the out of equilibrium state. They are usually expressed as expansions in temperature and velocity derivatives, the coefficients of which characterize the transport phenomena occurring in the fluid. The transport coefficients can be determined either from the underlying microscopic theory, or phenomenologically. In first-order hydrodynamics

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{3.4}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{c^2} a_\nu\right), \tag{3.5}$$

where [10]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{3.6}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \tfrac{1}{c^2} u_{(\mu} a_{\nu)} - \tfrac{1}{d}\Theta h_{\mu\nu}, \tag{3.7}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \tfrac{1}{c^2} u_{[\mu} a_{\nu]}, \tag{3.8}$$

are the acceleration, the expansion, the shear and the vorticity of the velocity field, with η, ζ the shear and bulk viscosities, and κ the thermal conductivity. In the above expressions, $h_{\mu\nu}$ is the projector onto the space transverse to the velocity field, and one similarly defines the longitudinal projector $U_{\mu\nu}$:

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}, \quad U_{\mu\nu} = -\frac{u_\mu u_\nu}{c^2}. \tag{3.9}$$

In three spacetime dimensions, the Hall viscosity appears as well in $\tau_{(1)\mu\nu}$:

$$-\zeta_{\mathrm{H}} \frac{u^\sigma}{c} \eta_{\sigma\lambda(\mu}\sigma_{\nu)\rho} g^{\lambda\rho}, \tag{3.10}$$

with $\eta_{\sigma\lambda\mu} = \sqrt{-g}\,\epsilon_{\sigma\lambda\mu}$.

[9] We omit here the chemical potential as we assume no independent conserved current.
[10] Our conventions for (anti-) symmetrization are $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

In view of the subsequent steps of our analysis, an important question arises at this stage, which concerns the behaviour of q_i and τ_{ij} with respect to the velocity of light. Answering this question requires a microscopic understanding of the fluid *i.e.* a many-body (quantum-field-theory and statistical-mechanics) determination of the transport coefficients. In the absence of this knowledge, we may consider a large-c or small-c expansion of these quantities, in powers of c^2 – irrespective of the derivative expansion. In the same spirit, we could also work out similar expansions for each of the functions entering the metrics (2.17) or (2.35), as these possibly carry deep relativistic dynamics. The advantage of such an exhaustive analysis would be to set-up general conditions on a relativistic fluid and its spacetime environment for a large-c or a small-c regime to make sense. As a drawback, this approach would blur the universality of the equations we want to set. We will therefore adopt a more pragmatic attitude and assume that Ω, b_i, w^j and a_{ij} are c-independent. Regarding the viscous stress tensor τ_{ij}, we will assume the following behaviours:

$$\tau_{ij} = -\Sigma^{\mathrm{G}}{}_{ij} \tag{3.11}$$

or

$$\tau^{ij} = -\frac{\Sigma^{\mathrm{C}ij}}{c^2} - \Xi^{ij}. \tag{3.12}$$

The first is appropriate for the Galilean limit. It is standard and considered *e.g.* in [1], where $\Sigma^{\mathrm{G}}{}_{ij}$ is named σ'_{ij}. For the Carrollian dynamics, our choice is inspired by flat-spacetime holography (see [16]). Similarly, for the heat current, we will adopt

$$q_i = Q^{\mathrm{G}}{}_i, \tag{3.13}$$

$$q^i = Q^{\mathrm{C}i} + c^2\pi^i, \tag{3.14}$$

in Galilean and Carrollian dynamics, respectively. Although kinematically poorer – because at rest, Carrollian fluids carry a richer internal information than their Galilean pendants since both the heat current and the viscous tensor are doubled in the above ansatz. Observe the position of the spatial indices, different for the two cases under consideration. They are designed to be covariant under different classes of diffeomorphisms.

One should finally notice that, in writing the energy–momentum tensor (3.2), we have not made any assumption regarding the hydrodynamic frame, which is therefore left generic.[11] There are two reasons for this. The first is the absence of a conserved relativistic current, which makes hydrodynamic-frame conditions delicate. Further subtleties arise when studying the system in special limits such as the Galilean, where the relativistic arbitrariness for the velocity field is lost, due to the decoupling of mass and energy. This is the second reason.

[11] The freedom of choosing the hydrodynamic frame was raised in [1]. Modern discussions can be found in [9, 26, 27] (see also [28]).

3.2 Galilean fluid dynamics from Zermelo background

The essence of the classical limit

We will consider in the following the ordinary non-relativistic limit of fluid equations, formally reached at infinite c. The physical validity of this situation is based on two assumptions.

The first is kinematical: it presumes that the global velocity of the fluid with respect to the observer is small compared to c. This is easily implemented using the Zermelo form of the metric (2.17), where the control parameter for the validity of the classical limit is $\left|\frac{\mathbf{v}-\mathbf{w}}{c}\right|$. We find

$$\begin{cases} u^0 = \gamma c = \dfrac{c}{\Omega} + \mathrm{O}\left(1/c\right), \quad u_0 = -c\Omega + \mathrm{O}\left(1/c\right), \\ u^i = \gamma v^i = \dfrac{v^i}{\Omega} + \mathrm{O}\left(1/c^2\right), \quad u_i = \dfrac{v_i - w_i}{\Omega} + \mathrm{O}\left(1/c^2\right). \end{cases} \tag{3.15}$$

The second is microscopic. The internal particle motion should also be Galilean, in other words the energy density should be large compared to the pressure: $\varepsilon \gg p$. This sets restrictions on the equation of state, as not every equation of state is compatible with such a microscopic assumption.[12]

An important consequence of the microscopic assumption is the separation of mass and energy, now both independently conserved. It is customary to introduce the following:

- ϱ the usual mass per unit of volume (mass density);
- ϱ_0 the usual mass per unit of proper volume (rest-mass density);
- e the internal energy per unit of mass;
- h the enthalpy per unit of mass.

These local thermodynamic quantities are related as

$$\begin{cases} \varepsilon = \left(e + c^2\right) \varrho_0, \\ h = e + \frac{p}{\varrho}, \\ \varrho_0 = \dfrac{\varrho}{\Omega\gamma} = \varrho\sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2} \approx \varrho - \frac{\varrho}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2, \end{cases} \tag{3.16}$$

where we have used Eq. (2.20) for the Lorentz factor γ, and expanded it for small $\left|\frac{\mathbf{v}-\mathbf{w}}{c}\right|$.

[12] For example, the conformal equation of state, $\varepsilon = dp$ is not compatible with the non-relativistic limit at hand.

The structure of the equations

The fluid equations are the conservation (3.1) of the energy–momentum tensor (3.2), in the background (2.17). It is computationally wise to split these equations as:

$$\nabla_\mu T^{\mu 0} = 0, \quad \nabla_\mu T^\mu{}_i = 0. \tag{3.17}$$

Indeed, applying a Galilean diffeomorphism (2.9), (2.18), the time components up and space components down transform faithfully and irreducibly. On the divergence of the energy–momentum tensor we find:

$$\nabla'_\mu T'^{\mu 0} = J \nabla_\mu T^{\mu 0}, \quad \nabla'_\mu T'^\mu{}_i = J^{-1l}{}_i \nabla_\mu T^\mu{}_l. \tag{3.18}$$

Hence, the two sets of equations (3.17) do not mix[13] and have furthermore a d-dimensional covariant transformation, which is our goal for the Galilean fluid dynamics.

The expressions displayed so far are fully relativistic. The next step is to consider the large-c regime. In this regime, Eqs. (3.17) can be expanded in powers of $1/c$. This expansion must be performed with care as the time equation needs an extra c factor with respect to the next d spatial equations because it describes the evolution of energy, which is a momentum multiplied by c. We find:[14]

$$c\nabla_\mu T^{\mu 0} = c^2 \frac{\mathcal{C}}{\Omega} + \frac{\mathcal{E}}{\Omega} + \mathrm{O}\left(\frac{1}{c^2}\right), \tag{3.19}$$

$$\nabla_\mu T^\mu{}_i = \mathcal{M}_i + \mathrm{O}\left(\frac{1}{c^2}\right). \tag{3.20}$$

At infinite c this leads to $d+2$ equations (rather than $d+1$, since in the Galilean limit, mass and energy are separately conserved) for ϱ, e, p and v^i:

- continuity equation (mass conservation) $\mathcal{C} = 0$;
- energy conservation $\mathcal{E} = 0$;
- momentum conservation $\mathcal{M}_i = 0$;

this system is completed with the equation of state $p = p(e, \varrho)$.

It is important to stress that Galilean diffeomorphisms (2.9), (2.10) do not involve c, and consequently they do not mix the various terms in the expansions (3.19) and (3.20). All $d+2$

[13] They do mix for general diffeomorphisms though.

[14] Had we considered $\Omega = \Omega(t, \mathbf{x})$, the divergence $\nabla_\mu T^\mu{}_i$ would have exhibited an extra, dominant term in the large-c limit: $c^2 \partial_i \ln \Omega$. The spatial conservation equation, $\nabla_\mu T^\mu{}_i = 0$, would then automatically require the $\mathbf{x}$-independence for Ω. Notice also the rescaling by Ω in (3.19), which guarantees that $\mathcal{C}$ and $\mathcal{E}$ are invariants under Galilean diffeomorphisms, see (3.35).

fluid equations reached this way on general backgrounds[15] are guaranteed to be covariant under Galilean diffeomorphisms, and this was one motivation of our work.

The dissipative tensors in Zermelo background

Before displaying the advertised equations, we would like to elaborate on the two tensors which capture the deviation of the real fluid with respect to the perfect one: the heat current and the viscous stress tensor.

Orthogonality conditions (3.3) allow to express every component of these tensors in terms of q_i and τ_{ij}. We assume here the Zermelo form of the metric (2.17), and a fluid velocity field as in (2.19), (2.20). We find

$$q_0 = -\frac{v^i q_i}{c}, \quad q^0 = \frac{\left(v^i - w^i\right) q_i}{c\Omega^2}, \quad q^i = a^{ij} q_j + \frac{w^i \left(v^j - w^j\right) q_j}{c^2 \Omega^2}. \tag{3.21}$$

Similarly, the components of the stress tensor are obtained from the τ_{ij}s. For example:

$$\tau_{00} = \frac{v^k v^l \tau_{kl}}{c^2}, \quad \tau_{0j} = -\frac{v^k \tau_{kj}}{c}, \quad \tau^0{}_j = -\frac{\left(v^k - w^k\right) \tau_{kj}}{c\Omega^2}, \quad \tau^{00} = \frac{\left(v^k - w^k\right)\left(v^l - w^l\right) \tau_{kl}}{c^2 \Omega^4}, \ldots \tag{3.22}$$

We now define $Q^{\mathrm{G}}{}_i = q_i$ as anticipated in (3.13), and

$$Q^{\mathrm{G}i} = a^{ij} Q^{\mathrm{G}}{}_j. \tag{3.23}$$

Similarly, calling for $\Sigma^{\mathrm{G}}{}_{ij}$ introduced in (3.11), we define

$$\Sigma^{\mathrm{G}}{}_i{}^j = \Sigma^{\mathrm{G}}{}_{ik} a^{kj}, \quad \Sigma^{\mathrm{G}ij} = a^{ik} \Sigma^{\mathrm{G}}{}_k{}^j. \tag{3.24}$$

Using the generic transformation rules of q_μ and $\tau_{\mu\nu}$ under spacetime diffeomorphisms, we find that $\boldsymbol{Q}^{\mathrm{G}}$ and $\boldsymbol{\Sigma}^{\mathrm{G}}$ introduced above, appearing as classical c-independent objects, transform as they should, namely as d-dimensional tensors under Galilean diffeomorphisms (2.9), (2.18):

$$Q^{\mathrm{G}\prime}{}_i = Q^{\mathrm{G}}{}_k J^{-1k}{}_i, \qquad Q^{\mathrm{G}\prime i} = J^i_k Q^{\mathrm{G}k}, \tag{3.25}$$

$$\Sigma^{\mathrm{G}\prime}{}_{ij} = J^{-1k}{}_i J^{-1l}{}_j \Sigma^{\mathrm{G}}{}_{kl}, \qquad \Sigma^{\mathrm{G}\prime}{}_i{}^j = J^{-1k}{}_i \Sigma^{\mathrm{G}}{}_k{}^l J^j_l, \quad \Sigma^{\mathrm{G}\prime ij} = \Sigma^{\mathrm{G}kl} J^i_k J^j_l. \tag{3.26}$$

Continuity and energy conservation

Using Eq. (3.2) for the energy–momentum tensor $T^{\mu\nu}$ with $g^{\mu\nu}$ and u^μ given in (2.17) and (2.19), using Eqs. (3.21), (3.23) for the heat current and (3.22), (3.24) for the stress tensor as

[15] We stress again that here, as for instance in [29, 30], Galilean fluids evolve on general, curved and time-dependent spaces $\mathscr{S}$, as opposed to other works on non-relativistic fluid dynamics (see *e.g.* [31]).

well as the definitions (3.16), we can perform the large-c expansion of the relativistic energy conservation equation (3.19). This requires the expansion of the Christoffel symbols, displayed in App. A.1.

We find the following at $\mathrm{O}(c^2)$:

$$\mathcal{C} = \frac{\partial_t \sqrt{a}\,\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i, \tag{3.27}$$

where a stands for the determinant of the d-dimensional metric $a_{ij}(t,\mathbf{x})$, and ∇_i is the Levi–Civita covariant derivative associated with $a_{ij}(t,\mathbf{x})$ and Christoffel symbols given in (A.9). The standard continuity equation $\mathcal{C} = 0$ is thus recovered. It is customary to decompose $\mathcal{C}$ in (3.27) as

$$\frac{\partial_t \sqrt{a}\,\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i = \frac{1}{\Omega}\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho\theta^{\mathrm{G}}, \tag{3.28}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + v^i\nabla_i \tag{3.29}$$

is the *material derivative*, and

$$\theta^{\mathrm{G}} = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \nabla_i v^i\right) \tag{3.30}$$

the *effective Galilean fluid expansion*. The latter combines the divergence of the fluid congruence with the logarithmic expansion of the volume form to produce a genuine scalar under Galilean diffeomorphisms (2.9), (2.10) (see Eqs. (2.15) and (A.17)). The material derivative (3.29), in the form $\frac{1}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}$, is also an "invariant" when acting on a scalar function. This is due to (2.12), (A.12) and (A.13). When acting on arbitrary tensors, it should be supplemented with the appropriate $\mathbf{w}$-connection terms, as shown in the appendix, Eq. (A.24).

At the next $\mathrm{O}(c^0)$ order, we obtain:

$$\begin{aligned}\mathcal{E} &= \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\,\varrho\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right)\right) + \frac{1}{\Omega}\nabla_i\left(\varrho v^i\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right)\right)\\ &\quad + \frac{1}{\Omega}\nabla_i\left(\left(v^j-w^j\right)\left(p\delta^i_j - \Sigma^{\mathrm{G}}{}_j^{\ i}\right)\right) + \nabla_i Q^{\mathrm{G}i} + \frac{1}{\Omega}\Pi^{\mathrm{G}ij}\left(\nabla_i w_j + \frac{1}{2}\partial_t a_{ij}\right)\end{aligned} \tag{3.31}$$

$$\begin{aligned}&= \frac{\varrho}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right) + \frac{1}{\Omega}\nabla_i\left(p\left(v^i-w^i\right)\right) + \nabla_i Q^{\mathrm{G}i}\\ &\quad - \frac{1}{\Omega}\nabla_i\left(\left(v^j-w^j\right)\Sigma^{\mathrm{G}}{}_j^{\ i}\right) + \frac{1}{\Omega}\Pi^{\mathrm{G}ij}\left(\nabla_i w_j + \frac{1}{2}\partial_t a_{ij}\right),\end{aligned} \tag{3.32}$$

where the alternative expression (3.32) is obtained from (3.31) using the continuity equation $\mathcal{C} = 0$. Here we introduced

$$\Pi^{\mathrm{G}ij} = \varrho\frac{\left(v^i-w^i\right)\left(v^j-w^j\right)}{\Omega^2} + pa^{ij} - \Sigma^{\mathrm{G}ij}, \tag{3.33}$$

the components of the Galilean energy–momentum tensor, following [1]. They are expressed in terms of the fluid velocity, measured in an inertial-like frame, *i.e.* $\mathbf{v}-\mathbf{w}$, and transform under Galilean diffeomorphisms (2.9), (2.10) as a genuine rank-two d-dimensional tensor on $\mathscr{S}$ (one uses (2.11), (2.12), (2.13), (2.15), and (3.26)):

$$\Pi^{\mathrm{G}ij\prime} = J^i_k J^j_l \Pi^{\mathrm{G}kl}. \tag{3.34}$$

Equation $\mathcal{E}=0$ is the Galilean energy conservation equation for a viscous fluid in motion on arbitrary, time-dependent d-dimensional space $\mathscr{S}$, and observed from an arbitrary frame (moving at velocity $-\mathbf{w}(t,\mathbf{x})$ with respect to a local inertial frame). In a short while, we will recast this equation in a suitable form for recognizing the underlying phenomena. Notice that both friction and thermal conduction occur, driven by the viscous stress tensor $\boldsymbol{\Sigma}^{\mathrm{G}}$ and the heat current $\boldsymbol{Q}^{\mathrm{G}}$. As opposed to the energy-conservation equation at hand, the continuity (mass-conservation) equation depends neither on the motion of the observer ($\mathbf{w}$) nor on the friction properties of the fluid. This is expected because energy is frame-dependent while mass it is not.

One can check that under Galilean diffeomorphisms (2.9), (2.10):

$$\mathcal{C}' = \mathcal{C}, \quad \mathcal{E}' = \mathcal{E}. \tag{3.35}$$

In order to show this, it is convenient to recognize some well-behaved blocks in the expressions at hand, based on the quoted transformation rules. We have gathered this information in App. A.1, Eqs. (A.16)–(A.19). For (3.35), we also need (3.25), (3.26).

Euler equation

Following the same pattern, we can process the large-c behaviour of the relativistic momentum-conservation equations. Along with (3.20) we find:

$$\mathcal{M}_i = \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\,\varrho\frac{v_i-w_i}{\Omega}\right) + \frac{1}{\Omega}\nabla_j\left(\varrho w^j\left(\frac{v_i-w_i}{\Omega}\right)\right) + \frac{\varrho}{\Omega}\left(\frac{v^j-w^j}{\Omega}\right)\nabla_i w_j + \nabla_j \Pi^{\mathrm{G}\,j}_{\ \ i} \tag{3.36}$$

with $\Pi^{\mathrm{G}\,j}_{\ \ i}$ as in (3.33). The equation $\mathcal{M}_i = 0$ is the ultimate generalization of the standard Euler equation, displayed *e.g.* in Ref. [1]. It is remarkably simple. The second and third terms in (3.36) contribute to inertial forces (Coriolis, centrifugal etc.), and are usually absent in Euclidean space with inertial frames. Together with the first term, they provide the components of a one-form on $\mathscr{S}$ transforming as $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ (see (A.21), (A.22)). This is also how $\mathcal{M}_i$ behave under Galilean diffeomorphisms (2.9), (2.10):

$$\mathcal{M}'_i = J^{-1\,l}_{\ \ \ i}\mathcal{M}_l. \tag{3.37}$$

The Euler equation (3.36) contains the *acceleration* $\boldsymbol{\gamma}^{\mathrm{G}} = \gamma^{\mathrm{G}}{}_i \mathrm{d}x^i$ of the Galilean fluid. This is defined covariantly as

$$a_i = \gamma^{\mathrm{G}}{}_i + \mathrm{O}\left({}^{1}/_{c^2}\right) \tag{3.38}$$

with a_i the spatial components of the relativistic fluid acceleration as in (3.6). We find:

$$\Omega^2 \gamma^{\mathrm{G}}{}_i = \Omega \frac{\mathrm{d}^{v_i}/_{\Omega}}{\mathrm{d}t} - \Omega \partial_t {}^{w_i}/_{\Omega} - \frac{1}{2}\partial_i \mathbf{w}^2 - v^j \left(\partial_j w_i - \partial_i w_j\right) \tag{3.39}$$

with ${}^{\mathrm{d}}/_{\mathrm{d}t}$ defined in (3.29). In this expression, $\gamma^{\mathrm{G}}{}_i$ appear as the components of the acceleration in the local inertial frame and $\frac{\mathrm{d}^{v_i}/_{\Omega}}{\Omega \mathrm{d}t}$ are the components of the effectively measured acceleration in the coordinate frame at hand. In the right hand side, the second term is the dragging acceleration, the third accounts for the centrifugal acceleration, and the last is Coriolis contribution. We can alternatively write (3.39) as

$$\gamma^{\mathrm{G}}{}_i = \frac{\mathrm{d}^{(v_i - w_i)}/_{\Omega}}{\Omega \mathrm{d}t} - \frac{1}{2}\partial_i \frac{\mathbf{w}^2}{\Omega^2} + \frac{v^j}{\Omega} \nabla_i \frac{w_j}{\Omega} = \frac{\mathrm{D}^{(v_i - w_i)}/_{\Omega}}{\Omega \mathrm{d}t}, \tag{3.40}$$

where we used the *Galilean covariant time-derivative* (A.25) in the second equality.

By construction, the $\gamma^{\mathrm{G}}{}_i$s transform as components of a genuine d-dimensional form and $\gamma^{\mathrm{G}i} = a^{ij}\gamma^{\mathrm{G}}{}_j$ as a vector, under Galilean diffeomorphisms (2.9), (2.10):

$$\gamma^{\mathrm{G}\prime}_{i} = J^{-1l}{}_i \gamma^{\mathrm{G}}{}_l. \tag{3.41}$$

One can also check explicitly the covariance of (3.39) using (A.22). Using $\gamma^{\mathrm{G}}{}_i$ in (3.39) and the expression (3.33) for the Galilean energy–momentum tensor, we can recast $\mathcal{M}_i$ in (3.36) *à la* Euler:

$$\mathcal{M}_i = \varrho \gamma^{\mathrm{G}}{}_i + \partial_i p - \nabla_j \Sigma^{\mathrm{G}\,j}{}_i. \tag{3.42}$$

Energy and entropy

The momentum equation $\mathcal{M}_i = 0$ together with continuity equation $\mathcal{C} = 0$ can also be used in order to provide a sharper expression for $\mathcal{E}$ given in (3.31), and leading to:

$$\frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\,\varrho\left(e + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right)\right) = -\nabla_i \Pi^{\mathrm{G}i} - \frac{1}{2\Omega}\Pi^{\mathrm{G}ij}\partial_t a_{ij} + \varrho\frac{v_j - w_j}{\Omega^2}\partial_t \frac{w^j}{\Omega}. \tag{3.43}$$

In this equation, $\varrho\left(e + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right)$ is the total energy density of the fluid in the natural, non-inertial frame. The energy density has three contributions: $e\varrho$ as internal energy, the kinetic energy ${}^{\varrho \mathbf{v}^2}/_{2\Omega^2}$, and the potential energy of inertial forces ${}^{-\varrho \mathbf{w}^2}/_{2\Omega^2}$ (see (2.6) for the free particle paradigm). Furthermore

$$\Pi^{\mathrm{G}i} = \varrho\frac{v^i}{\Omega}\left(h + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right) + Q^{\mathrm{G}i} - \frac{v^j}{\Omega}\Sigma^{\mathrm{G}\,i}_{\ \ j} \tag{3.44}$$

appears as the *Galilean energy flux*. It receives contributions from the enthalpy, the kinetic and inertial-potential energies, as well as from dissipative processes: thermal conduction and friction, with the corresponding heat current $\boldsymbol{Q}^{\mathrm{G}}$ and viscous stress current $-\mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}/\Omega$. The general energy conservation equation $\mathcal{E} = 0$ has now a simple interpretation: the time variation of energy in a local domain is due to the energy flux through the frontier plus the external work due to the time dependence of a_{ij} and w^i (as for the free particle (2.7)).

Dissipative processes create entropy. One can readily determine the variation of the latter by recasting the energy variation in a manner slightly different than (3.43). For that we compute $\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i$ with (3.31), (3.40), (3.42). We find, using continuity and (3.30):

$$\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i = \frac{\varrho}{\Omega}\frac{\mathrm{d}e}{\mathrm{d}t} + p\theta^{\mathrm{G}} + \nabla_i Q^{\mathrm{G}i} - \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right). \tag{3.45}$$

In this expression, we can trade the energy per mass e, for the entropy per mass s, obeying

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v = T\mathrm{d}s + \frac{p}{\varrho^2}\mathrm{d}\varrho, \tag{3.46}$$

where $v = 1/\varrho$. Substituting (3.46) in (3.45), and trading $\mathrm{d}\varrho/\mathrm{d}t$ for $-\Omega\varrho\theta^{\mathrm{G}}$ (continuity), we obtain finally, owing to $\mathcal{E} = \mathcal{M}_i = 0$:

$$\frac{\varrho T}{\Omega}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^{\mathrm{G}i}. \tag{3.47}$$

The entropy is not conserved as a consequence of friction and heat conduction, which encode dissipative processes. The latter are globally captured in a *generalized dissipation function*

$$\psi = \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^{\mathrm{G}i}, \tag{3.48}$$

appearing both in energy and entropy equations (3.45), (3.47). Observe that ψ depends explicitly on Christoffel symbols as well as on the time variation of the metric. Hence time dependence and inertial forces contribute the dissipation phenomena.[16]

[16] The effect of inertial forces on dissipation has been recently studied by simulation of flows on curved static films without heat current (*i.e.* $d = 2$, $\Omega = 1$, $\mathbf{w} = 0$, $\partial_t a_{ij} = 0$, $\boldsymbol{Q}^{\mathrm{G}} = 0$) [8]. One might consider performing similar simulations or experiments for probing the more general sources of dissipation present in (3.48).

First-order Galilean hydrodynamics and incompressible fluids

The viscous stress tensor $\boldsymbol{\Sigma}^{\mathrm{G}}$ and the heat current $\boldsymbol{Q}^{\mathrm{G}}$ are constructed phenomenologically as velocity and temperature derivative expansions. Since these objects transform tensorially under Galilean diffeomorphisms (see (3.25), (3.26)), they must be expressed in terms of tensorial derivative quantities.

At first order, we have θ^{G} defined in (3.30), which is an invariant, and

$$\frac{1}{\Omega}\left(\nabla_{(k}v_{l)}+\frac{1}{2}\partial_t a_{kl}\right), \tag{3.49}$$

which is a rank-two symmetric tensor (see (A.19)). We can therefore set

$$\Sigma^{\mathrm{G}}_{(1)ij}=2\eta^{\mathrm{G}}\xi^{\mathrm{G}}{}_{ij}+\zeta^{\mathrm{G}}a_{ij}\theta^{\mathrm{G}}, \tag{3.50}$$

$$Q^{\mathrm{G}}_{(1)i}=-\kappa^{\mathrm{G}}\partial_i T. \tag{3.51}$$

The transport coefficients are as usual the shear viscosity η^{G}, coupled to the *Galilean shear*,

$$\xi^{\mathrm{G}}{}_{ij}=\frac{1}{\Omega}\left(\nabla_{(i}v_{j)}+\frac{1}{2}\partial_t a_{ij}\right)-\frac{1}{d}a_{ij}\theta^{\mathrm{G}}, \tag{3.52}$$

which receives also contributions from the derivative of the metric; the bulk viscosity ζ^{G}, coupled to the Galilean expansion, and the thermal conductivity κ^{G} coupled to the temperature gradient.

Using the definitions of relativistic expansion and shear (3.6), (3.7), we can find their behaviour at large c in the Zermelo background:

$$\sigma_{ij}=\xi^{\mathrm{G}}{}_{ij}+\mathrm{O}\left(1/c^2\right), \tag{3.53}$$

$$\Theta=\theta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right). \tag{3.54}$$

For completeness we also display the leading behaviour of the vorticity (3.8), even though it plays no rôle in first-order hydrodynamics:

$$\omega_{ij}=\frac{1}{\Omega}\left(\partial_{[i}(v-w)_{j]}\right)+\mathrm{O}\left(1/c^2\right). \tag{3.55}$$

Since furthermore the transverse projector (3.9) is $h_{ij}=a_{ij}+\mathrm{O}\left(1/c^2\right)$, using (3.4) and (3.5) together with (3.11) and (3.38), we find indeed (3.50) and (3.51) (by definition $Q^{\mathrm{G}}{}_i=q_i$). It is important to stress at this point that transport coefficients are determined as modes of microscopic correlation functions, and are therefore sensitive to the velocity of light. In writing (3.11), we have assumed the following large-c behaviour:

$$\eta=\eta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right),\quad \zeta=\zeta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right),\quad \kappa=\kappa^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right). \tag{3.56}$$

The case $d = 2$ is peculiar because $\Sigma^{\mathrm{G}}_{(1)ij}$ admits an extra term:

$$\zeta^{\mathrm{G}}_{\mathrm{H}} \eta_{k(i} \xi^{\mathrm{G}}{}_{j)l} a^{kl} = \frac{\zeta^{\mathrm{G}}_{\mathrm{H}}}{2\Omega} \left(\eta_{k(i} \nabla_{j)} v^k + \eta_{k(i} a_{j)l} \left(\nabla^k v^l - \frac{\partial_t \sqrt{a}\, a^{kl}}{\sqrt{a}} - a^{kl} \nabla_m v^m \right) \right) \tag{3.57}$$

with $\eta_{kl} = \sqrt{a}\,\epsilon_{kl}$. This is indeed (up to a global sign) the infinite-c limit of the relativistic Hall-viscosity contribution in three spacetime dimensions given in (3.10), assuming again $\zeta_{\mathrm{H}} = \zeta^{\mathrm{G}}_{\mathrm{H}} + \mathrm{O}\left(1/c^2\right)$.

We can now combine the first-derivative contribution (3.50) of the viscous stress tensor with expression (3.42) for $\mathcal{M}_i$ in order to obtain the momentum conservation equation $\mathcal{M}_i = 0$ of first-order Galilean hydrodynamics. We obtain

$$\varrho \gamma^{\mathrm{G}}{}_i + \partial_i p - \frac{\eta^{\mathrm{G}}}{\Omega} \left(\Delta v_i + r_i^{\ j} v_j + a_{ik} a^{jl} \partial_t \gamma^k_{jl} \right) - \left(\zeta^{\mathrm{G}} + \frac{d-2}{d} \eta^{\mathrm{G}} \right) \partial_i \theta^{\mathrm{G}} = 0, \tag{3.58}$$

where $\Delta = \nabla^i \nabla_i$ is the Laplacian operator in d dimensions and r_{ij} the Ricci tensor of the d-dimensional Levi–Civita connection γ^k_{ij}. Similarly, substituting (3.50), (3.51) and (3.52) in (3.47), we find the entropy equation in first-order hydrodynamics on general backgrounds:

$$\frac{\varrho T}{\Omega} \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2\eta^{\mathrm{G}}}{\Omega^2} \left(\left(\nabla^i v^j\right) \left(\nabla_i v_j\right) + \left(\nabla^i v^j\right) \partial_t a_{ij} - \frac{1}{4} \left(\partial_t a^{ij}\right) \left(\partial_t a_{ij}\right) \right) + \left(\zeta^{\mathrm{G}} - \frac{2\eta^{\mathrm{G}}}{d} \right) \left(\theta^{\mathrm{G}}\right)^2 + \kappa^{\mathrm{G}} \Delta T, \tag{3.59}$$

where we assumed κ^{G} constant (otherwise the last term would read $\nabla^i(\kappa^{\mathrm{G}} \nabla_i T)$).

A special class of Galilean fluids deserves further analysis. These are the *incompressible fluids* for which $\varrho(t, \mathbf{x})$ obeys

$$\frac{\mathrm{d}\varrho(t, \mathbf{x})}{\mathrm{d}t} = 0 \tag{3.60}$$

with $\mathrm{d}/\mathrm{d}t$ the material derivative defined in (3.29). Using the expressions (3.27) and (3.28), we recast the incompressibility requirement as the vanishing of the effective fluid expansion:

$$\theta^{\mathrm{G}} = 0. \tag{3.61}$$

In this case, the bulk viscosity drops from the stress tensor (3.50) and the Galilean shear (3.52) simplifies. The first-order hydrodynamics momentum equation for an incompressible fluid thus reads:

$$\varrho \frac{\mathrm{d}^{v_i}/\Omega}{\Omega\, \mathrm{d}t} = \varrho \frac{\mathrm{d}^{w_i}/\Omega}{\Omega\, \mathrm{d}t} + \frac{\varrho}{2} \partial_i \frac{\mathbf{w}^2}{\Omega^2} - \varrho \frac{v^j}{\Omega} \nabla_i \frac{w_j}{\Omega} - \partial_i p + \frac{\eta^{\mathrm{G}}}{\Omega} \left(\Delta v_i + r_i^{\ j} v_j + a_{ik} a^{jl} \partial_t \gamma^k_{jl} \right). \tag{3.62}$$

We immediately recognize in this expression the generalized *covariant Navier–Stokes equation*, valid for incompressible fluids on any space $\mathscr{S}$, observed from an arbitrary frame. The first three terms in the right-hand side are contributions of frame inertial forces, the fourth is the pressure force, and next come the friction forces at first-order derivative. For Euclidean space with $\Omega = 1$ and $\mathbf{w} = 0$ we recover the textbook form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\mathbf{grad}\, p}{\varrho} + \frac{\eta^{\mathrm{G}}}{\varrho}\Delta\mathbf{v}. \tag{3.63}$$

3.3 Carrollian fluid dynamics from Randers–Papapetrou background

Preliminary remarks

As Carrollian particles, Carrollian fluids have no motion. From a relativistic perspective this is an observer-dependent statement, since boosts can turn on velocity. In the limit of vanishing velocity of light, however, these transformations are no longer permitted. Hence, being at rest becomes a genuinely intrinsic feature.

The fluid velocity must be set to zero faster than c in order to avoid blow-ups in the energy–momentum conservation. The appropriate scaling, ensuring a non-trivial kinematic contribution is

$$v^i = c^2\Omega\beta^i + \mathrm{O}\left(c^4\right), \tag{3.64}$$

where $v^i = {}^{u^i}/_{\gamma}$. This leaves the Carrollian fluid with a kinematic variable $\boldsymbol{\beta} = \beta^i\partial_i$ of inverse-velocity dimension, as in (2.34) for the one-body Carrollian dynamics studied in Sec. 2.2 – reason why we keep the same symbol. In order to reach covariant Carrollian fluid equations by expanding the relativistic fluid equations at small c, we need to define the β^is in such a way that they transform as components of a genuine Carrollian vector under (2.27), (2.36) already at finite c. This is achieved by setting

$$v^i = \frac{c^2\Omega\beta^i}{1 + c^2\beta^j b_j} \Leftrightarrow \beta^i = \frac{v^i}{c^2\Omega\left(1 - \frac{v^j b_j}{\Omega}\right)}, \tag{3.65}$$

from which one checks that[17]

$$\beta^{i\prime} = J^i_j\beta^j. \tag{3.66}$$

The full fluid congruence reads then:

$$\begin{cases} u^0 = \gamma c = \dfrac{c}{\Omega}\dfrac{1 + c^2\boldsymbol{\beta}\cdot\boldsymbol{b}}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = \dfrac{c}{\Omega} + \mathrm{O}\left(c^3\right), \quad u_0 = -\dfrac{c\Omega}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = -c\Omega + \mathrm{O}\left(c^3\right), \\ u^i = \gamma v^i = \dfrac{c^2\beta^i}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = c^2\beta^i + \mathrm{O}\left(c^4\right), \quad u_i = \dfrac{c^2\left(b_i + \beta_i\right)}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = c^2\left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right), \end{cases} \tag{3.67}$$

[17] This is easily proven by observing that $\beta_i + b_i = -\frac{\Omega u_i}{c u_0}$. We define as usual $b^i = a^{ij}b_j$, $\beta_i = a_{ij}\beta^j$, $v_i = a_{ij}v^j$, $\boldsymbol{b}^2 = b_i b^i$, $\boldsymbol{\beta}^2 = \beta_i\beta^i$ and $\boldsymbol{b}\cdot\boldsymbol{\beta} = b_i\beta^i$.

where the Lorentz factor has been obtained by imposing the usual normalization $\|\mathrm{u}\|^2 = -c^2$:

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \boldsymbol{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}} = \frac{1}{\Omega} \left(1 + \frac{c^2}{2} \boldsymbol{\beta} \cdot (\boldsymbol{\beta} + 2\boldsymbol{b}) + \mathrm{O}\left(c^4\right) \right). \tag{3.68}$$

In the relativistic regime, *i.e.* before taking the zero-c limit, in the Randers–Papapetrou background (2.35) the perfect part of the energy–momentum tensor reads then:

$$\begin{cases} T_{\text{perf}}{}^{0}{}_{0} = -\varepsilon - c^2(\varepsilon + p)\beta^k (b_k + \beta_k) + \mathrm{O}\left(c^4\right), \\ c\Omega T_{\text{perf}}{}^{0}{}_{i} = c^2(\varepsilon + p)(b_i + \beta_i) + \mathrm{O}\left(c^4\right), \\ \frac{c}{\Omega} T_{\text{perf}}{}^{j}{}_{0} = -c^2(\varepsilon + p)\beta^j + \mathrm{O}\left(c^4\right), \\ T_{\text{perf}}{}^{j}{}_{i} = p\delta^j_i + c^2(\varepsilon + p)\beta^j (b_i + \beta_i) + \mathrm{O}\left(c^4\right). \end{cases} \tag{3.69}$$

The non-perfect part is encoded in Eqs. (3.2), (3.12) and (3.14). Notice, on the one hand, that for vanishing β^i, these expressions are exact at finite c: most of the terms of order c^2 vanish as do all non-displayed higher-order contributions in c^2; on the other hand, for vanishing c, one recovers the perfect energy–momentum of a fluid at rest due to the simultaneous vanishing of v^i as a consequence of (3.64).

The eventual absence of motion, macroscopic or microscopic, and the shrinking of the light-cone raise many fundamental questions regarding the origin of pressure, temperature, thermalization, entropy etc. One may wonder in particular what causes viscosity and thermal conduction, what replaces the temperature derivative expansion of q_i, what justifies its behaviour (3.12). Even the propagation of a signal such as sound, if possible, should be reconsidered. It is tempting to claim that all this physics will be mostly of geometric nature rather than many-body statistics, because as we will see the only kinematic Carrollian-fluid variable $\boldsymbol{\beta}$ enters partly the dynamics.

We have no definite answers to all these questions though, and will not discuss these important issues here, which might possibly require to elaborate on space-filling branes as microscopic objects – see Sec. 2.2. Our approach will be kinematical, aiming at writing the fundamental equations, covariant under Carrollian diffeomorphisms (2.27), (2.36), starting from the relativistic equations (3.1). Alternative paths may exist, allowing to built some Carrollian dynamics without using the zero-c limit of a relativistic fluid.[18]

[18] In this spirit, one should quote the attempt made in [32], inspired by the membrane paradigm – admittedly suited for reaching Galilean rather than ultra-relativistic fluid dynamics, as well as Ref. [33], mostly focused on the structure of the energy–momentum tensor of perfect fluids (3.69), which also touches on Carrollian symmetry.

The structure of the equations

The relativistic equations (conservation of the energy–momentum tensor) should now be presented as

$$\nabla_\mu T^\mu{}_0 = 0, \quad \nabla_\mu T^{\mu i} = 0. \tag{3.70}$$

Under Carrollian diffeomorphisms (2.27), (2.36), the divergence of the energy–momentum tensor transforms as:

$$\nabla'_\mu T'^\mu{}_0 = \frac{1}{J}\nabla_\mu T^\mu{}_0, \quad \nabla'_\mu T'^{\mu i} = J^i_l \nabla_\mu T^{\mu l}. \tag{3.71}$$

In analogy with the Galilean case (3.17), the two sets of equations (3.70) have separately a d-dimensional covariant transformation. This is part of the agenda for the Carrollian dynamics.

Equations (3.70) are relativistic. Using the general energy–momentum tensor (3.2) with perfect part (3.69) and (3.12) as stress tensor, we find generally:

$$\frac{c}{\Omega}\nabla_\mu T^\mu{}_0 = \frac{1}{c^2}\mathcal{F} + \mathcal{E} + \mathrm{O}\left(c^2\right), \tag{3.72}$$

$$\nabla_\mu T^{\mu i} = \frac{1}{c^2}\mathcal{H}^i + \mathcal{G}^i + \mathrm{O}\left(c^2\right). \tag{3.73}$$

For zero β^i, these expressions are *exact* with extra terms of order c^2 only, and requiring they vanish leads to the $d+1$ fully relativistic fluid equations. With $\beta^i \neq 0$, (3.72) and (3.73) are genuinely infinite series. Thanks to the validity of (3.66) at finite c, Carrollian diffeomorphisms do not mix the different orders of these series, making each term Carrollian-covariant. Here, we are interested in the zero-c limit, and in this case Eqs. (3.72) and (3.73) split into $2+2d$ distinct equations:

- energy conservation $\mathcal{E} = 0$;
- momentum conservation $\mathcal{G}^i = 0$;
- constraint equations $\mathcal{F} = 0$ and $\mathcal{H}^i = 0$.

All of these are covariant under Carrollian diffeomorphisms (2.27), (2.36).

The Carrollian fluid, obtained as Carrollian limit of a relativistic fluid in the appropriate (Randers–Papapetrou) background, is described in terms of the d β^is, and the two variables p and ε.[19] The latter are related through an equation of state and the energy-conservation equation $\mathcal{E} = 0$. As we will see soon, the other $2d+1$ equations are setting consistency constraints among the $2d$ components of the heat currents ($Q^{\mathrm{C}}{}_i$ and π_i), the $d(d+1)$ components of the viscous stress tensors ($\Sigma^{\mathrm{C}}{}_{ij}$ and Ξ_{ij}), the inverse-velocity components β^i and the geometric

[19] The proper energy density cannot be split in mass density and energy per mass, because the limit at hand is ultra-relativistic. Observe also that $\boldsymbol{b}$ is not a fluid variable but a Carrollian-frame parameter as was $\mathbf{w}$ in the Galilean case. The fluid kinematical variable is $\boldsymbol{\beta}$, playing the rôle $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ had in the usual non-relativistic case.

environment. Geometry is therefore expected to interfere more actively in the dynamics of Carrollian fluids, than it did for Galilean hydrodynamics. Some of the aforementioned constraints are possibly rooted to more fundamental microscopic/geometric properties, yet to be unravelled. Their usage will be illustrated in Sec. 4.2.

The dissipative tensors in Randers–Papapetrou background

For a relativistic fluid in the Randers–Papapetrou background (2.35), using the velocity field in (3.64) and (3.67) and the components q^i, the transversality conditions (3.3) lead to

$$q^0 = \frac{c}{\Omega}\left(b_i + \beta_i\right) q^i, \quad q_0 = -c\Omega\beta_i q^i, \quad q_i = \left(a_{ij} + c^2 b_i \beta_j\right) q^j. \tag{3.74}$$

Similarly, the components of the viscous stress tensor are obtained from the τ^{ij}s. For example:

$$\begin{gathered}\tau^{00} = \frac{c^2}{\Omega^2}\left(b_k + \beta_k\right)\left(b_l + \beta_l\right)\tau^{kl}, \quad \tau^{0i} = \frac{c}{\Omega}\left(b_i + \beta_i\right)\tau^{ik}, \quad \tau_{00} = c^2\Omega^2\beta_k\beta_l\tau^{kl}, \\ \tau_{0i} = -c\Omega\beta_j\left(a_{ik} + c^2 b_i\beta_k\right)\tau^{jk}, \quad \tau_{ij} = \left(a_{ik} + c^2 b_i\beta_k\right)\left(a_{jl} + c^2 b_j\beta_l\right)\tau^{kl}, \ldots\end{gathered} \tag{3.75}$$

Under Carrollian diffeomorphisms (2.27), (2.36), we obtain the following transformation rules

$$q'^i = q^j J^i_j, \quad \tau'^{ij} = \tau^{kl} J^i_k J^j_l. \tag{3.76}$$

This suggests to use q^i as components for the Carrolian d-dimensional heat current decomposed as $Q^{\mathrm{C}i} + c^2\pi^i$ (see (3.14)), and τ^{ij} for the Carrolian d-dimensional viscous stress tensors $\Sigma^{\mathrm{C}ij}$ and Ξ^{ij} defined in (3.12). We introduce as usual

$$Q^{\mathrm{C}}{}_i = a_{ij} Q^{\mathrm{C}j}, \quad \Sigma^{\mathrm{C}}{}_i{}^j = a_{ik}\Sigma^{\mathrm{C}kj}, \quad \Sigma^{\mathrm{C}}{}_{ij} = a_{jk}\Sigma^{\mathrm{C}}{}_i{}^k, \tag{3.77}$$

$$\pi_i = a_{ij}\pi^j, \quad \Xi_i^{\ j} = a_{ik}\Xi^{kj}, \quad \Xi_{ij} = a_{jk}\Xi_i^{\ k}. \tag{3.78}$$

Using the generic transformations (3.76) under Carrollian diffeomorphisms (2.27), (2.28), we find that the above quantities transform as they should, for being eligible as d-dimensional tensors:

$$Q^{\mathrm{C}\prime}_i = Q^{\mathrm{C}}{}_j J^{-1j}{}_i, \quad Q^{\mathrm{C}\prime i} = J^i_j Q^{\mathrm{C}j}, \tag{3.79}$$

$$\Sigma^{\mathrm{C}\prime}{}_{ij} = J^{-1k}{}_i J^{-1l}{}_j \Sigma^{\mathrm{C}}{}_{kl}, \quad \Sigma^{\mathrm{C}\prime}{}_i{}^j = J^{-1k}{}_i \Sigma^{\mathrm{C}}{}_k{}^l J^j_l, \quad \Sigma^{\mathrm{C}\prime ij} = \Sigma^{\mathrm{C}kl} J^i_k J^j_l, \tag{3.80}$$

and similarly for π_i and Ξ_{jk}.

Scalar equations

The computation of the spacetime divergence in (3.72) is straightforward and leads to the following:

$$\begin{aligned}
\mathcal{E} &= -\left(\frac{1}{\Omega}\partial_t + \frac{d+1}{d}\theta^{\mathrm{C}}\right)\left(\varepsilon + 2\beta_i Q^{\mathrm{C}i} - \beta_i\beta_j\Sigma^{\mathrm{C}ij}\right) + \frac{1}{d}\theta^{\mathrm{C}}\left(\Xi^i{}_i - \beta_i\beta_j\Sigma^{\mathrm{C}ij} + \varepsilon - dp\right) \\
&\quad - \left(\hat{\nabla}_i + 2\varphi_i\right)\left(Q^{\mathrm{C}i} - \beta_j\Sigma^{\mathrm{C}ij}\right) - \left(2Q^{\mathrm{C}i}\beta^j - \Xi^{ij}\right)\xi^{\mathrm{C}}{}_{ij}, \qquad (3.81)\\
\mathcal{F} &= \Sigma^{\mathrm{C}ij}\xi^{\mathrm{C}}_{ij} + \frac{1}{d}\Sigma^{\mathrm{C}i}{}_i\theta^{\mathrm{C}}, \qquad (3.82)
\end{aligned}$$

where we have introduced a new covariant derivative $\hat{\nabla}_i$, as defined in the appendix, Eqs. (A.45)–(A.53). It is based on a new torsionless and metric-compatible connection (see (A.61)–(A.65)) dubbed *Levi–Civita–Carroll*, which plays for Carrollian geometry the rôle of ordinary Levi–Civita connection for ordinary geometry, *i.e.* it allows to built derivatives covariant under Carrollian diffeomorphisms (2.27), (2.28). Some further properties regarding the curvature of this connection are displayed in (A.66)–(A.78). A deeper investigation of this structure is out of place here. In (3.81) and (3.82) we have moreover defined

$$\begin{aligned}
\varphi_i &= \frac{1}{\Omega}\left(\partial_t b_i + \partial_i\Omega\right), \qquad (3.83)\\
\theta^{\mathrm{C}} &= \frac{1}{\Omega}\partial_t \ln\sqrt{a}. \qquad (3.84)
\end{aligned}$$

These expressions describe a form and a scalar (see (A.42) and (A.41) for their transformation rules under Carrollian diffeomorphisms). They play the rôle of *inertial acceleration* and *expansion* for the Carrollian fluid. These are both geometrical and the qualifier “inertial” refers to the frame (*i.e.* b_i and Ω) origin. We shall see in a moment that there is an extra contribution to the Carrollian fluid acceleration due to the kinematical observable β_i, but none for the expansion (see (3.95), (3.96)). As already stated and readily seen by its equations, most of the fluid properties are of geometrical nature. One similarly defines an *inertial vorticity two-form* with components

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}, \qquad (3.85)$$

and the traceless and symmetric *shear tensor*

$$\xi^{\mathrm{C}}{}_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d}a_{ij}\partial_t \ln\sqrt{a}\right). \qquad (3.86)$$

These quantities will be related in a short while to the ordinary relativistic counterparts (see (3.98) and (3.97)). The former receives a fluid kinematical contribution, as opposed to the latter. Eventually, we can elegantly check that

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{F}' = \mathcal{F} \tag{3.87}$$

(we use for that Eqs. (2.31), (3.79), (3.80), (A.42), (A.43), (A.50)–(A.59)).

Equation $\mathcal{F} = 0$ sets a geometrical constraint on the Carrollian stress tensor $\boldsymbol{\Sigma}^{\mathrm{C}}$, whereas $\mathcal{E} = 0$ is the energy conservation. Using (3.81), the latter can be recast as follows:

$$\left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right) e_{\mathrm{e}} = -\left(\hat{\nabla}_i + 2\varphi_i\right) \Pi^{\mathrm{C}i} - \Pi^{\mathrm{C}ij}\left(\xi^{\mathrm{C}}{}_{ij} + \frac{1}{d}\theta^{\mathrm{C}} a_{ij}\right), \tag{3.88}$$

and in this form it bares some resemblance with the Galilean homologous equation (3.43). It exhibits three Carrollian tensors, which capture the Carrollian energy exchanges:

$$e_{\mathrm{e}} = \varepsilon + 2\beta_i Q^{\mathrm{C}i} - \beta_i\beta_j\Sigma^{\mathrm{C}ij}, \quad \Pi^{\mathrm{C}i} = Q^{\mathrm{C}i} - \beta_j\Sigma^{\mathrm{C}ij}, \quad \Pi^{\mathrm{C}ij} = Q^{\mathrm{C}i}\beta^j + \beta^i Q^{\mathrm{C}j} + p a^{ij} - \Xi^{ij}. \tag{3.89}$$

The first is a scalar e_{e}, which can be interpreted as an *effective Carrollian energy density* (observe the absence of kinetic energy, expected from the vanishing velocity). Its time variation, including the dilution/contraction effects due to the expansion, is driven by the gradient of a *Carrollian energy flux*, which is the vector $\Pi^{\mathrm{C}i}$, and by the coupling of the shear to a *Carrollian energy–momentum tensor* $\Pi^{\mathrm{C}ij}$.

Vector equations

The vectorial part of the divergence is obtained from (3.73) and has two pieces. The first reads:

$$\begin{aligned} \mathcal{G}_j \;\; = \;\; & \left(\hat{\nabla}_i + \varphi_i\right) \Pi^{\mathrm{C}i}{}_j + \varphi_j e_{\mathrm{e}} + 2\Pi^{\mathrm{C}i}\varpi_{ij} + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right)\left(\pi_j + \beta_j\left(e_{\mathrm{e}} - 2\beta_i\Pi^{\mathrm{C}i} - \beta_i\beta_k\Sigma^{\mathrm{C}ik}\right)\right) \\ & + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right)\left(\beta^k\left(\Pi^{\mathrm{C}}{}_{kj} - \frac{1}{2}\beta_k\Pi^{\mathrm{C}}{}_j - \frac{1}{2}\beta_k\beta^i\Sigma^{\mathrm{C}}{}_{ij}\right)\right). \end{aligned} \tag{3.90}$$

The second is as follows:

$$\mathcal{H}_j = -\left(\hat{\nabla}_i + \varphi_i\right) \Sigma^{\mathrm{C}i}{}_j + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right) \Pi^{\mathrm{C}}{}_j. \tag{3.91}$$

Equation $\mathcal{G}_j = 0$ involves ε, p and their temporal and/or spatial derivatives, $\boldsymbol{\beta}$, the heat current $\boldsymbol{Q}^{\mathrm{C}}$, and $\boldsymbol{\Xi}$, expressed in terms of the effective energy density e_{e}, the Carrollian energy flux and energy–momentum tensor $\boldsymbol{\Pi}^{\mathrm{C}}$, as well as $\boldsymbol{\pi}$ and $\boldsymbol{\Sigma}^{\mathrm{C}}$. It is a momentum conservation. Notice also the coupling of the energy flux to the inertial vorticity. Equation $\mathcal{H}_j = 0$ depends neither on ε nor on p. This is an equation for the Carrollian energy flux $\boldsymbol{\Pi}^{\mathrm{C}}$ and the viscous stress tensor $\boldsymbol{\Sigma}^{\mathrm{C}}$, of geometrical nature as it involves the metric $\boldsymbol{a}$, the Carrollian "frame velocity" $\boldsymbol{b}$ and the inertial acceleration $\boldsymbol{\varphi}$.

Under Carrollian diffeomorphisms (2.27), (2.28), using the already quoted equations, (2.31), (3.79), (3.80), and (A.42)–(A.60), we obtain:

$$\mathcal{G}'^i = J^i_j \mathcal{G}^j, \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j. \tag{3.92}$$

One should observe at this point that the energy–momentum tensor and energy flux associated with a Carrollian fluid and defined in (3.89) are merely a repackaging of part of the dynamical data. They do not capture all perfect and friction quantities, as it happens for Galilean fluids, Eqs. (3.33) and (3.44). Equation $\mathcal{F} = 0$, as well as the vector equations need indeed more information than the energy–momentum tensor and energy flux. There is pressure, energy density and "velocity", on the one hand, and on the other hand, we find the two heat currents and the two viscous stress tensors. The zero-c limit produces a decoupling in the equations, sustained by the scaling assumption (3.12). This is the reason why $\mathcal{H}_j = 0$ appears as an equation for the dissipative pieces of data only, while the non-dissipative ones mix with the heat currents inside $\mathcal{G}_j = 0$.

Carrollian perfect fluids

We would like to end this chapter with a remark on the case of perfect fluids, namely fluids with vanishing dissipative tensors. For those, the dynamical variables are ε, p and β_i, with $e_{\mathrm{e}} = \varepsilon$, $\Pi^{\mathrm{C}}{}_j = 0$ and $\Pi^{\mathrm{C}}{}_{ij} = p a_{ij}$. In this case, $\mathcal{F} = \mathcal{H}^i = 0$ identically, and

$$\mathcal{E} = -\frac{1}{\Omega}\partial_t \varepsilon - (\varepsilon + p)\theta^{\mathrm{C}}, \tag{3.93}$$

$$\mathcal{G}_j = (\varepsilon + p)\left(\varphi_j + \gamma^{\mathrm{C}}{}_j + \beta_j \theta^{\mathrm{C}}\right) + \frac{\beta_j}{\Omega}\partial_t(\varepsilon + p) + \hat{\partial}_j p. \tag{3.94}$$

On the one hand, non-trivial energy exchanges can only result from time-dependence of the metric and pressure gradients. The latter, on the other hand, are bound to non-trivial $\boldsymbol{\beta}$, $\boldsymbol{\gamma}^{\mathrm{C}}$, $\boldsymbol{b}$ and Ω. Here $\gamma^{\mathrm{C}}{}_j$ is the kinematical acceleration defined later in (3.99).

For perfect fluids, only $\mathcal{E}$ and $\mathcal{G}_i$ survive in the relativistic divergence of the energy–momentum tensor, Eqs. (3.72) and (3.73). Furthermore, for zero $\boldsymbol{\beta}$ these are actually the only terms, at finite c. Hence, the relativistic equations are not affected by the vanishing-c limit, and coincide with the Carrollian ones: $\mathcal{E} = 0$ and $\mathcal{G}_i = 0$. As a consequence, the Carrollian nature of a fluid at $\boldsymbol{\beta} = 0$ can only emerge through interactions. This is to be opposed to the Galilean situation, since Galilean perfect fluids are definitely different from relativistic perfect fluids, even at rest.

First-order Carrollian hydrodynamics

In order to acquire a better perspective of Carrollian fluid dynamics, we can study the first-order in derivative expansion of its viscous tensors and heat currents. The first-derivative relativistic kinematical tensors as acceleration and expansion (3.6), shear (3.7), and vorticity (3.8), for a fluid with velocity vanishing as (3.64) when $c \to 0$ in Randers–Papapetrou background (2.35) read (the only independent components are the spatial ones):

$$a_i = \frac{c^2}{\Omega}\left(\partial_t\left(b_i+\beta_i\right)+\partial_i\Omega\right)+\mathrm{O}\left(c^4\right)=c^2\left(\varphi_i+\gamma^{\mathrm{C}}{}_i\right)+\mathrm{O}\left(c^4\right), \tag{3.95}$$

$$\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}+\mathrm{O}\left(c^2\right)=\theta^{\mathrm{C}}+\mathrm{O}\left(c^2\right), \tag{3.96}$$

$$\sigma_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij}-\frac{1}{d}a_{ij}\partial_t\ln\sqrt{a}\right)+\mathrm{O}\left(c^2\right)=\xi^{\mathrm{C}}{}_{ij}+\mathrm{O}\left(c^2\right), \tag{3.97}$$

$$\omega_{ij} = c^2\left(\partial_{[i}b_{j]}+\frac{1}{\Omega}b_{[i}\partial_{j]}\Omega+\frac{1}{\Omega}b_{[i}\partial_t b_{j]}+w_{ij}\right)+\mathrm{O}\left(c^4\right)=c^2\left(\varpi_{ij}+w_{ij}\right)+\mathrm{O}\left(c^4\right). \tag{3.98}$$

We find the corresponding Carrollian expansion θ^{C} and shear $\xi^{\mathrm{C}}{}_{ij}$, as already anticipated in (3.84) and (3.86). These quantities are purely geometric and originate from the time dependence of the d-dimensional spatial metric. Similarly, the relativistic acceleration and vorticity allow to define the already introduced Carrollian, inertial acceleration φ_i and vorticity ϖ_{ij}, as well as the kinematical acceleration $\gamma^{\mathrm{C}}{}_i$ and kinematical vorticity w_{ij} defined as:

$$\gamma^{\mathrm{C}}{}_i = \frac{1}{\Omega}\partial_t\beta_i, \tag{3.99}$$

$$w_{ij} = \hat{\partial}_{[i}\beta_{j]}+\beta_{[i}\varphi_{j]}+\beta_{[i}\gamma^{\mathrm{C}}{}_{j]}. \tag{3.100}$$

Starting from the first-order relativistic viscous tensor (3.4) and heat current (3.5), in order to comply with the behaviours (3.12) and the definition of the Carrollian heat currents (3.14), we must assume that (up to possible higher orders in c^2)

$$\eta = \tilde{\eta}+\frac{\eta^{\mathrm{C}}}{c^2}, \quad \zeta=\tilde{\zeta}+\frac{\zeta^{\mathrm{C}}}{c^2}, \quad \kappa = c^2\tilde{\kappa}++\kappa^{\mathrm{C}}. \tag{3.101}$$

Hence, putting these equations together, we find

$$\Sigma^{\mathrm{C}}_{(1)ij} = 2\eta^{\mathrm{C}}\xi^{\mathrm{C}}{}_{ij}+\zeta^{\mathrm{C}}\theta^{\mathrm{C}}a_{ij}, \tag{3.102}$$

$$\begin{aligned} Q^{\mathrm{C}}_{(1)i} &= -\frac{\kappa^{\mathrm{C}}}{\Omega}\left(\partial_t(b_iT)+\beta_i\partial_tT+\partial_i(\Omega T)\right) \\ &= -\kappa^{\mathrm{C}}\left(\hat{\partial}_iT+T\left(\varphi_i+\gamma^{\mathrm{C}}{}_i\right)\right), \end{aligned} \tag{3.103}$$

and similarly for $\Xi_{(1)ij}$ and $\pi_{(1)i}$. These quantities will include respectively terms like $2\tilde{\eta}\xi^{\mathrm{C}}{}_{ij} + \tilde{\zeta}\theta^{\mathrm{C}}a_{ij}$ and $-\tilde{\kappa}\left(\hat{\partial}_i T + T\left(\varphi_i + \gamma^{\mathrm{C}}{}_i\right)\right)$, plus extra terms coupled to η^{C}, ζ^{C} and κ^{C}, and originating from higher-order contributions in the c^2-expansion of the relativistic shear, acceleration and expansion. Notice that these are absent for vanishing β^i because in this case (3.95)–(3.98) are exact.

All the above expressions are covariant under Carrollian diffeomorphisms (2.27), (2.28) (see formulas (A.40)–(A.43) in appendix). The friction phenomena are geometric and due to time evolution of the background metric a_{ij}. The heat conduction, depends also on a temperature, which has not been defined in Carrollian thermodynamics due to the absence of kinetic theory.

In the two-dimensional case one should take into account the Hall viscosity (3.10) in the relativistic viscous tensor at first order. Assuming again $\zeta_{\mathrm{H}} = \zeta^{\mathrm{C}}_{\mathrm{H}}/c^2 + \tilde{\zeta}_{\mathrm{H}}$, the extra term to be added to $\Sigma^{\mathrm{C}}_{(1)ij}$ in (3.102) reads:

$$\zeta^{\mathrm{C}}_{\mathrm{H}}\sqrt{a}\,\epsilon_{k(i}\xi^{\mathrm{C}}{}_{j)l}a^{kl}, \tag{3.104}$$

and similarly for $\Xi_{(1)ij}$ with transport coefficients $\tilde{\zeta}_{\mathrm{H}}$ and $\zeta^{\mathrm{C}}_{\mathrm{H}}$ as already explained.

The final first-order Carrollian equations are obtained by substituting $\Sigma^{\mathrm{C}}_{(1)ij}$ and $Q^{\mathrm{C}}_{(1)i}$ given in (3.102) and (3.103), and similarly for $\Xi_{(1)ij}$, and $\pi_{(1)i}$, inside the general expressions for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}_i$ and $\mathcal{H}_i$, Eqs. (3.81), (3.82), (3.90) and (3.91).

Conformal Carrollian fluids

Carrollian fluids are ultra-relativistic and are thus compatible with conformal symmetry. For conformal relativistic fluids the energy–momentum tensor (3.2) is traceless and this requires

$$\varepsilon = dp, \quad \tau^{\mu}{}_{\mu} = 0. \tag{3.105}$$

In the Carrollian limit, the latter reads:

$$\Xi^i{}_i = \beta_i\beta_j\Sigma^{\mathrm{C}ij}, \quad \Sigma^{\mathrm{C}i}{}_i = 0. \tag{3.106}$$

In particular, we find $e_{\mathrm{e}} = \Pi^{\mathrm{C}i}{}_i$.

The dynamics of conformal fluids is covariant under Weyl transformations. Those act on the fluid variables as

$$\varepsilon \to \mathcal{B}^{d+1}\varepsilon, \quad \pi_i \to \mathcal{B}^d\pi_i, \quad Q^{\mathrm{C}}{}_i \to \mathcal{B}^d Q^{\mathrm{C}}{}_i, \quad \Xi_{ij} \to \mathcal{B}^{d-1}\Xi_{ij}, \quad \Sigma^{\mathrm{C}}{}_{ij} \to \mathcal{B}^{d-1}\Sigma^{\mathrm{C}}{}_{ij}, \tag{3.107}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The elements of the Carrollian geometry behave as follows:

$$a_{ij} \to \frac{1}{\mathcal{B}^2}a_{ij}, \quad b_i \to \frac{1}{\mathcal{B}}b_i, \quad \Omega \to \frac{1}{\mathcal{B}}\Omega, \tag{3.108}$$

and similarly for the kinematical variable β_i, the inertial and kinematical vorticity (3.85) and the shear (3.86):

$$\beta_i \to \frac{1}{\mathcal{B}}\beta_i, \quad \varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}, \quad w_{ij} \to \frac{1}{\mathcal{B}}w_{ij}, \quad \xi^{\mathrm{C}}{}_{ij} \to \frac{1}{\mathcal{B}}\xi^{\mathrm{C}}{}_{ij}. \tag{3.109}$$

The Carrollian inertial and kinematical accelerations (3.83) and (3.99), and the Carrollian expansion (3.84) transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \gamma^{\mathrm{C}}{}_i \to \gamma^{\mathrm{C}}{}_i - \frac{\beta_i}{\Omega}\partial_t \ln \mathcal{B}, \quad \theta^{\mathrm{C}} \to \mathcal{B}\theta^{\mathrm{C}} - \frac{d}{\Omega}\partial_t \mathcal{B}. \tag{3.110}$$

The first and the latter enable to define Weyl–Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$, as discussed in App. A.2, Eqs. (A.82)–(A.93). With these derivatives, Carrollian expressions (3.81), (3.82), (3.90) and (3.91) read for a conformal fluid:

$$\begin{aligned}
\mathcal{E} &= -\frac{1}{\Omega}\hat{\mathscr{D}}_t e_{\mathrm{e}} - \hat{\mathscr{D}}_i \Pi^{\mathrm{C}i} - \Pi^{\mathrm{C}ij}\xi^{\mathrm{C}}{}_{ij}, && (3.111)\\
\mathcal{F} &= \Sigma^{\mathrm{C}ij}\xi^{\mathrm{C}}{}_{ij}, && (3.112)\\
\mathcal{G}_j &= \hat{\mathscr{D}}_i \Pi^{\mathrm{C}i}{}_j + 2\Pi^{\mathrm{C}i}\varpi_{ij} + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\left(\pi_i + \beta_i\left(e_{\mathrm{e}} - 2\beta_k \Pi^{\mathrm{C}k} - \beta_k\beta_l \Sigma^{\mathrm{C}kl}\right)\right)\\
&\quad + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\left(\beta^k\left(\Pi^{\mathrm{C}}{}_{ki} - \frac{1}{2}\beta_k \Pi^{\mathrm{C}}{}_i - \frac{1}{2}\beta_k\beta^l \Sigma^{\mathrm{C}}{}_{li}\right)\right), && (3.113)\\
\mathcal{H}_j &= -\hat{\mathscr{D}}_i \Sigma^{\mathrm{C}i}{}_j + \frac{1}{\Omega}\hat{\mathscr{D}}_t \Pi^{\mathrm{C}}{}_j + \Pi^{\mathrm{C}}{}_i \xi^{\mathrm{C}i}{}_j. && (3.114)
\end{aligned}$$

These equations are Weyl-covariant of weights $d+2$, $d+2$, $d+1$ and $d+1$.

The case of conformal Carrollian perfect fluids is remarkably simple. As quoted earlier $\mathcal{F} = \mathcal{H}^i = 0$, and here

$$\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t \varepsilon, \quad \mathcal{G}_j = \frac{1}{d}\hat{\mathscr{D}}_j \varepsilon + \frac{d+1}{d}\left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\varepsilon\beta_i. \tag{3.115}$$

For these fluids the energy density is covariantly constant with respect to the Weyl–Carroll time derivative.

3.4 A self-dual fluid

A duality relationship between the Zermelo and the Randers–Papapetrou background metrics exist and can be stated as follows [15]: the contravariant form of Zermelo matches the covariant expression of Randers–Papapetrou and vice-versa (see Eqs. (A.1) and (A.28)).

This property is actually closely related to the duality among the Galilean and Carrollian contractions of the Poincaré group [12], and has many simple manifestations. For example, the reduction of a spacetime vector representation with respect to Galilean diffeomorphisms

(2.9), (2.10), (2.18) is performed with the components V^0 and V_i. Indeed, these transform as

$$V'^0 = JV^0, \quad V'_i = V_k J^{-1k}{}_i. \tag{3.116}$$

When reducing under Carrollian diffeomorphisms (2.27), (2.28), (2.36), one should instead use V_0 and V^i since

$$V'_0 = \frac{1}{J} V_0, \quad V'^i = J^i_k V^k. \tag{3.117}$$

The remarkable values $w^i = b_i = 0$ and $\Omega = 1$ define a sort of self-dual background. If furthermore we require the fluid to be at rest, no distinction survives between *perfect* Galilean and Carrollian fluids, as one readily checks that their equations are identical. The velocity of light is immaterial in this case. As soon as the system is driven away from perfection, this property does not hold any longer, because interactions are sensitive to c.

4 Examples

We will now illustrate our general formalism with examples for Galilean and Carrollian fluids. The latter is the first instance of a fluid obeying exact Carrollian dynamics. It is important both mathematically, as it makes contact with Calabi flows, and physically, for it is relevant in gravity and holography.

4.1 Galilean fluids

We provide here two applications: the flat space in rotating frame, which is well known and has the virtue of giving confidence to our methods, and the inflating space, combining both time-dependence and non-flatness of the host $\mathscr{S}$.

Euclidean three-dimensional space in rotating frame

We will present the hydrodynamical equations for a non-perfect fluid moving in Euclidean space E_3 with Cartesian coordinates, and observed from a uniformly rotating frame (see (2.8)):

$$a_{ij} = \delta_{ij}, \quad \Omega = 1, \quad \mathbf{w}(\mathbf{x}) = \mathbf{x} \times \boldsymbol{\omega}. \tag{4.1}$$

For this fluid, the continuity equation is simply

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho\, \mathbf{div}\, \mathbf{v} = 0. \tag{4.2}$$

The Euler equation in first-order hydrodynamics, Eq. (3.58) reads:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = (\boldsymbol{\omega}\times\mathbf{x})\times\boldsymbol{\omega} + 2\mathbf{v}\times\boldsymbol{\omega} - \frac{\mathbf{grad}\,p}{\varrho} + \frac{\eta^{\mathrm{G}}}{\varrho}\Delta\mathbf{v} + \frac{1}{\varrho}\left(\zeta^{\mathrm{G}} + \frac{\eta^{\mathrm{G}}}{3}\right)\mathbf{grad}(\mathbf{div}\,\mathbf{v}), \tag{4.3}$$

and we recognize the various, already spelled contributions to the dynamics. This equation has been obtained and used in many instances, see *e.g.* [7, 34, 35]. We also find the energy conservation equation (3.43):

$$\partial_t\left(\varrho\left(e + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right)\right) = -\mathbf{div}\boldsymbol{\Pi}^{\mathrm{G}}, \tag{4.4}$$

with

$$\boldsymbol{\Pi}^{\mathrm{G}} = \varrho\mathbf{v}\left(h + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right) - \kappa^{\mathrm{G}}\,\mathbf{grad}\,T - \mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}_{(1)} \tag{4.5}$$

and

$$\Sigma^{\mathrm{G}}_{(1)ij} = \eta^{\mathrm{G}}\left(\partial_i v_j + \partial_j v_i\right) + \left(\zeta^{\mathrm{G}} - \frac{2}{3}\eta^{\mathrm{G}}\right)\delta_{ij}\partial_k v^k. \tag{4.6}$$

Alternatively, using (3.32), the energy equation reads:

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right) = -\mathbf{div}p\mathbf{v} + \kappa^{\mathrm{G}}\Delta T + \mathbf{div}\left(\mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}_{(1)}\right). \tag{4.7}$$

The temporal variation of the total energy per mass is given by the divergences of the pressure, the thermal conduction and the viscous stress fluxes.

Inflating space

The dynamics of a non-perfect fluid moving on an inflating space can be studied considering:

$$a_{ij}(t,\mathbf{x}) = \exp\left(\alpha(t)\right)\tilde{a}_{ij}(\mathbf{x}), \quad \Omega = 1, \quad \mathbf{w} = 0. \tag{4.8}$$

The space dimension d is arbitrary here, therefore:

$$\ln\sqrt{a} = d\frac{\alpha}{2} + \ln\sqrt{\tilde{a}}. \tag{4.9}$$

The fluid equations obtained from (3.27), (3.32) and (3.42) become

$$\partial_t\varrho + \frac{\alpha'}{2}d\varrho + \mathbf{div}\varrho\mathbf{v} = 0, \tag{4.10}$$

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{\mathbf{v}^2}{2}\right) + \frac{\alpha'}{2}\left(\varrho\mathbf{v}^2 + dp - \mathrm{Tr}\boldsymbol{\Sigma}^{\mathrm{G}}\right) + \mathbf{div}\left(p\mathbf{v} + \mathbf{Q}^{\mathrm{G}} - \mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}\right) = 0, \tag{4.11}$$

$$\varrho\frac{\mathrm{d}v^i}{\mathrm{d}t} + \alpha'\varrho v^i + \nabla^i p - \nabla_j\Sigma^{\mathrm{G}ij} = 0. \tag{4.12}$$

where $\alpha' = {}^{\mathrm{d}\alpha}/_{\mathrm{d}t}$ and $\mathrm{Tr}\,\mathbf{\Sigma}^{\mathrm{G}} = a^{ij}\Sigma^{\mathrm{G}}{}_{ij}$.

The continuity equation (4.10) has an extra term proportional to ϱ. This reflects the change of density due to α'. For a static fluid one finds the familiar result $\varrho = \varrho_0 \mathrm{e}^{-d\alpha/2}$: for a space expanding in time, the density is getting diluted. In Euler's equation (4.12), a similar term creates a force proportional to the velocity field. For positive α', time dependence acts effectively like a friction. A similar conclusion is drawn from the energy conservation equation (4.11).

4.2 Two-dimensional Carrollian fluids and the Robinson–Trautman dynamics

Consider now a two-dimensional surface $\mathscr{S}$, endowed with a complex chart $(\zeta, \bar{\zeta})$ and a time-dependent metric of the form

$$\mathrm{d}\ell^2 = \frac{2}{P(t,\zeta,\bar{\zeta})^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}. \tag{4.13}$$

In this case the Carrollian shear $\boldsymbol{\xi}^{\mathrm{C}}$ (3.86) vanishes. We assume that the Carrollian frame has $\boldsymbol{b} = 0$ and $\Omega = 1$, and that the Carrollian kinematical variable $\boldsymbol{\beta}$ also vanishes. Hence, the Carrollian inertial acceleration $\boldsymbol{\varphi}$ (3.83) and inertial vorticity $\boldsymbol{\varpi}$ (3.85) vanish together with the kinematical acceleration $\boldsymbol{\gamma}^{\mathrm{C}}$ (3.99) and kinematical vorticity $\boldsymbol{w}$ (3.100). We further assume that $\boldsymbol{\pi}$ and $\mathbf{\Xi}$ vanish, so that the friction and heat-transport phenomena are captured exclusively by $\boldsymbol{Q}^{\mathrm{C}}$ and $\mathbf{\Sigma}^{\mathrm{C}}$. Hence $e_{\mathrm{e}} = \varepsilon$, $\Pi^{\mathrm{C}}{}_j = Q^{\mathrm{C}}{}_j$ and $\Pi^{\mathrm{C}}{}_{ij} = p a_{ij}$.

We will here study a conformal Carrollian fluid. In this case (see (3.106)), the Gibbs–Duhem equation reads

$$\varepsilon(t,\zeta,\bar{\zeta}) = 2p(t,\zeta,\bar{\zeta}), \tag{4.14}$$

and the viscous tensor is traceless:

$$\Sigma^{\mathrm{C}\zeta\bar{\zeta}} = 0. \tag{4.15}$$

The generic set of equations of motion for the Carrollian fluid at hand is (see (3.111), (3.113), (3.114))

$$\mathcal{E} = 3\varepsilon\partial_t \ln P - \partial_t\varepsilon - \mathbf{div}\,\boldsymbol{Q}^{\mathrm{C}} = 0, \tag{4.16}$$

$$\boldsymbol{\mathcal{G}} = \mathbf{grad}\, p = 0, \tag{4.17}$$

$$\boldsymbol{\mathcal{H}} = \partial_t \boldsymbol{Q}^{\mathrm{C}} - 2\boldsymbol{Q}^{\mathrm{C}}\partial_t \ln P - \mathbf{div}\mathbf{\Sigma}^{\mathrm{C}} = 0, \tag{4.18}$$

together with Eq. (3.112), $\mathcal{F} = 0$, identically satisfied due to the absence of shear. Equations (4.16), (4.17) and (4.18) are covariant under Weyl transformations mapping $P(t,\zeta,\bar{\zeta})$ onto $\mathcal{B}(t,\zeta,\bar{\zeta})P(t,\zeta,\bar{\zeta})$ with $\mathcal{B}(t,\zeta,\bar{\zeta})$ an arbitrary function.

The momentum equation (4.17) states that the pressure p is space-independent, which is not a surprise for a fluid at $\boldsymbol{\beta} = 0$ in a Carrollian frame with vanishing $\boldsymbol{b}$ and constant Ω. The same holds for the energy, due to the equation of state.

In order to proceed we must introduce some further assumptions regarding the heat current and the viscous stress tensor. These quantities are rooted to the unknown microscopic properties of the Carrollian fluids. As already mentioned earlier in Sec. 3.3, due to the absence of motion even at a microscopic level, it is tempting to assign a geometric rather than a statistical or kinetic origin to Carrollian thermodynamics. We may therefore define the *Carrollian temperature* as

$$\kappa^{\mathrm{C}} T(t,\zeta,\bar{\zeta}) = \left\langle \kappa^{\mathrm{C}} T \right\rangle (t) + \kappa' K(t,\zeta,\bar{\zeta}) - \kappa' \left\langle K \right\rangle (t), \tag{4.19}$$

where K the Gaussian curvature of (4.13):

$$K = \Delta \ln P \tag{4.20}$$

with $\Delta = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta}$ the ordinary two-dimensional Laplacian operator. The thermal conductivity κ^{C} is not constant in general because the identification with the curvature scalar endows the product $\kappa^{\mathrm{C}} T$ with a conformal weight 2, whereas the temperature T has weight 1. We also introduced a constant κ' for matching the dimensions. In expression (4.19), $\left\langle \kappa^{\mathrm{C}} T \right\rangle (t)$ is an *a priori* arbitrary time-dependent reference temperature (times thermal conductivity), and the brackets are meant to average over $\mathscr{S}$:[20]

$$\langle f \rangle (t) = \frac{1}{A} \int_{\mathscr{S}} \frac{\mathrm{d}^2\zeta}{P^2} f(t,\zeta,\bar{\zeta}), \quad A = \int_{\mathscr{S}} \frac{\mathrm{d}^2\zeta}{P^2}. \tag{4.21}$$

Equipped with a temperature, we define next the heat current as its gradient

$$\boldsymbol{Q}^{\mathrm{C}} = -\mathbf{grad}\, \kappa^{\mathrm{C}} T = -\kappa' \,\mathbf{grad}\, K, \tag{4.22}$$

following first-order Carrollian hydrodynamics, Eq. (3.103). Here, we assume this expression be exact, *i.e.* without higher-derivative contributions. With these definitions, the heat equation (4.16) for the Carrollian fluid at hand reads:

$$3\varepsilon \partial_t \ln P - \partial_t \varepsilon + \kappa' \Delta K = 0, \tag{4.23}$$

where we have used the equation of state (4.14). This is a dynamical equation for $P(t,\zeta,\bar{\zeta})$, given $\varepsilon(t)$. Carrollian dynamics, within the framework set by our definitions of temperature and heat current, is therefore purely geometrical and describes the evolution of the hosting space $\mathscr{S}$ rather than the fluid itself. This is not a surprise because the fluid does not move. Going in the Carrollian limit from a relativistic set-up, amounts to trading the dynamics of the fluid for that of the supporting geometry.

[20] Here $\mathrm{d}^2\zeta = -i\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$. If $\mathscr{S}$ is non-compact a limiting procedure is required for defining the integrals.

We must finally impose Eq. (4.18). As we mentioned in the general discussion of Sec. 3.3, this is not an evolution equation, but instead a constraint among the heat current, the viscous stress tensor and the ambient geometry. Thus, we can integrate it using (4.22). We find

$$\mathbf{\Sigma}^{\mathrm{C}} = -\frac{2\kappa'}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right), \tag{4.24}$$

up to a divergence-free, trace-free symmetric tensor. The viscous stress tensor for the Carrollian fluid at hand is therefore geometric, as is the heat current, and both appear as third-order derivatives of the metric. Actually, the effective expansion generally defined for Carrollian fluids as in (3.96), reads here:

$$\theta^{\mathrm{C}} = -2\partial_t \ln P. \tag{4.25}$$

It enables to view $\mathbf{\Sigma}^{\mathrm{C}}$ as a velocity third derivative through the writing

$$\Sigma^{\mathrm{C}}{}_{ij} = \kappa'\left(\nabla_i\nabla_j\theta^{\mathrm{C}} - \frac{1}{2}a_{ij}\nabla^k\nabla_k\theta^{\mathrm{C}}\right). \tag{4.26}$$

Notice that in the two-dimensional background under consideration (4.13), the viscous tensor $\mathbf{\Sigma}^{\mathrm{C}}$ could not have received an η^{C}-induced first-order derivative correction as in (3.102) because the Carrollian shear ξ^{C}_{ij} given in (3.97) vanishes here identically. However, since the Carrollian expansion θ^{C} is non-zero, the absence of first-order derivative correction (3.102) implies that for the fluid at hand $\zeta^{\mathrm{C}} = 0$.

Equation (4.23), which is at the heart of two-dimensional conformal Carrollian fluid dynamics, is actually known as Robinson–Trautman. It emerges when solving four-dimensional Einstein equations, assuming the existence of a null, geodesic and shearless congruence [36]. In vacuum or in the presence of a cosmological constant, Goldberg–Sachs theorems state that the corresponding spacetime is algebraically special and the whole dynamics boils down to the Robinson–Trautman equation with $\varepsilon(t) = 4\kappa' M(t)$ and $\kappa' = 1/16\pi G$ (using (4.20)):

$$\Delta\Delta \ln P + 12M\partial_t \ln P - 4\partial_t M = 0. \tag{4.27}$$

In that framework, the time dependence of the mass function $M(t)$ can be reabsorbed by an appropriate coordinate transformation (see *e.g.* [37]) and Robinson–Trautman equation becomes then

$$2\partial_{\bar\zeta}\partial_\zeta P^2\partial_{\bar\zeta}\partial_\zeta \ln P = 3M\partial_t\left(\frac{1}{P^2}\right) \tag{4.28}$$

with M constant related to the Bondi mass. This is a parabolic equation describing a Calabi flow on a two-surface [38].

The reason why Robinson–Trautman appears both as a heat equation in conformal Carrollian fluids and as a remnant of four-dimensional Einstein equations is the holographic relationship between gravity and fluid dynamics. The two-dimensional conformal Carrollian fluid studied here originates from flat Robinson–Trautman spacetime holography [16]. Similarly Robinson–Trautman equation is the heat equation for $2+1$-dimensional relativistic boundary fluids emerging holographically from four-dimensional anti-de Sitter Robinson–Trautman spacetimes [28].

5 Conclusions

We can summarize our method and results as follows.

A general relativistic spacetime metric is covariant under diffeomorphisms. When put in Zermelo form, the data $\Omega(t)$, $w^i(t,\mathbf{x})$ and $a_{ij}(t,\mathbf{x})$ transform under Galilean diffeomorphisms $t'=t'(t)$ and $\mathbf{x}'=\mathbf{x}'(t,\mathbf{x})$ as they should to comply with the infinite-c non-relativistic expectations. This observation is made by analyzing the relativistic particle motion and its classical limit. It provides the appropriate framework for studying the general non-relativistic Galilean fluid dynamics as an infinite-c limit of the relativistic one. In this manner, we have obtained the general equations *i.e.* continuity, energy-conservation and Euler, valid on any spatial background, potentially time-dependent, and observed from an arbitrary frame. These equations transform covariantly under Galilean diffeomorphisms.

Alternatively, one can study relativistic instantonic space-filling branes and the small-c behaviour of their dynamics. The latter is invariant under Carrollian diffeomorphisms $t'=t'(t,\mathbf{x})$ and $\mathbf{x}'=\mathbf{x}'(\mathbf{x})$, and Randers–Papapetrou form is the best designed spacetime metric because the data $\Omega(t,\mathbf{x})$, $b_i(t,\mathbf{x})$ and $a_{ij}(t,\mathbf{x})$ transform as expected from the non-relativistic limit (which is actually ultra-relativistic). In Randers–Papapetrou backgrounds one can study relativistic fluids and their Carrollian limit at vanishing velocity of light. This limit exhibits a new connection, which naturally fits into the emerging Carrollian geometry. One obtains in this way the general equations for the Carrollian fluids, manifestly covariant under Carrollian diffeomorphisms.

Several comments are in order here.

The Carrollian set we have reached is made of two scalar and two vector equations. The first scalar is an energy conservation, whereas the first vector is a momentum conservation. As there is no motion (due to $c=0$), there is no velocity field. Nonetheless there is a kinematical fluid variable (an "inverse velocity") accompanied by the pressure and energy density, related through an equation of state. We also find two heat currents and two viscous stress tensors. The Carrollian-fluid data cannot be naturally encapsulated all together in an energy–momentum tensor or an energy flux, as it happens in the Galilean case. Half of the equations concern exclusively the heat currents and the viscous stress tensors, relating them intimately to the ambient geometry and the Carrollian frame. We should stress here that we have made a specific assumption on the small-c behaviour of the relativistic viscous stress tensor and heat current, or equivalently of the transport coefficients. The number and the structure of the equations finally obtained reflects this unavoidable ansatz, inspired from the holographic Carrollian fluids met in flat-space gravity/fluid correspondence [16].[21] Going further in understanding this ansatz, and the physics behind the equations of motion, would require a microscopic analysis of Carrollian fluids.

Despite the absence of velocity field in Carrollian hydrodynamics, the concept of derivative expansion still holds. At each order one can define covariant tensors build on time and space derivatives of a_{ij}, b_i and β_i, as we met at first order with the shear and the expansion. The heat current and the viscous stress tensor can be expanded in these tensors, introducing phenomenological transport coefficients of increasing order.

Regarding Carrollian hydrodynamics, one could exploit a radically different perspective. Instead of defining a Carrollian fluid as the zero-c limit of a relativistic fluid in some Randers–Papapetrou background, one could simply try to build a fluid-like – *i.e.* continuous – generalization of an instantonic d-brane, directly within a Carrollian structure. This would promote the "inverse velocity" $\partial_i t$ of the elementary d-brane described by $t = t(\mathbf{x})$ into an "inverse velocity field" reminiscent of $\beta_i + b_i$ and transforming as in (2.32) under a Carrollian diffeomorphism. This could be the starting point for designing the dynamics of this new continuous Carrollian medium. Irrespective of the viewpoint chosen for describing Carrollian continuous media, zero-c limit of ordinary relativistic fluids or d-brane continuums, a great deal of fundamental thermodynamics, kinetic theory, derivative expansions, equilibrium and transport dynamics remains to be unravelled.

In conclusion of our general work, we have presented some examples. Those on Galilean hydrodynamics illustrate the power of the formalism for handling general, time-dependent and curved host spaces, potentially observed from non-inertial frames. The example of two-dimensional Carrollian fluid is interesting because it introduces the concept of geometric temperature and treats dissipative phenomena exactly *i.e.* by solving explicitly all the equations but one, finally brought in the canonical form of a Calabi flow on the two-dimensional surface. The Carrollian fluid dynamics translates into a dynamics for the geometry. This example has important implications in asymptotically flat holography [16] of Robinson–Trautman spacetimes.

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[21] Concrete examples of exact Carrollian fluids are described in this reference.

A Christoffel symbols, transformations and connections

We provide here a toolbox for working out the Galilean and Carrollian limits in the Zermelo and Randers–Papapetrou backgrounds, and checking the covariance properties of the set of equations reached by this method. These properties are bound to the emergence of novel Galilean and Carrollian connections, and covariant derivatives, which are discussed together with the associated curvature tensors. In the Carrollian case, an extra conformal connection is also presented, relevant when studying conformal Carrollian fluids.

A.1 Zermelo metric

Christoffel symbols

The Zermelo metric (2.17) has components (in the coframe $\left\{\mathrm{d}x^0 = c\mathrm{d}t, \mathrm{d}x^i\right\}$):

$$g^{Z}_{\mu\nu} \to \begin{pmatrix} -\Omega^2 + \frac{\mathbf{w}^2}{c^2} & -\frac{w_k}{c} \\ -\frac{w_i}{c} & a_{ik} \end{pmatrix}, \quad g^{Z\mu\nu} \to \frac{1}{\Omega^2}\begin{pmatrix} -1 & -\frac{w^j}{c} \\ -\frac{w^i}{c} & \Omega^2 a^{ij} - \frac{w^i w^j}{c^2} \end{pmatrix}, \tag{A.1}$$

where $w_k = a_{kj}w^j$. Its determinant reads:

$$\sqrt{-g} = \Omega\sqrt{a}, \tag{A.2}$$

where a is the determinant of a_{ij}. We remind that Ω depends on time only, whereas a_{ij} and w_i also depend on space.

The Christoffel symbols are easily computed. We are interested in their large-c behaviour for which one obtains the following:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{c}\partial_t \ln\Omega + +\frac{w^i}{2c^3\Omega^2}\left(\partial_i \mathbf{w}^2 + w^j\partial_t a_{ij}\right) + \mathrm{O}\left(1/c^5\right), && \text{(A.3)}\\
\Gamma^0_{0i} &= -\frac{1}{2c^2\Omega^2}\left(w_j\partial_i w^j + w^j\partial_j w_i + w^j\partial_t a_{ij}\right) + \mathrm{O}\left(1/c^4\right), && \text{(A.4)}\\
\Gamma^0_{ij} &= \frac{1}{c\Omega^2}\left(\frac{1}{2}\left(\partial_i w_j + \partial_j w_i + \partial_t a_{ij}\right) - w_k\gamma^k_{ij}\right), && \text{(A.5)}\\
\Gamma^i_{00} &= \frac{1}{c^2}\left(w^i\partial_t\ln\Omega - a^{ik}\left(\partial_t w_k + \partial_k\frac{\mathbf{w}^2}{2}\right)\right) + \mathrm{O}\left(1/c^4\right), && \text{(A.6)}\\
\Gamma^i_{j0} &= \frac{a^{ik}}{2c}\left(\partial_k w_j - \partial_j w_k + \partial_t a_{jk}\right) + \mathrm{O}\left(1/c^3\right), && \text{(A.7)}\\
\Gamma^i_{jk} &= \gamma^i_{jk} + \mathrm{O}\left(1/c^2\right), && \text{(A.8)}
\end{aligned}$$

where

$$\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right) \tag{A.9}$$

are the Christoffel symbols for the d-dimensional metric a_{ij}. Note also

$$\Gamma^{\mu}_{\mu 0} = \frac{1}{c}\partial_t \ln\left(\sqrt{a}\,\Omega\right), \quad \Gamma^{\mu}_{\mu i} = \partial_i \ln\sqrt{a}. \tag{A.10}$$

With these data it is possible to compute the divergence of the fluid energy–momentum tensor (3.19) and (3.20).

Covariance

In order to check the covariance (3.35) and (3.37),

$$\mathcal{C}' = \mathcal{C}, \quad \mathcal{E}' = \mathcal{E} \quad \mathcal{M}'_i = J^{-1l}{}_i \mathcal{M}_l,$$

for the Galilean fluid dynamics under Galilean diffeomorphisms (2.9)

$$t' = t'(t) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(t,\mathbf{x}),$$

with Jacobian functions (2.10)

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t,\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t,\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j},$$

we can use several simple covariant blocks. We first remind (2.11), (2.12), (2.13), (2.15):

$$a'_{ij} = a_{kl} J^{-1k}{}_i J^{-1l}{}_j, \quad v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right), \quad w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right), \quad \Omega' = \frac{\Omega}{J},$$

implying in particular

$$v'_k = \frac{J^{-1i}{}_k}{J}\left(v_i + a_{ij} J^{-1j}{}_l j^l\right), \quad w'_k = \frac{J^{-1i}{}_k}{J}\left(w_i + a_{ij} J^{-1j}{}_l j^l\right) \tag{A.11}$$

with

$$\partial'_t = \frac{1}{J}\left(\partial_t - j^k J^{-1i}{}_k \partial_i\right), \tag{A.12}$$

$$\partial'_j = J^{-1i}{}_j \partial_i. \tag{A.13}$$

Consider now A^k and B^k, the components of fields transforming like v^k or w^k (gauge-like transformation) and V^k a field transforming like $\frac{v^k - w^k}{\Omega}$ *i.e.* like a genuine vector:

$$A'^k = \frac{1}{J}\left(J^k_i A^i + j^k\right), \quad B'^k = \frac{1}{J}\left(J^k_i B^i + j^k\right), \quad V'^k = J^k_i V^i. \tag{A.14}$$

Consider also a scalar and a rank-two tensor

$$\Phi' = \Phi, \quad S'_{ij} = S_{kl} J^{-1k}{}_{i} J^{-1l}{}_{j}. \tag{A.15}$$

The basic transformation rules are as follows:

$$\begin{aligned}
\frac{A'^k - B'^k}{\Omega'} &= J^k_i \frac{A^i - B^i}{\Omega}, & \text{(A.16)}\\
\frac{1}{\sqrt{a'}}\partial'_t\left(\sqrt{a'}\Phi'\right) + \nabla'_i\left(\Phi' A'^i\right) &= \frac{1}{J}\left(\frac{1}{\sqrt{a}}\partial_t\left(\sqrt{a}\Phi\right) + \nabla_i\left(\Phi A^i\right)\right), & \text{(A.17)}\\
\nabla'_i V'^i &= \nabla_i V^i, & \text{(A.18)}\\
\nabla'_{(i}A'_{j)} + \frac{1}{2}\partial'_t a'_{ij} &= \frac{1}{J}\left(\nabla_{(k}A_{l)} + \frac{1}{2}\partial_t a_{kl}\right) J^{-1k}{}_{i} J^{-1l}{}_{j}, & \text{(A.19)}\\
\nabla'^{(i}A'^{j)} - \frac{1}{2}\partial'_t a'^{ij} &= \frac{1}{J}\left(\nabla^{(k}A^{l)} - \frac{1}{2}\partial_t a^{kl}\right) J^i_k J^j_l, & \text{(A.20)}\\
\nabla'_i S'^{ij} &= J^j_l \nabla_i S^{il}, & \text{(A.21)}\\
\frac{1}{\Omega'}\left(\partial'_t V'_i + A'^j \nabla'_j V'_i + V'_j \nabla'_i B'^j\right) &= \frac{J^{-1k}{}_{i}}{\Omega}\left(\partial_t V_k + A^j \nabla_j V_k + V_j \nabla_k B^j\right), & \text{(A.22)}\\
\Delta' A'_i + r'^m_i A'_m + a'_{ik} a'^{mn} \partial'_t \gamma'^k_{mn} &= \frac{J^{-1j}{}_{i}}{J}\left(\Delta A_j + r_j{}^m A_m + a_{jk} a^{mn} \partial_t \gamma^k_{mn}\right). & \text{(A.23)}
\end{aligned}$$

In the above expressions, ∇_i, Δ and r_{ij} are associated with the d-dimensional Levi–Civita connection γ^i_{jk} displayed in (A.9).

As a final comment regarding Galilean covariance properties, we would like to stress that the action of ∂_t spoils the transformation rules displayed in (A.14) and (A.15). This is both due to the transformation property of the partial time derivative (A.12), and to the time dependence of the Jacobian matrix J^i_j. A Galilean covariant time-derivative can be introduced, acting as follows on a vector:[22]

$$\frac{1}{\Omega}\frac{\mathrm{D}V^i}{\mathrm{d}t} = \frac{1}{\Omega}\left[\left(\partial_t + v^j\nabla_j\right)V^i - V^j\nabla_j w^i\right] = \frac{1}{\Omega}\frac{\mathrm{d}V^i}{\mathrm{d}t} - \frac{1}{\Omega}V^j\nabla_j w^i, \tag{A.24}$$

and resulting in a genuine vector under Galilean diffeomorphisms. Here, the frame velocity w^k plays the rôle of a connection, and the Galilean covariant time-derivative generalizes the material derivative $\mathrm{d}/\mathrm{d}t$ introduced in (3.29). The latter is covariant only when acting on scalar functions f, hence we set $\frac{\mathrm{D}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$. Expression (A.24) is easily extended for tensors of arbitrary rank using the Leibniz rule, as *e.g.* for one-forms:

$$\frac{1}{\Omega}\frac{\mathrm{D}V_i}{\mathrm{d}t} = \frac{1}{\Omega}\frac{\mathrm{d}V_i}{\mathrm{d}t} + \frac{1}{\Omega}V_j\nabla_i w^j. \tag{A.25}$$

[22] For a detailed and general presentation of Galilean affine connections see [23, 24].

Notice that the Galilean covariant time-derivative at hand is not "metric compatible":

$$\frac{1}{\Omega}\frac{\mathrm{D}a_{ij}}{\mathrm{d}t} = \frac{1}{\Omega}\left(\partial_t a_{ij} + 2\nabla_{(i}w_{j)}\right). \tag{A.26}$$

This result is actually expected because a covariant time-derivative of the metric should be interpreted as an extrinsic curvature. Indeed, expression (A.26) divided by $2c$ is exactly identified with the spatial components K_{ij} of constant-t hypersurfaces extrinsic curvature in the Zermelo background (2.17), (A.1).

The commutator of covariant time and space derivatives reveals a new piece of curvature, which appears in Galilean geometries, on top of the standard Riemann tensor associated with the spatial covariant derivative ∇_i. It is encapsulated in a one-form $\mathrm{d}\theta^{\mathrm{G}}$, as one observes from:

$$\left[\frac{1}{\Omega}\frac{\mathrm{D}}{\mathrm{d}t}, \nabla_i\right] V^i = V^i \partial_i \theta^{\mathrm{G}} + \nabla_j \left(V^i \nabla_i \left(\frac{w^j - v^j}{\Omega}\right)\right), \tag{A.27}$$

where θ^{G} is a scalar function introduced in (3.30) as the Galilean effective expansion:

$$\theta^{\mathrm{G}} = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \nabla_i v^i\right).$$

This extra piece of curvature should not come as a surprise. It is a Galilean remnant of some ordinary components of Riemannian curvature in the original Zermelo spacetime.

A.2 Randers–Papapetrou metric

Christoffel symbols

The Randers–Papapetrou metric (2.35) has components (in the coframe $\left\{\mathrm{d}x^0 = c\mathrm{d}t, \mathrm{d}x^i\right\}$):

$$g^{\mathrm{RP}}_{\mu\nu} \to \begin{pmatrix} -\Omega^2 & c\Omega b_j \\ c\Omega b_i & a_{ij} - c^2 b_i b_j \end{pmatrix}, \quad g^{\mathrm{RP}\mu\nu} \to \frac{1}{\Omega^2}\begin{pmatrix} -1 + c^2\boldsymbol{b}^2 & c\Omega b^k \\ c\Omega b^i & \Omega^2 a^{ik} \end{pmatrix}, \tag{A.28}$$

where $b^k = a^{kj}b_j$. The metric determinant is again given in (A.2):

$$\sqrt{-g} = \Omega\sqrt{a}. \tag{A.29}$$

Here, Ω, a_{ij} and b_i depend on time t and space $\mathbf{x}$.

The Christoffel symbols are computed exactly in the present case:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{c}\partial_t \ln\Omega + c\left(b^i\partial_i\Omega + \frac{1}{2}\left(\partial_t \boldsymbol{b}^2 - b_i b_j \partial_t a^{ij}\right)\right), && \text{(A.30)}\\
\Gamma^0_{0i} &= \left(1-\frac{1}{2}c^2\boldsymbol{b}^2\right)\partial_i\ln\Omega + \frac{1}{2}c^2 b^j\left(\partial_i b_j - \partial_j b_i - b_i\partial_j\ln\Omega\right)\\
&\quad + \frac{1}{2\Omega}b^j\partial_t\left(a_{ij} - c^2 b_i b_j\right), && \text{(A.31)}\\
\Gamma^0_{ij} &= -\frac{c}{2\Omega}\left(\partial_i b_j + \partial_j b_i + c^2 b^k\left(b_i\left(\partial_j b_k - \partial_k b_j\right) + b_j\left(\partial_i b_k - \partial_k b_i\right)\right)\right)\\
&\quad + \frac{c b_k}{\Omega}\gamma^k_{ij} + \frac{1-c^2\boldsymbol{b}^2}{2\Omega^2}\left(\frac{1}{c}\partial_t a_{ij} - c b_j\left(\partial_t b_i + \partial_i\Omega\right) - c b_i\left(\partial_t b_j + \partial_j\Omega\right)\right), && \text{(A.32)}\\
\Gamma^i_{00} &= \Omega a^{ij}\left(\partial_t b_j + \partial_j\Omega\right), && \text{(A.33)}\\
\Gamma^i_{j0} &= \frac{1}{2c}a^{ik}\left(\partial_t\left(a_{kj} - c^2 b_k b_j\right) + c^2\Omega\left(\partial_j b_k - \partial_k b_j\right) - c^2\left(b_k\partial_j\Omega + b_j\partial_k\Omega\right)\right), && \text{(A.34)}\\
\Gamma^i_{jk} &= \frac{c^2}{2}\left(\frac{b^i}{\Omega}\left(b_j\left(\partial_t b_k + \partial_k\Omega\right) + b_k\left(\partial_t b_j + \partial_j\Omega\right)\right) - a^{il}\left(b_j\left(\partial_k b_l - \partial_l b_k\right) + b_k\left(\partial_j b_l - \partial_l b_j\right)\right)\right)\\
&\quad + \gamma^i_{jk} - \frac{b^i}{2\Omega}\partial_t a_{jk}, && \text{(A.35)}
\end{aligned}$$

where γ^k_{ij} are the d-dimensional Christoffel symbols:

$$\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right). \tag{A.36}$$

Note also

$$\Gamma^\mu_{\mu 0} = \frac{1}{c}\partial_t \ln\left(\sqrt{a}\,\Omega\right), \quad \Gamma^\mu_{\mu i} = \partial_i \ln\left(\sqrt{a}\,\Omega\right). \tag{A.37}$$

With these data it is possible to compute the divergence of the fluid energy–momentum tensor (3.72) and (3.73).

Covariance and the Levi–Civita–Carroll connection

In order to check the covariance (3.87) and (3.92),

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{F}' = \mathcal{F}, \quad \mathcal{G}'^i = J^i_j\mathcal{G}^j, \quad \mathcal{H}'^i = J^i_j\mathcal{H}^j$$

for the Carrollian fluid dynamics under Carrollian diffeomorphisms (2.27)

$$t' = t'(t,\mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}),$$

with Jacobian functions (2.28)

$$J(t,\mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t,\mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j},$$

we can use several simple covariant blocks. We first remind (2.15), (2.31), (2.33):

$$a'_{ij} = a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{\ \ k}, \quad \Omega' = \frac{\Omega}{J},$$

and

$$\partial'_t = \frac{1}{J}\partial_t, \tag{A.38}$$

$$\partial'_j = J^{-1i}_{\ \ j}\left(\partial_i - \frac{j_i}{J}\partial_t\right). \tag{A.39}$$

From the above transformation rules we obtains:

$$\frac{1}{\Omega'}\partial'_t a'_{ij} = \frac{1}{\Omega}\partial_t a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \tag{A.40}$$

$$\frac{1}{\Omega'}\partial'_t \ln\sqrt{a'} = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \tag{A.41}$$

$$\partial'_t b'_i + \partial'_i \Omega' = \frac{1}{J} J^{-1j}_{\ \ i}\left(\partial_t b_j + \partial_j \Omega\right), \tag{A.42}$$

$$\hat{\partial}'_i = J^{-1j}_{\ \ i}\hat{\partial}_j, \tag{A.43}$$

where we have defined

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t. \tag{A.44}$$

In view of the basic rules (A.38), (A.39) and (A.40)–(A.43), it is tempting to introduce a new connection for Carrollian geometry that we will call *Levi–Civita–Carroll*, whose coefficients will be generalizations of the Christoffel symbols (A.36):

$$\begin{aligned}\hat{\gamma}^i_{jk} &= \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right)\\ &= \frac{a^{il}}{2}\left(\left(\partial_j + \tfrac{b_j}{\Omega}\partial_t\right) a_{lk} + \left(\partial_k + \tfrac{b_k}{\Omega}\partial_t\right) a_{lj} - \left(\partial_l + \tfrac{b_l}{\Omega}\partial_t\right) a_{jk}\right)\\ &= \gamma^i_{jk} + c^i_{jk}\end{aligned} \tag{A.45}$$

with γ^i_{jk} and $\hat{\partial}_i$ defined in (A.36) and (A.44). We will refer to those as *Christoffel–Carroll* symbols. They transform under Carrollian diffeomorphisms as ordinary Christoffel symbols under ordinary diffeomorphisms:

$$\hat{\gamma}'^k_{ij} = J^k_n J^{-1l}_{\ \ i} J^{-1m}_{\ \ j}\hat{\gamma}^n_{lm} - J^{-1l}_{\ \ i} J^{-1n}_{\ \ j}\partial_l J^k_n. \tag{A.46}$$

The emergence of this new set of connection coefficients should not be a surprise. Indeed one readily shows that

$$h_i^{\ \mu}\Gamma^k_{\mu\nu} h^\nu_{\ j} = \hat{\gamma}^k_{ij}, \tag{A.47}$$

where $\Gamma^{k}_{\mu\nu}$ are the $d+1$-dimensional Randers–Papapetrou Christoffel symbols (A.30)–(A.35), and $h_{\nu}{}^{\mu}$ the projector orthogonal to $\mathrm{u} = \partial_t/\Omega$ (as in (3.9), (3.67)).

The Levi–Civita–Carroll covariant derivative acts symbolically as

$$\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}} = \boldsymbol{\partial} + \frac{\boldsymbol{b}}{\Omega}\partial_t + \boldsymbol{\gamma} + \boldsymbol{c} = \boldsymbol{\nabla} + \frac{\boldsymbol{b}}{\Omega}\partial_t + \boldsymbol{c}. \tag{A.48}$$

For example, consider Φ, V^k and S_{kl}, the components of a scalar, a vector, and rank-two symmetric tensor:

$$\Phi' = \Phi, \quad V'^i = J^i_j V^j, \quad S'_{ij} = S_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \tag{A.49}$$

the action of this new covariant derivative is

$$\hat{\partial}_i \Phi = \partial_i \Phi + \frac{b_i}{\Omega}\partial_t \Phi, \tag{A.50}$$

$$\begin{aligned} \hat{\nabla}_i V^j &= \partial_i V^j + \frac{b_i}{\Omega}\partial_t V^j + \hat{\gamma}^j_{il} V^l \\ &= \nabla_i V^j + \frac{b_i}{\Omega}\partial_t V^j + c^j_{il} V^l, \end{aligned} \tag{A.51}$$

$$\hat{\nabla}_i V^i = \frac{1}{\sqrt{a}}\hat{\partial}_i\left(\sqrt{a}V^i\right) \tag{A.52}$$

$$\begin{aligned} \hat{\nabla}_i S_{jk} &= \partial_i S_{jk} + \frac{b_i}{\Omega}\partial_t S_{jk} - \hat{\gamma}^l_{ij} S_{lk} - \hat{\gamma}^l_{ik} S_{jl} \\ &= \nabla_i S_{jk} + \frac{b_i}{\Omega}\partial_t S_{jk} - c^l_{ij} S_{lk} - c^l_{ik} S_{jl}. \end{aligned} \tag{A.53}$$

All these transform as genuine tensors, namely:

$$\hat{\partial}'_i \Phi' = J^{-1j}_{\ \ i}\hat{\partial}_j \Phi, \tag{A.54}$$

$$\hat{\nabla}'_i V'^j = J^{-1k}_{\ \ i} J^j_l \hat{\nabla}_k V^l, \tag{A.55}$$

$$\hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i, \tag{A.56}$$

$$\hat{\nabla}'_i S'_{jk} = J^{-1m}_{\ \ i} J^{-1n}_{\ \ j} J^{-1l}_{\ \ k} \hat{\nabla}_m S_{nl}. \tag{A.57}$$

Further elementary transformation rules are as follows:

$$\frac{1}{\Omega'}\partial'_t \Phi' = \frac{1}{\Omega}\partial_t \Phi, \quad \frac{1}{\Omega'}\partial'_t V'^i = J^i_j \frac{1}{\Omega}\partial_t V^j, \quad \frac{1}{\Omega'}\partial'_t S'^{ij} = J^i_k J^j_l \frac{1}{\Omega}\partial_t S^{kl}, \tag{A.58}$$

as well as

$$\nabla'_i V'^i + \frac{b'_i}{\Omega'\sqrt{a'}}\partial'_t\left(\sqrt{a'}V'^i\right) = \hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i = \nabla_i V^i + \frac{b_i}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}V^i\right), \tag{A.59}$$

and

$$\nabla'_k S'^{ki} + \frac{b'_k}{\Omega'\sqrt{a'}}\left(\partial'_t\left(\sqrt{a'}S'^{ki}\right) - \sqrt{a'}S'^k{}_j\partial'_t a'^{ij}\right) - \frac{b'^i}{2\Omega'}S'^{kl}\partial'_t a'_{kl} = \hat{\nabla}'_k S'^{ki} =$$
$$= J^i_j \hat{\nabla}_k S^{kj} = J^i_j\left(\nabla_k S^{kj} + \frac{b_k}{\Omega\sqrt{a}}\left(\partial_t\left(S^{kj}\sqrt{a}\right) - \sqrt{a}S^k{}_l\partial_t a^{jl}\right) - \frac{b^j}{2\Omega}S^{kl}\partial_t a_{kl}\right). \tag{A.60}$$

Curvature, effective torsion and further properties of the Levi–Civita–Carroll connection

The Levi–Civita–Carroll connection is metric,

$$\hat{\nabla}_i a_{jk} = 0. \tag{A.61}$$

Furthermore, the usual torsion tensor vanishes:[23]

$$\hat{t}^k{}_{ij} = 2\hat{\gamma}^k_{[ij]} = 0. \tag{A.62}$$

However, the new ordinary (as opposed to covariant) derivatives $\hat{\partial}_i$ defined in (A.44) do not commute. Indeed, acting on any arbitrary function they lead to

$$\left[\hat{\partial}_i, \hat{\partial}_j\right]\Phi = \frac{2}{\Omega}\varpi_{ij}\partial_t\Phi, \tag{A.63}$$

where ϖ_{ij} are the components of the Carrollian vorticity defined in (3.85) (explicitly in (3.98)) using the Carrollian acceleration φ_i (3.83):

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}, \quad \varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i\Omega\right). \tag{A.64}$$

Therefore, the Levi–Civita–Carroll connection has an *effective torsion* as one can see from

$$\left[\hat{\nabla}_i, \hat{\nabla}_j\right]\Phi = \varpi_{ij}\frac{2}{\Omega}\partial_t\Phi, \tag{A.65}$$

where Φ is a scalar.

Similarly, one can compute the commutator of the Levi–Civita–Carroll covariant derivatives acting on a vector field:

$$\begin{aligned}\left[\hat{\nabla}_k, \hat{\nabla}_l\right]V^i &= \left(\hat{\partial}_k\hat{\gamma}^i_{lj} - \hat{\partial}_l\hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km}\hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm}\hat{\gamma}^m_{kj}\right)V^j + \left[\hat{\partial}_k, \hat{\partial}_l\right]V^i \\ &= \hat{r}^i{}_{jkl}V^j + \varpi_{kl}\frac{2}{\Omega}\partial_t V^i.\end{aligned} \tag{A.66}$$

In this expression we have defined $\hat{r}^i{}_{jkl}$, which are by construction components of a genuine tensor under Carrollian diffeomorphisms in d dimensions. This should be called the *Riemann–Carroll* tensor. It is made of several pieces, among which $\partial_k\gamma^i_{lj} - \partial_l\gamma^i_{kj} + \gamma^i_{km}\gamma^m_{lj} -$

[23]Discussions on Carrollian affine connections can be found *e.g.* in [24, 39, 40]. In particular, Ref. [24] provides a general classification of connections with or without torsion.

$\gamma^i_{lm}\gamma^m_{kj}$, which is *not* covariant under Carrollian diffeomorphisms – it is under ordinary d-dimensional diffeomorphisms though. The Ricci–Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k{}_{ikj}, \quad \hat{r} = a^{ij}\hat{r}_{ij}. \tag{A.67}$$

Notice that the Ricci–Carroll tensor is *not* symmetric in general: $\hat{r}_{ij} \neq \hat{r}_{ji}$.

We would like to close this part with two remarks regarding Carrollian geometry and in particular Carrollian time. As readily seen in (A.58), acting on any object tensorial under Carrollian diffeomorphisms, the time derivative ∂_t provides another tensor. For this reason, it was not necessary to define any "temporal covariant derivative". Our first remark is that the ordinary time derivative has an unsatisfactory feature: its action on the metric does not vanish. One is tempted therefore to set a new time derivative $\hat{\partial}_t$ such that

$$\hat{\partial}_t a_{jk} = 0, \tag{A.68}$$

while keeping the transformation rule under Carrollian diffeomorphisms:

$$\hat{\partial}'_t = \frac{1}{J}\hat{\partial}_t. \tag{A.69}$$

This is achieved by introducing a "temporal Carrollian connection"

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega}a^{ik}\partial_t a_{kj}. \tag{A.70}$$

Calling this a connection is actually inappropriate because it transforms as a genuine tensor under Carrollian diffeomorphisms:

$$\hat{\gamma}'^k{}_j = J^k_n J^{-1m}{}_j \hat{\gamma}^n{}_m. \tag{A.71}$$

In fact, the trace of this object is the Carrollian expansion introduced in (3.84):

$$\theta^{\mathrm{C}} = \frac{1}{\Omega}\partial_t \ln\sqrt{a} = \hat{\gamma}^i{}_i, \tag{A.72}$$

whereas its traceless part is the Carrollian shear defined in (3.86):

$$\xi^{\mathrm{C}i}{}_j = \hat{\gamma}^i{}_j - \frac{1}{d}\delta^i_j\hat{\gamma}^i{}_i = \hat{\gamma}^i{}_j - \frac{1}{d}\delta^i_j\theta^{\mathrm{C}}. \tag{A.73}$$

The temporal connection $\hat{\gamma}^i{}_j$ appears also as the zero-c remnant of the mixed projected relativistic Randers–Papapetrou Christoffel symbols, as in (A.47):

$$\frac{c}{\Omega}u_0{}^\mu\Gamma^k_{\mu\nu}h^\nu{}_j = \hat{\gamma}^k{}_j. \tag{A.74}$$

The action of $\hat{\partial}_t$ on scalars is simply ∂_t:

$$\hat{\partial}_t \Phi = \partial_t \Phi, \tag{A.75}$$

whereas on vectors or forms it is defined as

$$\frac{1}{\Omega}\hat{\partial}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \quad \frac{1}{\Omega}\hat{\partial}_t V_i = \frac{1}{\Omega}\partial_t V_i - \hat{\gamma}^j{}_i V_j. \tag{A.76}$$

Leibniz rule generalizes the latter to any tensor and allows to demonstrate the property (A.68). Indices can now be raised and lowered with the metric passing through $\hat{\partial}_t$.

The above Riemann–Carroll curvature tensor of a Carrollian geometry appears actually as the zero-c limit of the spatial components of the ordinary Riemann curvature in the Randers–Papapetrou background.[24] In the same spirit, one may also wonder what the Carrollian limit is for the temporal components of the relativistic Randers–Papapetrou curvature, and this is our second and last remark. In order to answer this question, we must compute the commutator of time and space covariant derivatives acting on scalar and vector fields, as in (A.65) and (A.66). We find:

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\partial}_i\right]\Phi = \left(\varphi_i \frac{1}{\Omega}\partial_t - \hat{\gamma}^j{}_i \hat{\partial}_j\right)\Phi, \tag{A.77}$$

and

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\nabla}_i\right] V^i = \left(\hat{\partial}_i \theta^{\mathrm{C}} - \hat{\nabla}_j \hat{\gamma}^j{}_i\right) V^i + \left(\theta^{\mathrm{C}} \delta^j_i - \hat{\gamma}^j{}_i\right)\varphi_j V^i + \left(\varphi_i \frac{1}{\Omega}\hat{\partial}_t - \hat{\gamma}^j{}_i \hat{\nabla}_j\right) V^i \tag{A.78}$$

with φ_i and θ^{C} the Carrollian acceleration and expansion (A.64), (A.72). We can define from this expression the components of a time-curvature Carrollian form:

$$\hat{r}_i = \frac{1}{d}\left(\hat{\nabla}_j \hat{\gamma}^j{}_i - \hat{\partial}_i \theta^{\mathrm{C}}\right) = \frac{1}{d}\left(\hat{\nabla}_j \hat{\xi}^{\mathrm{C}j}{}_i + \frac{1-d}{d}\hat{\partial}_i \theta^{\mathrm{C}}\right). \tag{A.79}$$

Using ϖ_{kl}, $\hat{r}_i$ and time derivative in the framework at hand, many new curvature-like (*i.e.* two-derivative) tensorial objects can be defined. We will not elaborate any longer on these issues, which would naturally fit in a more thorough analysis of Carrollian geometry.

[24] This statement is accurate but comes without a proof. Evaluating the zero-c (or infinite-c, as we would do in the Galilean counterpart) limit is a subtle task because in this kind of limits several components of the curvature usually diverge (see *e.g.* [16], where the rôle of curvature is prominent). From the perspective of the final geometry this does not produce any harm because the involved components decouple.

The Weyl–Carroll connection

The Levi–Civita–Carroll covariant derivatives $\hat{\boldsymbol{\nabla}}$ and $\hat{\partial}_t$ defined in (A.48), (A.75) and (A.76) for Carrollian geometry are not covariant with respect to Weyl transformations (3.108),

$$a_{ij} \to \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \to \frac{1}{\mathcal{B}} b_i, \quad \Omega \to \frac{1}{\mathcal{B}} \Omega. \tag{A.80}$$

We can define *Weyl–Carroll* covariant spatial and time derivatives using the Carrollian acceleration φ_i defined in (A.64) and the Carrollian expansion (A.72), which transform as connections (see (3.109)):

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta^{\mathrm{C}} \to \mathcal{B}\theta^{\mathrm{C}} - \frac{d}{\Omega}\partial_t \mathcal{B}. \tag{A.81}$$

For a weight-w scalar function Φ, *i.e.* a function scaling with $\mathcal{B}^w$ under (A.80), we introduce

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w\varphi_j \Phi, \tag{A.82}$$

such that under a Weyl transformation

$$\hat{\mathscr{D}}_j \Phi \to \mathcal{B}^w \hat{\mathscr{D}}_j \Phi. \tag{A.83}$$

Similarly, for a vector with weight-w components V^l:

$$\hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i. \tag{A.84}$$

The action on any other tensor is obtained using the Leibniz rule, as in example for rank-two tensors:

$$\hat{\mathscr{D}}_j t_{kl} = \hat{\nabla}_j t_{kl} + (w+2)\varphi_j t_{kl} + \varphi_k t_{jl} + \varphi_l t_{kj} - a_{jl} t_{ki}\varphi^i - a_{jk} t_{il}\varphi^i. \tag{A.85}$$

The Weyl–Carroll spatial derivative does not modify the weight of the tensor it acts on. Furthermore, it is metric as (a_{kl} has weight -2):

$$\hat{\mathscr{D}}_j a_{kl} = 0. \tag{A.86}$$

It has an effective torsion because

$$\left[\hat{\mathscr{D}}_i, \hat{\mathscr{D}}_j\right] \Phi = \frac{2}{\Omega}\varpi_{ij}\hat{\mathscr{D}}_t \Phi + w\Omega_{ij}\Phi, \tag{A.87}$$

although this expression does not contain terms of the type $\hat{\mathscr{D}}_k \Phi$. We have introduced here

$$\Omega_{ij} = \varphi_{ij} - \frac{2}{d}\varpi_{ij}\theta^{\mathrm{C}}, \tag{A.88}$$

where ϖ_{ij} are the components of the Carrollian vorticity defined in (A.64), and

$$\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i. \tag{A.89}$$

Both Ω_{ij} and ϖ_{ij} are components of genuine Carrollian two-forms, and Weyl-covariant of weight 0 and -1. However, φ_{ij} are not Weyl-covariant, although they are also by construction components of a good Carrollian two-form.

In Eq. (A.87), we have used a Weyl–Carroll derivative with respect to time $\hat{\mathscr{D}}_t$. Its action on a weight-w function Φ is defined as:

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi = \frac{1}{\Omega}\hat{\partial}_t\Phi + \frac{w}{d}\theta^{\mathrm{C}}\Phi = \frac{1}{\Omega}\partial_t\Phi + \frac{w}{d}\theta^{\mathrm{C}}\Phi, \tag{A.90}$$

which is a scalar of weight $w+1$ under (A.80):

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi \to \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi. \tag{A.91}$$

Accordingly, on a weight-w vector the action of the Weyl–Carroll time derivative is

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\hat{\partial}_t V^l + \frac{w-1}{d}\theta^{\mathrm{C}}V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta^{\mathrm{C}}V^l + \xi^{\mathrm{C}l}{}_i V^i. \tag{A.92}$$

These are the components of a genuine Carrollian vector of weight $w+1$ (the tensor $\xi^{\mathrm{C}l}{}_i$ is Weyl-covariant of weight 1). We have used (A.75), (A.76) and (A.73) for the second equalities in (A.90) and (A.92). The same pattern applies for any tensor by Leibniz rule, and in particular:

$$\hat{\mathscr{D}}_t a_{kl} = 0. \tag{A.93}$$

We will close the present appendix with the Weyl–Carroll curvature tensors, obtained by studying the commutation of Weyl–Carroll covariant derivatives acting on vectors. We find

$$\left[\hat{\mathscr{D}}_k, \hat{\mathscr{D}}_l\right] V^i = \left(\hat{\mathscr{R}}^i{}_{jkl} - 2\xi^{\mathrm{C}i}{}_j\varpi_{kl}\right)V^j + \varpi_{kl}\frac{2}{\Omega}\hat{\mathscr{D}}_t V^i + w\Omega_{kl}V^i, \tag{A.94}$$

where

$$\begin{aligned} \hat{\mathscr{R}}^i{}_{jkl} \;=\; & \hat{r}^i{}_{jkl} - \delta^i_j\varphi_{kl} - a_{jk}\hat{\nabla}_l\varphi^i + a_{jl}\hat{\nabla}_k\varphi^i + \delta^i_k\hat{\nabla}_l\varphi_j - \delta^i_l\hat{\nabla}_k\varphi_j \\ & + \varphi^i\left(\varphi_k a_{jl} - \varphi_l a_{jk}\right) - \left(\delta^i_k a_{jl} - \delta^i_l a_{jk}\right)\varphi_m\varphi^m + \left(\delta^i_k\varphi_l - \delta^i_l\varphi_k\right)\varphi_j \end{aligned} \tag{A.95}$$

are the components of the Riemann–Weyl–Carroll weight-0 tensor, from which we define

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^k{}_{ikj}, \quad \hat{\mathscr{R}} = a^{ij}\hat{\mathscr{R}}_{ij}. \tag{A.96}$$

Notice that the Ricci–Weyl–Carroll tensor is *not* symmetric in general: $\hat{\mathscr{R}}_{ij} \neq \hat{\mathscr{R}}_{ji}$.

Eventually, we quote

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]\Phi = w\hat{\mathscr{R}}_i\Phi - \xi^{\mathrm{C}j}{}_i\hat{\mathscr{D}}_j\Phi \tag{A.97}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]V^i = (w-d)\hat{\mathscr{R}}_i V^i - V^i\hat{\mathscr{D}}_j\xi^{\mathrm{C}j}{}_i - \xi^{\mathrm{C}j}{}_i\hat{\mathscr{D}}_j V^i, \tag{A.98}$$

with

$$\hat{\mathscr{R}}_i = \hat{r}_i + \frac{1}{\Omega}\hat{\partial}_t\varphi_i - \frac{1}{d}\hat{\nabla}_j\hat{\gamma}^j{}_i + \xi^{\mathrm{C}j}{}_i\varphi_j = \frac{1}{\Omega}\partial_t\varphi_i - \frac{1}{d}\left(\hat{\partial}_i + \varphi_i\right)\theta^{\mathrm{C}} \tag{A.99}$$

the components of a Weyl-covariant weight-1 Carrollian curvature one-form, where $\hat{r}_i$ is given in (A.79).

References

[1] L. D. Landau et E. M. Lifchitz, *Physique Théorique*, Vol. 6 *Mécanique des fluides*, Editions Mir, Moscou, 1969.

[2] M. Vinokur, *A new formulation of the conservation equations of fluid dynamics*, Technical Report NASA-TM-X-62415 (1974), National Aeronautics and Space Administration: Ames Research Center, Moffett Field, CA.

[3] L. M. Avis, *A spacetime tensor formulation for continuum mechanics in general curvilinear, moving, and deforming coordinate systems*, Technical Report NASA-TR-R-462 (1976), National Aeronautics and Space Administration: Langley Research Center, Hampton, VA.

[4] R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Courier Dover Publications: Mineola, NY (1989).

[5] H. A. Carlson, G. Berkooz and J. L. Lumley, *Direct numerical simulation of flow in a channel with complex, time-dependent wall geometries: a pseudospectral method*, J. Comput. Phys. **121** (1995) 155.

[6] H. Luo and T. R. Bewley, *On the contravariant form of the Navier–Stokes equations in time-dependent curvilinear coordinate systems*, J. Comput. Phys. **199** (2004) 355.

[7] M. Charron, A. Zadra and C. Girard, *Four-dimensional tensor equations for a classical fluid in an external gravitational field*, Q.J.R. Meteorol. Soc. **140** (2014) 908.

[8] J.-D. Debus, M. Mendoza, S. Succi, H. J. Herrmann, *Energy dissipation in flows through curved spaces*, Scientific reports **7**, 42350 (2017), arXiv:1511.08031 [physics.flu-dyn].

[9] L. Rezzolla and O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, Oxford UK, 2013.

[10] K. Jensen and A. Karch, *Revisiting non-relativistic limits*, JHEP **1504** (2015) 155, arXiv:1412.2738 [hep-th].

[11] J.-M. Lévy-Leblond, *Une nouvelle limite non-relativiste du groupe de Poincaré*, A. Inst. Henri Poincaré **III** (1965) 1.

[12] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,* Class. Quant. Grav. **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[13] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll groups and BMS symmetry,* Class. Quant. Grav. **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[14] E. Bergshoeff, J. Gomis and G. Longhi, *Dynamics of Carroll particles*, Class. Quant. Grav. **31** (2014) 205009, arXiv:1405.2264 [hep-th].

[15] G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick and M. C. Werner, *Stationary metrics and optical Zermelo–Randers–Finsler geometry,* Phys. Rev. **D79** (2009) 044022, arXiv:0811.2877 [gr-qc].

[16] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Flat holography and Carrollian fluids*, arXiv:1802.06809 [hep-th].

[17] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear fluid dynamics from gravity,* JHEP **0802** (2008) 045, arXiv:0712.2456 [hep-th].

[18] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT,* JHEP **0810** (2008) 063, arXiv:0806.4602 [hep-th].

[19] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, *Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions*, JHEP **0812** (2008) 116, arXiv:0809.4272 [hep-th].

[20] V. E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, arXiv:1107.5780 [hep-th].

[21] E. Cartan, *Sur les variétés à connexion affine, et la théorie de la relativité généralisée* (première partie), Ann. École norm. **41** (1924) 1.

[22] C. Duval and P. A. Horvathy, *Non-relativistic conformal symmetries and Newton–Cartan structures,* J. Phys. **A42** (2009) 465206, arXiv:0904.0531 [math-ph].

[23] X. Bekaert and K. Morand, *Connections and dynamical trajectories in generalised Newton–Cartan gravity I. An intrinsic view*, J. Math. Phys. **57** (2016) 022507, arXiv:1412.8212 [hep--th].

[24] X. Bekaert and K. Morand, *Connections and dynamical trajectories in generalised Newton–Cartan gravity II. An ambient perspective*, arXiv:1505.03739 [hep-th].

[25] G. Festuccia, D. Hansen, J. Hartong and N. A. Obers, *Torsional Newton–Cartan geometry from the Nœther procedure,* Phys. Rev. **D94** (2016) 105023, arXiv:1607.01926 [hep-th].

[26] P. Romatschke, *New developments in relativistic viscous hydrodynamics*, Int. J. Mod. Phys. **E19** (2010) 1, arXiv:0902.3663 [hep-th].

[27] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, J. Phys. **A45** (2012) 473001, arXiv:1205.5040 [hep-th].

[28] L. Ciambelli, A. C. Petkou, P. M. Petropoulos and K. Siampos, *The Robinson–Trautman spacetime and its holographic fluid*, PoS **CORFU2016** (2016) 076, arXiv:1707.02995 [hep--th].

[29] K. Jensen, *Aspects of hot Galilean field theory*, JHEP **1504**, 123 (2015), arXiv:1411.7024 [hep-th].

[30] M. Geracie, K. Prabhu and M. M. Roberts, *Fields and fluids on curved non-relativistic spacetimes*, JHEP **1508** (2015) 042, arXiv:1503.02680 [hep-th].

[31] N. Banerjee, S. Dutta, A. Jain and D. Roychowdhury, *Entropy current for non-relativistic fluid,* JHEP **1408** (2014) 037, arXiv:1405.5687 [hep-th].

[32] R. F. Penna, *BMS_3 invariant fluid dynamics at null infinity,* Class. Quant. Grav. **35** (2018) 044002, arXiv:1708.08470 [hep-th].

[33] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma and S. Vandoren, *Perfect fluids*, arXiv:1710.04708 [hep-th].

[34] J. Pedolský, *Geophysical fluid dynamics*, Springer, Berlin, 1987.

[35] O. Regev, O.M. Umurhan and P.A. Yecko, *Modern fluid dynamics for physics and astrophysics*, Graduate Texts in Physics, Springer, Berlin, 2016.

[36] I. Robinson and A. Trautman, *Spherical gravitational waves*, Phys. Rev. Lett. **4** (1960) 431; *Some spherical gravitational waves in general relativity*, Proc. Roy. Soc. **A265** (1962) 463.

[37] J. B. Griffiths and J. Podolský, *Exact space–times in Einstein's general relativity*, Cambridge monographs on mathematical physics, 2009, Cambridge UK.

[38] K. P. Tod, *Analogues of the past horizon in the Robinson–Trautman metrics*, Class. Quant. Grav. **6** (1989) 1159.

[39] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll groups*, J. Phys. **A47** (2014) 335204, arXiv:1403.4213 [hep-th].

[40] J. Hartong, *Gauging the Carroll algebra and ultra-relativistic gravity*, JHEP **1508** (2015) 069, arXiv:1505.05011 [hep-th].

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Flat holography and Carrollian fluids

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Abstract

We show that a holographic description of four-dimensional asymptotically locally flat spacetimes is reached smoothly from the zero-cosmological-constant limit of anti-de Sitter holography. To this end, we use the derivative expansion of fluid/gravity correspondence. From the boundary perspective, the vanishing of the bulk cosmological constant appears as the zero velocity of light limit. This sets how Carrollian geometry emerges in flat holography. The new boundary data are a two-dimensional spatial surface, identified with the null infinity of the bulk Ricci-flat spacetime, accompanied with a Carrollian time and equipped with a Carrollian structure, plus the dynamical observables of a conformal Carrollian fluid. These are the energy, the viscous stress tensors and the heat currents, whereas the Carrollian geometry is gathered by a two-dimensional spatial metric, a frame connection and a scale factor. The reconstruction of Ricci-flat spacetimes from Carrollian boundary data is conducted with a flat derivative expansion, resummed in a closed form in Eddington–Finkelstein gauge under further integrability conditions inherited from the ancestor anti-de Sitter set-up. These conditions are hinged on a duality relationship among fluid friction tensors and Cotton-like geometric data. We illustrate these results in the case of conformal Carrollian perfect fluids and Robinson–Trautman viscous hydrodynamics. The former are dual to the asymptotically flat Kerr–Taub–NUT family, while the latter leads to the homonymous class of algebraically special Ricci-flat spacetimes.

Contents

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1 Introduction

Ever since its conception, there have been many attempts to extend the original holographic anti-de Sitter correspondence along various directions, including asymptotically flat or de Sitter bulk spacetimes. Since the genuine microscopic correspondence based on type IIB string and maximally supersymmetric Yang–Mills theory is deeply rooted in the anti-de Sitter background, phenomenological extensions such as fluid/gravity correspondence have been considered as more promising for reaching a flat spacetime generalization.

The mathematical foundations of holography are based on the existence of the Fefferman–Graham expansion for asymptotically anti-de Sitter Einstein spaces [1, 2]. Indeed, on the one hand, putting an asymptotically anti-de Sitter Einstein metric in the Fefferman–Graham gauge allows to extract the two independent boundary data *i.e.* the boundary metric and the conserved boundary conformal energy–momentum tensor. On the other hand, given a pair of suitable boundary data the Fefferman–Graham expansion makes it possible to reconstruct, order by order, an Einstein space.

More recently, fluid/gravity correspondence has provided an alternative to Fefferman–Graham, known as derivative expansion [3–6]. It is inspired from the fluid derivative expansion (see *e.g.* [7, 8]), and is implemented in Eddington–Finkelstein coordinates. The metric of an Einstein spacetime is expanded in a light-like direction and the information on the boundary fluid is made available in a slightly different manner, involving explicitly a velocity field whose derivatives set the order of the expansion. Conversely, the boundary fluid data, including the fluid's congruence, allow to reconstruct an exact bulk Einstein spacetime.

Although less robust mathematically, the derivative expansion has several advantages over Fefferman–Graham. Firstly, under some particular conditions it can be resummed leading to algebraically special Einstein spacetimes in a closed form [9–14]. Such a resummation is very unlikely, if at all possible, in the context of Fefferman–Graham. Secondly, boundary geometrical terms appear packaged at specific orders in the derivative expansion, which is performed in Eddington–Finkelstein gauge. These terms feature precisely whether the bulk is asymptotically globally or locally anti-de Sitter. Thirdly, and contrary to Fefferman–Graham again, the derivative expansion admits a consistent limit of vanishing scalar curvature. Hence it appears to be applicable to Ricci-flat spacetimes and emerges as a valuable tool for setting up flat holography. Such a smooth behaviour is not generic, as in most coordinate systems switching off the scalar curvature for an Einstein space leads to plain Minkowski spacetime.[1]

The observations above suggest that it is relevant to wonder whether a Ricci-flat spacetime admits a dual fluid description. This can be recast into two sharp questions:

1. Which surface $\mathscr{S}$ would replace the AdS conformal boundary $\mathscr{I}$, and what is the geometry that this new boundary should be equipped with?

2. Which are the degrees of freedom hosted by $\mathscr{S}$ and succeeding the relativistic-fluid energy–momentum tensor, and what is the dynamics these degrees of freedom obey?

Many proposals have been made for answering these questions. Most of them were inspired by the seminal work [17, 18], where Navier–Stokes equations were shown to capture the dynamics of black-hole horizon perturbations. This result is taken as the crucial evidence regarding the deep relation between gravity, without cosmological constant, and fluid dynamics.

A more recent approach has associated Ricci-flat spacetimes in $d+1$ dimensions with d-dimensional fluids [19–24]. This is based on the observation that the Brown–York energy–momentum tensor on a Rindler hypersurface of a flat metric has the form of a perfect fluid [25]. In this particular framework, one can consider a non-relativistic limit, thus showing

[1] This phenomenon is well known in supergravity, when studying the gravity decoupling limit of scalar manifolds. For this limit to be non-trivial, one has to chose an appropriate gauge (see [15, 16] for a recent discussion and references).

that the Navier–Stokes equations coincide with Einstein's equations on the Rindler hypersurface. Paradoxically, it has simultaneously been argued that all information can be stored in a relativistic d-dimensional fluid.

Outside the realm of fluid interpretation, and on the more mathematical side of the problem, some solid works regarding flat holography are [26–28] (see also [29]). The dual theories reside at null infinity emphasizing the importance of the null-like formalisms of [30–32]. In this line of thought, results where also reached focusing on the expected symmetries, in particular for the specific case of three-dimensional bulk versus two-dimensional boundary [33–39].[2] These achievements *are not* unconditionally transferable to four or higher dimensions, and can possibly infer inaccurate expectations due to features holding exclusively in three dimensions.

The above wanderings between relativistic and non-relativistic fluid dynamics in relation with Ricci-flat spacetimes are partly due to the incomplete understanding on the rôle played by the null infinity. On the one hand, it has been recognized that the Ricci-flat limit is related to some contraction of the Poincaré algebra [33–37, 40, 41]. On the other hand, this observation was tempered by a potential confusion among the Carrollian algebra and its dual contraction, the conformal Galilean algebra, as they both lead to the decoupling of time. This phenomenon was exacerbated by the equivalence of these two algebras in two dimensions, and has somehow obscured the expectations on the nature and the dynamics of the relevant boundary degrees of freedom. Hence, although the idea of localizing the latter on the spatial surface at null infinity was suggested (as *e.g.* in [42–45]), their description has often been accustomed to the relativistic-fluid or the conformal-field-theory approaches, based on the revered energy–momentum tensor and its conservation law.[3]

From this short discussion, it is clear that the attempts implemented so far follow different directions without clear overlap and common views. Although implicitly addressed in the literature, the above two questions have not been convincingly answered, and the treatment of boundary theories in the zero cosmological constant limit remains nowadays tangled.

In this work we make a precise statement, which clarifies unquestionably the situation. Our starting point is a four-dimensional bulk Einstein spacetime with $\Lambda = -3k^2$, dual to a boundary relativistic fluid. In this set-up, we consider the $k \to 0$ limit, which has the following features:

- The derivative expansion is generically well behaved. We will call its limit the *flat derivative expansion*. Under specified conditions it can be resummed in a closed form.
- Inside the boundary metric, and in the complete boundary fluid dynamics, k plays the

[2] Reference [37] is the first where a consistent and non-trivial $k \to 0$ limit was taken, mapping the entire family of three-dimensional Einstein spacetimes (locally AdS) to the family of Ricci–flat solutions (locally flat).

[3] This is manifest in the very recent work of Ref. [46].

rôle of *velocity of light*. Its vanishing is thus a *Carrollian limit*.

- The boundary is the two-dimensional *spatial* surface $\mathscr{S}$ emerging as the future null infinity of the limiting Ricci-flat bulk spacetime. It replaces the AdS conformal boundary and is endowed with a *Carrollian geometry i.e.* is covariant under *Carrollian diffeomorphisms*.

- The degrees of freedom hosted by this surface are captured by a *conformal Carrollian fluid* : energy density and pressure related by a conformal equation of state, heat currents and traceless viscous stress tensors. These macroscopic degrees of freedom obey *conformal Carrollian fluid dynamics*.

Any two-dimensional conformal Carrollian fluid hosted by an arbitrary spatial surface $\mathscr{S}$, and obeying conformal Carrollian fluid dynamics on this surface, is therefore mapped onto a Ricci-flat four-dimensional spacetime using the flat derivative expansion. The latter is invariant under boundary Weyl transformations. Under a set of resummability conditions involving the Carrollian fluid and its host $\mathscr{S}$, this derivative expansion allows to reconstruct exactly algebraically special Ricci-flat spacetimes. The results summarized above answer in the most accurate manner the two questions listed earlier.

Carrollian symmetry has sporadically attracted attention following the pioneering work or Ref. [47], where the Carroll group emerged as a new contraction of the Poincaré group: the ultra-relativistic contraction, dual to the usual non-relativistic one leading to the Galilean group. Its conformal extensions were explored latterly [48–51], showing in particular its relationship to the BMS group, which encodes the asymptotic symmetries of asymptotically flat spacetimes along a null direction [53–56].[4]

It is therefore quite natural to investigate on possible relationships between Carrollian asymptotic structure and flat holography and, by the logic of fluid/gravity correspondence, to foresee the emergence of Carrollian hydrodynamics rather than any other, relativistic or Galilean fluid. Nonetheless searches so far have been oriented towards the near-horizon membrane paradigm, trying to comply with the inevitable BMS symmetries as in [59, 60]. The power of the derivative expansion and its flexibility to handle the zero-k limit has been somehow dismissed. This expansion stands precisely at the heart of our method. Its actual implementation requires a comprehensive approach to Carrollian hydrodynamics, as it emanates from the ultra-relativistic limit of relativistic fluid dynamics, made recently available in [52].

The aim of the present work is to provide a detailed analysis of the various statements presented above, and exhibit a precise expression for the Ricci-flat line element as reconstructed from the boundary Carrollian geometry and Carrollian fluid dynamics. As already

[4] Carroll symmetry has also been explored in connection to the tensionless-string limit, see *e.g.* [57, 58].

stated, the tool for understanding and implementing operationally these ideas is the derivative expansion and, under conditions, its resummed version. For this reason, Sec. 2 is devoted to its thorough description in the framework of ordinary anti-de Sitter fluid/gravity holography. This chapter includes the conditions, stated in a novel fashion with respect to [12, 13], for the expansion to be resummed in a closed form, representing generally an Einstein spacetime of algebraically special Petrov type.

In Sec. 3 we discuss how the Carrollian geometry emerges at null infinity and describe in detail conformal Carrollian hydrodynamics following [52]. The formulation of the Ricci-flat derivative expansion is undertaken in Sec. 4. Here we discuss the important issue of resumming in a closed form the generic expansion. This requires the investigation of another uncharted territory: the higher-derivative curvature-like Carrollian tensors. The Carrollian geometry on the spatial boundary $\mathscr{S}$ is naturally equipped with a (conformal) Carrollian connection, which comes with various curvature tensors presented in Sec. 3. The relevant object for discussing the resummability in the anti-de Sitter case is the Cotton tensor, as reviewed in Sec. 2. It turns out that this tensor has well-defined Carrollian descendants, which we determine and exploit. With those, the resummability conditions are well-posed and set the framework for obtaining exact Ricci-flat spacetimes in a closed form from conformal-Carrollian-fluid data.

In order to illustrate our results, we provide examples starting from Sec. 3 and pursuing systematically in Sec. 5. Generic Carrollian perfect fluids are meticulously studied and shown to be dual to the general Ricci-flat Kerr–Taub–NUT family. The non perfect Carrollian fluid called Robinson–Trautman fluid is discussed both as the limiting Robinson–Trautman relativistic fluid (Sec. 3), and alternatively from Carrollian first principles (Sec. 5, following [52]). It is shown to be dual to the Ricci-flat Robinson–Trautman spacetime, of which the line element is obtained thanks to our flat resummation procedure.

One of the resummability requirements is the absence of shear for the Carrollian fluid. This is a geometric quantity, which, if absent, makes possible for using holomorphic coordinates. In App. A, we gather the relevant formulas in this class of coordinates.

2 Fluid/gravity in asymptotically locally AdS spacetimes

We present here an executive summary of the holographic reconstruction of four-dimensional asymptotically locally anti-de Sitter spacetimes from three-dimensional relativistic boundary fluid dynamics. The tool we use is the fluid-velocity derivative expansion. We show that exact Einstein spacetimes written in a closed form can arise by resumming this expansion. It appears that the key conditions allowing for such an explicit resummation are the absence of shear in the fluid flow, as well as the relationship among the non-perfect components of the fluid energy–momentum tensor (*i.e.* the heat current and the viscous stress tensor) and the boundary Cotton tensor.

2.1 The derivative expansion

The spirit

Due to the Fefferman–Graham ambient metric construction [61], asymptotically locally anti-de Sitter four-dimensional spacetimes are determined by a set of independent boundary data, namely a three-dimensional metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ and a rank-2 tensor $\mathrm{T} = T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, symmetric ($T_{\mu\nu} = T_{\nu\mu}$), traceless ($T^\mu{}_\mu = 0$) and conserved:

$$\nabla^\mu T_{\mu\nu} = 0. \tag{2.1}$$

Perhaps the most well known subclass of asymptotically locally AdS spacetimes are those whose boundary metrics are conformally flat (see *e.g.* [62, 63]). These are asymptotically *globally* anti-de Sitter. The asymptotic symmetries of such spacetimes comprise the finite dimensional conformal group, *i.e.* $SO(3,2)$ in four dimensions [64], and AdS/CFT is at work giving rise to a boundary conformal field theory. Then, the rank-2 tensor $T_{\mu\nu}$ is interpreted as the expectation value over a boundary quantum state of the conformal-field-theory energy–momentum tensor. Whenever hydrodynamic regime is applicable, this approach gives rise to the so-called fluid/gravity correspondence and all its important spinoffs (see [4] for a review).

For a long time, all the work on fluid/gravity correspondence was confined to asymptotically globally AdS spacetimes, hence to holographic boundary fluids that flow on conformally flat backgrounds. In a series of works [9–14] we have extended the fluid/gravity correspondence into the realm of asymptotically locally AdS_4 spacetimes. In the following, we present and summarize our salient findings.

The energy–momentum tensor

Given the energy–momentum tensor of the boundary fluid and assuming that it represents a state in a hydrodynamic regime, one should be able to pick a boundary congruence u, playing the rôle of fluid velocity. Normalizing the latter as[5] $\|\mathrm{u}\|^2 = -k^2$ we can in general decompose the energy–momentum tensor as

$$T_{\mu\nu} = (\varepsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2}. \tag{2.2}$$

[5] This unconventional normalization ensures that the derivative expansion is well-behaved in the $k \to 0$ limit. In the language of fluids, it naturally incorporates the scaling introduced in [37] – see footnote 2.

We assume local thermodynamic equilibrium with p the local pressure and ε the local energy density:

$$\varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu. \tag{2.3}$$

A local-equilibrium thermodynamic equation of state $p = p(T)$ is also needed for completing the system, and we omit the chemical potential as no independent conserved current, *i.e.* no gauge field in the bulk, is considered here.

The symmetric viscous stress tensor $\tau_{\mu\nu}$ and the heat current q_μ are purely transverse:

$$u^\mu \tau_{\mu\nu} = 0, \quad u^\mu q_\mu = 0, \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{2.4}$$

For a conformal fluid in 3 dimensions

$$\varepsilon = 2p, \quad \tau^\mu{}_\mu = 0. \tag{2.5}$$

The quantities at hand are usually expressed as expansions in temperature and velocity derivatives, the coefficients of which characterize the transport phenomena occurring in the fluid. In first-order hydrodynamics

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{2.6}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{k^2} a_\nu \right), \tag{2.7}$$

where $h_{\mu\nu}$ is the projector onto the space transverse to the velocity field:

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{k^2} + g_{\mu\nu}, \tag{2.8}$$

and[6]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{2.9}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \tfrac{1}{k^2} u_{(\mu} a_{\nu)} - \tfrac{1}{2}\Theta h_{\mu\nu}, \tag{2.10}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \tfrac{1}{k^2} u_{[\mu} a_{\nu]}, \tag{2.11}$$

are the acceleration (transverse), the expansion, the shear and the vorticity (both rank-two tensors are transverse and traceless). As usual, η, ζ are the shear and bulk viscosities, and κ is the thermal conductivity.

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2}\omega_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}\left(\mathrm{du} + \frac{1}{k^2} \mathrm{u} \wedge \mathrm{a} \right), \tag{2.12}$$

[6] Our conventions for (anti-) symmetrization are: $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

as well as the Hodge–Poincaré dual of this form, which is proportional to u (we are in $2+1$ dimensions):

$$k\gamma \mathrm{u} = \star\omega \quad \Leftrightarrow \quad k\gamma u_\mu = \frac{1}{2}\eta_{\mu\nu\sigma}\omega^{\nu\sigma}, \tag{2.13}$$

where $\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma}$. In this expression γ is a scalar, that can also be expressed as

$$\gamma^2 = \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu}. \tag{2.14}$$

In three spacetime dimensions and in the presence of a vector field, one naturally defines a fully antisymmetric two-index tensor as

$$\eta_{\mu\nu} = -\frac{u^\rho}{k}\eta_{\rho\mu\nu}, \tag{2.15}$$

obeying

$$\eta_{\mu\sigma}\eta_\nu{}^\sigma = h_{\mu\nu}. \tag{2.16}$$

With this tensor the vorticity reads:

$$\omega_{\mu\nu} = k^2\gamma\eta_{\mu\nu}. \tag{2.17}$$

Weyl covariance, Weyl connection and the Cotton tensor

In the case when the boundary metric $g_{\mu\nu}$ is conformally flat, it was shown that using the above set of boundary data it is possible to reconstruct the four-dimensional bulk Einstein spacetime order by order in derivatives of the velocity field [3–6]. The guideline for the spacetime reconstruction based on the derivative expansion is *Weyl covariance*: the bulk geometry should be insensitive to a conformal rescaling of the boundary metric (weight -2)

$$\mathrm{d}s^2 \to \frac{\mathrm{d}s^2}{\mathcal{B}^2}, \tag{2.18}$$

which should correspond to a bulk diffeomorphism and be reabsorbed into a redefinition of the radial coordinate: $r \to \mathcal{B}r$. At the same time, u_μ is traded for $u_\mu/\mathcal{B}$ (velocity one-form), $\omega_{\mu\nu}$ for $\omega_{\mu\nu}/\mathcal{B}$ (vorticity two-form) and $T_{\mu\nu}$ for $\mathcal{B}T_{\mu\nu}$. As a consequence, the pressure and energy density have weight 3, the heat-current q_μ weight 2, and the viscous stress tensor $\tau_{\mu\nu}$ weight 1.

Covariantization with respect to rescaling requires to introduce a Weyl connection one-form:[7]

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \frac{\Theta}{2}\mathrm{u}\right), \tag{2.19}$$

which transforms as $\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathcal{B}$. Ordinary covariant derivatives ∇ are thus traded

[7] The explicit form of A is obtained by demanding $\mathscr{D}_\mu u^\mu = 0$ and $u^\lambda \mathscr{D}_\lambda u_\mu = 0$.

for Weyl covariant ones $\mathscr{D} = \nabla + w\,\mathrm{A}$, w being the conformal weight of the tensor under consideration. We provide for concreteness the Weyl covariant derivative of a weight-w form v_μ:

$$\mathscr{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho. \tag{2.20}$$

The Weyl covariant derivative is metric with effective torsion:

$$\mathscr{D}_\rho g_{\mu\nu} = 0, \tag{2.21}$$

$$\left(\mathscr{D}_\mu\mathscr{D}_\nu - \mathscr{D}_\nu\mathscr{D}_\mu\right) f = wfF_{\mu\nu}, \tag{2.22}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.23}$$

is Weyl-invariant.

Commuting the Weyl-covariant derivatives acting on vectors, as usual one defines the Weyl covariant Riemann tensor

$$\left(\mathscr{D}_\mu\mathscr{D}_\nu - \mathscr{D}_\nu\mathscr{D}_\mu\right) V^\rho = \mathscr{R}^\rho{}_{\sigma\mu\nu}V^\sigma + wV^\rho F_{\mu\nu} \tag{2.24}$$

(V^ρ are weight-w) and the usual subsequent quantities. In three spacetime dimensions, the covariant Ricci (weight 0) and the scalar (weight 2) curvatures read:

$$\mathscr{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu + g_{\mu\nu}\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right) - F_{\mu\nu}, \tag{2.25}$$

$$\mathscr{R} = R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu. \tag{2.26}$$

The Weyl-invariant Schouten tensor[8] is

$$\mathscr{S}_{\mu\nu} = \mathscr{R}_{\mu\nu} - \frac{1}{4}\mathscr{R}g_{\mu\nu} = S_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu - \frac{1}{2}A_\lambda A^\lambda g_{\mu\nu} - F_{\mu\nu}. \tag{2.27}$$

Other Weyl-covariant velocity-related quantities are

$$\mathscr{D}_\mu u_\nu = \nabla_\mu u_\nu + \frac{1}{k^2}u_\mu a_\nu - \frac{\Theta}{2}h_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.28}$$

$$\mathscr{D}_\nu \omega^\nu{}_\mu = \nabla_\nu \omega^\nu{}_\mu, \tag{2.29}$$

$$\mathscr{D}_\nu \eta^\nu{}_\mu = 2\gamma u_\mu, \tag{2.30}$$

$$u^\lambda \mathscr{R}_{\lambda\mu} = \mathscr{D}_\lambda\left(\sigma^\lambda{}_\mu - \omega^\lambda{}_\mu\right) - u^\lambda F_{\lambda\mu}, \tag{2.31}$$

of weights -1, 1, 0 and 1 (the scalar vorticity γ has weight 1).

The remarkable addition to the fluid/gravity dictionary came with the realization that

[8] The ordinary Schouten tensor in three spacetime dimensions is given by $R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$.

the derivative expansion can be used to reconstruct Einstein metrics which are asymptotically locally AdS. For the latter, the boundary metric has a non zero Cotton tensor [9–13]. The Cotton tensor is generically a three-index tensor with mixed symmetries. In three dimensions, which is the case for our boundary geometry, the Cotton tensor can be dualized into a two-index, symmetric and traceless tensor. It is defined as

$$C_{\mu\nu} = \eta_{\mu}{}^{\rho\sigma}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right) = \eta_{\mu}{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right). \tag{2.32}$$

The Cotton tensor is Weyl-covariant of weight 1 (*i.e.* transforms as $C_{\mu\nu} \to \mathcal{B}C_{\mu\nu}$), and is *identically* conserved:

$$\mathscr{D}_\rho C^\rho{}_\nu = \nabla_\rho C^\rho{}_\nu = 0, \tag{2.33}$$

sharing thereby all properties of the energy–momentum tensor. Following (2.2) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the fluid velocity u:[9]

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}. \tag{2.34}$$

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$c = \frac{1}{k^3}C_{\mu\nu}u^\mu u^\nu, \tag{2.35}$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor* $c_{\mu\nu}$ and the *Cotton current* c_μ (weights 1 and 2, respectively) are purely transverse:

$$c_\mu{}^\mu = 0, \quad u^\mu c_{\mu\nu} = 0, \quad u^\mu c_\mu = 0, \tag{2.36}$$

and obey

$$c_{\mu\nu} = -kh^\rho{}_\mu h^\sigma{}_\nu C_{\rho\sigma} + \frac{ck^2}{2}h_{\mu\nu}, \quad c_\nu = -cu_\nu - \frac{u^\mu C_{\mu\nu}}{k}. \tag{2.37}$$

One can use the definition (2.32) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$c = \frac{1}{k^2}u^\nu\eta^{\sigma\rho}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right), \tag{2.38}$$

$$c_\nu = \eta^{\rho\sigma}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right) - cu_\nu, \tag{2.39}$$

$$c_{\mu\nu} = -h^\lambda{}_\mu\left(k\eta_\nu{}^{\rho\sigma} - u_\nu\eta^{\rho\sigma}\right)\mathscr{D}_\rho\left(\mathscr{S}_{\lambda\sigma} + F_{\lambda\sigma}\right) + \frac{ck^2}{2}h_{\mu\nu}. \tag{2.40}$$

[9]Notice that the energy–momentum tensor has an extra factor of k with respect to the Cotton tensor, see (2.60), due to their different dimensions.

The bulk Einstein derivative expansion

Given the ingredients above, the leading terms in a $1/r$ expansion for a four-dimensional Einstein metric are of the form:[10]

$$\begin{aligned} \mathrm{d}s^2_{\text{bulk}} &= 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2\mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} \\ &+ \frac{\mathrm{u}^2}{k^4 r^2}\left(1 - \frac{1}{2k^4 r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^4}\right) \\ &+ \text{terms with } \sigma, \sigma^2, \nabla\sigma, \ldots + \mathrm{O}\left(\mathscr{D}^4\mathrm{u}\right). \end{aligned} \tag{2.41}$$

In this expression

- S is a Weyl-invariant tensor:

$$\mathrm{S} = S_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\mathrm{u}\mathscr{D}_\nu \omega^\nu{}_\mu \mathrm{d}x^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{u}^2\frac{\mathscr{R}}{2}; \tag{2.42}$$

- the boundary metric is parametrized *à la* Randers–Papapetrou:

$$\mathrm{d}s^2 = -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j; \tag{2.43}$$

- the boundary conformal fluid velocity field and the corresponding one form are

$$\mathrm{u} = \frac{1}{\Omega}\partial_t \quad \Leftrightarrow \quad \mathrm{u} = -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right), \tag{2.44}$$

  i.e. the fluid is at rest in the frame associated with the coordinates in (2.43) – this is not a limitation, as one can always choose a local frame where the fluid is at rest, in which the metric reads (2.43) (with Ω, b_i and a_{ij} functions of all coordinates);

- $\omega_{\mu\nu}$ is the vorticity of u as given in (2.11), which reads:

$$\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{k^2}{2}\left(\partial_i b_j + \frac{1}{\Omega}b_i\partial_j\Omega + \frac{1}{\Omega}b_i\partial_t b_j\right)\mathrm{d}x^i \wedge \mathrm{d}x^j; \tag{2.45}$$

- $\gamma^2 = \frac{1}{2}a^{ik}a^{jl}\left(\partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]}\right)\left(\partial_{[k}b_{l]} + \frac{1}{\Omega}b_{[k}\partial_{l]}\Omega + \frac{1}{\Omega}b_{[k}\partial_t b_{l]}\right);$

[10] We have traded here the usual advanced-time coordinate used in the quoted literature on fluid/gravity correspondence for the retarded time, spelled t (see (2.44)).

- the expansion and acceleration are

$$\begin{aligned} \Theta &= \frac{1}{\Omega}\partial_t \ln \sqrt{a}, & (2.46)\\ \mathrm{a} &= k^2\left(\partial_i \ln\Omega + \frac{1}{\Omega}\partial_t b_i\right)\mathrm{d}x^i, & (2.47) \end{aligned}$$

  leading to the Weyl connection

$$\mathrm{A} = \frac{1}{\Omega}\left(\partial_i\Omega + \partial_t b_i - \frac{1}{2}b_i\partial_t \ln\sqrt{a}\right)\mathrm{d}x^i + \frac{1}{2}\partial_t \ln\sqrt{a}\,\mathrm{d}t, \tag{2.48}$$

  with a the determinant of a_{ij};

- $\frac{1}{k^2}T_{\mu\nu}u^\mu u^\nu$ is the energy density ε of the fluid (see (2.3)), and in the Randers–Papapetrou frame associated with (2.43), (2.44), q_0, τ_{00} , $\tau_{0i} = \tau_{i0}$ entering in (2.2) all vanish due to (2.4);

- $\frac{1}{2k^4}C_{\lambda\mu}u^\lambda\eta^{\mu\nu\sigma}\omega_{\nu\sigma} = c\gamma$, where we have used (2.13) and (2.35), and similarly $c_0 = c_{00} = c_{0i} = c_{i0} = 0$ as a consequence of (2.36) with (2.43), (2.44);

- σ, σ^2, $\nabla\sigma$ stand for the shear of u and combinations of it, as computed from (2.10):

$$\sigma = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right)\mathrm{d}x^i\mathrm{d}x^j. \tag{2.49}$$

We have not exhibited explicitly shear-related terms because we will ultimately assume the absence of shear for our congruence. This raises the important issue of choosing the fluid velocity field, not necessary in the Fefferman–Graham expansion, but fundamental here. In relativistic fluids, the absence of sharp distinction between heat and matter fluxes leaves a freedom in setting the velocity field. This choice of *hydrodynamic frame* is not completely arbitrary though, and one should stress some reservations, which are often dismissed, in particular in the already quoted fluid/gravity literature.

As was originally exposed in [65] and extensively discussed *e.g.* in [7], the fluid-velocity ambiguity is well posed in the presence of a conserved current J, naturally decomposed into a longitudinal perfect piece and a transverse part:

$$J^\mu = \varrho u^\mu + j^\mu. \tag{2.50}$$

The velocity freedom originates from the redundancy in the heat current q and the non-perfect piece of the matter current j. One may therefore set $\mathrm{j} = 0$ and reach the Eckart frame. Alternatively $\mathrm{q} = 0$ defines the Landau–Lifshitz frame. In the absence of matter current, nothing guarantees that one can still move to the Landau–Lifshitz frame, and setting $\mathrm{q} = 0$ appears as a constraint on the fluid, rather than a choice of frame for describing arbitrary fluids. This important issue was recently discussed in the framework of holography [66], from which it is clear that setting $\mathrm{q}=0$ in the absence of a conserved current would simply inhibit certain classes of Einstein spaces to emerge holographically from boundary data, and possibly blur the physical phenomena occurring in the fluids under consideration. Consequently, we will not make any such assumption, keeping the heat current as part of the physical data.

We would like to close this section with an important comment on asymptotics. The reconstructed bulk spacetime can be asymptotically locally or globally anti-de Sitter. This property is read off directly inside terms appearing at designated orders in the radial expansion, and built over specific boundary tensors. For $d+1$-dimensional boundaries, the boundary energy–momentum contribution first appears at order $1/r^{d-1}$, whereas the boundary Cotton tensor[11] emerges at order $1/r^2$. This behaviour is rooted in the Eddington–Finkelstein gauge used in (2.41), but appears also in the slightly different Bondi gauge. It is however absent in the Fefferman–Graham coordinates, where the Cotton cannot be possibly isolated in the expansion.

2.2 The resummation of AdS spacetimes

Resummation and exact Einstein spacetimes in closed form

In order to further probe the derivative expansion (2.41), we will impose the fluid velocity congruence be shearless. This choice has the virtue of reducing considerably the number of terms compatible with conformal invariance in (2.41), and potentially making this expansion resummable, thus leading to an Einstein metric written in a closed form. Nevertheless, this shearless condition reduces the class of Einstein spacetimes that can be reconstructed holographically to the algebraically special ones [10–14]. Going beyond this class is an open problem that we will not address here.

Following [6, 10–14], it is tempting to try a resummation of (2.41) using the following substitution:

$$1-\frac{\gamma^2}{r^2}\to\frac{r^2}{\rho^2} \tag{2.51}$$

with

$$\rho^2=r^2+\gamma^2. \tag{2.52}$$

The resummed expansion would then read

$$\boxed{\mathrm{d}s^2_{\text{res. Einstein}}=2\frac{\mathrm{u}}{k^2}(\mathrm{d}r+r\mathrm{A})+r^2\mathrm{d}s^2+\frac{\mathrm{S}}{k^4}+\frac{\mathrm{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r+c\gamma\right),} \tag{2.53}$$

which is indeed written in a closed form. Under the conditions listed below, the metric (2.53)

[11] Actually, the object appearing in generic dimension is the Weyl divergence of the boundary Weyl tensor, which contains also the Cotton tensor (see [67] for a preliminary discussion on this point).

defines the line element of an *exact* Einstein space with $\Lambda = -3k^2$.

- *The congruence u is shearless.* This requires (see (2.49))

$$\partial_t a_{ij} = a_{ij} \partial_t \ln \sqrt{a}. \tag{2.54}$$

Actually (2.54) is equivalent to ask that the two-dimensional spatial section $\mathscr{S}$ defined at every time t and equipped with the metric $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is conformally flat. This may come as a surprise because every two-dimensional metric is conformally flat. However, a_{ij} generally depends on space $\mathbf{x}$ *and* time t, and the transformation required to bring it in a form proportional to the flat-space metric might depend on time. This would spoil the three-dimensional structure (2.43) and alter the *a priori* given u. Hence, $\mathrm{d}\ell^2$ is conformally flat within the three-dimensional spacetime (2.43) under the condition that the transformation used to reach the explicit conformally flat form be of the type $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. This exists if and only if (2.54) is satisfied.[12] Under this condition, one can always choose $\zeta = \zeta(\mathbf{x})$, $\bar{\zeta} = \bar{\zeta}(\mathbf{x})$ such that

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \tag{2.55}$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function. Even though this does not hold for arbitrary $\mathrm{u} = \partial_t/\Omega$, one can show that there exists always a congruence for which it does [68], and this will be chosen for the rest of the paper.

- *The heat current of the boundary fluid introduced in* (2.2) *and* (2.4) *is identified with the transverse-dual of the Cotton current defined in* (2.34) *and* (2.37). The Cotton current being transverse to u, it defines a field on the conformally flat two-surface $\mathscr{S}$, the existence of which is guaranteed by the absence of shear. This surface is endowed with a natural hodge duality mapping a vector onto another, which can in turn be lifted back to the three-dimensional spacetime as a new transverse vector. This whole process is taken care of by the action of $\eta^\nu{}_\mu$ defined in (2.15):

$$\boxed{q_\mu = \frac{1}{8\pi G}\eta^\nu{}_\mu c_\nu = \frac{1}{8\pi G}\eta^\nu{}_\mu \eta^{\rho\sigma}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right),} \tag{2.56}$$

where we used (2.39) in the last expression. Using holomorphic and antiholomorphic coordinates $\zeta, \bar{\zeta}$ as in (2.55)[13] leads to $\eta^\zeta{}_\zeta = \mathrm{i}$ and $\eta^{\bar{\zeta}}{}_{\bar{\zeta}} = -\mathrm{i}$, and thus

$$\mathrm{q} = \frac{\mathrm{i}}{8\pi G}\left(c_\zeta \mathrm{d}\zeta - c_{\bar{\zeta}}\mathrm{d}\bar{\zeta}\right). \tag{2.57}$$

[12] A peculiar subclass where this works is when ∂_t is a Killing field.

[13] Orientation is chosen such that in the coordinate frame $\eta_{0\zeta\bar{\zeta}} = \sqrt{-g}\epsilon_{0\zeta\bar{\zeta}} = \frac{\mathrm{i}\Omega}{P^2}$, where $x^0 = kt$.

- *The viscous stress tensor of the boundary conformal fluid introduced in* (2.2) *is identified with the transverse-dual of the Cotton stress tensor defined in* (2.34) *and* (2.37). Following the same pattern as for the heat current, we obtain:

$$\boxed{\begin{aligned} \tau_{\mu\nu} &= -\frac{1}{8\pi G k^2}\eta^{\rho}{}_{\mu}c_{\rho\nu} \\ &= \frac{1}{8\pi G k^2}\left(-\frac{1}{2}u^{\lambda}\eta_{\mu\nu}\eta^{\rho\sigma}+\eta^{\lambda}_{\mu}\left(k\eta_{\nu}{}^{\rho\sigma}-u_{\nu}\eta^{\rho\sigma}\right)\right)\mathscr{D}_{\rho}\left(\mathscr{S}_{\lambda\sigma}+F_{\lambda\sigma}\right),\end{aligned}} \tag{2.58}$$

where we also used (2.40) in the last equality. The viscous stress tensor $\tau_{\mu\nu}$ is transverse symmetric and traceless because these are the properties of the Cotton stress tensor $c_{\mu\nu}$. Similarly, we find in complex coordinates:

$$\tau = -\frac{\mathrm{i}}{8\pi G k^2}\left(c_{\zeta\zeta}\mathrm{d}\zeta^2 - c_{\bar{\zeta}\bar{\zeta}}\mathrm{d}\bar{\zeta}^2\right). \tag{2.59}$$

- *The energy–momentum tensor defined in* (2.2) *with $p = \varepsilon/2$, heat current as in* (2.56) *and viscous stress tensor as in* (2.58) *must be conserved, i.e. obey Eq.* (2.1). These are differential constraints that from a bulk perspective can be thought of as a generalization of the Gauss law.

Identifying parts of the energy–momentum tensor with the Cotton tensor may be viewed as setting integrability conditions, similar to the electric–magnetic duality conditions in electromagnetism, or in Euclidean gravitational dynamics. As opposed to the latter, it is here implemented in a rather unconventional manner, on the conformal boundary.

It is important to emphasize that the conservation equations (2.1) concern *all* boundary data. On the fluid side the only remaining unknown piece is the energy density $\varepsilon(x)$, whereas for the boundary metric $\Omega(x)$, $b_i(x)$ and $a_{ij}(x)$ are available and must obey (2.1), together with $\varepsilon(x)$. Given these ingredients, (2.1) turns out to be precisely the set of equations obtained by demanding bulk Einstein equations be satisfied with the metric (2.53). This observation is at the heart of our analysis.

The bulk algebraic structure and the physics of the boundary fluid

The pillars of our approach are (i) the requirement of a shearless fluid congruence and (ii) the identification of the non-perfect energy–momentum tensor pieces with the corresponding Cotton components by transverse dualization.

What does motivate these choices? The answer to this question is rooted to the Weyl tensor and to the remarkable integrability properties its structure can provide to the system.

Let us firstly recall that from the bulk perspective, u is a manifestly null congruence associated with the vector ∂_r. One can show (see [13]) that this bulk congruence is also *geodesic* and *shear-free*. Therefore, accordingly to the generalizations of the Goldberg–Sachs theorem, if the bulk metric (2.41) is an Einstein space, then it is algebraically special, *i.e.* of

Petrov type II, III, D, N or O. Owing to the close relationship between the algebraic structure and the integrability properties of Einstein equations, it is clear why the absence of shear in the fluid congruence plays such an instrumental rôle in making the tentatively resummed expression (2.53) an exact Einstein space.

The structure of the bulk Weyl tensor makes it possible to go deeper in foreseeing how the boundary data should be tuned in order for the resummation to be successful. Indeed the Weyl tensor can be expanded for large-r, and the dominant term ($1/r^3$) exhibits the following combination of the boundary energy–momentum and Cotton tensors [69–73]:

$$T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{\mathrm{i}}{8\pi G k} C_{\mu\nu}, \tag{2.60}$$

satisfying a conservation equation, analogue to (2.1)

$$\nabla^{\mu} T^{\pm}_{\mu\nu} = 0. \tag{2.61}$$

For algebraically special spaces, these complex-conjugate tensors simplify considerably (see detailed discussions in [10–14]), and this suggests the transverse duality enforced between the Cotton and the energy–momentum non-perfect components. Using (2.57) and (2.59), we find indeed for the tensor T^{+} in complex coordinates:

$$\mathrm{T}^{+} = \left(\varepsilon + \frac{\mathrm{i}c}{8\pi G}\right)\left(\frac{\mathrm{u}^2}{k^2} + \frac{1}{2}\mathrm{d}\ell^2\right) + \frac{\mathrm{i}}{4\pi G k^2}\left(2c_{\zeta}\mathrm{d}\zeta\mathrm{u} - c_{\zeta\zeta}\mathrm{d}\zeta^2\right), \tag{2.62}$$

and similarly for T^{-} obtained by complex conjugation with

$$\varepsilon_{\pm} = \varepsilon \pm \frac{\mathrm{i}c}{8\pi G}. \tag{2.63}$$

The bulk Weyl tensor and consequently the Petrov class of the bulk Einstein space are encoded in the three complex functions of the boundary coordinates: ε_{+}, c_{ζ} and $c_{\zeta\zeta}$.

The proposed resummation procedure, based on boundary relativistic fluid dynamics of non-perfect fluids with heat current and stress tensor designed from the boundary Cotton tensor, allows to reconstruct all algebraically special four-dimensional Einstein spaces. The simplest correspond to a Cotton tensor of the perfect form [10]. The complete class of Plebański–Demiański family [74] requires non-trivial b_i with two commuting Killing fields [13], while vanishing b_i without isometry leads to the Robinson–Trautman Einstein spaces [12]. For the latter, the heat current and the stress tensor obtained from the Cotton by the transverse duality read:

$$
\begin{aligned}
\mathrm{q} &= -\frac{1}{16\pi G}\left(\partial_\zeta K \mathrm{d}\zeta + \partial_{\bar\zeta} K \mathrm{d}\bar\zeta\right), && (2.64)\\
\tau &= \frac{1}{8\pi G k^2 P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right), && (2.65)
\end{aligned}
$$

where $K = 2P^2\partial_{\bar\zeta}\partial_\zeta \ln P$ is the Gaussian curvature of (2.55). With these data the conservation of the energy–momentum tensor (2.1) enforces the absence of spatial dependence in $\varepsilon = 2p$, and leads to a single independent equation, the heat equation:

$$
12M\partial_t \ln P + \Delta K = 4\partial_t M. \tag{2.66}
$$

This is the Robinson–Trautman equation, here expressed in terms of $M(t) = 4\pi G\varepsilon(t)$.

The boundary fluids emerging in the systems considered here have a specific physical behaviour. This behaviour is inherited from the boundary geometry, since their excursion away from perfection is encoded in the Cotton tensor via the transverse duality. In the hydrodynamic frame at hand, this implies in particular that the derivative expansion of the energy–momentum tensor terminates at third order. Discussing this side of the holography is not part of our agenda. We shall only stress that such an analysis does not require to change hydrodynamic frame. Following [66], it is possible to show that the frame at hand is the Eckart frame. Trying to discard the heat current in order to reach a Landau–Lifshitz-like frame (as in [75–78] for Robinson–Trautman) is questionable, as already mentioned earlier, because of the absence of conserved current, and distorts the physical phenomena occurring in the holographic conformal fluid.

3 The Ricci-flat limit I: Carrollian geometry and Carrollian fluids

The Ricci-flat limit is achieved at vanishing k. Although no conformal boundary exists in this case, a two-dimensional spatial conformal structure emerges at null infinity. Since the Einstein bulk spacetime derivative expansion is performed along null tubes, it provides the appropriate arena for studying both the nature of the two-dimensional "boundary" and the dynamics of the degrees of freedom it hosts as "holographic duals" to the bulk Ricci-flat spacetime.

3.1 The Carrollian boundary geometry

The emergence of a boundary

For vanishing k, time decouples in the boundary geometry (2.43). There exist two decoupling limits, associated with two distinct contractions of the Poincaré group: the Galilean, reached at infinite velocity of light and referred to as "non-relativistic", and the Carrollian, emerging at zero velocity of light [47] – often called "ultra-relativistic". In (2.43), k plays effectively the rôle of velocity of light and $k \to 0$ is indeed a *Carrollian limit*.

This very elementary observation sets precisely and unambiguously the fate of asymptotically flat holography: *the reconstruction of four-dimensional Ricci-flat spacetimes is based on Carrollian boundary geometry*.

The appearance of Carrollian symmetry, or better, conformal Carrollian symmetry at null infinity of asymptotically flat spacetimes is not new [48–51]. It has attracted attention in the framework of flat holography, mostly from the algebraic side [79, 80], or in relation with its dual geometry emerging in the Galilean limit, known as Newton–Cartan (see [81]). The novelties we bring in the present work are twofold. On the one hand, the Carrollian geometry emerging at null infinity is generally non-flat, *i.e.* it is not isometric under the Carroll group, but under a more general group associated with a time-dependent positive-definite spatial metric and a Carrollian time arrow, this general Carrollian geometry being covariant under a subgroup of the diffeomorphisms dubbed Carrollian diffeomorphisms. On the other hand, the Carrollian surface is the natural host for a Carrollian fluid, zero-k limit of the relativistic boundary fluid dual to the original Einstein space of which we consider the flat limit. This Carrollian fluid must be considered as the holographic dual of a Ricci-flat spacetime, and its dynamics (studied in Sec. 3.2) as the dual of gravitational bulk dynamics at zero cosmological constant. From the hydrodynamical viewpoint, this gives a radically new perspective on the subject of flat holography.

The Carrollian geometry: connection and curvature

The Carrollian geometry consists of a spatial surface $\mathscr{S}$ endowed with a positive-definite metric

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.1}$$

and a Carrollian time $t \in \mathbb{R}$.[14] The metric on $\mathscr{S}$ is generically time-dependent: $a_{ij} = a_{ij}(t, \mathbf{x})$. Much like a Galilean space is observed from a spatial frame moving with respect to a local inertial frame with velocity $\mathbf{w}$, a Carrollian frame is described by a form $\boldsymbol{b} = b_i(t, \mathbf{x})\,\mathrm{d}x^i$. The latter is *not* a velocity because in Carrollian spacetimes motion is forbidden. It is rather an inverse velocity, describing a "temporal frame" and plays a dual rôle. A scalar $\Omega(t, \mathbf{x})$ is also introduced (as in the Galilean case, see [52] – this reference will be useful along the present section), as it may naturally arise from the $k \to 0$ limit.

[14] We are genuinely describing a spacetime $\mathbb{R} \times \mathscr{S}$ endowed with a Carrollian structure, and this is actually how the boundary geometry should be spelled. In order to make the distinction with the relativistic pseudo-Riemannian three-dimensional spacetime boundary $\mathscr{I}$ of AdS bulks, we quote only the spatial surface $\mathscr{S}$ when referring to the Carrollian boundary geometry of a Ricci-flat bulk spacetime. For a complete description of such geometries we recommend [82].

We define the Carrollian diffeomorphisms as

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{3.2}$$

with Jacobian functions

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{3.3}$$

Those are the diffeomorphisms adapted to the Carrollian geometry since under such transformations, $\mathrm{d}\ell^2$ remains a positive-definite metric (it does not produce terms involving $\mathrm{d}t'$). Indeed,

$$a'_{ij} = a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{\ \ k}, \quad \Omega' = \frac{\Omega}{J}, \tag{3.4}$$

whereas the time and space derivatives become

$$\partial'_t = \frac{1}{J}\partial_t, \quad \partial'_j = J^{-1i}_{\ \ j}\left(\partial_i - \frac{j_i}{J}\partial_t\right). \tag{3.5}$$

We will show in a short while that the Carrollian fluid equations are precisely covariant under this particular set of diffeomorphisms.

Expression (3.5) shows that the ordinary exterior derivative of a scalar function does not transform as a form. To overcome this issue, it is desirable to introduce a Carrollian derivative as

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t, \tag{3.6}$$

transforming as

$$\hat{\partial}'_i = J^{-1j}_{\ \ i}\hat{\partial}_j. \tag{3.7}$$

Acting on scalars this provides a form, whereas for any other tensor it must be covariantized by introducing a new connection for Carrollian geometry, called *Levi–Civita–Carroll* connection, whose coefficients are the *Christoffel–Carroll* symbols,[15]

$$\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right) = \gamma^i_{jk} + c^i_{jk}. \tag{3.8}$$

The Levi–Civita–Carroll covariant derivative acts symbolically as $\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}}$. It is metric and torsionless: $\hat{\nabla}_i a_{jk} = 0$, $\hat{t}^k_{\ ij} = 2\hat{\gamma}^k_{[ij]} = 0$. There is however an effective torsion, since the derivatives $\hat{\nabla}_i$ do not commute, even when acting of scalar functions Φ – where they are identical to $\hat{\partial}_i$:

$$[\hat{\nabla}_i, \hat{\nabla}_j]\Phi = \frac{2}{\Omega}\varpi_{ij}\partial_t\Phi. \tag{3.9}$$

[15] We remind that the ordinary Christoffel symbols are $\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right)$.

Here ϖ_{ij} is a two-form identified as the Carrollian vorticity defined using the Carrollian acceleration one-form φ_i:

$$\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right) = \partial_t \frac{b_i}{\Omega} + \hat{\partial}_i \ln \Omega, \tag{3.10}$$

$$\varpi_{ij} = \partial_{[i} b_{j]} + b_{[i}\varphi_{j]} = \frac{\Omega}{2}\left(\hat{\partial}_i \frac{b_j}{\Omega} - \hat{\partial}_j \frac{b_i}{\Omega}\right). \tag{3.11}$$

Since the original relativistic fluid is at rest, the kinematical "inverse-velocity" variable potentially present in the Carrollian limit vanishes.[16] Hence the various kinematical quantities such as the vorticity and the acceleration are purely geometric and originate from the temporal Carrollian frame used to describe the surface $\mathscr{S}$. As we will see later, they turn out to be $k \to 0$ counterparts of their relativistic homologues defined in (2.9), (2.10), (2.11) (see also (3.14) for the expansion and shear).

The time derivative transforms as in (3.5), and acting on any tensor under Carrollian diffeomorphisms, it provides another tensor. This ordinary time derivative has nonetheless an unsatisfactory feature: its action on the metric does not vanish. One is tempted therefore to set a new time derivative $\hat{\partial}_t$ such that $\hat{\partial}_t a_{jk} = 0$, while keeping the transformation rule under Carrollian diffeomorphisms: $\hat{\partial}'_t = \frac{1}{J}\hat{\partial}_t$. This is achieved by introducing a "temporal Carrollian connection"

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega} a^{ik}\partial_t a_{kj}, \tag{3.12}$$

which allows us to define the time covariant derivative on a vector field:

$$\frac{1}{\Omega}\hat{\partial}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \tag{3.13}$$

while on a scalar the action is as the ordinary time derivative: $\hat{\partial}_t \Phi = \partial_t \Phi$. Leibniz rule allows extending the action of this derivative to any tensor.

Calling $\hat{\gamma}^i{}_j$ a connection is actually misleading because it transforms as a genuine tensor under Carrollian diffeomorphisms: $\hat{\gamma}'^k{}_j = J^k_n J^{-1m}{}_j \hat{\gamma}^n{}_m$. Its trace and traceless parts have a well-defined kinematical interpretation, as the expansion and shear, completing the acceleration and vorticity introduced earlier in (3.10), (3.11):

$$\theta = \hat{\gamma}^i{}_i = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \quad \xi^i{}_j = \hat{\gamma}^i{}_j - \frac{1}{2}\delta^i_j \theta = \frac{1}{2\Omega} a^{ik}\left(\partial_t a_{kj} - a_{kj}\partial_t \ln\sqrt{a}\right). \tag{3.14}$$

We can define the curvature associated with a connection, by computing the commutator

[16] A Carrollian fluid is always at rest, but could generally be obtained from a relativistic fluid moving at $v^i = k^2\beta^i + \mathrm{O}\left(k^4\right)$. In this case, the "inverse velocity" β^i would contribute to the kinematics and the dynamics of the fluid (see [52]). Here, $v^i = 0$ before the limit $k \to 0$ is taken, so $\beta^i = 0$.

of covariant derivatives acting on a vector field. We find

$$\left[\hat{\nabla}_k, \hat{\nabla}_l\right] V^i = \hat{r}^i{}_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i, \tag{3.15}$$

where

$$\hat{r}^i{}_{jkl} = \hat{\partial}_k \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj} \tag{3.16}$$

is a genuine tensor under Carrollian diffeomorphisms, the Riemann–Carroll tensor.

As usual, the Ricci–Carroll tensor is

$$\hat{r}_{ij} = \hat{r}^k{}_{ikj}. \tag{3.17}$$

It is *not* symmetric in general ($\hat{r}_{ij} \neq \hat{r}_{ji}$) and carries four independent components:

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K} a_{ij} + \hat{A} \eta_{ij}. \tag{3.18}$$

In this expression $\hat{s}_{ij}$ is symmetric and traceless, whereas[17]

$$\hat{K} = \frac{1}{2} a^{ij} \hat{r}_{ij} = \frac{1}{2} \hat{r}, \quad \hat{A} = \frac{1}{2} \eta^{ij} \hat{r}_{ij} = *\varpi\theta \tag{3.19}$$

are the scalar-electric and scalar-magnetic Gauss–Carroll curvatures, with

$$* \varpi = \frac{1}{2} \eta^{ij} \varpi_{ij}. \tag{3.20}$$

Since time and space are intimately related in Carrollian geometry, curvature extends also in time. This can be seen by computing the covariant time and space derivatives commutator:

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\nabla}_i\right] V^i = -2\hat{r}_i V^i + \left(\theta \delta^j_i - \hat{\gamma}^j{}_i\right) \varphi_j V^i + \left(\varphi_i \frac{1}{\Omega} \hat{\partial}_t - \hat{\gamma}^j{}_i \hat{\nabla}_j\right) V^i. \tag{3.21}$$

A Carroll curvature one-form emerges thus as

$$\hat{r}_i = \frac{1}{2} \left(\hat{\nabla}_j \xi^j{}_i - \frac{1}{2} \hat{\partial}_i \theta\right). \tag{3.22}$$

The Ricci–Carroll curvature tensor $\hat{r}_{ij}$ and the Carroll curvature one-form $\hat{r}_i$ are actually the Carrollian vanishing-k contraction of the ordinary Ricci tensor $R_{\mu\nu}$ associated with the original three-dimensional pseudo-Riemannian AdS boundary $\mathscr{I}$, of Randers–Papapetrou type (2.43). The identification of the various pieces is however a subtle task because in this

[17] We use $\eta_{ij} = \sqrt{a}\epsilon_{ij}$, which matches, in the zero-k limit, with the spatial components of the $\eta_{\mu\nu}$ introduced in (2.15). To avoid confusion we also quote that $\eta^{il}\eta_{jl} = \delta^i_j$ and $\eta^{ij}\eta_{ij} = 2$.

kind of limit, where the size of one dimension shrinks, the curvature usually develops divergences. From the perspective of the final Carrollian geometry this does not produce any harm because the involved components decouple.

The metric (3.1) of the Carrollian geometry on $\mathscr{S}$ may or may not be recast in conformally flat form (2.55) using Carrollian diffeomorphisms (3.2), (3.3). A necessary and sufficient condition is the vanishing of the Carrollian shear ξ_{ij}, displayed in (3.14). Assuming this holds, one proves that the traceless and symmetric piece of the Ricci-Carroll tensor is zero,

$$\hat{s}_{ij} = 0. \tag{3.23}$$

We gather in App. A various expressions when holomorphic coordinates are used and the metric is given in conformally flat form. The absence of shear will be imposed again in Sec. 4, where it plays a crucial rôle in the resummation of the derivative expansion.

The conformal Carrollian geometry

In the present set-up, the spatial surface $\mathscr{S}$ appears as the null infinity of the resulting Ricci-flat geometry *i.e.* as $\mathscr{I}^+$. This is not surprising. The bulk congruence tangent to ∂_r is lightlike. Hence the holographic limit $r \to \infty$ is lightlike, already at finite k, which is a well known feature of the derivative expansion, expressed by construction in Eddington–Finkelstein-like coordinates [3,4,6]. What is specific about $k = 0$ is the decoupling of time.

The geometry of $\mathscr{I}^+$ is equipped with a conformal class of metrics rather than with a metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to study conformal Carrollian geometry as opposed to plain Carrollian geometry (see [48]).

The action of Weyl transformations on the elements of the Carrollian geometry on a surface $\mathscr{S}$ is inherited from (2.18):

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}, \quad b_i \to \frac{b_i}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \tag{3.24}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The Carrollian vorticity (3.11) and shear (3.14) transform covariantly under (3.24): $\varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}$, $\xi_{ij} \to \frac{1}{\mathcal{B}}\xi_{ij}$. However, the Levi–Civita–Carroll covariant derivatives $\hat{\boldsymbol{\nabla}}$ and $\hat{\partial}_t$ defined previously for Carrollian geometry are not covariant under (3.24). Following [52], they must be replaced with Weyl–Carroll covariant spatial and time derivatives built on the Carrollian acceleration φ_i (3.10) and the Carrollian expansion (3.14), which transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{2}{\Omega}\partial_t \mathcal{B}. \tag{3.25}$$

In particular, these can be combined in[18]

$$\alpha_i = \varphi_i - \frac{\theta}{2} b_i, \tag{3.26}$$

transforming under Weyl rescaling as:

$$\alpha_i \to \alpha_i - \partial_i \ln \mathcal{B}. \tag{3.27}$$

The Weyl–Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ are defined according to the pattern (2.19), (2.20). They obey

$$\hat{\mathscr{D}}_j a_{kl} = 0, \quad \hat{\mathscr{D}}_t a_{kl} = 0. \tag{3.28}$$

For a weight-w scalar function Φ, or a weight-w vector V^i, *i.e.* scaling with $\mathcal{B}^w$ under (3.24), we introduce

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad \hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i, \tag{3.29}$$

which leave the weight unaltered. Similarly, we define

$$\frac{1}{\Omega} \hat{\mathscr{D}}_t \Phi = \frac{1}{\Omega} \hat{\partial}_t \Phi + \frac{w}{2} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{2} \theta \Phi, \tag{3.30}$$

and

$$\frac{1}{\Omega} \hat{\mathscr{D}}_t V^l = \frac{1}{\Omega} \hat{\partial}_t V^l + \frac{w-1}{2} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{2} \theta V^l + \xi^l{}_i V^i, \tag{3.31}$$

where $\frac{1}{\Omega} \hat{\mathscr{D}}_t$ increases the weight by one unit. The action of $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ on any other tensor is obtained using the Leibniz rule.

The Weyl–Carroll connection is torsion-free because

$$\left[\hat{\mathscr{D}}_i, \hat{\mathscr{D}}_j\right] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathscr{D}}_t \Phi + w \left(\varphi_{ij} - \varpi_{ij} \theta\right) \Phi \tag{3.32}$$

does not contain terms of the type $\hat{\mathscr{D}}_k \Phi$. Here $\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i$ is a Carrollian two-form, not conformal though. Connection (3.32) is accompanied with its own curvature tensors, which emerge in the commutation of Weyl–Carroll covariant derivatives acting *e.g.* on vectors:

$$\left[\hat{\mathscr{D}}_k, \hat{\mathscr{D}}_l\right] V^i = \left(\hat{\mathscr{R}}^i{}_{jkl} - 2\xi^i{}_j \varpi_{kl}\right) V^j + \varpi_{kl} \frac{2}{\Omega} \hat{\mathscr{D}}_t V^i + w \left(\varphi_{kl} - \varpi_{kl} \theta\right) V^i. \tag{3.33}$$

The combination $\varphi_{kl} - \varpi_{kl} \theta$ forms a weight-0 conformal two-form, whose dual $*\varphi - *\varpi\theta$ is

[18] Contrary to φ_i, α_i is not a Carrollian one-form, *i.e.* it does not transform covariantly under Carrollian diffeomorphisms (3.2).

conformal of weight 2 ($*\varpi$ is defined in (3.20) and similarly $*\varphi = \frac{1}{2}\eta^{ij}\varphi_{ij}$). Moreover

$$\begin{aligned}\hat{\mathscr{R}}^{i}{}_{jkl} &= \hat{r}^{i}{}_{jkl} - \delta^{i}_{j}\varphi_{kl} - a_{jk}\hat{\nabla}_{l}\varphi^{i} + a_{jl}\hat{\nabla}_{k}\varphi^{i} + \delta^{i}_{k}\hat{\nabla}_{l}\varphi_{j} - \delta^{i}_{l}\hat{\nabla}_{k}\varphi_{j} \\ &\quad +\varphi^{i}\left(\varphi_{k}a_{jl} - \varphi_{l}a_{jk}\right) - \left(\delta^{i}_{k}a_{jl} - \delta^{i}_{l}a_{jk}\right)\varphi_{m}\varphi^{m} + \left(\delta^{i}_{k}\varphi_{l} - \delta^{i}_{l}\varphi_{k}\right)\varphi_{j} \end{aligned} \tag{3.34}$$

is the Riemann–Weyl–Carroll weight-0 tensor, from which we define

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^{k}{}_{ikj} = \hat{r}_{ij} + a_{ij}\hat{\nabla}_{k}\varphi^{k} - \varphi_{ij}. \tag{3.35}$$

We also quote

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_{t},\hat{\mathscr{D}}_{i}\right]\Phi = w\hat{\mathscr{R}}_{i}\Phi - \xi^{j}{}_{i}\hat{\mathscr{D}}_{j}\Phi \tag{3.36}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_{t},\hat{\mathscr{D}}_{i}\right]V^{i} = (w-2)\hat{\mathscr{R}}_{i}V^{i} - V^{i}\hat{\mathscr{D}}_{j}\xi^{j}{}_{i} - \xi^{j}{}_{i}\hat{\mathscr{D}}_{j}V^{i}, \tag{3.37}$$

with

$$\hat{\mathscr{R}}_{i} = \hat{r}_{i} + \frac{1}{\Omega}\hat{\partial}_{t}\varphi_{i} - \frac{1}{2}\hat{\nabla}_{j}\hat{\gamma}^{j}{}_{i} + \xi^{j}{}_{i}\varphi_{j} = \frac{1}{\Omega}\partial_{t}\varphi_{i} - \frac{1}{2}\left(\hat{\partial}_{i} + \varphi_{i}\right)\theta. \tag{3.38}$$

This is a Weyl-covariant weight-1 curvature one-form, where $\hat{r}_i$ is given in (3.22).

The Ricci–Weyl–Carroll tensor (3.35) is *not* symmetric in general: $\hat{\mathscr{R}}_{ij} \neq \hat{\mathscr{R}}_{ji}$. Using (3.17) we can recast it as

$$\hat{\mathscr{R}}_{ij} = \hat{s}_{ij} + \hat{\mathscr{K}}a_{ij} + \hat{\mathscr{A}}\eta_{ij}, \tag{3.39}$$

where we have introduced the Weyl-covariant scalar-electric and scalar-magnetic Gauss–Carroll curvatures

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_{k}\varphi^{k}, \quad \hat{\mathscr{A}} = \frac{1}{2}\eta^{ij}\hat{\mathscr{R}}_{ij} = \hat{A} - *\varphi \tag{3.40}$$

both of weight 2.

Before closing the present section, it is desirable to make a clarification: Weyl transformations (3.24) should not be confused with the action of the conformal Carroll group, which is a subset of Carrollian diffeomorphisms defined as[19]

$$\mathbf{CCarr}_{2}\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^{2},\mathrm{u}\right) = \left\{\phi \in \mathrm{Diff}(\mathbb{R}\times\mathscr{S}), \quad \mathrm{d}\ell^{2} \xrightarrow{\phi} \mathrm{e}^{-2\Phi}\mathrm{d}\ell^{2} \quad \mathrm{u} \xrightarrow{\phi} e^{\Phi}\mathrm{u}\right\}, \tag{3.41}$$

where $\Phi \in \mathcal{C}^{\infty}(\mathbb{R}\times\mathscr{S})$, $\mathrm{d}\ell^2$ is the spatial metric on $\mathscr{S}$ as in (3.1), and $\mathrm{u} = \frac{1}{\Omega}\partial_t$ the Carrollian time arrow. This group is actually the zero-k contraction of $\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)$, the group of conformal isometries of the original finite-k relativistic metric $\mathrm{d}s^2$ on the boundary $\mathscr{I}$ of the

[19]The subscript 2 stands for level-2 conformal Carroll group. For a detailed discussion, see [49] .

corresponding AdS bulk:

$$\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)=\left\{\phi\in\mathrm{Diff}(\mathscr{I}),\quad \mathrm{d}s^2\xrightarrow{\phi}\mathrm{e}^{-2\Phi}\mathrm{d}s^2\right\}\tag{3.42}$$

with $\Phi\in\mathcal{C}^\infty(\mathscr{I})$. Indeed, consider the Lie algebra of conformal symmetries of $\mathrm{d}s^2$, denoted $\mathfrak{cisom}\left(\mathscr{I},\mathrm{d}s^2\right)$ and spanned by vector fields $\mathrm{X}=X^0\partial_0+X^i\partial_i$ such that

$$\mathscr{L}_{\mathrm{X}}\mathrm{d}s^2=-2\lambda\mathrm{d}s^2\tag{3.43}$$

for some function λ on $\mathscr{I}$. In order to perform the zero-k contraction we write the generators as $\mathrm{X}=kX^t\partial_0+X^i\partial_i$ (here $x^0=kt$, thus $X^0=kX^t$) and the metric $\mathrm{d}s^2$ in the Randers–Papapetrou form (2.43). At zero k Eq. (3.43) splits into:[20]

$$\mathscr{L}_{\mathrm{X}}\mathrm{u}=\lambda\mathrm{u},\quad \mathscr{L}_{\mathrm{X}}\mathrm{d}\ell^2=-2\lambda\mathrm{d}\ell^2.\tag{3.44}$$

These are the equations the field X must satisfy for belonging to $\mathfrak{ccarr}_2\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^2,\mathrm{u}\right)$, the Lie algebra of the corresponding conformal Carroll group. This confirms that

$$\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)\underset{k\to0}{\longrightarrow}\mathbf{CCarr}_2\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^2,\mathrm{u}\right).\tag{3.45}$$

At last, if $\mathscr{S}$ is chosen to be the two-sphere and $\mathrm{d}\ell^2$ the round metric, it can be shown (see [49]) that the corresponding conformal Carroll group is precisely the BMS(4) group, which describes the asymptotic symmetries of an asymptotically flat $3+1$-dimensional metric.

3.2 Carrollian conformal fluid dynamics

Physical data and hydrodynamic equations

More on the physics underlying the Carrollian limit can be found in [52], with emphasis on hydrodynamics. This is precisely what we need here, since the original asymptotically AdS bulk Einstein spacetime is the holographic dual of a relativistic fluid hosted by its $2+1$-dimensional boundary. This relativistic fluid satisfying Eq. (2.1), will obey Carrollian dynamics at vanishing k. Even though the fluid has no velocity, it has non-trivial hydrodynamics based on the following data:

- the energy density $\varepsilon(t,\mathbf{x})$ and the pressure $p(t,\mathbf{x})$, related here through a conformal equation of state $\varepsilon=2p$;

[20] In coordinates, defining $\chi=\Omega X^t-b_jX^j$, these equations are written as:

$$\frac{1}{\Omega}\partial_t\chi+\varphi_jX^j=-\lambda,\quad \frac{1}{\Omega}\partial_tX^i=0,\quad \hat{\nabla}^{(i}X^{j)}+\chi\left(\xi^{ij}+\frac{1}{2}a^{ij}\theta\right)=-\lambda a^{ij},$$

which are manifestly covariant under Carrollian diffeomorphisms.

- the heat currents $\boldsymbol{Q} = Q_i(t,\mathbf{x})\mathrm{d}x^i$ and $\boldsymbol{\pi} = \pi_i(t,\mathbf{x})\mathrm{d}x^i$;
- the viscous stress tensors $\boldsymbol{\Sigma} = \Sigma_{ij}(t,\mathbf{x})\,\mathrm{d}x^i\mathrm{d}x^j$ and $\boldsymbol{\Xi} = \Xi_{ij}(t,\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j$.

The latter quantities are inherited from the relativistic ones (see (2.2)) as the following limits:

$$Q_i = \lim_{k\to 0} q_i, \quad \pi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(q_i - Q_i\right), \tag{3.46}$$

$$\Sigma_{ij} = -\lim_{k\to 0} k^2\tau_{ij}, \quad \Xi_{ij} = -\lim_{k\to 0}\left(\tau_{ij} + \tfrac{1}{k^2}\Sigma_{ij}\right). \tag{3.47}$$

Compared with the corresponding ones in the Galilean fluids, they are doubled because two orders seem to be required for describing the Carrollian dynamics. They obey

$$\Sigma_{ij} = \Sigma_{ji}, \quad \Sigma^i{}_i = 0, \quad \Xi_{ij} = \Xi_{ji}, \quad \Xi^i{}_i = 0. \tag{3.48}$$

The Carrollian energy and pressure are just the zero-k limits of the corresponding relativistic quantities. In order to avoid symbols inflation, we have kept the same notation, ε and p.

All these objects are Weyl-covariant with conformal weights 3 for the pressure and energy density, 2 for the heat currents, and 1 for the viscous stress tensors (when all indices are lowered). They are well-defined in all examples we know from holography. Ultimately they should be justified within a microscopic quantum/statistical approach, missing at present since the microscopic nature of a Carrollian fluid has not been investigated so far, except for [52], where some elementary issues were addressed.

Following this reference, the equations for a Carrollian fluid are as follows:

- a set of two scalar equations, both weight-4 Weyl-covariant:

$$-\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon - \hat{\mathscr{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0, \tag{3.49}$$

$$\Sigma^{ij}\xi_{ij} = 0; \tag{3.50}$$

- two vector equations, Weyl-covariant of weight 3:

$$\hat{\mathscr{D}}_j p + 2Q^i\varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t\pi_j - \hat{\mathscr{D}}_i\Xi^i{}_j + \pi_i\xi^i{}_j = 0, \tag{3.51}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t Q_j - \hat{\mathscr{D}}_i\Sigma^i{}_j + Q_i\xi^i{}_j = 0. \tag{3.52}$$

Equation (3.49) is the energy conservation, whereas (3.50) sets a geometrical constraint on the Carrollian viscous stress tensor Σ_{ij}. Equations (3.51) and (3.52) are dynamical equations involving the pressure $p = \varepsilon/2$, the heat currents Q_i and π_i, and the viscous stress tensors Σ_{ij} and Ξ_{ij}. They are reminiscent of a momentum conservation, although somewhat degenerate due to the absence of fluid velocity.

An example of Carrollian fluid

The simplest non-trivial example of a Carrollian fluid is obtained as the Carrollian limit of the relativistic Robinson–Trautman fluid, studied at the end of Sec. 2.2 (see also [66] and [52] for the relativistic and Carrollian approaches, respectively).

The geometric Carrollian data are in this case

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}, \tag{3.53}$$

$b_i = 0$ and $\Omega = 1$. Hence the Carrollian shear vanishes ($\xi_{ij} = 0$), whereas the expansion reads:

$$\theta = -2\partial_t \ln P. \tag{3.54}$$

Similarly $\varpi_{ij} = 0$, $\varphi_i = 0$, $\varphi_{ij} = 0$, and using results from App. A, we find

$$\hat{\mathscr{K}} = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta \ln P, \quad \hat{\mathscr{A}} = 0 \tag{3.55}$$

(in fact $\hat{\mathscr{K}} = \hat{K} = K$), while

$$\hat{\mathscr{R}}_{\bar{\zeta}} = \partial_{\bar{\zeta}} \partial_t \ln P, \quad \hat{\mathscr{R}}_{\bar{\zeta}} = \partial_{\bar{\zeta}} \partial_t \ln P. \tag{3.56}$$

From the relativistic heat current q and viscous stress tensor τ displayed in (2.64) and (2.65), we obtain the Carrollian descendants:[21]

$$\boldsymbol{Q} = -\tfrac{1}{16\pi G} \left(\partial_\zeta K d\zeta + \partial_{\bar{\zeta}} K d\bar{\zeta} \right), \quad \boldsymbol{\pi} = 0, \tag{3.57}$$

$$\boldsymbol{\Sigma} = -\tfrac{1}{8\pi G P^2} \left(\partial_\zeta \left(P^2 \partial_t \partial_\zeta \ln P \right) d\zeta^2 + \partial_{\bar{\zeta}} \left(P^2 \partial_t \partial_{\bar{\zeta}} \ln P \right) d\bar{\zeta}^2 \right), \quad \boldsymbol{\Xi} = 0. \tag{3.58}$$

Due to the absence of shear, the hydrodynamic equation (3.50) is identically satisfied, whereas (3.49), (3.51), (3.52) are recast as:

$$3\varepsilon \partial_t \ln P - \partial_t \varepsilon - \nabla_i Q^i = 0, \tag{3.59}$$

$$\partial_i p = 0, \tag{3.60}$$

$$\partial_t Q_i - 2Q_i \partial_t \ln P - \nabla_j \Sigma^j{}_i = 0. \tag{3.61}$$

In agreement with the relativistic Robinson–Trautman fluid, the pressure p (and so the energy density, since the fluid is conformal) must be space-independent. Furthermore, as expected from the relativistic case, Eq. (3.61) is satisfied with Q_i and Σ_{ij} given in (3.57) and (3.58). Hence we are left with a single non-trivial equation, Eq. (3.59), the heat equation of

[21] Notice a useful identity: $\partial_t \left(\frac{\partial_\zeta^2 P}{P} \right) = \frac{1}{P^2} \partial_\zeta \left(P^2 \partial_t \partial_\zeta \ln P \right)$.

the Carrollian fluid:

$$3\varepsilon\partial_t \ln P - \partial_t\varepsilon + \frac{1}{16\pi G}\Delta K = 0 \tag{3.62}$$

with $\Delta = \nabla_j \nabla^j$ the Laplacian operator on $\mathscr{S}$.

Equation (3.62) is exactly Robinson–Trautman's, Eq. (2.66). We note that the relativistic and the Carrolian dynamics lead to the same equations – and hence to the same solutions $\varepsilon = \varepsilon(t)$. This is specific to the case under consideration, and it is actually expected since the bulk Einstein equations for a geometry with a shearless and vorticity-free null congruence lead to the Robinson–Trautman equation, irrespective of the presence of a cosmological constant, $\Lambda = -3k^2$: asymptotically locally AdS or locally flat spacetimes lead to the same dynamics. This is not the case in general though, because there is no reason for the relativistic dynamics to be identical to the Carrollian (see [52] for a detailed account of this statement). For example, when switching on more data, as in the case of the Plebański–Demiański family, where all b_i, φ_i, ϖ_{ij}, as well as π_i and Ξ_{ij}, are on, the Carrollian equations are different from the relativistic ones.

4 The Ricci-flat limit II: derivative expansion and resummation

We can summarize our observations as follows. Any four-dimensional Ricci-flat spacetime is associated with a two-dimensional spatial surface, emerging at null infinity and equipped with a conformal Carrollian geometry. This geometry is the host of a Carrollian fluid, obeying Carrollian hydrodynamics. Thanks to the relativistic-fluid/AdS-gravity duality, one can also safely claim that, conversely, any Carrollian fluid evolving on a spatial surface with Carrollian geometry is associated with a Ricci-flat geometry. This conclusion is reached by considering the simultaneous zero-k limit of both sides of the quoted duality. In order to make this statement operative, this limit must be performed inside the derivative expansion. When the latter is resummable in the sense discussed in Sec. 2.2, the zero-k limit will also affect the resummability conditions, and translate them in terms of Carrollian fluid dynamics.

4.1 Back to the derivative expansion

Our starting point is the derivative expansion of an asymptotically locally AdS spacetime, Eq. (2.41). The fundamental question is whether the latter admits a smooth zero-k limit.

We have implicitly assumed that the Randers–Papapetrou data of the three-dimensional pseudo-Riemannian conformal boundary $\mathscr{I}$ associated with the original Einstein spacetime, a_{ij}, b_i and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new spatial two-dimensional boundary $\mathscr{S}$ emerging at $\mathscr{I}^+$.[22] Following again the

[22] Indeed our ultimate goal is to set up a derivative expansion (in a closed resummed form under appropriate

detailed analysis performed in [52], we can match the various three-dimensional Riemannian quantities with the corresponding two-dimensional Carrollian ones:

$$\mathrm{u} = -k^2 \left(\Omega \mathrm{d}t - \boldsymbol{b}\right) \tag{4.1}$$

and

$$\begin{aligned} \omega &= \tfrac{k^2}{2}\varpi_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j, \\ \gamma &= *\varpi, \\ \Theta &= \theta, \\ \mathrm{a} &= k^2\varphi_i \mathrm{d}x^i, \\ \mathrm{A} &= \alpha_i \mathrm{d}x^i + \tfrac{\theta}{2}\Omega\mathrm{d}t, \\ \sigma &= \xi_{ij}\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{4.2}$$

where the left-hand-side quantities are Riemannian (given in Eqs. (2.45), (2.46), (2.47), (2.48), (2.49)), and the right-hand-side ones Carrollian (see (3.10), (3.11), (3.14), (3.20)).

In the list (4.2), we have dealt with the first derivatives, *i.e.* connexion-related quantities. We move now to second-derivative objects and collect the tensors relevant for the derivative expansion, following the same pattern (Riemannian vs. Carrollian):

$$\mathscr{R} = \frac{1}{k^2}\xi_{ij}\xi^{ij} + 2\hat{\mathscr{K}} + 2k^2 * \varpi^2, \tag{4.3}$$

$$\omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = k^4\varpi_i{}^l\varpi_{lj}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.4}$$

$$\omega^{\mu\nu}\omega_{\mu\nu} = 2k^4 * \varpi^2, \tag{4.5}$$

$$\mathscr{D}_\nu\omega^\nu{}_\mu\mathrm{d}x^\mu = k^2\hat{\mathscr{D}}_j\varpi^j{}_i\mathrm{d}x^i - 2k^4 * \varpi^2\Omega\mathrm{d}t + 2k^4 * \varpi^2\boldsymbol{b}. \tag{4.6}$$

Using (2.42) this leads to

$$\mathrm{S} = -\frac{k^2}{2}\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2\xi_{ij}\xi^{ij} + k^4\boldsymbol{s} - 5k^6\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2 * \varpi^2 \tag{4.7}$$

with the Weyl-invariant tensor

$$\boldsymbol{s} = 2\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)\mathrm{d}x^i\eta^j{}_i\hat{\mathscr{D}}_j * \varpi + *\varpi^2\mathrm{d}\ell^2 - \hat{\mathscr{K}}\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2. \tag{4.8}$$

In the derivative expansion (2.41), two explicit divergences appear at vanishing k. The first originates from the first term of S, which is the shear contribution to the Weyl-covariant

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assumptions) for building up four-dimensional Ricci-flat spacetimes from a boundary Carrollian fluid, irrespective of its AdS origin. For this it is enough to assume a_{ij}, b_i and Ω k-independent (as in [52]), and use these data as fundamental blocks for the Ricci-flat reconstruction. It should be kept in mind, however, that for general Einstein spacetimes, these may depend on k with well-defined limit and subleading terms. Due to the absence of shear and to the particular structure of these solutions, the latter do not alter the Carrollian equations. This occurs for instance in Plebański–Demiański or in the Kerr–Taub–NUT sub-family, which will be discussed in Sec. 5.1. In the following, we avoid discussing this kind of sub-leading terms, hence saving further technical developments.

scalar curvature $\mathscr{R}$ of the three–dimensional AdS boundary (Eq. (4.3)).[23] The second divergence comes from the Cotton tensor and is also due to the shear. It is fortunate – and expected – that counterterms coming from equal-order (non-explicitly written) σ^2 contributions, cancel out these singular terms. This is suggestive that (2.41) is well-behaved at zero-k, showing that the reconstruction of Ricci-flat spacetimes works starting from two-dimensional Carrollian fluid data.

We will not embark here in proving finiteness at $k = 0$, but rather confine our analysis to situations without shear, as we discussed already in Sec. 2.2 for Einstein spacetimes. Vanishing σ in the pseudo-Riemannian boundary $\mathscr{I}$ implies indeed vanishing ξ_{ij} in the Carrollian (see (4.2)), and in this case, the divergent terms in S and C are absent. Of course, other divergences may occur from higher-order terms in the derivative expansion. To avoid dealing with these issues, we will focus on the resummed version of (2.41) *i.e.* (2.53), valid for algebraically special bulk geometries. This closed form is definitely smooth at zero k and reads:

$$\boxed{\mathrm{d}s^2_{\text{res. flat}} = -2\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)\left(\mathrm{d}r + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2\mathrm{d}\ell^2 + \boldsymbol{s} + \frac{\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)^2}{\rho^2}\left(8\pi G\varepsilon r + c * \varpi\right).} \tag{4.9}$$

Here

$$\rho^2 = r^2 + *\varpi^2, \tag{4.10}$$

$\mathrm{d}\ell^2$, Ω, $\boldsymbol{b} = b_i \mathrm{d}x^i$, $\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i$, θ and $*\varpi$ are the Carrollian geometric objects introduced earlier, while c and ε are the zero-k (finite) limits of the corresponding relativistic functions. Expression (4.9) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (2.53) was Einstein are fulfilled in the zero-k limit. These conditions are the set of Carrollian hydrodynamic equations (3.49), (3.50), (3.51) and (3.52), and the integrability conditions, as they emerge from (2.56) and (2.58) at vanishing k. Making the latter explicit is the scope of next section.

Notice eventually that the Ricci-flat line element (4.9) inherits Weyl invariance from its relativistic ancestor. The set of transformations (3.24), (3.25) and (3.27), supplemented with $*\varpi \to \mathcal{B} * \varpi$, $\varepsilon \to \mathcal{B}^3\varepsilon$ and $c \to \mathcal{B}^3 c$, can indeed be absorbed by setting $r \to \mathcal{B}r$ ($\boldsymbol{s}$ is Weyl invariant), resulting thus in the invariance of (4.9). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this is rooted to the location of the two-dimensional spatial boundary $\mathscr{S}$ at null infinity $\mathscr{I}^+$.

[23]This divergence is traced back in the Gauss–Codazzi equation relating the intrinsic and extrinsic curvatures of an embedded surface, to the intrinsic curvature of the host. When the size of a fiber shrinks, the extrinsic-curvature contribution diverges.

4.2 Resummation of the Ricci-flat derivative expansion

The Cotton tensor in Carrollian geometry

The Cotton tensor monitors from the boundary the global asymptotic structure of the bulk four-dimensional Einstein spacetime (for higher dimensions, the boundary Weyl tensor is also involved, see footnote 11). In order to proceed with our resummability analysis, we need to describe the zero-k limit of the Cotton tensor (2.32) and of its conservation equation (2.33).

As already mentioned, at vanishing k divergences do generally appear for some components of the Cotton tensor. These divergences are no longer present when (2.54) is satisfied (see footnote 23), *i.e.* in the absence of shear, which is precisely the assumption under which we are working with (4.9). Every piece of the three-dimensional relativistic Cotton tensor appearing in (2.34) has thus a well-defined limit. We therefore introduce

$$\chi_i = \lim_{k\to 0} c_i, \quad \psi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_i - \chi_i\right), \tag{4.11}$$

$$X_{ij} = \lim_{k\to 0} c_{ij}, \quad \Psi_{ij} = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_{ij} - X_{ij}\right). \tag{4.12}$$

The time components c_0, c_{00} and $c_{0i} = c_{i0}$ vanish already at finite k (due to (2.36)), and χ_i, ψ_i, X_{ij} and Ψ_{ij} are thus genuine Carrollian tensors transforming covariantly under Carrollian diffeomorphisms. Actually, in the absence of shear the Cotton current and stress tensor are given exactly (*i.e.* for finite k) by $c_i = \chi_i + k^2\psi_i$ and $c_{ij} = X_{ij} + k^2\Psi_{ij}$.

The scalar $c(t,\mathbf{x})$ is Weyl-covariant of weight 3 (like the energy density). As expected, it is expressed in terms of geometric Carrollian objects built on third-derivatives of the two-dimensional metric $\mathrm{d}\ell^2$, b_i and Ω:

$$c = \left(\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi. \tag{4.13}$$

Similarly, the forms χ_i and ψ_i, of weight 2, are

$$\chi_j = \frac{1}{2}\eta^l{}_j\hat{\mathscr{D}}_l\hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j\hat{\mathscr{A}} - 2 * \varpi\hat{\mathscr{R}}_j, \tag{4.14}$$

$$\psi_j = 3\eta^l{}_j\hat{\mathscr{D}}_l * \varpi^2. \tag{4.15}$$

Finally, the weight-1 symmetric and traceless rank-two tensors read:

$$X_{ij} = \frac{1}{2}\eta^l{}_j\hat{\mathscr{D}}_l\hat{\mathscr{R}}_i + \frac{1}{2}\eta^l{}_i\hat{\mathscr{D}}_j\hat{\mathscr{R}}_l, \tag{4.16}$$

$$\Psi_{ij} = \hat{\mathscr{D}}_i\hat{\mathscr{D}}_j * \varpi - \frac{1}{2}a_{ij}\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l * \varpi - \eta_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t * \varpi^2. \tag{4.17}$$

Observe that c and the subleading terms ψ_i and Ψ_{ij} are present only when the vorticity is non-vanishing ($*\varpi \neq 0$). All these are of gravito-magnetic nature.

The tensors c, χ_i, ψ_i, X_{ij} and Ψ_{ij} should be considered as the two-dimensional Carrollian resurgence of the three-dimensional Riemannian Cotton tensor. They should be referred to as Cotton descendants (there is no Cotton tensor in two dimensions anyway), and obey identities inherited at zero k from its conservation equation.[24] These are similar to the hydrodynamic equations (3.49), (3.50), (3.51) and (3.52), satisfied by the different pieces of the energy–momentum tensor ε, Q_i, π_i, Σ_{ij} and Ξ_{ij}, and translating its conservation. In the case at hand, the absence of shear trivializes (3.50) and discards the last term in the other three equations:

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0, \tag{4.18}$$

$$\frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j = 0, \tag{4.19}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j = 0. \tag{4.20}$$

One appreciates from these equations why it is important to keep the subleading corrections at vanishing k, both in the Cotton current c_μ and in the Cotton stress tensor $c_{\mu\nu}$. As for the energy–momentum tensor, ignoring them would simply lead to wrong Carrollian dynamics.

The resummability conditions

We are now ready to address the problem of resummability in Carrollian framework, for Ricci-flat spacetimes. In the relativistic case, where one describes relativistic hydrodynamics on the pseudo-Riemannian boundary of an asymptotically locally AdS spacetime, resummability – or integrability – equations are Eqs. (2.56) and (2.58). These determine the friction components of the fluid energy–momentum tensor in terms of geometric data, captured by the Cotton tensor (current and stress components), via a sort of gravitational electric–magnetic duality, transverse to the fluid congruence. Equipped with those, the fluid equations (2.1) guarantee that the bulk is Einstein, *i.e.* that bulk Einstein equations are satisfied.

Correspondingly, using (3.46), (3.47), (4.11) and (4.12), the zero-k limit of Eq. (2.56) sets up a duality relationship among the Carrollian-fluid heat current Q_i and the Carrollian-geometry third-derivative vector χ_i:

$$\boxed{Q_i = \frac{1}{8\pi G}\eta^j{}_i \chi_j = -\frac{1}{16\pi G}\left(\hat{\mathscr{D}}_i \mathscr{K} - \eta^j{}_i \hat{\mathscr{D}}_j \mathscr{A} + 4 * \varpi \eta^j{}_i \mathscr{R}_j\right),} \tag{4.21}$$

while Eqs. (2.58) allow to relate the Carrollian-fluid quantities Σ_{ij} and Ξ_{ij}, to the Carrollian-

[24] Observe that the Cotton tensor enters in Eq. (2.60) with an extra factor $1/k$, the origin of which is explained in footnote 9. Hence, the advisable prescription is to analyze the small-k limit of $\frac{1}{k}\nabla^\mu C_{\mu\nu} = 0$.

geometry ones X_{ij} and Ψ_{ij}:

$$\boxed{\Sigma_{ij} = \frac{1}{8\pi G}\eta^l{}_i X_{lj} = \frac{1}{16\pi G}\left(\eta^k{}_j\eta^l{}_i\hat{\mathscr{D}}_k\hat{\mathscr{R}}_l - \hat{\mathscr{D}}_j\hat{\mathscr{R}}_i\right),} \tag{4.22}$$

and

$$\boxed{\Xi_{ij} = \frac{1}{8\pi G}\eta^l{}_i\Psi_{lj} = \frac{1}{8\pi G}\left(\eta^l{}_i\hat{\mathscr{D}}_l\hat{\mathscr{D}}_j * \varpi + \frac{1}{2}\eta_{ij}\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l * \varpi - a_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t * \varpi^2\right).} \tag{4.23}$$

One readily shows that (3.48) is satisfied as a consequence of the symmetry and tracelessness of X_{ij} and Ψ_{ij}.

One can finally recast the Carrollian hydrodynamic equations (3.49), (3.50), (3.51) and (3.52) for the fluid under consideration. Recalling that the shear is assumed to vanish,

$$\xi_{ij} = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right) = 0, \tag{4.24}$$

Eq. (3.50) is trivialized. Furthermore, Eq. (3.52) is automatically satisfied with Q_j and $\Sigma^i{}_j$ given above, thanks also to Eq. (4.20). We are therefore left with two equations for the energy density ε and the heat current π_i:

- one scalar equation from (3.49):

$$\boxed{-\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{16\pi G}\hat{\mathscr{D}}^i\left(\hat{\mathscr{D}}_i\hat{\mathscr{K}} - \eta^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}} + 4 * \varpi\eta^j{}_i\hat{\mathscr{R}}_j\right) = 0;} \tag{4.25}$$

- one vector equation from (3.51):

$$\boxed{\hat{\mathscr{D}}_j\varepsilon + 4 * \varpi\eta^i{}_j Q_i + \frac{2}{\Omega}\hat{\mathscr{D}}_t\pi_j - 2\hat{\mathscr{D}}_i\Xi^i{}_j = 0} \tag{4.26}$$

  with Q_i and $\Xi^i{}_j$ given in (4.21) and (4.23).

These last two equations are Carrollian equations, describing time and space evolution of the fluid energy and heat current, as a consequence of transport phenomena like heat conduction and friction. These phenomena have been identified by duality to geometric quantities, and one recognizes distinct gravito-electric (like $\hat{\mathscr{K}}$) and gravito-magnetic contributions (like $\hat{\mathscr{A}}$). It should also be stressed that not all the terms are independent and one can reshuffle them using identities relating the Carrollian curvature elements. In the absence of shear, (3.23) holds and all information about $\hat{\mathscr{R}}_{ij}$ in (3.39) is stored in $\hat{\mathscr{K}}$ and $\hat{\mathscr{A}}$, while other geometrical data are supplied by $\hat{\mathscr{R}}_i$ in (3.38). All these obey

$$\begin{aligned}
\tfrac{2}{\Omega}\hat{\mathscr{D}}_t * \varpi + \hat{\mathscr{A}} &= 0,\\
\tfrac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{K}} - a^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j &= 0,\\
\tfrac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{A}} + \eta^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j &= 0,
\end{aligned} \tag{4.27}$$

which originate from three-dimensional Riemannian Bianchi identities and emerge along the k-to-zero limit.

Summarizing

Our analysis of the zero-k limit in the derivative expansion (2.53), valid assuming the absence of shear, has the following salient features.

- As the general derivative expansion (2.41), this limit reveals a two-dimensional spatial boundary $\mathscr{S}$ located at $\mathscr{I}^+$. It is endowed with a Carrollian geometry, encoded in a_{ij}, b_i and Ω, all functions of t and $\mathbf{x}$. This is inherited from the conformal three-dimensional pseudo-Riemannian boundary $\mathscr{I}$ of the original Einstein space.

- The Carrollian boundary $\mathscr{S}$ is the host of a Carrollian fluid, obtained as the limit of a relativistic fluid, and described in terms of its energy density ε, and its friction tensors Q_i, π_i, Σ_{ij} and Ξ_{ij}.

- When the friction tensors Q_i, Σ_{ij} and Ξ_{ij} of the Carrollian fluid are given in terms of the geometric objects χ_i, X_{ij} and Ψ_{ij} using (4.21), (4.22) and (4.23), and when the energy density ε and the current π_i obey the hydrodynamic equations (4.25) and (4.26), the limiting resummed derivative expansion (4.9) is an exact Ricci-flat spacetime.

- The bulk spacetime is in general asymptotically locally flat. This property is encoded in the zero-k limit of the Cotton tensor, *i.e.* in the Cotton Carrollian descendants c, χ_i and X_{ij}.

The bulk Ricci-flat spacetime obtained following the above procedure is algebraically special. We indeed observe that the bulk congruence ∂_r is null. Moreover, it is geodesic and shear-free. To prove this last statement, we rewrite the metric (4.9) in terms of a null tetrad $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$:

$$\mathrm{d}s^2_{\text{res. flat}} = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}\,, \quad \mathbf{k}\cdot\mathbf{l} = -1\,, \quad \mathbf{m}\cdot\bar{\mathbf{m}} = 1\,, \tag{4.28}$$

where $\mathbf{k} = -(\Omega \mathrm{d}t - \boldsymbol{b})$ is the dual of ∂_r and

$$\mathbf{l} = -\mathrm{d}r - r\boldsymbol{\alpha} - \frac{r\theta\Omega}{2}\mathrm{d}t + \frac{\boldsymbol{\psi}}{6 * \varpi} + \frac{\Omega \mathrm{d}t - \boldsymbol{b}}{2\rho^2}\left(8\pi G\varepsilon r + c * \varpi - \rho^2\hat{\mathscr{K}}\right), \tag{4.29}$$

(here $\boldsymbol{\psi} = \psi_i \mathrm{d}x^i$), along with

$$2\mathbf{m}\bar{\mathbf{m}} = \rho^2 \mathrm{d}\ell^2 . \tag{4.30}$$

Using the above results and repeating the analysis of App. A.2 in [13], we find that ∂_r is shear-free due to (4.24).

According to the Goldberg–Sachs theorem, the bulk spacetime (4.9) is therefore of Petrov type II, III, D, N or O. The precise type is encoded in the Carrollian tensors $\varepsilon^{\pm}$, $Q_i^{\pm}$ and $\Sigma_{ij}^{\pm}$

$$\begin{aligned} \varepsilon^{\pm} &= \varepsilon \pm \tfrac{\mathrm{i}}{8\pi G} c, \\ Q_i^{\pm} &= Q_i \pm \tfrac{\mathrm{i}}{8\pi G} \chi_i, \\ \Sigma_{ij}^{\pm} &= \Sigma_{ij} \pm \tfrac{\mathrm{i}}{8\pi G} X_{ij}. \end{aligned} \tag{4.31}$$

Working again in holomorphic coordinates, we find the compact result

$$\boldsymbol{Q}^{+} = \frac{\mathrm{i}}{4\pi G} \chi_\zeta \mathrm{d}\zeta, \tag{4.32}$$

$$\boldsymbol{\Sigma}^{+} = \frac{\mathrm{i}}{4\pi G} X_{\zeta\zeta} \mathrm{d}\zeta^2, \tag{4.33}$$

and their complex-conjugates $\boldsymbol{Q}^{-}$ and $\boldsymbol{\Sigma}^{-}$. These Carrollian geometric tensors encompass the information on the canonical complex functions describing the Weyl-tensor decomposition in terms of principal null directions – usually referred to as $\Psi_a, a = 0, \ldots, 4$.

5 Examples

There is a plethora of Carrollian fluids that can be studied. We will analyze here the class of *perfect conformal fluids*, and will complete the discussion of Sec. 3.2 on the *Carrollian Robinson–Trautman fluid*. In each case, assuming the integrability conditions (4.21), (4.22) and (4.23) are fulfilled and the hydrodynamic equations (4.25) and (4.26) are obeyed, a Ricci-flat spacetime is reconstructed from the Carrollian spatial boundary $\mathscr{S}$ at $\mathscr{I}^+$. More examples exist like the Plebański–Demiański or the Weyl axisymmetric solutions, assuming extra symmetries (but not necessarily stationarity) for a viscous Carrollian fluid. These would require a more involved presentation.

5.1 Stationary Carrollian perfect fluids and Ricci-flat Kerr–Taub–NUT families

We would like to illustrate our findings and reconstruct from purely Carrollian fluid dynamics the family of Kerr–Taub–NUT stationary Ricci-flat black holes. We pick for that the following geometric data: $a_{ij}(\mathbf{x})$, $b_i(\mathbf{x})$ and $\Omega = 1$. Stationarity is implemented in these fluids by requiring that all the quantities involved are time independent.

Under this assumption, the Carrollian shear ξ_{ij} vanishes together with the Carrollian expansion θ, whereas constant Ω makes the Carrollian acceleration φ_i vanish as well (Eq. (3.10)). Consequently

$$\hat{\mathscr{A}} = 0, \quad \hat{\mathscr{R}}_i = 0, \tag{5.1}$$

and we are left with non-trivial curvature and vorticity:

$$\hat{\mathscr{K}} = \hat{K} = K, \quad \varpi_{ij} = \partial_{[i}b_{j]} = \eta_{ij} * \varpi. \tag{5.2}$$

The Weyl–Carroll spatial covariant derivative $\hat{\mathscr{D}}_i$ reduces to the ordinary covariant derivative ∇_i, whereas the action of the Weyl–Carroll temporal covariant derivative $\hat{\mathscr{D}}_t$ vanishes.

We further assume that the Carrollian fluid is perfect: Q_i, π_i, Σ_{ij} and Ξ_{ij} vanish. This assumption is made according to the pattern of Ref. [10], where the asymptotically AdS Kerr–Taub–NUT spacetimes were studied starting from relativistic perfect fluids. Due to the duality relationships (4.21), (4.22) and (4.23) among the friction tensors of the Carrollian fluid and the geometric quantities χ_i, X_{ij} and Ψ_{ij}, the latter must also vanish. Using (4.14), (4.16) and (4.17), this sets the following simple geometric constraints:

$$\chi_i = 0 \Leftrightarrow \partial_i K = 0, \tag{5.3}$$

and

$$\Psi_{ij} = 0 \Leftrightarrow \left(\nabla_i\nabla_j - \frac{1}{2}a_{ij}\nabla_l\nabla^l\right) * \varpi = 0, \tag{5.4}$$

whereas X_{ij} vanishes identically without bringing any further restriction. These are equations for the metric $a_{ij}(\mathbf{x})$ and the scalar vorticity $*\varpi$, from which we can extract $b_i(\mathbf{x})$. Using (4.13), we also learn that

$$c = (\Delta + 2K) * \varpi, \tag{5.5}$$

where $\Delta = \nabla_l\nabla^l$ is the ordinary Laplacian operator on $\mathscr{S}$. The last piece of the geometrical data, (4.15), it is non-vanishing and reads:

$$\psi_j = 3\eta^l{}_j\partial_l * \varpi^2. \tag{5.6}$$

Finally, we must impose the fluid equations (4.25) and (4.26), leading to

$$\partial_t\varepsilon = 0, \quad \partial_i\varepsilon = 0. \tag{5.7}$$

The energy density ε of the Carrollian fluid is therefore a constant, which will be identified to the bulk mass parameter $M = 4\pi G\varepsilon$.

Every stationary Carrollian geometry encoded in $a_{ij}(\mathbf{x})$ and $b_i(\mathbf{x})$ with constant scalar curvature K hosts a conformal Carrollian perfect fluid with constant energy density, and is associated with the following exact Ricci-flat spacetime:

$$\mathrm{d}s^2_{\text{perf. fl.}} = -2\left(\mathrm{d}t - \boldsymbol{b}\right)\mathrm{d}r + \frac{2Mr + c * \varpi - K\rho^2}{\rho^2}\left(\mathrm{d}t - \boldsymbol{b}\right)^2 + \left(\mathrm{d}t - \boldsymbol{b}\right)\frac{\boldsymbol{\psi}}{3 * \varpi} + \rho^2 \mathrm{d}\ell^2, \quad (5.8)$$

where $\rho^2 = r^2 + *\varpi^2$. The vorticity $*\varpi$ is determined by Eq. (5.4), solved on a constant-curvature background.

Using holomorphic coordinates (see App. A), a constant-curvature metric on $\mathscr{S}$ reads:

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta} \quad (5.9)$$

with

$$P = 1 + \frac{K}{2}\zeta\bar{\zeta}, \quad K = 0, \pm 1, \quad (5.10)$$

corresponding to S^2 and E_2 or H_2 (sphere and Euclidean or hyperbolic planes). Using these expressions we can integrate (5.4). The general solution depends on three real, arbitrary parameters, n, a and ℓ:

$$* \varpi = n + a - \frac{2a}{P} + \frac{\ell}{P}\left(1 - |K|\right)\zeta\bar{\zeta}. \quad (5.11)$$

The parameter ℓ is relevant in the flat case exclusively. We can further integrate (3.11) and find thus

$$\boldsymbol{b} = \frac{\mathrm{i}}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\left(1 - |K|\right)\zeta\bar{\zeta}\right)\left(\bar{\zeta}\mathrm{d}\zeta - \zeta \mathrm{d}\bar{\zeta}\right). \quad (5.12)$$

It is straightforward to determine the last pieces entering the bulk resumed metric (5.8):

$$c = 2Kn + 2\ell\left(1 - |K|\right) \quad (5.13)$$

and

$$\frac{\boldsymbol{\psi}}{3 * \varpi} = 2\eta^j{}_i \partial_j * \varpi \mathrm{d}x^i = 2\mathrm{i}\frac{Ka + \ell\left(1 - |K|\right)}{P^2}\left(\bar{\zeta}\mathrm{d}\zeta - \zeta \mathrm{d}\bar{\zeta}\right). \quad (5.14)$$

In order to reach a more familiar form for the line element (5.8), it is convenient to trade the complex-conjugate coordinates ζ and $\bar{\zeta}$ for their modulus[25] and argument

$$\zeta = Z e^{i\Phi}, \quad (5.15)$$

and move from Eddington–Finkelstein to Boyer–Lindquist by setting

$$\mathrm{d}t \to \mathrm{d}t - \frac{r^2 + (n-a)^2}{\Delta_r}\mathrm{d}r, \quad \mathrm{d}\Phi \to \mathrm{d}\Phi - \frac{Ka + \ell(1 - |K|)}{\Delta_r}\mathrm{d}r \quad (5.16)$$

[25] The modulus and its range depend on the curvature. It is commonly expressed as: $Z = \sqrt{2}\tan\frac{\Theta}{2}, 0 < \Theta < \pi$ for S^2; $Z = \frac{R}{\sqrt{2}}, 0 < R < +\infty$ for E_2; $Z = \sqrt{2}\tanh\frac{\Psi}{2}, 0 < \Psi < +\infty$ for H_2.

with

$$\Delta_r = -2Mr + K\left(r^2 + a^2 - n^2\right) + 2\ell(n-a)(|K|-1). \tag{5.17}$$

We obtain finally:

$$\begin{aligned} \mathrm{d}s^2_{\text{perf. fl.}} \quad = \quad & -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \frac{2}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\left(1-|K|\right)Z^2\right)Z^2\mathrm{d}\Phi\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 \\ & + \frac{2\rho^2}{P^2}\mathrm{d}Z^2 + \frac{2Z^2}{\rho^2 P^2}\left(\left(Ka + \ell\left(1-|K|\right)\right)\mathrm{d}t - \left(r^2 + (n-a)^2\right)\mathrm{d}\Phi\right)^2 \end{aligned} \tag{5.18}$$

with

$$P = 1 + \frac{K}{2}Z^2, \quad \rho^2 = r^2 + \left(n + a - \frac{2a}{P} + \frac{\ell}{P}\left(1-|K|\right)Z^2\right)^2. \tag{5.19}$$

This bulk metric is Ricci-flat for any value of the parameters M, n, a and ℓ with $K = 0, \pm 1$. For vanishing n, a and ℓ, and with $M > 0$ and $K = 1$, one recovers the standard asymptotically flat Schwarzschild solution with spherical horizon. For $K = 0$ or -1, this is no longer Schwarzschild, but rather a metric belonging to the A class (see e.g. [83]). The parameter a switches on rotation, while n is the standard nut charge. The parameter ℓ is also a rotational parameter available only in the flat-$\mathscr{S}$ case. Scanning over all these parameters, in combination with the mass and K, we recover the whole Kerr–Taub–NUT family of black holes, plus other, less familiar configurations, like the A-metric quoted above.

For the solutions at hand, the only potentially non-vanishing Carrollian boundary Cotton descendants are c and $\boldsymbol{\psi}$, displayed in (5.13) and (5.14). The first is non-vanishing for asymptotically locally flat spacetimes, and this requires non-zero n or ℓ. The second measures the bulk twist. In every case the metric (5.18) is Petrov type D.

We would like to conclude the example of Carrollian conformal perfect fluids with a comment regarding the isometries of the associated resummed Ricci-flat spacetimes with line element (5.18). For vanishing a and ℓ, there are *four* isometry generators and the field is in this case a stationary gravito-electric and/or gravito-magnetic monopole (mass and nut parameters M, n). Constant-r hypersurfaces are homogeneous spaces in this case. The number of Killing fields is reduced to *two* (∂_t and ∂_Φ) whenever any of the rotational parameters a or ℓ is non-zero. These parameters make the gravitational field dipolar.

The bulk isometries are generally inherited from the boundary symmetries, *i.e.* the symmetries of the Carrollian geometry and the Carrollian fluid. The time-like Killing field ∂_t is clearly rooted to the stationarity of the boundary data. The space-like ones have legs on ∂_Φ and ∂_Z, and are associated to further boundary symmetries. From a Riemannian viewpoint, the metric (5.9) with (5.10) on the two-dimensional boundary surface $\mathscr{S}$ admits three Killing vector fields:

$$\begin{aligned}
\mathbf{X}_1 &= \mathrm{i}\left(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}\right), && (5.20)\\
\mathbf{X}_2 &= \mathrm{i}\left(\left(1-\frac{K}{2}\zeta^2\right)\partial_\zeta - \left(1-\frac{K}{2}\bar{\zeta}^2\right)\partial_{\bar{\zeta}}\right), && (5.21)\\
\mathbf{X}_3 &= \left(1+\frac{K}{2}\zeta^2\right)\partial_\zeta + \left(1+\frac{K}{2}\bar{\zeta}^2\right)\partial_{\bar{\zeta}}, && (5.22)
\end{aligned}$$

closing in $\mathfrak{so}(3)$, $\mathfrak{e}_2$ and $\mathfrak{so}(2,1)$ algebras for $K = +1, 0$ and -1 respectively. The Carrollian structure is however richer as it hinges on the set $\{a_{ij}, b_i, \Omega\}$. Hence, not all Riemannian isometries generated by a Killing field $\mathbf{X}$ of $\mathscr{S}$ are necessarily promoted to Carrollian symmetries. For the latter, it is natural to further require the Carrollian vorticity be invariant:

$$\mathscr{L}_{\mathrm{X}} * \varpi = \mathbf{X}(*\varpi) = 0. \tag{5.23}$$

Condition (5.23) is fulfilled for all fields $\mathbf{X}_A$ $(A = 1, 2, 3)$ in (5.20), (5.21) and (5.22), only as long as $a = \ell = 0$, since $*\varpi = n$. Otherwise $*\varpi$ is non-constant and only $\mathbf{X}_1 = \mathrm{i}\left(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}\right) = \partial_\Phi$ leaves it invariant. This is in line with the bulk isometry properties discussed earlier, while it provides a Carrollian-boundary manifestation of the rigidity theorem.

5.2 Vorticity-free Carrollian fluid and the Ricci-flat Robinson–Trautman

The zero-k limit of the relativistic Robinson–Trautman fluid presented in Sec. (3.2) (Eqs. (3.53)–(3.56)) is in agreement with the direct Carrollian approach of Sec. 4.2. Indeed, it is straightforward to check that the general formulas (4.13)–(4.17) give $c = 0$ together with

$$\boldsymbol{\chi} = \frac{\mathrm{i}}{2}\left(\partial_\zeta K \mathrm{d}\zeta - \partial_{\bar{\zeta}} K \mathrm{d}\bar{\zeta}\right), \quad \mathbf{X} = \frac{\mathrm{i}}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 - \partial_{\bar{\zeta}}\left(P^2\partial_t\partial_{\bar{\zeta}} \ln P\right)\mathrm{d}\bar{\zeta}^2\right), \tag{5.24}$$

while $\psi_i = 0 = \Psi_{ij}$. These expressions satisfy (4.18)–(4.20), and the duality relations (4.21), (4.22) and (4.23) lead to the friction components of the energy–momentum tensor Q_i, Σ_{ij} and Ξ_{ij}, precisely as they appear in (3.57), (3.58). The general hydrodynamic equations (4.25), (4.26), are solved with[26] $\pi_i = 0$ and $\varepsilon = \varepsilon(t)$ satisfying (3.59), *i.e.* Robinson–Trautman's (3.62).

Our goal is to present here the resummation of the derivative expansion (4.9) into a Ricci-flat spacetime dual to the fluid at hand. The basic feature of the latter is that $b_i = 0$ and $\Omega = 1$, hence it is vorticity-free – on top of being shearless. With these data, using (4.9), we find

$$\mathrm{d}s^2_{\mathrm{RT}} = -2\mathrm{d}t\left(\mathrm{d}r + H\mathrm{d}t\right) + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \tag{5.25}$$

[26] Since π_i is not related to the geometry by duality as the other friction and heat tensors, it can *a priori* assume any value. It is part of the Carrollian Robinson–Trautman fluid definition to set it to zero.

where

$$2H = -2r\partial_t \ln P + K - \frac{2M(t)}{r}, \tag{5.26}$$

with $K = 2P^2\partial_{\bar\zeta}\partial_\zeta \ln P$ the Gaussian curvature of (3.53). This metric is Ricci-flat provided the energy density $\varepsilon(t) = M(t)/4\pi G$ and the function $P = P(t,\zeta,\bar\zeta)$ satisfy (3.62). These are algebraically special spacetimes of all types, as opposed to the Kerr–Taub–NUT family studied earlier (Schwarzschild solution is common to these two families). Furthermore they never have twist ($\boldsymbol{\psi} = \boldsymbol{\Psi} = 0$) and are generically asymptotically locally but not globally flat due to $\boldsymbol{\chi}$ and $\mathbf{X}$.

The specific Petrov type of Robinson–Trautman solutions is determined by analyzing the tensors (4.31), or (4.32) and (4.33) in holomorphic coordinates:

$$\varepsilon^+ = \frac{M(t)}{4\pi G}, \quad \mathbf{Q}^+ = -\frac{1}{8\pi G}\partial_\zeta K \mathrm{d}\zeta, \quad \boldsymbol{\Sigma}^+ = -\frac{1}{4\pi G P^2}\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2. \tag{5.27}$$

We find the following classification (see [12]):

II generic;

III with $\varepsilon^+ = 0$ and $\nabla_i Q^{+i} = 0$;

N with $\varepsilon^+ = 0$ and $Q^+_i = 0$;

D with $2Q^+_i Q^+_j = 3\varepsilon^+\Sigma^+_{ij}$ and vanishing traceless part of $\nabla_{(i}Q^+_{j)}$.

6 Conclusions

The main message of our work is that starting with the standard AdS holography, there is a well-defined zero-cosmological-constant limit that relates asymptotically flat spacetimes to Carrollian fluids living on their null boundaries.

In order to unravel this relationship and make it operative for studying holographic duals, we used the derivative expansion. Originally designed for asymptotically anti-de Sitter spacetimes with cosmological constant $\Lambda = -3k^2$, this expansion provides their line element in terms of the conformal boundary data: a pseudo-Riemannian metric and a relativistic fluid. It is expressed in Eddington–Finkelstein coordinates, where the zero-k limit is unambiguous: it maps the pseudo-Riemannian boundary $\mathscr{I}$ onto a Carrollian geometry $\mathbb{R} \times \mathscr{S}$, and the conformal relativistic fluid becomes Carrollian.

The emergence of the conformal Carrollian symmetry in the Ricci-flat asymptotic is not a surprise, as we have extensively discussed in the introduction. In particular, the BMS group has been used for investigating the asymptotically flat dual dynamics. What is remarkable is the efficiency of the derivative expansion to implement the limiting procedure and deliver a genuine holographic relationship between Ricci-flat spacetimes and conformal Carrollian fluids. These are defined on $\mathscr{S}$ but their dynamics is rooted in $\mathbb{R} \times \mathscr{S}$.

Even though proving that the derivative expansion is unconditionally well-behaved in the limit under consideration is still part of our agenda, we have demonstrated this property in the instance where it is resummable.

The resummability of the derivative expansion has been studied in our earlier works about anti-de Sitter fluid/gravity correspondence. It has two features:

- the shear of the fluid congruence vanishes;
- the heat current and the viscous stress tensor are determined from the Cotton current and stress tensor components via a transverse (with respect to the velocity) duality.

The first considerably simplifies the expansion. Together with the second, it ultimately dictates the structure of the bulk Weyl tensor, making the Einstein spacetime of special Petrov type. The conservation of the energy–momentum tensor is the only requirement left for the bulk be Einstein. It involves the energy density (*i.e.* the only fluid observable left undetermined) and various geometric data in the form of partial differential equations (as is the Robinson–Trautman for the vorticity-free situation).

This pattern survives the zero-k limit, taken in a frame where the relativistic fluid is at rest. The corresponding Carrollian fluid – at rest *by law* – is required to be shearless, but has otherwise acceleration, vorticity and expansion. Since the fluid is at rest, these are geometric data, as are the descendants of the Cotton tensor used again to formulate the duality that determines the dissipative components of the Carrollian fluid.

The study of the Cotton tensor and its Carrollian limit is central in our analysis. In Carrollian geometry (conformal in the case under consideration) it opens the pandora box of the classification of curvature tensors, which we have marginally discussed here. Our observation is that the Cotton tensor grants the zero-k limiting Carrollian geometry on $\mathscr{S}$ with a scalar, two vectors and two symmetric, traceless tensors, satisfying a set of identities inherited from the original conservation equation.

In a similar fashion, the relativistic energy–momentum tensor descends in a scalar (the energy density), two heat currents and two viscous stress tensors. This doubling is suggested by that of the Cotton. The physics behind it is yet to be discovered, as it requires a microscopic approach to Carrollian fluids, missing at present. Irrespective of its microscopic origin, however, this is an essential result of our work, in contrast with previous attempts. Not only we can state that the fluid holographically dual to a Ricci-flat spacetime is neither relativistic, nor Galilean, but we can also exhibit for the actually Carrollian fluid the fundamental observables and the equations they obey.[27] These are quite convoluted, and

[27] From this perspective, trying to design four-dimensional flat holography using two-dimensional conformal field theory described in terms of a conserved two-dimensional energy–momentum tensor [42–44] looks inappropriate.

whenever satisfied, the resummed metric is Ricci-flat.

Our analysis, amply illustrated by two distinct examples departing from Carrollian hydrodynamics and ending on widely used Ricci-flat spacetimes, raises many questions, which deserve a comprehensive survey.

As already acknowledged, the Cotton Carrollian descendants enter the holographic reconstruction of a Ricci-flat spacetime, along with the energy–momentum data. It would be rewarding to explore the information stored in these objects, which may carry the boundary interpretation of the Bondi news tensor as well as of the asymptotic charges one can extract from the latter.

We should stress at this point that Cotton and energy–momentum data (and the charges they transport) play dual rôles. The nut and the mass provide the best paradigm of this statement. Altogether they raise the question on the thermodynamic interpretation of magnetic charges. Although we cannot propose a definite answer to this question, the tools of fluid/gravity holography (either AdS or flat) may turn helpful. This is tangible in the case of algebraically special Einstein solutions, where the underlying integrability conditions set a deep relationship between geometry and energy–momentum *i.e.* between geometry and local thermodynamics. To make this statement more concrete, observe the heat current as constructed using the integrability conditions, Eq. (4.21):

$$Q_i = -\frac{1}{16\pi G}\left(\hat{\mathscr{D}}_i\hat{\mathscr{K}} - \eta^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}} + 4 * \varpi\eta^j{}_i\hat{\mathscr{R}}_j\right).$$

In the absence of magnetic charges, only the first term is present and it is tempting to set a relationship between the temperature and the gravito-electric curvature scalar $\hat{\mathscr{K}}$. This was precisely discussed in the AdS framework when studying the Robinson–Trautman relativistic fluid, in Ref. [66]. Magnetic charges switch on the other terms, exhibiting natural thermodynamic potentials, again related with curvature components ($\hat{\mathscr{A}}$ and $\hat{\mathscr{R}}_j$).

We would like to conclude with a remark. On the one hand, we have shown that the boundary fluids holographically dual to Ricci-flat spacetimes are of Carrollian nature. On the other hand, the stretched horizon in the membrane paradigm seems to be rather described in terms of Galilean hydrodynamics [17,18,84]. Whether and how these two pictures could been related is certainly worth refining.

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A Carrollian boundary geometry in holomorphic coordinates

Using Carrollian diffeomorphisms (3.2), the metric (3.1) of the Carrollian geometry on the two-dimensional surface $\mathscr{S}$ can be recast in conformally flat form,

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \tag{A.1}$$

with $P = P(t,\zeta,\bar{\zeta})$ a real function, under the necessary and sufficient condition that the Carrollian shear ξ_{ij} displayed in (3.14) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data: $\Omega(t,\zeta,\bar{\zeta})$ and

$$\boldsymbol{b} = b_\zeta(t,\zeta,\bar{\zeta})\,\mathrm{d}\zeta + b_{\bar{\zeta}}(t,\zeta,\bar{\zeta})\,\mathrm{d}\bar{\zeta} \tag{A.2}$$

with $b_{\bar{\zeta}}(t,\zeta,\bar{\zeta}) = \bar{b}_\zeta(t,\zeta,\bar{\zeta})$. Our choice of orientation is inherited from the one adopted for the relativistic boundary (see footnote 13) with $a_{\zeta\bar{\zeta}} = 1/P^2$ is[28]

$$\eta_{\zeta\bar{\zeta}} = -\frac{\mathrm{i}}{P^2}. \tag{A.3}$$

The first-derivative Carrollian tensors are the acceleration (3.10), the expansion (3.14) and the scalar vorticity (3.20):

$$\varphi_\zeta = \partial_t \frac{b_\zeta}{\Omega} + \hat{\partial}_\zeta \ln\Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln\Omega, \tag{A.4}$$

$$\theta = -\frac{2}{\Omega}\partial_t \ln P, \quad *\varpi = \frac{\mathrm{i}\Omega P^2}{2}\left(\hat{\partial}_\zeta \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}}\frac{b_\zeta}{\Omega}\right) \tag{A.5}$$

with

$$\hat{\partial}_\zeta = \partial_\zeta + \frac{b_\zeta}{\Omega}\partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega}\partial_t. \tag{A.6}$$

[28]This amounts to setting $\sqrt{a} = \mathrm{i}/P^2$ in coordinate frame and $\epsilon_{\zeta\bar{\zeta}} = -1$.

Curvature scalars and vector are second-derivative (see (3.19), (3.22)):[29]

$$\hat{K} = P^2\left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta} + \hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}}\right)\ln P, \quad \hat{A} = \mathrm{i}P^2\left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta} - \hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}}\right)\ln P, \tag{A.7}$$

$$\hat{r}_{\zeta} = \frac{1}{2}\hat{\partial}_{\zeta}\left(\frac{1}{\Omega}\partial_t \ln P\right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2}\hat{\partial}_{\bar{\zeta}}\left(\frac{1}{\Omega}\partial_t \ln P\right), \tag{A.8}$$

and we also quote:

$$*\varphi = \mathrm{i}P^2\left(\hat{\partial}_{\zeta}\varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}}\varphi_{\zeta}\right), \tag{A.9}$$

$$\hat{\nabla}_k\varphi^k = P^2\left[\hat{\partial}_{\zeta}\partial_t\tfrac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}}\partial_t\tfrac{b_{\zeta}}{\Omega} + \left(\hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta}\right)\ln\Omega\right]. \tag{A.10}$$

Regarding conformal Carrollian tensors we remind the weight-2 curvature scalars (3.40):

$$\hat{\mathscr{K}} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathscr{A}} = \hat{A} - *\varphi, \tag{A.11}$$

and the weight-1 curvature one-form (3.38):

$$\hat{\mathscr{R}}_{\zeta} = \frac{1}{\Omega}\partial_t\varphi_{\zeta} - \frac{1}{2}\left(\hat{\partial}_{\zeta} + \varphi_{\zeta}\right)\theta, \quad \hat{\mathscr{R}}_{\bar{\zeta}} = \frac{1}{\Omega}\partial_t\varphi_{\bar{\zeta}} - \frac{1}{2}\left(\hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}}\right)\theta. \tag{A.12}$$

The three-derivative Cotton descendants displayed in (4.13)–(4.17) are a scalar

$$c = \left(\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi \tag{A.13}$$

of weight 3 ($*\varpi$ is of weght 1), two vectors

$$\chi_{\zeta} = \tfrac{\mathrm{i}}{2}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{K}} + \tfrac{1}{2}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{A}} - 2*\varpi\hat{\mathscr{R}}_{\zeta}, \quad \chi_{\bar{\zeta}} = -\tfrac{\mathrm{i}}{2}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{K}} + \tfrac{1}{2}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{A}} - 2*\varpi\hat{\mathscr{R}}_{\bar{\zeta}}, \tag{A.14}$$

$$\psi_{\zeta} = 3\mathrm{i}\hat{\mathscr{D}}_{\zeta} * \varpi^2, \quad \psi_{\bar{\zeta}} = -3\mathrm{i}\hat{\mathscr{D}}_{\bar{\zeta}} * \varpi^2, \tag{A.15}$$

of weight 2, and two symmetric and traceless tensors

$$X_{\zeta\zeta} = \mathrm{i}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{R}}_{\zeta}, \quad X_{\bar{\zeta}\bar{\zeta}} = -\mathrm{i}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{R}}_{\bar{\zeta}}, \tag{A.16}$$

$$\Psi_{\zeta\zeta} = \hat{\mathscr{D}}_{\zeta}\hat{\mathscr{D}}_{\zeta} * \varpi, \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{D}}_{\bar{\zeta}} * \varpi, \tag{A.17}$$

of weight 1. Notice that in holomorphic coordinates a symmetric and traceless tensor S_{ij} has only diagonal entries: $S_{\zeta\bar{\zeta}} = 0 = S_{\bar{\zeta}\zeta}$.

We also remind for convenience some expressions for the determination of Weyl–Carroll

[29]We also quote for completeness (useful *e.g.* in Eq. (A.11)):

$$\hat{K} = K + P^2\left[\partial_{\zeta}\frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}}\frac{b_{\zeta}}{\Omega} + \partial_t\frac{b_{\zeta}b_{\bar{\zeta}}}{\Omega^2} + 2\frac{b_{\bar{\zeta}}}{\Omega}\partial_{\zeta} + 2\frac{b_{\zeta}}{\Omega}\partial_{\bar{\zeta}} + 2\frac{b_{\zeta}b_{\bar{\zeta}}}{\Omega^2}\partial_t\right]\partial_t \ln P$$

with $K = 2P^2\partial_{\bar{\zeta}}\partial_{\zeta}\ln P$ the ordinary Gaussian curvature of the two-dimensional metric (A.1).

covariant derivatives. If Φ is a weight-w scalar function

$$\hat{\mathscr{D}}_\zeta\Phi = \hat{\partial}_\zeta\Phi + w\varphi_\zeta\Phi, \quad \hat{\mathscr{D}}_{\bar{\zeta}}\Phi = \hat{\partial}_{\bar{\zeta}}\Phi + w\varphi_{\bar{\zeta}}\Phi. \tag{A.18}$$

For weight-w form components V_ζ and $V_{\bar{\zeta}}$ the Weyl–Carroll derivatives read:

$$\hat{\mathscr{D}}_\zeta V_\zeta = \hat{\nabla}_\zeta V_\zeta + (w+2)\varphi_\zeta V_\zeta, \quad \hat{\mathscr{D}}_{\bar{\zeta}} V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} + (w+2)\varphi_{\bar{\zeta}} V_{\bar{\zeta}}, \tag{A.19}$$

$$\hat{\mathscr{D}}_\zeta V_{\bar{\zeta}} = \hat{\nabla}_\zeta V_{\bar{\zeta}} + w\varphi_\zeta V_{\bar{\zeta}}, \quad \hat{\mathscr{D}}_{\bar{\zeta}} V_\zeta = \hat{\nabla}_{\bar{\zeta}} V_\zeta + w\varphi_{\bar{\zeta}} V_\zeta, \tag{A.20}$$

while the Carrollian covariant derivatives are simply:

$$\hat{\nabla}_\zeta V_\zeta = \frac{1}{P^2}\hat{\partial}_\zeta\left(P^2 V_\zeta\right), \quad \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} = \frac{1}{P^2}\hat{\partial}_{\bar{\zeta}}\left(P^2 V_{\bar{\zeta}}\right), \tag{A.21}$$

$$\hat{\nabla}_\zeta V_{\bar{\zeta}} = \hat{\partial}_\zeta V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}} V_\zeta = \hat{\partial}_{\bar{\zeta}} V_\zeta. \tag{A.22}$$

Finally,

$$\hat{\mathscr{D}}_k\hat{\mathscr{D}}^k\Phi = P^2\left(\hat{\partial}_\zeta\hat{\partial}_{\bar{\zeta}}\Phi + \hat{\partial}_{\bar{\zeta}}\hat{\partial}_\zeta\Phi + w\Phi\left(\hat{\partial}_\zeta\varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}}\varphi_\zeta\right) + 2w\left(\varphi_\zeta\hat{\partial}_{\bar{\zeta}}\Phi + \varphi_{\bar{\zeta}}\hat{\partial}_\zeta\Phi + w\varphi_\zeta\varphi_{\bar{\zeta}}\Phi\right)\right). \tag{A.23}$$

References

[1] C. Fefferman and C. R. Graham, *Conformal invariants*, in *Elie Cartan et les mathématiques d'aujourd'hui*, Astérisque, 1985, numéro hors série Soc. Math. France, Paris, 95.

[2] C. Fefferman and C. R. Graham, *The ambient metric*, arXiv:0710.0919 [math.DG].

[3] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear fluid dynamics from gravity*, JHEP **0802** (2008) 045, arXiv:0712.2456 [hep-th].

[4] V. E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, arXiv:1107.5780 [hep-th].

[5] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*, JHEP **0810** (2008) 063, arXiv:0806.4602 [hep-th].

[6] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, *Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions*, JHEP **0812** (2008) 116, arXiv:0809.4272 [hep-th].

[7] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, J. Phys. **A45** (2012) 473001, arXiv:1205.5040 [hep-th].

[8] P. Romatschke, *New developments in relativistic viscous hydrodynamics*, Int. J. Mod. Phys. **E19** (2010) 1, arXiv:0902.3663 [hep-th].

[9] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli and K. Siampos, *Vorticity in holographic fluids*, Proc. of Science **CORFU11** (2012) 076, arXiv:1206.4351 [hep-th].

[10] A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli and K. Siampos, *Holographic perfect fluidity, Cotton energy–momentum duality and transport properties*, JHEP **1404** (2014) 136, arXiv:1309.2310 [hep-th].

[11] P. M. Petropoulos, *Gravitational duality, topologically massive gravity and holographic fluids*, Lect. Notes Phys. **892** (2015) 331, arXiv:1406.2328 [hep-th].

[12] J. Gath, A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Petrov Classification and holographic reconstruction of spacetime*, JHEP **1509** (2015) 005, arXiv:1506.04813 [hep-th].

[13] P. M. Petropoulos and K. Siampos, *Integrability, Einstein spaces and holographic fluids*, proceedings of the workshop About various kinds of interactions in honour of the 65th birthday of Professor Philippe Spindel, N. Boulanger and S. Detournay, eds., Mons 2017, arXiv:1510.06456 [hep-th].

[14] A. C. Petkou, P. M. Petropoulos and K. Siampos, *Geroch group for Einstein spaces and holographic integrability*, Proc. of Science **PLANCK 2015** (2015) 104, arXiv:1512.04970 [hep-th].

[15] I. Antoniadis, J.-P. Derendinger, P. M. Petropoulos and K. Siampos, *Isometries, gaugings and $\mathcal{N} = 2$ supergravity decoupling*, JHEP **1611** (2016) 169, arXiv:1611.00964 [hep-th].

[16] S. Alexandrov, S. Banerjee and P. Longhi, *Rigid limit for hypermultiplets and five-dimensional gauge theories*, JHEP **1801** (2018) 156, arXiv:1710.10665 [hep-th].

[17] T. Damour, *Black-hole eddy currents*, Phys. Rev. **D18** (1978) 3598.

[18] T. Damour, *Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs*, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris VI, 1979.

[19] S. de Haro, K. Skenderis and S. N. Solodukhin, *Gravity in warped compactifications and the holographic stress tensor*, Class. Quant. Grav. **18** (2001) 3171, arXiv:0011230 [hep-th].

[20] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, *From Navier–Stokes to Einstein*, JHEP **1207** (2012) 146, arXiv:1101.2451 [hep-th].

[21] G. Compère, P. McFadden, K. Skenderis and M. Taylor, *The holographic fluid dual to vacuum Einstein gravity*, JHEP **1107** (2011) 050, arXiv:1103.3022 [hep-th].

[22] G. Compère, P. McFadden, K. Skenderis and M. Taylor, *The relativistic fluid dual to vacuum Einstein gravity*, JHEP **1203** (2012) 076, arXiv:1201.2678 [hep-th].

[23] M. Caldarelli, J. Camps, B. Goutéraux and K. Skenderis, *AdS/Ricci-flat correspondence*, JHEP **1404** (2014) 071, arXiv:1312.7874 [hep-th].

[24] N. Pinzani-Fokeeva and M. Taylor, *Towards a general fluid/gravity correspondence*, Phys. Rev. **D91** (2015) 044001, arXiv:1401.5975 [hep-th].

[25] C. Eling, A. Meyer and Y. Oz, *The relativistic Rindler hydrodynamics*, JHEP **1205** (2012) 116, arXiv:1201.2705 [hep-th].

[26] G. Arcioni and C. Dappiaggi, *Exploring the holographic principle in asymptotically flat spacetimes via the BMS group*, Nucl. Phys. **B674** (2003) 553, arXiv:0306142 [hep-th].

[27] G. Arcioni and C. Dappiaggi, *Holography in asymptotically flat spacetimes and the BMS group*, Class. Quant. Grav. **21** (2003) 5655, arXiv:0312186 [hep-th].

[28] C. Dappiaggi, V. Moretti and N. Pinamonti, *Rigorous steps towards holography in asymptotically flat spacetimes*, Rev. Math. Phy. **18**, 349 (2006), arXiv:0506069 [gr-qc].

[29] J. de Boer and S. N. Solodukhin, *A holographic reduction of Minkowski spacetime*, Nucl. Phys. **B665** (2003) 545-593, arXiv:0303006 [hep-th].

[30] E. T. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962) 566-578.

[31] T. M. Adamo, E. T. Newman and C. N. Kozameh, *Null geodesic congruences, asymptotically flat spacetimes and their physical interpretation*, Living Rev. Relativity **15:1** (2012), arXiv:0906.2155 [gr-qc].

[32] T. Mädler and J. Winicour, *Bondi-Sachs formalism*, Scholarpedia **11** (2016) 33528, arXiv:1609.01731 [gr-qc].

[33] A. Bagchi and R. Gopakumar, *Galilean conformal algebras and AdS/CFT*, JHEP **0907** (2009) 037, arXiv:0902.1385 [hep-th].

[34] A. Bagchi, *The BMS/GCA correspondence*, Phys. Rev. Lett. **105** (2010) 171601, arXiv:1006.3354 [hep-th].

[35] A. Bagchi and R. Fareghbal, *BMS/GCA redux: towards flat-space holography from non-relativistic symmetries*, JHEP **1210** (2012) 092, arXiv:1203.5795 [hep-th].

[36] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence,* JHEP **1005** (2010) 062, arXiv:1001.1541 [hep-th].

[37] G. Barnich, A. Gomberoff and H. A. Gonzalez, *The flat limit of three-dimensional asymptotically anti-de Sitter spacetimes*, Phys. Rev. **D86** (2012) 024020, arXiv:1204.3288 [gr-qc].

[38] A. Bagchi, R. Basu, D. Grumiller and M. Riegler, *Entanglement entropy in Galilean conformal field theories and flat holography*, Phys. Rev. Lett. **114** (2015), 111602, arXiv:1410.4089 [hep-th].

[39] J. Hartong, *Holographic reconstruction of 3D flat spacetime*, JHEP **1610** (2016) 104, arXiv:1511.01387 [hep-th].

[40] K. Jensen and A. Karch, *Revisiting non-relativistic limits*, JHEP **1504** (2015) 155, arXiv:1412.2738 [hep-th].

[41] O. Baghchesaraei, R. Fareghbal and Y. Izadi, *Flat-space holography and stress tensor of Kerr black hole*, Phys. Lett. **B760** (2016) 713, arXiv:1603.04137 [hep-th].

[42] T. He, P. Mitra and A. Strominger, *2D Kac–Moody symmetry of 4D Yang–Mills theory*, JHEP **1610** (2016) 137, arXiv:1503.02663 [hep-th].

[43] D. Kapec, P. Mitra, A.-M. Raclariu and A. Strominger, *A 2D stress tensor for 4D gravity*, Phys. Rev. Lett. **119** (2017) 121601, arXiv:1609.00282 [hep-th].

[44] S. Pasterski, S.-H. Shao and A. Strominger, *Flat space amplitudes and conformal symmetry of the celestial sphere*, Phys. Rev. **D96** (2017) 065026, arXiv:1701.00049 [hep-th].

[45] D. Kapec and P. Mitra, *A d-dimensional stress tensor for* $Mink_{d+2}$ *gravity*, arXiv:1711.04371 [hep-th].

[46] R. Fareghbal and I. Mohammadi, *Flat-space holography and correlators of Robinson–Trautman stress tensor*, arXiv:1802.05445 [hep-th].

[47] J.-M. Lévy-Leblond, *Une nouvelle limite non-relativiste du groupe de Poincaré*, A. Inst. Henri Poincaré **III** (1965) 1.

[48] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,* Class. Quant. Grav. **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[49] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll groups and BMS symmetry,* Class. Quant. Grav. **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[50] C. Duval and P. A. Horvathy, *Non-relativistic conformal symmetries and Newton–Cartan structures,* J. Phys. **A42** (2009) 465206, arXiv:0904.0531 [math-ph].

[51] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll group*, J. Phys. **A47** (2014) 335204, arXiv:1403.4213 [hep-th].

[52] L. Ciambelli, C. Marteau, A.C. Petkou, P. M. Petropoulos and K. Siampos, *Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids*, arXiv:1802.05286 [hep- -th].

[53] H. Bondi, M. G. van der Burg, and A. W. Metzner, *Gravitational waves in general relativity, VII. Waves from axi-symmetric isolated systems*, Proc. Roy. Soc. Lond. **A269** (1962) 21.

[54] R. K. Sachs, *Asymptotic symmetries in gravitational theories*, Phys. Rev. **128** (1962) 2851-2864.

[55] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat 4-dimensional spacetimes at null infinity revisited*, Phys. Rev. Lett. **105** (2010) 111103, arXiv:0909.2617 [gr-qc].

[56] A. Ashtekar, *Geometry and physics of null infinity*, in *One hundred years of general relativity*, edited by L. Bieri and S. T. Yau (International press, Boston, 2015) 99, arXiv:1409.1800 [gr-qc].

[57] A. Bagchi, S. Chakrabortty and P. Parekh, *Tensionless strings from worldsheet symmetries*, JHEP **1601** (2016) 158, arXiv:1507.04361 [hep-th].

[58] B. Cardona, J. Gomis and J. M. Pons, *Dynamics of Carroll strings*, JHEP **1607** (2016) 050, arXiv:1605.05483 [hep-th].

[59] R. F. Penna, *BMS invariance and the membrane paradigm*, JHEP **1603** (2016) 023, arXiv:1508.06577 [hep-th].

[60] R. F. Penna, *Near-horizon BMS symmetries as fluid symmetries*, JHEP **1710** (2017) 049, arXiv:1703.07382 [hep-th].

[61] M. T. Anderson, *Geometric aspects of the AdS/CFT correspondence*, IRMA Lect. Math. Theor. Phys. **8** (2005) 1, arXiv:0403087 [hep-th].

[62] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS space-times*, JHEP **0508** (2005) 004, arXiv:0505190 [hep-th].

[63] S. Fischetti, W. Kelly and D. Marolf, *Conserved charges in asymptotically (locally) AdS spacetimes*, in Springer *Handbook of spacetime*, A. Ashtekar and V. Petkov, eds. (2014) 381, arXiv:1211.6347 [gr-qc].

[64] A. Ashtekar and S. Das, *Asymptotically anti-de Sitter spacetimes: conserved quantities*, Class. Quant. Grav. **17** (2000) L.17-L.30, arXiv:9911230 [hep-th].

[65] L. D. Landau et E. M. Lifchitz, *Physique Théorique*, Vol. 6 *Mécanique des fluides*, MIR 1969.

[66] L. Ciambelli, A. C. Petkou, P. M. Petropoulos and K. Siampos, *The Robinson–Trautman spacetime and its holographic fluid*, PoS **CORFU2016** (2016) 076, arXiv:1707.02995 [hep--th].

[67] M. Humbert, *Holographic reconstruction in higher dimension*, internship report, Ecole normale supérieure, Ecole polytechnique, 2017.

[68] B. Coll, J. Llosa and D. Soler, *Three-dimensional metrics as deformations of a constant curvature metric*, Gen. Rel. Grav. **34** (2002) 269, arXiv:0104070 [gr-qc].

[69] D. S. Mansi, A. C. Petkou and G. Tagliabue, *Gravity in the* $3+1$*-split formalism I: holography as an initial value problem*, Class. Quant. Grav. **26** (2009) 045008, arXiv:0808.1212 [hep-th].

[70] D. S. Mansi, A. C. Petkou and G. Tagliabue, *Gravity in the* $3+1$*-split formalism II: self-duality and the emergence of the gravitational Chern–Simons in the boundary*, Class. Quant. Grav. **26** (2009) 045009, arXiv:0808.1213 [hep-th].

[71] S. de Haro, *Dual gravitons in* AdS_4/CFT_3 *and the holographic Cotton tensor*, JHEP **0901** (2009) 042, arXiv:0808.2054 [hep-th].

[72] I. Bakas, *Energy–momentum/Cotton tensor duality for* AdS_4 *black holes*, JHEP **0901** (2009) 003, arXiv:0809.4852 [hep-th].

[73] O. Miskovic and R. Olea, *Topological regularization and self-duality in four-dimensional anti-de Sitter gravity*, Phys. Rev. **D79** (2009) 124020, arXiv:0902.2082 [hep-th].

[74] J. F. Plebański and M. Demiański, *Rotating, charged, and uniformly accelerating mass in general relativity*, Annals Phys. **98** (1976) 98.

[75] G. B. de Freitas and H. S. Reall, *Algebraically special solutions in AdS/CFT*, JHEP **1406** (2014) 148, arXiv:1403.3537 [hep-th].

[76] I. Bakas and K. Skenderis, *Non-equilibrium dynamics and* AdS_4 *Robinson–Trautman*, JHEP **1408** (2014) 056, arXiv:1404.4824 [hep-th].

[77] I. Bakas, K. Skenderis and B. Withers, *Self-similar equilibration of strongly interacting systems from holography*, Phys. Rev. **D93** (2016) 101902, arXiv:1512.09151 [hep-th].

[78] K. Skenderis and B. Withers, *Robinson–Trautman spacetimes and gauge/gravity duality*, PoS CORFU **2016** (2017) 097, arXiv:1703.10865 [hep-th].

[79] R. Fareghbal, A. Naseh and S. Rouhani, *Aspects of ultra-relativistic field theories via flat-space holography*, Phys. Lett. **B771** (2017) 189-193, arXiv:1511.01774 [hep-th].

[80] A. Bagchi, R. Basu, A. Kakkar and A. Mehra, *Flat holography: aspects of the dual field theory*, JHEP **1612** (2016) 147, arXiv:1609.06203 [hep-th].

[81] E. Cartan, *Sur les variétés à connexion affine, et la théorie de la relativité généralisée* (première partie), Ann. École norm. **41** (1924) 1.

[82] X. Bekaert and K. Morand, *Connections and dynamical trajectories in generalised Newton–Cartan gravity II. An ambient perspective*, arXiv:1505.03739 [hep-th].

[83] J. B. Griffiths and J. Podolský, *Exact space–times in Einstein's general relativity*, Cambridge Monographs on Mathematical Physics, 2009.

[84] R. H. Price and K. S. Thorne, *Membrane viewpoint on black holes: properties and evolution of the stretched horizon*, Phys. Rev. **D33** (1986) 915.

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Carrollian conservation laws and Ricci-flat gravity

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Abstract

We construct the Carrollian equivalent of the relativistic energy–momentum tensor, based on variation of the action with respect to the elementary fields of the Carrollian geometry. We prove that, exactly like in the relativistic case, it satisfies conservation equations that are imposed by general Carrollian covariance. In the flat case we recover the usual non-symmetric energy–momentum tensor obtained using Nœther procedure. We show how Carrollian conservation equations emerge taking the ultra-relativistic limit of the relativistic ones. We introduce Carrollian Killing vectors and build associated conserved charges. We finally apply our results to asymptotically flat gravity, where we interpret the boundary equations of motion as ultra-relativistic Carrollian conservation laws, and observe that the surface charges obtained through covariant phase-space formalism match the ones we defined earlier.

Contents

1 Introduction

The Carroll group was firstly introduced in [1] as a contraction of the Poincaré group for vanishing speed of light and this is referred to as the *ultra-relativistic* limit. The main feature is that, as opposed to the Galilean case, this group allows for boosts only in the time direction: space is absolute.

We could wonder what happens when we take the zero-c limit of a relativistic general-covariant theory. The resulting theory ends up being covariant only under a subset of the diffeormorphisms, as illustrated in [2] , the so-called *Carrollian diffeormorphisms*

$$t' = t'(t, \mathbf{x}), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \tag{1.1}$$

The ultra-relativistic limit breaks the spacetime metric into three independent data, a scalar density, a connection and a spatial metric. These geometric fields are nicely interpreted as constituents of a *Carrollian geometry*, as we will show in Sec. 2. Now considering an action defined on such a geometry, covariant under (1.1), we are facing a problem in defining the energy–momentum tensor. Indeed, in general-covariant theories it is obtained as the variation of the action with respect to the metric. This requires the existence of a regular metric *i.e.* of a pseudo-Riemannian manifold, but in the Carrollian case, as we mentioned, there is no spacetime non-degenerate metric. Therefore, we must introduce new objects. The core of Sec. 2 will be dedicated to the definition of these new objects, dubbed *Carrollian momenta*, and obtained as the variation of the action with respect to the 3 geometric fields mentioned above.

General covariance usually ensures that the energy–momentum tensor is conserved. In the context of Carroll-covariant theories, we will derive similar conservation equations for the Carrollian momenta. In order to gain confidence with these new definitions, we will study a simple Carrollian action, and show that, on a flat geometrical background, the Carrollian momenta are packaged in a spacetime energy–momentum tensor which coincides with the Nœther current associated with spacetime translations. This will be done in Sec. 3.

We will further discuss the intrinsic Carrollian nature of the ultra-relativistic limit. Indeed, in Sec. 4, starting from the conservation equations of an energy–momentum tensor, covariant under all changes of coordinates, we reach conservation laws that look strikingly similar to the ones we derived for the Carrollian momenta, which are covariant only under (1.1).

In general-covariant theories, the existence of a Killing vector allows to build a conserved current by projecting the energy–momentum tensor on the Killing field. This ultimately leads to a conserved charge. After briefly introducing the notion of conserved current in the Carrollian context, we define in Sec. 5 the Carrollian Killing vectors and build their associated currents and charges.

There are by now different instances in which the Carrollian framework has found applications. For instance, it has been used in electromagnetism [3] and to discuss the so-called Carroll strings [4]. The last part of this paper is devoted to yet another application of the Carrollian framework: flat holography. The latter is a holographic correspondence between a theory of asymptotically flat gravity and a non-gravitational theory leaving on its boundary, see [5–12] for recent progresses in this direction. Asymptotically anti-de-Sitter spacetimes enjoy a timelike pseudo-Riemannian boundary and the associated metric sources its dual operator: the boundary energy–momentum tensor. For asymptotically flat spacetimes, the dual theory leaves on the null infinity $\mathcal{I}^+$. Nevertheless this surface does not carry the same geometrical structure, it is a null hypersurface thus equipped with a Carrollian geometry [9] and this will be the source for the Carrollian momenta. The conservation of the latter will be shown to correspond to the gravitational dynamics in the bulk.[1] As a cross check, it has been shown [14] that the conformal Carroll group has a particular realization which is nothing but the Bondi–Metzner–Sachs (BMS) group [15]: the symmetries associated with a Carrollian structure match the asymptotic symmetries of the bulk.

In Secs. 6.1 and 6.2 we focus on the Carrollian theory on $\mathcal{I}^+$ and its relevance for gravitational asymptotically flat duals in 3 and 4 dimensions, and in Sec. 6.3 we study explicit solutions, namely the Robinson–Trautman and the Kerr–Taub–NUT families.

2 Carrollian momenta

We start with a brief reminder on the energy–momentum tensor in the relativistic case, and then define its counterpart, that we call *Carrollian momenta*, on a general Carrollian background. This requires the study of Carrollian geometry and covariance, which will be eventually the guideline for obtaining the conservation equations of these momenta. We also extend our results for a scale invariant theory (Weyl invariant) and write the conservation equations in a Weyl-covariant way. Finally, we focus on the flat case and show how, in this case only, one can promote the Carrollian momenta to a "non-symmetric energy–momentum tensor".

2.1 A relativistic synopsis

In a relativistic theory, the energy–momentum tensor is usually defined as

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.1}$$

[1] Some attention has been recently given to the interpretation of the bulk dynamics in terms of null conservation laws, see *e.g.* [13].

For a general-covariant theory, it is easy to prove that it is conserved. Indeed, considering the variation of the action under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, we have

$$\delta_\xi S = \int \mathrm{d}^{d+1}x \left(\frac{\delta S}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta S}{\delta \phi} \delta_\xi \phi \right) + \text{b. t.}, \tag{2.2}$$

where $d+1$ is the spacetime dimension and ϕ stands for the various other fields of the theory. We assume that we are on-shell so $\frac{\delta S}{\delta \phi} = 0$. Moreover, δ_ξ is the Lie derivative, which for a Levi Civita reads

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{2.3}$$

We thus obtain

$$\delta_\xi S = -\int \mathrm{d}^{d+1}x \sqrt{-g} T^{\mu\nu} \nabla_\mu \xi_\nu = \int \mathrm{d}^{d+1}x \sqrt{-g} \nabla_\mu T^{\mu\nu} \xi_\nu + \text{b. t.}. \tag{2.4}$$

If the theory is general-covariant, $\delta_\xi S = 0$ for all ξ. From this we deduce that $\nabla_\mu T^{\mu\nu}$ vanishes on shell, which is the usual conservation law of the energy–momentum tensor.

2.2 Carrollian geometry

We briefly introduce here the Carrollian geometry, as it emerges from an ultra-relativistic ($c \to 0$) limit of the relativistic metric. It has been shown in [2, 12] that the conservation equations of a relativistic energy–momentum tensor, covariant under all diffeomorphisms, lead, in the $c \to 0$ limit, to equations covariant under a subset called *Carrollian diffeomorphisms*

$$t' = t'(t, \mathbf{x}), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \tag{2.5}$$

An adequate parametrization for taking this limit is the so-called Randers–Papapetrou, in which the various components transform nicely under this subset of diffeomorphisms. The metric takes the form[2]

$$g = \begin{pmatrix} -\Omega^2 & c\Omega b_i \\ c\Omega b_j & a_{ij} - c^2 b_i b_j \end{pmatrix}_{\{c\mathrm{d}t, \mathrm{d}x^i\}} \tag{2.6}$$

where $i = \{1, \ldots, d\}$. Indeed, under (2.5)

$$a'_{ij} = a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1i}_{\ \ k}, \quad \Omega' = \frac{\Omega}{J}, \tag{2.7}$$

where $J^k_i = \frac{\partial x'^k}{\partial x^i}$, $j_i = \frac{\partial t'}{\partial x^i}$ and $J = \frac{\partial t'}{\partial t}$. In the $c \to 0$ limit the metric becomes degenerate, hence we cannot package the different metric fields in a spacetime tensor $g_{\mu\nu}$, but instead we have to treat those three fields separately: time and space decouple as (2.5) clearly suggests. We

[2]Every metric can be parametrized in this way. The alternative parametrization, known as Zermelo, turns out to be useful for the Galilean limit (see [2, 16]).

therefore trade the metric $g_{\mu\nu}$ for the time lapse $\Omega(t,\mathbf{x})$, connection $b_i(t,\mathbf{x})$ and spatial metric $a_{ij}(t,\mathbf{x})$,[3] which we refer to as Carrollian metric fields, defining a Carrollian geometry. On the derivatives, (2.5) infers

$$\partial'_t = \frac{1}{J}\partial_t, \quad \partial'_i = J_i^{-1k}\left(\partial_k - \frac{j_k}{J}\partial_t\right), \tag{2.8}$$

which implies that the spatial derivative is not a Carrollian tensor and the temporal one is a density. Therefore we introduce the Carroll-covariant derivatives $\frac{1}{\Omega}\partial_t$ and $\hat{\nabla}_i$. In the temporal one the role of Ω as a time lapse is clear, and the spatial one is defined through its action on scalars as

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t. \tag{2.9}$$

On Carrollian tensors, it acts as usual with the following Christoffel symbols

$$\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right). \tag{2.10}$$

By construction, $\hat{\partial}_i$ transforms as a Carrollian tensor

$$\hat{\partial}'_i = J_i^{-1k}\hat{\partial}_k, \tag{2.11}$$

and thus we also see clearly the role of b_i as connection. Out of the Carrollian metric fields, we can build first-order derivative geometrical objects

$$\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right), \tag{2.12}$$

$$\theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \tag{2.13}$$

$$\xi_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d}a_{ij}\partial_t \ln\sqrt{a}\right), \tag{2.14}$$

$$\varpi_{ij} = \partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]}. \tag{2.15}$$

They are all Carrollian tensors and they encode the non-flatness of the Carrollian geometrical structure we are defining. They will turn out very useful in writing the conservation equations of the Carrollian momenta defined in the next section.

[3] Hence, we will use a_{ij} to raise and lower spatial indexes in the Carrollian geometry.

2.3 Carrollian momenta

We define the Carrollian equivalent of the energy–momentum tensor as the three following pieces of data:

$$\mathcal{O} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta\Omega}, \quad \mathcal{B}^i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i} \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}. \tag{2.16}$$

Here $\Omega\sqrt{a}$ is the Carrollian counterpart of the relativistic $\sqrt{-g}$ and the variations are taken with respect to the 3 fields that replace the metric in the Carrollian setting. From now on, we call (2.16) the *Carrollian momenta*. Before continuing, notice that these quantities transform under Carrollian diffeomorphisms as

$$\mathcal{O}' = J\mathcal{O} - \mathcal{B}^i j_i, \quad \mathcal{B}^{i\prime} = J^i_j \mathcal{B}^j, \quad \text{and} \quad \mathcal{A}^{ij\prime} = J^i_k J^j_l \mathcal{A}^{kl}. \tag{2.17}$$

The spatial vector $\mathcal{B}^i$ and matrix $\mathcal{A}^{ij}$ are indeed Carrollian tensors. However, $\mathcal{O}$ is not a scalar and, as we will see and use, it is wiser to introduce the scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i\mathcal{B}^i$.

Given a Carroll-covariant theory, the action is invariant under Carrollian diffeomorphisms, generated by the spacetime vector ξ

$$\delta_\xi S = 0, \quad \xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i. \tag{2.18}$$

Notice that ξ^i only depends on $\mathbf{x}$, this is the infinitesimal translation of (2.5). Under such an infinitesimal coordinate transformation we have

$$\delta_\xi S = \int \mathrm{d}^{d+1}x \left(\frac{\delta S}{\delta\Omega}\delta_\xi\Omega + \frac{\delta S}{\delta b_i}\delta_\xi b_i + \frac{\delta S}{\delta a_{ij}}\delta_\xi a_{ij} + \frac{\delta S}{\delta\phi}\delta_\xi\phi \right) + \text{b.t.}, \tag{2.19}$$

and the on-shell condition ensures $\frac{\delta S}{\delta\phi} = 0$. We need to compute $\delta_\xi\Omega$, $\delta_\xi b_i$ and $\delta_\xi a_{ij}$. In order to do so we compute the infinitesimal version of (2.7). If $x'^\mu = x^\mu - \xi^\mu$, then

$$\delta_\xi\Omega = \xi(\Omega) + \Omega\partial_t\xi^t, \tag{2.20}$$
$$\delta_\xi b_i = \xi(b_i) - \Omega\partial_i\xi^t + b_j\partial_i\xi^j, \tag{2.21}$$
$$\delta_\xi a_{ij} = \xi(a_{ij}) + \partial_i\xi^k a_{kj} + \partial_j\xi^k a_{ik}, \tag{2.22}$$

where $\xi(f) \equiv \xi^t\partial_t f + \xi^i\partial_i f$. We would like to write these transformations in terms of manifestly Carroll-covariant objects, so we define $X = \Omega\xi^t - b_i\xi^i$. By noticing that the components of a spacetime vector transform as

$$\xi^{t\prime} = J\xi^t + j_i\xi^i, \quad \xi^{i\prime} = J^i_k\xi^k, \tag{2.23}$$

it is straightforward to show that X is the right combination for obtaining a scalar. We thus rewrite (2.20), (2.21) and (2.22) in terms of X, ξ^i and the Carrollian geometrical tensors introduced above

$$\begin{aligned}
\delta_\xi \Omega &= \partial_t X + \Omega \varphi_j \xi^j, && (2.24)\\
\delta_\xi b_i &= -\hat{\partial}_i X + \varphi_i X - 2\varpi_{ij}\xi^j + \frac{b_i}{\Omega}\left(\partial_t X + \Omega\varphi_j \xi^j\right), && (2.25)\\
\delta_\xi a_{ij} &= \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega}\partial_t a_{ij}. && (2.26)
\end{aligned}$$

This rewriting hints toward Carrollian covariance, as it replaces ξ^t with X. Therefore, we obtain $\delta_\xi S = \delta_X S + \delta_{\xi^i} S$ with

$$\begin{aligned}
\delta_X S &= \int \mathrm{d}^{d+1}x \Omega \sqrt{a}\left(\mathcal{O}\partial_t X - \mathcal{B}^i \hat{\partial}_i X + \mathcal{B}^i \varphi_i X + \mathcal{B}^i \frac{b_i}{\Omega}\partial_t X + \mathcal{A}^{ij}\frac{X}{\Omega}\partial_t a_{ij}\right), && (2.27)\\
\delta_{\xi^i} S &= \int \mathrm{d}^{d+1}x \Omega \sqrt{a}\left(\mathcal{O}\Omega\varphi_j\xi^j - 2\mathcal{B}^i \varpi_{ij}\xi^j + \mathcal{B}^i b_i \varphi_j \xi^j + 2\mathcal{A}^{ij}\hat{\nabla}_i \xi_j\right). && (2.28)
\end{aligned}$$

Finally, demanding $\delta_X S$ and $\delta_{\xi^i} S$ be zero separately and manipulating them, we obtain two conservation equations which are manifestly Carroll-covariant:[4]

$$\begin{aligned}
\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} &= 0, && (2.29)\\
2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i \varpi_{ij} - \mathcal{E}\varphi_j &= 0, && (2.30)
\end{aligned}$$

where we used the already introduced scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i \mathcal{B}^i$.

Let us briefly summarize. By strict comparison with the relativistic situation, we have defined the momenta of our Carrollian theory to be the variation of the action under the geometrical set of data that characterizes the background. Exploiting the underlying Carrollian symmetry we reached a set of two equations which encode the conservation properties of the momenta. As expected, these equations are fully Carroll-covariant.

2.4 Weyl covariance

At the relativistic level, Weyl invariance merges when the theory is invariant under a rescaling $g_{\mu\nu} \to \frac{g_{\mu\nu}}{\mathcal{B}^2}$ for any $\mathcal{B}$ function of spacetime coordinates.[5] The transformations of Ω, b_i and a_{ij} under Weyl rescaling are deduced from the relativistic Randers–Papapetrou metric (2.6)

$$\Omega \to \frac{\Omega}{\mathcal{B}}, \quad b_i \to \frac{b_i}{\mathcal{B}} \quad \text{and} \quad a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}. \qquad (2.31)$$

[4] A useful relation is $\mathcal{B}^i \hat{\partial}_i X = -X\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{B}^i$, valid up to total derivatives and for any scalar X and vector $\mathcal{B}^i$.

[5] This conformal symmetry has important consequences in hydrodynamical holographic theories, [17, 18].

If the action is invariant under such transformations,

$$\delta_\lambda S = \int \mathrm{d}^{d+1}x\Omega\sqrt{a}\left(\mathcal{O}\delta_\lambda\Omega + \mathcal{B}^i\delta_\lambda b_i + \mathcal{A}^{ij}\delta_\lambda a_{ij}\right) = \int \mathrm{d}^{d+1}x\Omega\sqrt{a}\lambda\left(\mathcal{O}\Omega + \mathcal{B}^i b_i + 2\mathcal{A}^{ij}a_{ij}\right) \tag{2.32}$$

has to vanish for every $\lambda(t,\mathbf{x})$. Therefore

$$\delta_\lambda S = 0 \quad \Rightarrow \quad \mathcal{E} = -2\mathcal{A}_i^i. \tag{2.33}$$

We will refer to this condition as the *conformal state equation*, it is the equivalent of the tracelessness of the energy–momentum tensor in the relativistic case. From (2.31) we deduce the following transformations of the Carrollian momenta

$$\mathcal{O} \to \mathcal{B}^{d+2}\mathcal{O}, \quad \mathcal{B}^i \to \mathcal{B}^{d+2}\mathcal{B}^i \quad \text{and} \quad \mathcal{A}^{ij} \to \mathcal{B}^{d+3}\mathcal{A}^{ij}. \tag{2.34}$$

This implies also $\mathcal{E} \to \mathcal{B}^{d+1}\mathcal{E}$.

We would like to write the conservation equations in a manifestly Weyl-covariant form. To do so, we decompose $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$ with Ξ^{ij} traceless, such that the constraint (2.33) becomes $\mathcal{E} = d\mathcal{P}$. This enable us rewriting (2.29) and (2.30) as

$$\begin{aligned}
\left(\frac{1}{\Omega}\partial_t + \frac{d+1}{d}\theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \Xi^{ij}\xi_{ij} &= 0, &(2.35)\\
\left(\hat{\nabla}_i + \varphi_i\right)\Xi^i_j - \frac{1}{d}\left(\hat{\partial}_j + (d+1)\varphi_j\right)\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} &= 0. &(2.36)
\end{aligned}$$

The Carrollian derivatives are not covariant under Weyl rescaling, since the latter brings extra shift terms. In order to reach manifestly Weyl-Carroll-covariant equations, we can upgrade the Carroll derivatives to Weyl-Carroll ones. Among the Carrollian first derivative tensors introduced above, φ_i and θ are Weyl connections as

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln\mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{d}{\Omega}\partial_t\mathcal{B}. \tag{2.37}$$

Therefore, they can be used for defining the Weyl-Carroll derivative. For a weight-w scalar function Φ, *i.e.* a function scaling with $\mathcal{B}^w$ under Weyl, and a weight-w vector, the Weyl-Carroll spatial and temporal derivatives are defined as

$$\begin{aligned}
\hat{\mathscr{D}}_j\Phi &= \hat{\partial}_j\Phi + w\varphi_j\Phi, &(2.38)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi &= \frac{1}{\Omega}\partial_t\Phi + \frac{w}{d}\theta\Phi, &(2.39)\\
\hat{\mathscr{D}}_j V^l &= \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i\varphi_i, &(2.40)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l &= \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta V^l + \xi^l{}_i V^i, &(2.41)
\end{aligned}$$

such that under a Weyl transformation

$$\hat{\mathscr{D}}_j\Phi \rightarrow \mathcal{B}^w\hat{\mathscr{D}}_j\Phi, \tag{2.42}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi \rightarrow \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi, \tag{2.43}$$

$$\hat{\mathscr{D}}_jV^l \rightarrow \mathcal{B}^w\hat{\mathscr{D}}_jV^l, \tag{2.44}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_tV^l \rightarrow \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_tV^l. \tag{2.45}$$

The action on any other tensor is obtained using the Leibniz rule.

Eventually, we can write (2.35) and (2.36) using these derivatives as

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\mathcal{E} - \hat{\mathscr{D}}_i\mathcal{B}^i - \Xi^{ij}\xi_{ij} = 0, \tag{2.46}$$

$$-\frac{1}{d}\hat{\mathscr{D}}_j\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} + \hat{\mathscr{D}}_i\Xi^i_j = 0. \tag{2.47}$$

Not only these equations are now very compact, they are also manifestly Weyl-Carroll-covariant.

2.5 The flat case

So far we have worked on general Carrollian geometry, *i.e.* we did not impose any particular value of Ω, b_i and a_{ij}. We now restrict our attention to the flat Carrollian background.[6]

At the relativistic level, the Poincaré group is defined as the set of coordinate transformations that leave the Minkowski metric invariant. By strict analogy, the Carroll group is defined as the set of transformations that preserve the Carrollian flatness, [16]. Therefore, the Carroll group corresponds to the transformations satisfying

$$\partial_t \to \partial_t, \quad \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \to \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad b_{0i} \to R^j_i\left(b_{0j} + \beta_j\right), \tag{2.48}$$

with b_{0i} constant. The resulting change of coordinates is

$$t' = t + \beta_i x^i + t_0, \quad x'^i = R^i_j x^j + x^i_0, \tag{2.49}$$

where $t_0 \in \mathbb{R}$, $\{x^i_0, \beta_i\} \in \mathbb{R}^d$ and $R^i_j \in \mathrm{O}(d)$. This group is known in the literature as the Carroll group. [7]

[6]We refer here to flat Carrollian geometry as the geometry for which the Carroll group is an isometry, see *e.g.* [16].

[7]The Carroll group was already shown to be the symmetry group of flat zero signature geometries in the precursory work [19].

Recasting (2.29) and (2.30) for $a_{ij}(t,\mathbf{x}) = \delta_{ij}$, $\Omega(t,\mathbf{x}) = 1$ and $b_i(t,\mathbf{x}) = b_{0i}$, we obtain

$$\begin{aligned} \partial_t \mathcal{O} - \partial_i \mathcal{B}^i &= 0, & (2.50)\\ 2\partial_i \mathcal{A}^i_j + 2b_{0i}\partial_t \mathcal{A}^i_j &= 0. & (2.51) \end{aligned}$$

The momenta appearing in these two equations can be packaged in a spacetime energy–momentum tensor (where spacetime does not mean relativistic)

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} \\ -\mathcal{B}^j & -2\mathcal{A}^{ij} \end{pmatrix}. \tag{2.52}$$

The usual conservation of this tensor $\partial_\mu T^{\mu\nu} = 0$ is ensured by the conservation equations of the momenta, namely (2.50) and (2.51). This tensor is not symmetric, but this should not come as a surprise: it is not defined throughout the variation of the action with respect to the spacetime metric (symmetric by construction), instead it is defined using the Carrollian metric fields.[8] Finally notice that this spacetime lifting procedure was possible here due to the flatness of the Carrollian geometry. In general backgrounds, this is not possible, and the very concept of spacetime energy–momentum tensor is ambiguous–whereas the Carrollian momenta are by construction well suited.

As a conclusive remark notice that the Carroll group contains spacetime translations, so if a theory is invariant under this group, there will be a set of $d+1$ Nœther currents associated with spacetime translations. Packaging them in a $d+1$-dimensional kind of Nœther energy–momentum tensor, enables us comparing it with (2.52), as we do in the next section.

3 A Carrollian scalar-field action

In order to probe our results, we start with the example of a single scalar field $\phi(t,\mathbf{x})$. We begin the study on a general Carrollian background and show that the momenta are conserved. Then, we restrict the geometry to the flat case, where spacetime translational invariance of the theory allows us to compare our energy–momentum tensor (defined only in the flat case, as in Sec. 2.5) to the conserved current computed using Nœther procedure. The two energy–momentum tensors will turn out to be equivalent up to divergence-free terms.

In order to ensure Carroll invariance of the scalar-field action, we need to trade the usual derivatives for the Carrollian ones. So we consider the action

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^{d+1}x\Omega\sqrt{a}\,a^{ij}\hat{\partial}_i\phi\hat{\partial}_j\phi = \int \mathrm{d}^{d+1}x\mathcal{L}, \tag{3.1}$$

[8] Although the construction is different, another example of non-symmetric Carrollian energy–momentum tensor can be found in [20].

which is manifestly covariant. The equations of motion are readily determined

$$\left(\hat{\nabla}_i + \varphi_i\right)\hat{\partial}^i\phi = 0. \tag{3.2}$$

The Carrollian momenta are

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi, && (3.3)\\ \mathcal{B}^i &= \frac{1}{\Omega}\partial_t\phi\hat{\partial}^i\phi, && (3.4)\\ \mathcal{A}^{ij} &= \frac{1}{2}\left(\frac{1}{2}a^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi - \hat{\partial}^i\phi\hat{\partial}^j\phi\right). && (3.5) \end{aligned}$$

These momenta are conserved on shell since the conservation equations (2.29) and (2.30) are automatically satisfied given the equations of motion (3.2). This last result shows unambiguously the relevance of these objects. Notice moreover that these momenta satisfy the conformal state equation (2.33) only for $d = 1$. In fact this action can be recovered from an ultra-relativistic limit of the free relativistic scalar theory, which is known to be conformal only in 2 spacetime dimensions.

We now impose the Carrollian background to be flat. In this case, the action (3.1) becomes

$$S[\phi] = \int \mathrm{d}^{d+1}x\mathcal{L} = \frac{1}{2}\int \mathrm{d}^{d+1}x\delta^{ij}\left(\partial_i + b_{0i}\partial_t\right)\phi\left(\partial_j + b_{0j}\partial_t\right)\phi, \tag{3.6}$$

which is invariant under spacetime translations. In the flat case, we can lift the Carrollian momenta into a spacetime energy–momentum tensor (2.52), which here takes the form

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi - b_{0i}\partial_t\phi\hat{\partial}^i\phi & -\frac{b_0^i}{2}\hat{\partial}^k\phi\hat{\partial}_k\phi + b_{0k}\hat{\partial}^k\phi\hat{\partial}^i\phi \\ -\partial_t\phi\hat{\partial}^i\phi & -\frac{1}{2}a^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi + \hat{\partial}^i\phi\hat{\partial}^j\phi \end{pmatrix}, \tag{3.7}$$

and it is conserved.

The action (3.6) is invariant under spacetime translations. As stated in the previous section, we therefore have $d + 1$ associated Nœther currents

$$\hat{T}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \tag{3.8}$$

which explicitly read:

$$\begin{aligned} \hat{T}^{tt} &= \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi - b_{0i}\hat{\partial}^i\phi\partial_t\phi, && (3.9)\\ \hat{T}^{it} &= -\hat{\partial}^i\phi\partial_t\phi, && (3.10)\\ \hat{T}^{ti} &= b_{0j}\hat{\partial}^j\phi\partial^i\phi, && (3.11)\\ \hat{T}^{ij} &= \hat{\partial}^i\phi\partial^j\phi - \frac{1}{2}\delta^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi. && (3.12) \end{aligned}$$

The conservation $\partial_\mu \hat{T}^{\mu\nu} = 0$, is achieved thanks to the equations of motion (3.2) for flat geometry $\hat{\partial}^i \hat{\partial}_i \phi = 0$.

We can now compare the energy–momentum tensor (3.7) with (3.9), (3.10), (3.11) and (3.12). We obtain

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + B^{\mu\nu}, \tag{3.13}$$

with

$$\begin{aligned} B^{tt} &= 0, & (3.14)\\ B^{it} &= 0, & (3.15)\\ B^{ti} &= -b_0^i b_0^j \hat{\partial}_j \phi \partial_t \phi + \frac{1}{2} b_0^i \hat{\partial}_k \phi \hat{\partial}^k \phi, & (3.16)\\ B^{ij} &= -b_0^j \partial_t \phi \hat{\partial}^i \phi. & (3.17) \end{aligned}$$

As anticipated, the tensor $B^{\mu\nu}$ is divergenceless on-shell $\partial_\mu B^{\mu\nu} = 0$, which implies that the two energy–momentum tensors carry the same physical information on the theory.

4 Ultra-relativistic limit: the emergence of Carrollian physics

In the previous sections, we have intrinsically defined the Carrollian momenta starting from the metric fields of a Carrollian geometry. The Carrollian geometry was inspired by an ultra-relativistic contraction of the relativistic metric. We will see now how the ultra-relativistic limit can be directly taken at the level of the conservation equation of the relativistic energy–momentum tensor. This limit provides a richer structure, with more equations and fields. This is neither surprising nor contradictory. It is suggested by the dual Galilean limit, [12]. Indeed, in the non-relativistic case, on top of the momentum and energy conservation, an extra equation arises, which is ultimately identified with the continuity equation. A similar phenomenon occurs in the Carrollian case: additional fields and equations survive in the limit, and this is controlled by our choice of c-dependence of the fields.

Given a vector field u^μ, normalized as $u^2 = -c^2$ with respect to the relativistic metric (2.6), the energy–momentum tensor can always be decomposed as[9]

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}) \frac{u^\mu u^\nu}{c^2} + \mathcal{P} g^{\mu\nu} + \tau^{\mu\nu} + \frac{q^\mu u^\nu}{c^2} + \frac{q^\nu u^\mu}{c^2}. \tag{4.1}$$

In the hydrodynamic interpretation, $\mathcal{E}$ and $\mathcal{P}$ are the energy density and pressure of the fluid, $g^{\mu\nu}$ is the spacetime metric and $\tau^{\mu\nu}$ and q^μ are the transverse dissipative tensors, named viscous stress tensor and heat current. We choose to adapt the velocity to the geometry $u^\mu = \left(\frac{c}{\Omega}, 0\right)$: the fluid is at rest. The advantage of this choice is that the dissipative tensors,

[9] Reminder of the conventions: $x^\mu = (x^0, x^i) = (ct, x^i)$.

since transverse, have only spatial independent components. Inspired by flat holography [12], we choose a particular scaling of these tensors in c, namely

$$\tau^{ij} = -\frac{\Sigma^{ij}}{c^2} - \Xi^{ij} \quad \text{and} \quad q^i = -\mathcal{B}^i + c^2\pi^i. \tag{4.2}$$

A more general dependence could have been considered. This would add new fields and new equations to the resulting Carrollian theory, whereas the present choice will be sufficient for the examples we want to analyze. Notice that the c-independent situation is recovered for $\Sigma^{ij} = 0 = \pi^i$. We now perform the zero-c limit of $\nabla_\mu T^{\mu\nu} = 0$. Defining again $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$, we obtain the following set of equations[10]

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} = 0, \tag{4.3}$$

$$2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i\varpi_{ij} - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j = 0, \tag{4.4}$$

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j + \left(\hat{\nabla}_i + \varphi_i\right)\Sigma^i_j = 0, \tag{4.5}$$

$$\Sigma^{ij}\xi_{ij} + \frac{\theta}{d}\Sigma^i_i = 0. \tag{4.6}$$

As advertised, we immediately recognize (4.5) and (4.6) as the Carrollian counterpart of the continuity equation: these are two consistency equations of the limit. Notice moreover how these equations reduce to the Carrollian equations (2.29) and (2.30) when the dissipative terms have no c-dependence, $\Sigma^{ij} = 0 = \pi^i$, together with the additional constraint $\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j = 0$. This result undoubtedly shows the nature of the ultra-relativistic limit: it is a Carrollian limit. Conversely, this analysis gives credit to our intrinsic Carrollian construction of the previous sections.

Summarizing, we have shown how the ultra-relativistic expansion gives rise to a leading Carrollian behavior. Furthermore, we have analyzed the extra inputs this limit brings and the associated conservation equations. It is remarkable how the Carrollian momenta intrinsically defined using Carrollian geometry match the ultra-relativistic limit.

We conclude with an aside important remark: we have taken the ultra-relativistic limit of the conservation equations because it would have been inconsistent to compute directly the limit of the energy–momentum tensor itself. Indeed we would have lost information on the fields which survive and the conservation equations they satisfy. This confirms that we have to give up the concept of spacetime energy–momentum tensor on general Carrollian backgrounds, as anticipated in [2] but sometimes disregarded in the current literature.

[10]This limit is performed using the decomposition (4.1) and the Randers–Papapetrou parametrization of the spacetime metric. For the detailed derivation of these equations, see [2].

5 Charges

This section is dedicated to the definition of charges in the Carrollian framework. Charges are conserved quantities associated with a symmetry of the theory. Relativistically, the latter can be generated by a Killing vector field. By projecting the energy–momentum tensor on the Killing vector, we obtain a conserved current. We will show here how to implement this procedure in the Carrollian case. In order to do so, we firstly derive charges starting from a conserved Carrollian current. Secondly, we define Carrollian Killing and conformal Killing vectors. Thirdly, we build conserved charges associated with conformal Killing vectors. This will be useful for the forthcoming examples involving asymptotically flat gravity. Finally, we give another example of Carrollian action and compute the charges to illustrate our results.

5.1 Conserved Carrollian current and associated charge

We show here a way to define a conserved charge starting from a conserved current. In this derivation we never impose the current to be associated with a Killing vector, therefore our construction is very general. Whenever we have a scalar $\mathcal{J}$ and a vector $\mathcal{J}^i$ satisfying

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{J} + \left(\hat{\nabla}_i + \varphi_i\right)\mathcal{J}^i = 0, \tag{5.1}$$

we can build the conserved charge

$$\mathcal{Q} = \int_{\Sigma_t} \mathrm{d}^d x \, \sqrt{a}\left(\mathcal{J} + b_i \mathcal{J}^i\right), \tag{5.2}$$

where Σ_t is a constant-time slice. A way to derive this formula is to start from the relativistic level: consider a conserved current J^μ, the charge is then

$$Q = \int_{\Sigma_t} \mathrm{d}^d x \, \sqrt{\sigma} n_\mu J^\mu. \tag{5.3}$$

Here n_μ is the unit vector normal to Σ_t and $\sigma_{\mu\nu}$ is the induced metric on Σ_t. In order to perform the zero-c limit, we decompose J^μ in an already Carroll-covariant basis

$$J = \mathcal{J}\left(\frac{c}{\Omega}\partial_0\right) + \mathcal{J}^i\left(\partial_i + \frac{cb_i}{\Omega}\partial_0\right). \tag{5.4}$$

Then, using the Randers–Papapetrou parametrization for the relativistic spacetime metric $\mathrm{d}s^2 = -c^2(\Omega \mathrm{d}t - b_i \mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$, we obtain

$$\sqrt{\sigma} = \sqrt{a} + \mathcal{O}\left(c^2\right), \quad n_0 = c\Omega + \mathcal{O}\left(c^3\right), \quad J^0 = \frac{c}{\Omega}\left(\mathcal{J} + b_i \mathcal{J}^i\right). \tag{5.5}$$

Therefore, we find $Q \underset{c\to 0}{\to} c^2 \mathcal{Q}$, showing the relevance of the proposed Carrollian charge (5.2).

5.2 Carrollian Killing vectors and associated conserved currents

A Killing vector is usually defined as a vector field that preserves the metric. Analogously, we define the Carrollian Killing vector ξ to be the vector satisfying[11]

$$\delta_\xi \Omega = 0 = \delta_\xi a_{ij}, \tag{5.6}$$

where δ_ξ is the Lie derivative. This gives rise to two Killing equations on ξ, which are exactly (2.24) and (2.26),[12]

$$\partial_t X + \Omega \varphi_j \xi^j = 0, \tag{5.7}$$

$$\hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega} \partial_t a_{ij} = 0, \tag{5.8}$$

where we recall $X = \Omega \xi^t - b_i \xi^i$. Notice that these equations do not actually depend on b_i.

The generalization to conformal Carrollian Killing vectors is straightforward. We call ξ a conformal Carrollian Killing vector if

$$\delta_\xi \Omega = \lambda \Omega \quad \text{and} \quad \delta_\xi a_{ij} = 2\lambda a_{ij}. \tag{5.9}$$

It obeys the following conformal Killing equations:

$$\partial_t X + \Omega \varphi_j \xi^j = \lambda \Omega, \tag{5.10}$$

$$\hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega} \partial_t a_{ij} = 2\lambda a_{ij}. \tag{5.11}$$

In particular from the last equation we obtain $\lambda = \frac{1}{d}\left(\hat{\nabla}_i \xi^i + \frac{X}{\Omega} \partial_t \ln \sqrt{a}\right)$. This general construction is very useful, as we will shortly confirm.

We now build a conserved current by projecting the Carrollian momenta on a Carrollian Killing vector, exactly like in the relativistic case. Indeed consider the following Carrollian current:

$$\mathcal{J} = \xi_i \mathcal{B}^i, \quad \mathcal{J}^i = \xi_j \Sigma^{ij}. \tag{5.12}$$

It is conserved provided ξ satisfies (5.8), and the Carrollian conservation equations (4.5) and (4.6) are verified. According to Sec. 5.1, the corresponding conserved charge is

$$\mathcal{Q}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\, \xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right), \tag{5.13}$$

This charge is also conserved when ξ satisfies (5.11), if we further impose the condition $\Sigma^i_i = 0$.

[11] This is the translation in our language of $\mathcal{L}_X g = 0$ and $\mathcal{L}_X \xi = 0$ of (III.6) in [16].

[12] On top of these equations, a Carrollian Killing vector has a time independent spatial part, *i.e.* $\partial_t \xi^i = 0$.

5.3 Charges for $\mathcal{B}^i = 0$

We will show in Sec. 6 that the equations describing the dynamics of asymptotically flat spacetimes in 3 and 4 dimensions can be related to Carrollian conservation laws for $\mathcal{B}^i = 0$. For this reason we focus here on this particular case and build other conserved currents associated with conformal Killing vectors. In Sec. 6 we will observe that the corresponding charges match the surface charges obtained through covariant phase-space formalism.

The Carrollian conservation equations obtained from the ultra-relativistic limit (4.3) and (4.4), for $\mathcal{B}^i = 0$, become

$$\begin{aligned} \left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} &= 0, \quad (5.14)\\ 2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j &= 0. \quad (5.15) \end{aligned}$$

We could have also reported the two equations on Σ^{ij}, (4.5) and (4.6), but they are immaterial here. Consider a Killing vector ξ, the following charge, up to boundary terms, is conserved

$$\mathcal{C}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\left(X\mathcal{E} - \xi^i\pi_i + 2b_i\xi^j\mathcal{A}^i_j\right), \quad (5.16)$$

assuming only (5.14) and (5.15). This charge is also conserved when ξ is a conformal Killing vector, if we further impose the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$. According to Sec. 5.1, the corresponding conserved current reads[13]

$$\mathcal{J} = X\mathcal{E} - \xi^i\pi_i, \quad \mathcal{J}^i = 2\xi^j\mathcal{A}^i_j. \quad (5.17)$$

It is interesting to investigate the flat case $a_{ij}(t,\mathbf{x}) = \delta_{ij}$, $\Omega(t,\mathbf{x}) = 1$ and $b_i(t,\mathbf{x}) = b_{0i}$. Here, (5.14) and (5.15) can be written as $\partial_\mu T^{\mu\nu} = 0$ with[14]

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} + \pi^i \\ 0 & -2\mathcal{A}^{ij} \end{pmatrix}, \quad (5.18)$$

and we notice that the charge, up to a divergenceless term, takes the usual form

$$\mathcal{C}^{\text{Flat}}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \left(\xi^t\mathcal{O} - \xi^i b_{0i}\mathcal{O} - \xi^i\pi_i + 2b_{0i}\xi^j\mathcal{A}^i_j\right) = -\int_{\Sigma_t} \mathrm{d}^d x T^{0\mu}\xi_\mu + \tilde{\mathcal{C}}_{\xi^i}, \quad (5.19)$$

with $\tilde{\mathcal{C}}_{\xi^i} = -\int_{\Sigma_t} \mathrm{d}^d x \xi^i b_{0i}\mathcal{O}$, which is separately conserved.

[13] Its conservation (5.1) is ensured thanks to the Killing equations together with (5.14) and (5.15).
[14] We recall that for $\mathcal{B}^i = 0$, $\mathcal{E} = \Omega\mathcal{O}$. Thus in the flat case $\mathcal{E} = \mathcal{O}$.

For ξ and η Killing vectors, we define the brackets

$$\begin{aligned} \{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) \right], \\ \{\mathcal{C}_\xi, \mathcal{C}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2 b_i \xi^j \mathcal{A}^i_j \right) \right]. \end{aligned} \tag{5.20}$$

Here δ_η is the Lie derivative acting on the metric fields and the momenta, but not on ξ^t and ξ^i. A lengthly computation (see appendix A) shows that the charges $\mathcal{Q}_\xi$ and $\mathcal{C}_\xi$ equipped with these brackets form two representations of the Carrollian Killing algebra:

$$\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \mathcal{Q}_{[\xi,\eta]} \quad \text{and} \quad \{\mathcal{C}_\xi, \mathcal{C}_\eta\} = \mathcal{C}_{[\xi,\eta]}. \tag{5.21}$$

We can extend these results to the conformal Killing algebra when imposing the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$ for the charge $\mathcal{C}_\xi$ and the condition $\Sigma^i_i = 0$ for the charge $\mathcal{Q}_\xi$.

5.4 Application to the scalar field

We close this section with an example of scalar-field action whose Carrollian momenta reproduce exactly the conservation equations described in Sec. 5.3. Consider a scalar field $\phi(t, \mathbf{x})$ and the following Carroll-covariant action:

$$S[\phi] = \frac{1}{2} \int \mathrm{d}^{d+1} x \sqrt{a} \frac{\dot{\phi}^2}{\Omega} = \int \mathrm{d}^{d+1} x \mathcal{L}, \tag{5.22}$$

where $\dot{\phi} = \partial_t \phi$. The equation of motion reads

$$\left(\frac{1}{\Omega} \partial_t + \theta \right) \left(\frac{\dot{\phi}}{\Omega} \right) = 0, \tag{5.23}$$

and we find the following Carrollian momenta through the variational definition (2.16)

$$\mathcal{E} = -\frac{1}{2\Omega^2} \dot{\phi}^2, \quad \mathcal{B}^i = 0 \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{4\Omega^2} \dot{\phi}^2 a^{ij}. \tag{5.24}$$

Carrollian conservation equations of the type (5.14) and (5.15) are satisfied provided $\pi_i = \frac{1}{\Omega} \dot{\phi} \hat{\partial}_i \phi$. In the flat case the energy–momentum tensor (5.18) computed earlier becomes:

$$T^{\mu\nu} = \begin{pmatrix} -\frac{1}{2}\dot{\phi}^2 & \frac{1}{2} b^i_0 \dot{\phi}^2 + \dot{\phi} \partial^i \phi \\ 0 & -\frac{1}{2} \dot{\phi}^2 \delta^{ij} \end{pmatrix}. \tag{5.25}$$

As in the other example of scalar-field action (Sec. 3), this object coincides with the Nœther current associated with spacetime translations, up to a divergenceless term.

We can now focus on the charges in the Hamiltonian formalism. Defining the conjugate momentum $\psi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\sqrt{a}}{\Omega}\dot{\phi}$, and writing the Carrollian momenta in terms of ϕ and ψ, we obtain

$$\mathcal{E} = -\frac{1}{2}\left(\frac{\psi}{\sqrt{a}}\right)^2, \quad \pi_i = \frac{\psi}{\sqrt{a}}\left(\partial_i\phi + b_i\frac{\psi}{\sqrt{a}}\right) \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{4}\left(\frac{\psi}{\sqrt{a}}\right)^2 a^{ij}. \tag{5.26}$$

Therefore, the charges (5.16) become

$$\mathcal{C}_\xi = -\int_{\Sigma_t} \mathrm{d}^d x \left(\frac{\xi^t}{2}\frac{\Omega}{\sqrt{a}}\psi^2 + \xi^i\psi\partial_i\phi\right). \tag{5.27}$$

These charges are expressed in Hamiltonian formalism. They are indeed conserved thanks to the equation of motion and together with the Poisson bracket they realize a representation of the Carrollian Killing algebra:

$$\{\mathcal{C}_\xi, \mathcal{C}_\eta\}_{\text{Poisson}} = \int_{\Sigma_t} \mathrm{d}^d x \left[\frac{\delta\mathcal{C}_\eta}{\delta\phi}\frac{\delta\mathcal{C}_\xi}{\delta\psi} - \frac{\delta\mathcal{C}_\xi}{\delta\phi}\frac{\delta\mathcal{C}_\eta}{\delta\psi}\right] = \mathcal{C}_{[\xi,\eta]}. \tag{5.28}$$

This result confirms that the charges (5.16) previously introduced are the correct ones. Finally, we notice that when $d = 1$ the conformal state equation (2.33) is satisfied and the representation can be extended to conformal Killing vectors.

6 Carrollian conservation laws in Ricci-flat gravity

We will now turn our attention to Ricci-flat gravity. When the bulk metric is expressed in an appropriate gauge, usually given by imposing the radial coordinate be null, Einstein equations can reduce in some instances to equations defined on null infinity $\mathcal{I}^+$.[15] Its null nature makes it a natural host for a Carrollian geometry and the gravitational dynamics will be shown to match with Carrollian conservation laws. This section can be considered as a precursor of a full asymptotically flat holographic scheme. Indeed, the putative dual boundary theory would be Carrollian and live on $\mathcal{I}^+$. This theory would be coupled to a Carrollian geometry and satisfy Carrollian conservation laws that we map here to the gravitational dynamics. In gravity, the covariant phase-space formalism allows to compute surface charges, those will be shown to be given exactly or partially by the conserved charges defined in Sec. 5.3, depending whether the gravitational solution has radiation or not. To compute the charges explicitly, we use the code [21].

[15] It will be the case for the three families of solutions we study in this section: the 3-dimensional asymptotically flat spacetimes, the weak field approximation of 4-dimensional asymptotically flat spacetimes in Bondi gauge and the Robinson Trautman solutions. The reduction of Einstein equations to equations on $\mathcal{I}^+$ would not be true, for example, for non-linearized 4-dimensional asymptotically flat gravity in Bondi gauge.

6.1 Asymptotically flat spacetimes in three dimensions

Three-dimensional asymptotically flat spacetimes are often studied in the Bondi gauge which, as we will shortly describe, imposes by definition the corresponding two-dimensional Carrollian manifold be flat. Here we want to show that we can source the geometric boundary fields, in order to create a general Carrollian structure [10].

Consider the following bulk metric

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -2\mathrm{u}\left(\mathrm{d}r + r\left(\varphi_x\mathrm{d}x + \theta\mathrm{u}\right)\right) + r^2 a_{xx}\mathrm{d}x^2 + 8\pi G\mathrm{u}\left(\mathcal{E}\mathrm{u} - \pi_x\mathrm{d}x\right). \tag{6.1}$$

The bulk coordinates are $\{u, r, x \in \mathbf{S}^1\}$, $\mathrm{u} = \Omega\mathrm{d}u - b_x\mathrm{d}x$, a_{xx} is the one-dimensional boundary spatial metric, $\mathcal{E}$ and π_x are the Carrollian momenta and θ and φ_x correspond to (2.13) and (2.12) defined earlier:

$$\theta = \frac{1}{\Omega}\partial_u \ln\sqrt{a_{xx}} \quad \text{and} \quad \varphi_x = \frac{1}{\Omega}\left(\partial_x\Omega + \partial_u b_x\right). \tag{6.2}$$

All the fields appearing in the bulk metric depend only on u and x. From this metric we can extract the corresponding Carrollian geometry on $\mathcal{I}^+ = \{r \to \infty\}$. The following procedure is general but we will use the specific case of three-dimensional asymptotically flat spacetimes as an illustration. Consider the conformal extension of (6.1)

$$\mathrm{d}\tilde{s}^2 = r^{-2}\mathrm{d}s^2, \tag{6.3}$$

the factor r^{-2} is present to regularize the metric on $\mathcal{I}^+$. We perform the change of variable $\omega = r^{-1}$ in the conformal metric, it becomes[16]

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -2\mathrm{u}\left(-\mathrm{d}\omega + \omega\left(\varphi_x\mathrm{d}x + \theta\mathrm{u}\right)\right) + a_{xx}\mathrm{d}x^2 + 8\pi G\omega^2\mathrm{u}\left(\mathcal{E}\mathrm{u} - \pi_x\mathrm{d}x\right). \tag{6.4}$$

We can deduce the Carrollian geometry on $\mathcal{I}^+$

$$\tilde{g}^{-1}\left(.,\mathrm{d}\omega\right)_{|\mathcal{I}^+} = \frac{1}{\Omega}\partial_u, \quad \mathrm{d}\tilde{s}^2_{|\mathcal{I}^+} = a_{xx}\mathrm{d}x^2 \quad \text{and} \quad \tilde{g}\left(.,\partial_\omega\right)_{|\mathcal{I}^+} = \Omega\mathrm{d}u - b_i\mathrm{d}x^i. \tag{6.5}$$

We now move to the dynamics. In the following, we restrict our attention to the bulk line element (6.1) with the additional geometrical constraint

$$\hat{\mathscr{D}}_x s^x = \hat{\nabla}_x s^x + 2\varphi_x s^x = 0, \tag{6.6}$$

where $s_x = \frac{1}{\Omega}\partial_u\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta$ is a Weyl-weight 1 two-derivative object. The Carrollian momenta do not appear in this equation, it is just a constraint on the boundary geometrical background as it involves only the Carrollian metric fields. Using this ansatz, Einstein

[16] The null asymptote is thus $\mathcal{I}^+ = \{\omega \to 0\}$.

equations reduce to

$$\left(\frac{1}{\Omega}\partial_u + 2\theta\right)\mathcal{E} = 0, \tag{6.7}$$

$$\left(\hat{\partial}_x + 2\varphi_x\right)\mathcal{E} + \left(\frac{1}{\Omega}\partial_u + \theta\right)\pi_x = 0. \tag{6.8}$$

We interpret them as the Carrollian conservation equations (4.3), (4.4), (4.5) and (4.6) for $\Sigma^{xx} = \mathcal{B}^x = 0$ and $\mathcal{E} = \mathcal{P}$ (conformal case). Furthermore Ξ^{xx} is automatically zero due to its tracelessness. Therefore, the gravitational dynamics of this metric ansatz coincides with the Carrollian conservation equations that fall into the case described in Sec. 5.3.[17]

We would like at this point to obtain the surface charges. We thus compute the asymptotic Killing vectors of $\mathrm{d}s^2$ whose leading orders in r^{-1} are

$$\hat{\xi}^r = -r\lambda(u,x) + \mathcal{O}(1), \quad \hat{\xi}^u = \xi^u(u,x) + \mathcal{O}(r^{-1}) \quad \text{and} \quad \hat{\xi}^x = \xi^x(x) + \mathcal{O}(r^{-1}). \tag{6.9}$$

Here $\lambda = \hat{\nabla}_x \xi^x + \frac{X}{\Omega}\partial_u \ln\sqrt{a_{xx}}$ and $\xi = \xi^u\partial_u + \xi^x\partial_x$ is a conformal Killing vector (*i.e.* satisfying (5.10) and (5.11)) of the corresponding Carrollian geometry $\{\Omega, a_{xx}, b_x\}$. We calculate the associated surface charge through covariant phase-space formalism and obtain that they are integrable and have exactly the same expression as the conserved charges defined in Sec. 5.3 out of purely Carrollian considerations

$$Q_{\hat{\xi}}[\mathrm{d}s^2] = \int_{\mathbf{S}^1} \mathrm{d}x \sqrt{a_{xx}}\left((\Omega\xi^u - 2b_x\xi^x)\mathcal{E} - \xi^x\pi_x\right) = \mathcal{C}_\xi. \tag{6.10}$$

There is no gravitational radiation in three dimensions, the charges are thus conserved. We will see that things are slightly different in four dimensions, where we have to consider the radiation at null infinity.

If we restrict our attention to the case $\Omega = 1$, $a_{xx} = 1$ and $b_x = 0$, we recover the usual Bondi gauge for asymptotically flat spacetimes and Carrollian conservation becomes

$$\partial_u\mathcal{E} = 0, \tag{6.11}$$

$$\partial_x\mathcal{E} = -\partial_u\pi_x. \tag{6.12}$$

This set-up was extensively studied for instance in [22]. Here, the solutions to the Carrollian Killing equations are exactly the $\mathfrak{bms}_3$ algebra vectors $\xi = \xi^u\partial_u + \xi^x\partial_x$ with $\xi^u = \partial_x\xi^x u + \alpha$, for any smooth functions $\xi^x(x)$ and $\alpha(x)$ on $\mathbf{S}^1$. Moreover the solutions to (6.11) and (6.12) are

$$\mathcal{E}(u,x) = \mathcal{E}_0(x) \quad \text{and} \quad \pi_x(u,x) = -\partial_x\mathcal{E}_0 u + \pi_0(x). \tag{6.13}$$

[17] With respect to Sec. 5.3, we trade here t with u, to empathize that it is a retarded time.

Hence, the charges become the usual ones

$$\mathcal{C}_\xi^{\text{Bondi}} = \int_{S^1} \mathrm{d}x \, (\alpha \mathcal{E}_0 - \xi^x \pi_0) \,, \tag{6.14}$$

which are manifestly conserved. These were obtained in [6, 22]. [18]

6.2 Linearized gravity in four dimensions

We can perform the same kind of analysis in the case of asymptotically flat spacetimes in four dimensions, where asymptotic charges have been computed. We show that the boundary equations of motion, which are the linearized Einstein equations after gauge fixing, can be interpreted as a Carrollian conservation, and that the asymptotic charges are also charges associated with conformal Carrollian Killing vectors.

The bulk metric is $g_{ab} = \eta_{ab} + h_{ab}$ with

$$\begin{aligned}
\eta &= -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \\
h_{uu} &= \frac{2}{r}m_B + \mathcal{O}\left(r^{-2}\right), \\
h_{uj} &= \frac{1}{2}\nabla^i C_{ij} + \frac{1}{r}N_j + \mathcal{O}\left(r^{-2}\right), \\
h_{ij} &= rC_{ij} + \mathcal{O}(1), \\
h_{ra} &= 0.
\end{aligned} \tag{6.15}$$

where $a = \{r, \mu\} = \{r, u, x^i\}$, $i = 1,2$. The perturbation h_{ab} is traceless, so $\gamma^{ij}C_{ij} = 0$, where γ^{ij} is the metric of the two-sphere and ∇_i the associated covariant derivative. We recognize the mass aspect m_B, the angular momentum aspect N_i and the gravitational wave aspect C_{ij}, all depending on u and x^i. In this gauge, the linearized Einstein equations become:[19]

$$\partial_u m_B = \frac{1}{4}\partial_u \nabla^i \nabla^j C_{ij}, \tag{6.16}$$

$$\partial_u N_i = \frac{2}{3}\partial_i m_B - \frac{1}{6}\left[(\Delta - 1)\nabla^j C_{ji} - \nabla_i \nabla^k \nabla^j C_{jk}\right]. \tag{6.17}$$

We first consider the case

$$\nabla^i \nabla^j C_{ij} = 0. \tag{6.18}$$

Then (6.16) and (6.17) admit a Carrollian interpretation and are recovered from (2.29) and

[18] To compare, we have to identify $\phi = x$, $\Xi(\phi) = -4\pi G \pi_0(x)$, $\Theta(\phi) = 8\pi G \mathcal{E}_0(x)$, $Y(\phi) = \xi^x(x)$ and $T(\phi) = \alpha(x)$.

[19] Solving empty linearized Einstein equations order by order in r^{-1} allows to express the various subleading coefficients in terms of m_B, C_{ij} and N_i. The only residual equations are then the ones that we present here.

(2.30) with the following metric data

$$\Omega = 1, \quad b_i = 0, \quad a_{ij} = \gamma_{ij}, \tag{6.19}$$

and Carrollian momenta

$$\Sigma^{ij} = \mathcal{B}^i = \Xi^i_i = 0, \tag{6.20}$$

$$\mathcal{E} = 4m_B, \quad \mathcal{A}^{ij} = -\frac{1}{2}\left(\frac{\mathcal{E}}{2}a^{ij} - \Xi^{ij}\right), \quad \pi^i = -3N^i, \quad \Xi^i_j = \frac{1}{2}(\Delta - 4)\, C^i_j, \tag{6.21}$$

where $\mathcal{E} = -2\mathcal{A}^i_i$ and $\Xi^i_i = 0$–we are in the conformal case. We obtain the following conservation equations:

$$\partial_u \mathcal{E} = 0, \tag{6.22}$$

$$\partial_u \pi_i + \nabla_j \left(\frac{\mathcal{E}}{2}\gamma^j_i - \Xi^j_i\right) = 0. \tag{6.23}$$

This type of Carrollian conservation falls again into the class described in Sec. 5.3.

The asymptotic Killing vectors $\hat{\xi} = \hat{\xi}^r\partial_r + \hat{\xi}^u\partial_u + \hat{\xi}^i\partial_i$ associated with the gauge (6.15) have the following leading order in r^{-1}

$$\hat{\xi}^r = -\lambda(\mathbf{x})r + \mathcal{O}(1), \quad \hat{\xi}^u = \xi^u(t,\mathbf{x}) + \mathcal{O}(r^{-1}) \quad \text{and} \quad \hat{\xi}^i = \xi^i(\mathbf{x}) + \mathcal{O}(r^{-1}), \tag{6.24}$$

where $\xi = \xi^u\partial_u + \xi^i\partial_i$ is a conformal Killing vector (*i.e.* satisfying (5.10) and (5.11)) of the Carrollian geometry given by $\{\Omega = 1, a_{ij} = \gamma_{ij}, b_i = 0\}$ and λ is the conformal factor. The solutions to the corresponding conformal Killing equations reproduce exactly the $\mathfrak{bms}_4$ algebra: $\xi^u = \frac{u}{2}\nabla_i\xi^i + \alpha(\mathbf{x})$, α being any function on $\mathbf{S}^2$, ξ^i a conformal Killing of $\mathbf{S}^2$ and $\lambda = \frac{1}{2}\nabla_i\xi^i$. We compute the corresponding surface charges. When $\nabla^i\nabla^j C_{ij} = 0$ they take the form

$$Q_{\hat{\xi}}[g] = \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\xi^u\mathcal{E} - \xi^i\pi_i\right) = C_\xi, \tag{6.25}$$

with $\mathcal{E}$ and π_i given by (6.21). We recognize again the charges defined from purely Carrollian considerations in Sec. 5.3, associated with the data (6.19), (6.20) and (6.21). These charges are automatically conserved. Physically, this is due to the fact that part of the effect of gravitational radiation has suppressed by demanding $\nabla^i\nabla^j C_{ij} = 0$. We will find shortly that relaxing this condition has an effect on the charge conservation.

Integrating (6.22) and (6.23) we obtain

$$\mathcal{E} = \mathcal{E}_0(\mathbf{x}), \quad \pi_i = -\frac{1}{2}\partial_i\mathcal{E}_0 u + \int \mathrm{d}u'\nabla_j\Xi^j_i + \pi_{0i}(\mathbf{x}). \tag{6.26}$$

The charges become

$$\begin{aligned}\mathcal{C}_\xi &= \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\left(\frac{\nabla_i\xi^i}{2}u+\alpha\right)\mathcal{E}_0-\xi^i\left(-\frac{1}{2}\partial_i\mathcal{E}_0 u+\int \mathrm{d}u'\nabla_j\Xi_i^j+\pi_{0i}\right)\right)\\ &= u\int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\left(\frac{1}{2}\nabla_i(\xi^i\mathcal{E}_0)\right)+\int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\left(\int \mathrm{d}u'\nabla_j\Xi_i^j+\pi_{0i}\right)\right)\\ &= \int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\pi_{0i}\right)-\int \mathrm{d}u'\int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\xi^i\nabla_j\Xi_i^j+\text{b.t.}\\ &= \int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\pi_{0i}\right)+\text{b.t.}.\end{aligned} \tag{6.27}$$

The last step follows from the fact that ξ^i is a conformal Killing vector on $\mathbf{S}^2$ and Ξ^i_j is traceless. We observe that $\mathcal{C}_\xi$ is now manifestly conserved.

When $\nabla^i\nabla^j C_{ij}\neq 0$, on the gravity side the radiation affects the surface charges and spoils their conservation. Therefore, these charges do not match those we defined earlier. This situation can be further investigated and recast in Carrollian language. To this end, we define $\sigma=\nabla^i\nabla^j C_{ij}$ and rewrite (6.16) and (6.17)

$$\partial_u\mathcal{E} = 0, \tag{6.28}$$

$$\partial_u\pi_i+\nabla_j\left(\mathcal{P}\gamma_i^j-\Xi_i^j\right) = 0. \tag{6.29}$$

Here, the metric fields are

$$\Omega=1,\quad b_i=0,\quad a_{ij}=\gamma_{ij}, \tag{6.30}$$

together with the Carrollian momenta

$$\Sigma^{ij}=\mathcal{B}^i=0, \tag{6.31}$$

$$\mathcal{E}=4m_B-\sigma,\quad \mathcal{P}=\frac{\mathcal{E}}{2}+\sigma,\quad \pi^i=-3N^i,\quad \Xi^i_j=\frac{1}{2}(\Delta-4)C^i_j. \tag{6.32}$$

Hence turning on σ can be interpreted as spoiling the conformal state equation: $\mathcal{E}=-2\left(\mathcal{A}^i_i+\sigma\right)$. It appears as a sort of *conformal anomaly* in the boundary theory. The surface charges become

$$Q_{\hat{\xi}}[g](u)=\int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\left(\xi^u(\mathcal{E}+\sigma)-\xi^i\pi_i\right), \tag{6.33}$$

and, as already stated, they are no longer conserved

$$\partial_u Q_{\hat{\xi}}[g]=\int_{\mathbf{S}^2}\mathrm{d}^2x\,\sqrt{\gamma}\,(\delta_\xi+\lambda)\,\sigma, \tag{6.34}$$

where δ_ξ is the usual Lie derivative and $\lambda=\frac{1}{2}\nabla_i\xi^i$ the conformal factor. These charges were obtained in [23].[20] For non linear gravity see [24], where the charges are now non-integrable.

[20] See the $n=2$ case of Sec. 3. Their charges coincide with (6.33) with $\alpha=T$, $\xi^i=v^i$, $\mathcal{E}_0=4\mathcal{M}$ and $\pi^i_0=-3\mathcal{N}^i$.

6.3 Black hole solutions: Robinson–Trautman and Kerr–Taub–NUT

For asymptotically AdS solutions, Einstein equations lead to the conservation of an energy–momentum tensor on the timelike boundary with the cosmological constant playing the role of the velocity of light [12]. Taking the flat limit in the bulk therefore corresponds to an ultra-relativistic limit on the boundary, and this is how Carrollian dynamics emerges. We illustrate this for the specific examples of Robinson–Trautman and Kerr–Taub–NUT, and analyze their charges.

Robinson–Trautman

The Robinson–Trautman ansatz is

$$\mathrm{d}s^2 = \frac{2r^2}{P^2}\mathrm{d}z\mathrm{d}\bar{z} - 2\mathrm{d}u\mathrm{d}r - \left(\Delta \ln P - 2r\partial_u \ln P - \frac{2m}{r}\right)\mathrm{d}u^2, \tag{6.35}$$

where m and P depend on the boundary coordinates $\{u,z,\bar{z}\}$. This metric is Ricci-flat provided the Robinson–Trautman equations are satisfied:

$$\begin{aligned}\Delta\Delta \ln P + 12M\partial_u \ln P - 4\partial_u M &= 0, &(6.36)\\ \partial_z M &= 0, &(6.37)\\ \partial_{\bar{z}} M &= 0, &(6.38)\end{aligned}$$

where we have defined $\Delta = \nabla^i\nabla_i$, for $i = \{z,\bar{z}\}$, and ∇_i is the Levi Civita covariant derivative of the spatial metric $a = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}$. These equations can be interpreted as Carrollian conservation laws (4.3), (4.4), (4.5) and (4.6) with the metric data $\Omega = 1$, $b_i = 0$ and $a = \frac{2}{P(u,z,\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z}$ and the Carrollian momenta

$$\Xi^{ij} = \pi^i = \Sigma^i_i = 0, \tag{6.39}$$

$$\mathcal{E} = 4M, \quad \mathcal{B}^i = \nabla^i K, \quad \mathcal{A}^{ij} = -Ma^{ij}, \quad \Sigma^{ij} = \nabla^i\nabla^j\theta - \frac{1}{2}a^{ij}\nabla^k\nabla_k\theta. \tag{6.40}$$

Here we have introduced the Gaussian curvature $K = \Delta \ln P$. Weyl covariance is ensured by the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$, together with $\Sigma^i_i = 0$. With this set of data, the conservation equations are

$$\begin{aligned}\left(\partial_u + \frac{3\theta}{2}\right)\mathcal{E} - \nabla_i\mathcal{B}^i &= 0, &(6.41)\\ \partial_j\mathcal{E} &= 0, &(6.42)\\ (\partial_u + \theta)\,\mathcal{B}_j + \nabla_i\Sigma^i_j &= 0, &(6.43)\\ \Sigma^{ij}\xi_{ij} + \frac{\theta}{d}\Sigma^i_i &= 0. &(6.44)\end{aligned}$$

Equations (4.5) and (4.6) do not appear in the Robinson–Trautman equations because they are geometrical constraints on the spatial metric, which are automatically satisfied when imposing $a = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}$.

We want to interpret the charges we have introduced in Secs. 5.2 and 5.3 for the Robinson–Trautman spacetime. To this end, we introduce a conformal Carrollian Killing vector ξ, with (5.10) and (5.11) here given by

$$\partial_u \xi^u = \lambda, \tag{6.45}$$

$$\nabla_i \xi_j + \nabla_j \xi_i + \xi^u \partial_u a_{ij} = 2\lambda a_{ij}. \tag{6.46}$$

The solution is the following vector[21]

$$\xi = \left(\sqrt{a}\right)^{\frac{1}{2}} \left(\alpha(\mathbf{x}) + \frac{1}{2}\int \mathrm{d}u \left(\sqrt{a}\right)^{-\frac{1}{2}} \nabla_i \xi^i\right)\partial_u + \xi^i(\mathbf{x})\partial_i, \tag{6.47}$$

where ξ^i is a spatial conformal Killing vector, *i.e.* it satisfies

$$\nabla_i \xi_j + \nabla_j \xi_i = \nabla_k \xi^k a_{ij}. \tag{6.48}$$

The associated charges (5.13) become

$$\mathcal{Q}_\xi = \int_{\mathbf{S}^2} \mathrm{d}^2 z \sqrt{a}\, \xi_j \mathcal{B}^j = \int_{\mathbf{S}^2} \mathrm{d}^2 z P^{-2} \left(\xi^z \partial_z K + \xi^{\bar{z}} \partial_{\bar{z}} K\right). \tag{6.49}$$

They are conserved by construction.

Even though the second family of charges (5.16) were defined only for $\mathcal{B}^i = 0$, we can nevertheless study what their expression is for the solution at hand. We find

$$\mathcal{C}_\xi = \int_{\mathbf{S}^2} \mathrm{d}^2 z \sqrt{a}\, \xi^u \mathcal{E} = \int_{\mathbf{S}^2} \mathrm{d}^2 z P^{-3} \left(\alpha(z,\bar{z}) + \frac{1}{2}\int \mathrm{d}u P \nabla_i \xi^i\right) 4M. \tag{6.50}$$

As expected, they are not generically conserved, and using (6.41) we find

$$\partial_u \mathcal{C}_\xi = -\int_{\mathbf{S}^2} \mathrm{d}^2 z \sqrt{a}\, \partial_i \xi^u \mathcal{B}^i. \tag{6.51}$$

Their conservation holds in two instances. The first, expected by construction, is when $\mathcal{B}_i = \partial_i K = 0$, and corresponds to a uniform curvature of the boundary sphere at all times. The second, which is a new condition, occurs when the conformal Killing vectors satisfy also $\partial_i \xi^u = 0$. This can be written as

$$\delta_\xi b_i = 0, \tag{6.52}$$

[21] The metric (6.35) is not in the Bondi gauge unless P is time independent. Therefore, the conformal Killing vector ξ does not satisfy the usual bms_4 algebra, but a generalized version of it.

when considering the Robinson–Trautman Carrollian geometry $\Omega = 1$, $b_i = 0$ and $a = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}$.[22]

Kerr–Taub–NUT family

The interesting feature of the Kerr–Taub–NUT family is that, although stationary, it has a non-trivial metric field b_i. Its line element, in $\{t, r, \theta, \phi\}$ coordinates, is given by

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}(\mathrm{d}t - \mathrm{b})^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \frac{\sin^2\theta}{\rho^2}\left(\alpha\mathrm{d}t - \left(r^2 + (n-\alpha)^2\right)\mathrm{d}\phi\right)^2, \tag{6.53}$$

where

$$\Delta_r = -2Mr + r^2 + \alpha^2 - n^2, \tag{6.54}$$

$$\rho^2 = r^2 + (n - \alpha\cos\theta)^2, \tag{6.55}$$

$$\mathrm{b} = \left(2n(\cos\theta - 1) + \alpha\sin^2\theta\right)\mathrm{d}\phi. \tag{6.56}$$

In this solution, M is interpreted as the black hole mass, α its angular parameter and n its NUT charge. The Carrollian geometrical data are $\Omega = 1$, b_i as in (6.56) and $a = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The bulk Einstein equations are satisfied for a constant mass. We can interpret this result as given by the following Carrollian data

$$\Xi^{ij} = \pi^i = \Sigma^{ij} = \mathcal{B}^i = 0 \quad \mathcal{E} = M \quad \mathcal{A}^{ij} = -\frac{M}{4}a^{ij}, \tag{6.57}$$

such that Carrollian conservation equations give straightforwardly M constant. From the hydrodynamical viewpoint, these data describe a perfect fluid.

The conformal Carrollian Killing equations can be solved with the result

$$\xi = \left(T(\mathbf{x}) + \frac{1}{2}t\nabla_i\xi^i\right)\partial_t + \xi^i(\mathbf{x})\partial_i. \tag{6.58}$$

where T is any smooth function on $\mathbf{S}^2$ and ξ^i a Killing vector of the sphere. This is precisely the bms_4 generator. The charges (5.13) are identically zero in this case. Conversely, the charges (5.16) are non-trivial

$$\mathcal{C}_\xi = M\int_{\mathbf{S}^2}\mathrm{d}\theta\mathrm{d}\phi\sin\theta\left(T - \frac{3}{2}\xi^i b_i\right). \tag{6.59}$$

They explicitly depend on the Kerr–Taub–NUT parameters thanks to the presence of the metric field b_i, and they are manifestly conserved.

[22] Actually, it is possible to show that, even when $\mathcal{B}^i \neq 0$, the charges (5.16) are generically conserved if the vectors ξ satisfy $\delta_\xi a_{ij} = 0$, $\delta_\xi\Omega = 0$ *and* $\delta_\xi b_i = 0$.

7 Conclusions

We are now ready to summarize our achievements.

In the framework of Carrollian dynamics we have defined Carrollian momenta as the variation of the action with respect to the Carrollian metric fields Ω, b_i, a_{ij}. These momenta obey conservation laws ensuing the invariance of the action under Carrollian diffeomorphisms. We have carefully stressed that this set of Carrollian momenta plays the role the energy–momentum tensor has in relativistic theories, since such an object cannot be defined in general Carrollian dynamics. In the very particular instance of flat Carrollian geometry, due to the existence of global symmetries, the on-shell Carrollian momenta are indistinguishable from the Nœther conserved currents. In this case they can be packaged in a non-symmetric spacetime energy–momentum tensor.

We have proven that the general conservation equations of the set of Carrollian momenta are recovered as the ultra-relativistic limit of the relativistic energy–momentum tensor conservation equations. This is expected and shows in passing that the Carrollian limit of the energy–momentum tensor outside its conservation equations is non sensible.

As usual in theories with local symmetries, volume conserved charges cannot be defined from plain conserved momenta. Killing fields are needed, in order to construct conserved currents and extract conserved charges, which encode the physical information stored in the fields at hand. We performed all these steps in a general Carrollian geometry, starting with the definition of the Killing vectors and proceeding with currents (projections of the Carrollian momenta) and charges.

All these concepts and techniques have been finally illustrated in concrete examples inspired from flat holography. Indeed, the null infinity of an asymptotically flat spacetime is a natural host for Carrollian geometry, and Carrollian conservation equations on $\mathcal{I}^+$ emerge as part of the bulk Einstein dynamics. More specifically, we have shown that in three bulk dimensions the Carrollian charges match the surface charges obtained from standard bulk methods. However, in four-dimensional linearized gravity, the presence of gravitational radiation spoils the conservation of surface charges. At the level of the Carrollian conservation equations, this is interpreted as a conformal anomaly, the radiation sourcing the anomalous factor. The subsequent analysis of the Robinson–Trautman and Kerr–Taub–NUT exact solutions nicely confirms these expectations and the interplay among the bulk and the boundary dynamics.

Our analysis triggers many questions. Among others, the two examples of exact Ricci-flat spacetimes treated here suggest to further investigate the Carrollian interpretation of four-dimensional gravity in full generality, *i.e.* without assuming linearity. More generally, this work may help in paving the road toward the Carrollian understanding of flat holography, already discussed in several instances in the literature.

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A Carrollian Charges algebra

We have defined two types of conserved charges in 5.2 and 5.3, $\mathcal{Q}_\xi$ and $\mathcal{C}_\xi$. The first one is conserved for any type of Carrollian conservation laws given by (4.3), (4.4), (4.5) and (4.6), while the second is conserved only when the Carrollian momenta $\mathcal{B}^i$ vanishes. We recall their expression:

$$\mathcal{Q}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\, \xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right) \quad \text{and} \quad \mathcal{C}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2 b_i \xi^j \mathcal{A}^i_j\right). \tag{A.1}$$

In this appendix we show that both of them are also representations of the (conformal) Carrollian Killing algebra.

Consider two Carrollian Killing vectors ξ and η. It is possible to decompose them in a coordinate basis,

$$\xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i \quad \text{and} \quad \eta = \eta^t(t,\mathbf{x})\partial_t + \eta^i(\mathbf{x})\partial_i, \tag{A.2}$$

or in a Carroll-covariant one,

$$\xi = \frac{X}{\Omega}\partial_t + \xi^i \hat{\partial}_i \quad \text{and} \quad \eta = \frac{Y}{\Omega}\partial_t + \eta^i \hat{\partial}_i, \tag{A.3}$$

where $X = \Omega\xi^t - b_i\xi^i$, $Y = \Omega\eta^t - b_i\eta^i$ and $\hat{\partial}_i$ is the Carroll-covariant spatial derivative defined in 2.2. The commutator of ξ and η is given by

$$\lambda \equiv [\xi, \eta] = \left(\xi^t\partial_t\eta^t - \eta^t\partial_t\xi^t + \xi^k\partial_k\eta^t - \eta^k\partial_k\xi^t\right)\partial_t + \left(\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i\right)\partial_i = \frac{L}{\Omega}\partial_t + \lambda^i\hat{\partial}_i. \tag{A.4}$$

For ξ and η Carrollian Killing vectors, we define the two following quantities

$$
\begin{aligned}
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) \right], \\
\{\mathcal{C}_\xi, \mathcal{C}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2 b_i \xi^j \mathcal{A}^i_j \right) \right],
\end{aligned} \tag{A.5}
$$

where δ_η is the Lie derivative w.r.t. η acting on the metric fields and the momenta, but not on ξ^t and ξ^i. We want to show that, up to boundary terms,

$$
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \mathcal{Q}_{[\xi,\eta]} \quad \text{and} \quad \{\mathcal{C}_\xi, \mathcal{C}_\eta\} = \mathcal{C}_{[\xi,\eta]}, \tag{A.6}
$$

the first result being true for any type of Carrollian conservation laws while the second one holds only when $\mathcal{B}^i = 0$.

We start with the first one, we have

$$
\begin{aligned}
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \Big[&\delta_\eta \sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) + \sqrt{a} (\delta_\eta a_{ik}) \xi^k \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) \\
&+ \sqrt{a}\xi_i \left(\delta_\eta \mathcal{B}^i + \delta_\eta b_j \Sigma^{ji} + b_j \delta_\eta \Sigma^{ji} \right) \Big].
\end{aligned} \tag{A.7}
$$

We compute the infinitesimal variations of the geometric fields and the Carrollian momenta:

$$
\delta_\eta a_{ik} = \eta^t \partial_t a_{ik} + \eta^j \partial_j a_{ik} + \partial_i \eta^j a_{kj} + \partial_k \eta^j a_{ij} = 0, \tag{A.8}
$$

$$
\delta_\eta \sqrt{a} = \eta^i \partial_i \sqrt{a} + \eta^t \partial_t \sqrt{a} + \partial_i \eta^i \sqrt{a} = 0, \tag{A.9}
$$

$$
\delta_\eta b_i = \eta^t \partial_t b_i + \eta^j \partial_j b_i - \Omega \partial_i \eta^t + b_j \partial_i \eta^j, \tag{A.10}
$$

$$
\delta_\eta \mathcal{B}^i = \eta^t \partial_t \mathcal{B}^i + \eta^j \partial_j \mathcal{B}^i - \mathcal{B}^j \partial_j \eta^i, \tag{A.11}
$$

$$
\delta_\eta \Sigma^{ij} = \eta^t \partial_t \Sigma^{ij} + \eta^k \partial_k \Sigma^{ij} - \Sigma^{kj} \partial_k \eta^i - \Sigma^{ik} \partial_k \eta^j. \tag{A.12}
$$

The variation of a_{ik} and $\sqrt{a}$ vanish because η is a Carrollian Killing vector. Then we eliminate every temporal derivative of the Carrollian momenta using the conservation laws (4.5) and (4.6). Finally performing integration by parts and using properties of the Carrollian Killing vectors (5.7) and (5.8), we suppress every spatial derivative of the Carrollian momenta to obtain:

$$
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a} \lambda_i \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) + \text{b.t.} = \mathcal{Q}_\lambda + \text{b.t.}. \tag{A.13}
$$

This proves that the charges $\mathcal{Q}_\xi$ form a representation of the Carrollian Killing algebra.

We now prove the second relation. We have

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \Big[\delta_\eta \sqrt{a}\left((\Omega\xi^t - b_i\xi^i)\mathcal{E} - \xi^i\pi_i + 2b_i\xi^j\mathcal{A}^i_j\right) \\ + \sqrt{a}\left((\delta_\eta\Omega\xi^t - \delta_\eta b_i\xi^i)\mathcal{E} + (\Omega\xi^t - b_i\xi^i)\delta_\eta\mathcal{E} - \xi^i\delta_\eta\pi_i + 2\delta_\eta b_i\xi^j\mathcal{A}^i_j + 2b_i\xi^j\delta_\eta\mathcal{A}^i_j\right)\Big]. \tag{A.14}$$

We compute the infinitesimal variations of the geometric fields and the Carrollian momenta:

$$\delta_\eta\Omega = \eta^t\partial_t\Omega + \eta^i\partial_i\Omega + \Omega\partial_t\eta^t = 0, \tag{A.15}$$

$$\delta_\eta\sqrt{a} = \eta^i\partial_i\sqrt{a} + \eta^t\partial_t\sqrt{a} + \partial_i\eta^i\sqrt{a} = 0, \tag{A.16}$$

$$\delta_\eta b_i = \eta^t\partial_t b_i + \eta^j\partial_j b_i - \Omega\partial_i\eta^t + b_j\partial_i\eta^j, \tag{A.17}$$

$$\delta_\eta\mathcal{E} = \eta^i\partial_i\mathcal{E} + \eta^t\partial_t\mathcal{E}, \tag{A.18}$$

$$\delta_\eta\pi_i = \eta^t\partial_t\pi_i + \eta^j\partial_j\pi_i + \pi_j\partial_i\eta^j, \tag{A.19}$$

$$\delta_\eta\mathcal{A}^i_j = \eta^t\partial_t\mathcal{A}^i_j + \eta^k\partial_k\mathcal{A}^i_j - \mathcal{A}^k_j\partial_k\eta^i + \mathcal{A}^i_k\partial_j\eta^k. \tag{A.20}$$

The variations of Ω and $\sqrt{a}$ are vanishing because η is a Carrollian Killing vector. Then we eliminate every temporal derivative of the Carrollian momenta using the conservation laws (5.14) and (5.15). Finally performing integration by parts and using properties of the Carrollian Killings, (5.7) and (5.8), we suppress every spatial derivative of the Carrollian momenta to obtain:

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\Big[\left(\Omega(\xi^t\partial_t\eta^t - \eta^t\partial_t\xi^t + \xi^k\partial_k\eta^t - \eta^k\partial_k\xi^t) - b_i(\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i)\right)\mathcal{E} \\ - (\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i)\pi_i + 2b_i(\xi^k\partial_k\eta^j - \eta^k\partial_k\xi^j)\mathcal{A}^i_j\Big] + \text{b.t.}, \tag{A.21}$$

which corresponds to

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\left(L\mathcal{E} - \xi^i\pi_i + 2b_i\lambda^j\mathcal{A}^i_j\right) + \text{b.t.} = \mathcal{C}_\lambda + \text{b.t.}. \tag{A.22}$$

Therefore, up to boundary terms, the charges $\mathcal{C}_\xi$ form a representation of the Carrollian Killing algebra.

We can extend the previous results to the conformal Carrollian Killing algebra when imposing $\Sigma^i_i = 0$ and the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$.

References

[1] J.-M. Lévy-Leblond, "Une nouvelle limite non-relativiste du groupe de poincaré," *Annales de l'I.H.P. Physique théorique* **3** no. 1, (1965) 1–12. http://eudml.org/doc/75509.

[2] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids," *Class. Quant. Grav.* **35** no. 16, (2018) 165001, arXiv:1802.05286 [hep-th].

[3] R. Basu and U. N. Chowdhury, "Dynamical structure of Carrollian Electrodynamics," *JHEP* **04** (2018) 111, arXiv:1802.09366 [hep-th].

[4] B. Cardona, J. Gomis, and J. M. Pons, "Dynamics of Carroll Strings," *JHEP* **07** (2016) 050, arXiv:1605.05483 [hep-th].

[5] C. Dappiaggi, V. Moretti, and N. Pinamonti, "Rigorous steps towards holography in asymptotically flat spacetimes," *Rev. Math. Phys.* **18** (2006) 349–416, arXiv:gr-qc/0506069 [gr-qc].

[6] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* **05** (2010) 062, arXiv:1001.1541 [hep-th].

[7] A. Bagchi, S. Detournay, R. Fareghbal, and J. Simón, "Holography of 3D Flat Cosmological Horizons," *Phys. Rev. Lett.* **110** no. 14, (2013) 141302, arXiv:1208.4372 [hep-th].

[8] R. Fareghbal and A. Naseh, "Flat-Space Energy-Momentum Tensor from BMS/GCA Correspondence," *JHEP* **03** (2014) 005, arXiv:1312.2109 [hep-th].

[9] J. Hartong, "Gauging the Carroll Algebra and Ultra-Relativistic Gravity," *JHEP* **08** (2015) 069, arXiv:1505.05011 [hep-th].

[10] J. Hartong, "Holographic Reconstruction of 3D Flat Space-Time," *JHEP* **10** (2016) 104, arXiv:1511.01387 [hep-th].

[11] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, "2D Stress Tensor for 4D Gravity," *Phys. Rev. Lett.* **119** no. 12, (2017) 121601, arXiv:1609.00282 [hep-th].

[12] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Flat holography and Carrollian fluids," *JHEP* **07** (2018) 165, arXiv:1802.06809 [hep-th].

[13] F. Hopfmüller and L. Freidel, "Null Conservation Laws for Gravity," *Phys. Rev.* **D97** no. 12, (2018) 124029, arXiv:1802.06135 [gr-qc].

[14] C. Duval, G. W. Gibbons, and P. A. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class. Quant. Grav.* **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[15] F. Alessio and G. Esposito, "On the structure and applications of the Bondi – Metzner – Sachs group," *Int. J. Geom. Meth. Mod. Phys.* **15** no. 02, (2018) 1830002, arXiv:1709.05134 [gr-qc].

[16] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, "Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time," *Class. Quant. Grav.* **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[17] R. Loganayagam, "Entropy Current in Conformal Hydrodynamics," *JHEP* **05** (2008) 087, arXiv:0801.3701 [hep-th].

[18] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," *JHEP* **12** (2008) 116, arXiv:0809.4272 [hep-th].

[19] M. Henneaux, "Geometry of zero signature spacetime," *Bull.Soc.Math.Belg. 31 47-63* (1979) .

[20] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, "Perfect Fluids," *SciPost Phys.* **5** (2018) 003, arXiv:1710.04708 [hep-th].

[21] G. Compère, "Surfacecharges,". http://www.ulb.ac.be/sciences/ptm/pmif/gcompere/package.html.

[22] G. Barnich, A. Gomberoff, and H. A. Gonzalez, "The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes," *Phys. Rev.* **D86** (2012) 024020, arXiv:1204.3288 [gr-qc].

[23] A. Campoleoni, D. Francia, and C. Heissenberg, "Asymptotic Charges at Null Infinity in Any Dimension," *Universe* **4** no. 3, (2018) 47, arXiv:1712.09591 [hep-th].

[24] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **12** (2011) 105, arXiv:1106.0213 [hep-th].

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Two-dimensional fluids and their holographic duals

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Abstract

We describe the dynamics of two-dimensional relativistic and Carrollian fluids. These are mapped holographically to three-dimensional locally anti-de Sitter and locally Minkowski spacetimes, respectively. To this end, we use Eddington–Finkelstein coordinates, and grant general curved two-dimensional geometries as hosts for hydrodynamics. This requires to handle the conformal anomaly, and the expressions obtained for the reconstructed bulk metrics incorporate non-conformal-fluid data. We also analyze the freedom of choosing arbitrarily the hydrodynamic frame for the description of relativistic fluids. This freedom breaks down in the dual gravitational picture, and fluid/gravity correspondence turns out to be sensitive to dissipation processes: the fluid heat current is a necessary ingredient for reconstructing all Bañados asymptotically anti-de Sitter solutions. The same feature emerges for Carrollian fluids, which enjoy a residual frame invariance, and their Barnich–Troessaert locally Minkowski duals. These statements are proven by computing the algebra of surface conserved charges in the fluid-reconstructed bulk three-dimensional spacetimes.

Contents

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1 Introduction

Fluid/gravity correspondence is a macroscopic spin-off of holography, originally mapping relativistic fluid configurations onto Einstein spacetimes. These are obtained in the form of a derivative expansion [1–4], inspired from the fluid homonymous expansion (see *e.g.* [5,6]), and implemented in Eddington–Finkelstein coordinates.

Compared to the Fefferman–Graham expansion [7,8], the derivative expansion has the following distinctive features:

- the spacetime metric is expanded using a null direction rather than a spatial one;
- the boundary data include a vector congruence, interpreted as the fluid velocity field, whose derivatives set the order of the expansion;
- the derivative expansion is generically well behaved in the bulk flat limit.

The third property has recently allowed to set up a derivative expansion for asymptotically flat spacetimes, establishing thereby, at least macroscopically, a holographic correspondence among Ricci-flat bulk solutions and boundary Carrollian hydrodynamics [9]. The second feature raises another important question, regarding the role played by the boundary fluid velocity.

The fluid velocity field is absent in the Fefferman–Graham expansion, which provides an Einstein bulk reconstruction solely based on the boundary metric and the boundary energy–momentum tensor. This should not be a surprise because the velocity field of a relativistic fluid is not a physical observable. To some extent it can be chosen freely, altering neither the energy–momentum tensor nor the entropy current, but only transforming the various pieces that enter the decomposition of these quantities with respect to its longitudinal and transverse directions [10].

However, the fluid congruence appears explicitly in the derivative expansion (it actually organizes the latter). Following the above logic, it should be possible to transform it while keeping unchanged the boundary metric and energy–momentum tensor, and this should not affect the reconstructed bulk metric. This reasoning is too naive, though. Indeed when writing the derivative expansion, some implicit gauge choice may be made, partly locking the form of the velocity. If this does not happen, the velocity transformation is expected to be reabsorbed by some appropriate bulk diffeomorphism. Such a diffeomorphism is possibly a large one, in which case the two fluid congruences definitely lead to two distinct dual spacetimes.

Analyzing the role of the velocity field in the fluid/gravity derivative expansion is not an easy task. Generically the derivative expansion is given in the form of a series, built on Weyl covariance, and furthermore assuming the Landau–Lifshitz frame, as in [1–4]. In this framework, it is difficult to disentangle the various contributions and investigate the behaviour under a congruence transformation. In some more specific classes, it is possible to resum the derivative expansion (see [11–15]), if we abandon the Landau–Lifshitz frame and impose integrability conditions relating the heat current and stress tensor to the boundary geometry. The latter blur the transformation properties under a change of fluid congruence.

One aim of the present work is to analyze the role of the fluid congruence in an instance where these problems are overcome. This occurs in three bulk dimensions because all expansions, Fefferman–Graham or derivative, are naturally truncated to a finite number of terms, and because asymptotically anti-de Sitter spacetimes are necessarily locally anti-de Sitter. A class of such spacetimes is known as *Bañados* solutions [16,17], labeled unambiguously with their conserved surface charges. Hence, showing that the fluid velocity cannot be chosen at wish, as naively expected, is within reach.

Ricci-flat spacetimes are dual to Carrollian hydrodynamics emerging at null infinity [18]. In some instances, Carrollian fluids possess a residual frame invariance involving a kinematical parameter reminiscent of the relativistic velocity field. The latter enters the flat derivative expansion, and it is legitimate to ask the same questions as for the anti-de Sitter spacetimes. Again, answering is possible in three dimensions, in which case all Ricci-flat spacetimes compatible with a set of fall-off conditions are described in [19], again in terms of their surface-charge algebra. These are locally Minkowskian and will be referred to as

Barnich–Troessaert solutions.

In order to undertake the above analysis we must rely on robust derivative expansions.[1] In other words, we need expressions that provide the bulk dual (Einstein or Ricci-flat) of an arbitrary fluid, hosted by any two-dimensional geometry. Such expressions are not available in full generality for the relativistic fluids, and are unknown for Carrollian fluids. Another goal we have pursued here is to settle them. For the Carrollian case, our fluid reconstruction of flat spacetimes resembles the general formulas given in BMS (Bondi–Metzner–Sachs) gauge in [19].[2] In the relativistic case, we exhibit a universal resummation formula, which turns out to be a BMS-like alternative to the existing Fefferman–Graham expression [17, 19]. Its prime virtue is to accommodate the conformal anomaly arising from the curvature of the boundary, which has a detectable counterpart in the Carrollian situation.

The output of the above analysis regarding the freedom in hydrodynamic frame confirms our suspicion. Indeed, computing the asymptotic charges,[3] we show that the holographic reconstruction of all AdS and flat spacetimes requires the boundary fluid (relativistic or Carrollian) have a non-vanishing heat current. In this instance, the charge algebra is either Virasoro or BMS with the expected central charges. Dismissing the heat current, the solutions carry surface charges obeying algebras of the same type, with vanishing central charges though. This is typical of non-spinning BTZ zero modes [23–25] and of their flat counterparts, including angular defects or excesses (see [26] for a global view on both situations).

In Sec. 2 we review two-dimensional relativistic conformal fluid dynamics, and expand its Carrollian limit, insisting on the hydrodynamic-frame invariance. Section 3 is devoted to the general method of holographic reconstruction of asymptotically AdS and flat spacetimes. This method is applied in Sec. 4 for flat two-dimensional boundary metrics, without loosing generality, and followed by the computation of charges, which enables us to reach a conclusive analysis on the solutions under investigation.

2 Two-dimensional fluids

2.1 Relativistic fluids

General properties

We consider a two-dimensional geometry $\mathscr{M}$ equipped with a metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. The dynamics of a relativistic fluid is captured by the energy–momentum tensor $\mathrm{T} = T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$,

[1] *Expansion* is an abuse of terminology in three dimensions because there, it is naturally truncated. We will often make it, and use the word *resummation* for simple sums .

[2] In three dimensions the derivative expansion, implemented in Eddington–Finkelstein coordinates, has falloffs similar to those of the BMS gauge. A slight difference will be stressed in due time. This is not true in higher dimension.

[3] Useful references for the analysis of asymptotic charges are *e.g.* [20, 21]. Our surface charge computations have been performed using the package [22], built using the conventions of the papers just quoted.

which is symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and generally obeys:

$$\nabla^\mu T_{\mu\nu} = f_\nu, \tag{2.1}$$

where f_ν is an external force density. Together with the equation of state (local thermodynamic equilibrium is assumed), this set of equations provide the hydrodynamic equations of motion. Normalizing the velocity congruence u as $\|\mathrm{u}\|^2 = -k^2$, we can in general decompose the energy–momentum tensor as

$$T_{\mu\nu} = (\varepsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2} \tag{2.2}$$

with p the local pressure and ε the local energy density:

$$\varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu. \tag{2.3}$$

The symmetric viscous stress tensor $\tau_{\mu\nu}$ and the heat current q_μ are purely transverse:

$$u^\mu \tau_{\mu\nu} = 0, \quad u^\mu q_\mu = 0, \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{2.4}$$

In two dimensions, the transverse direction with respect to u is entirely supported by the Hodge-dual $*$u:[4]

$$* u_\rho = u^\sigma \eta_{\sigma\rho}. \tag{2.5}$$

This dual congruence is space-like and normalized as $\| * \mathrm{u}\|^2 = k^2$. Therefore

$$\mathrm{q} = \chi * \mathrm{u} \quad \text{with} \quad \chi = -\frac{1}{k^2} * u^\mu T_{\mu\nu} u^\nu, \tag{2.6}$$

the local *heat density*, appearing here as the magnetic dual of the energy density. Similarly, the viscous stress tensor has a unique component encoded in the *viscous stress scalar* τ:[5]

$$\tau_{\mu\nu} = \tau h_{\mu\nu} \quad \text{with} \quad h_{\mu\nu} = \frac{1}{k^2} * u_\mu * u_\nu \tag{2.7}$$

the projector onto the space transverse to the velocity field. The trace reads: $T^\mu{}_\mu = p - \varepsilon + \tau$.

The pressure p and the viscous stress scalar τ appear in the fully transverse component of the energy–momentum tensor. Their sum is therefore the total stress. If the system is free and at *global* equilibrium, τ vanishes and the stress is given by the thermodynamic pressure p alone. Hence, the viscous stress scalar τ is usually expressed as an expansion in temperature and velocity gradients, and this distinguishes it from p. The same holds for the heat current

[4]Our conventions are: $\eta_{\sigma\rho} = \sqrt{g}\epsilon_{\sigma\rho}$ with $\epsilon_{01} = +1$. Hence $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$.

[5]This component of the energy–momentum tensor is also referred to as the *viscous bulk pressure*, or the *dynamic pressure*, or else the *non-equilibrium pressure*.

q. The coefficients of these expansions characterize the transport phenomena occurring in the fluid.

The shear and the vorticity vanish identically in two spacetime dimensions. The only non-vanishing first-derivative tensors of the velocity are the acceleration and the expansion

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{2.8}$$

and one defines similarly the expansion of the dual congruence as[6]

$$\Theta^* = \nabla_\mu * u^\mu, \tag{2.9}$$

which enables us expressing the acceleration:

$$a_\mu = \Theta^* * u_\mu. \tag{2.10}$$

In first-order hydrodynamics[7]

$$\tau_{(1)} = -\zeta\Theta, \tag{2.11}$$
$$\chi_{(1)} = -\frac{\kappa}{k^2}\left(*\mathrm{u}(T) + T\Theta^*\right). \tag{2.12}$$

As usual, ζ is the bulk viscosity and κ is the thermal conductivity – assumed constant in this expression.

It is convenient to use the orthonormal Cartan frame $\{\mathrm{u}/k, *\mathrm{u}/k\}$. Then the metric reads:

$$\mathrm{d}s^2 = \frac{1}{k^2}\left(-\mathrm{u}^2 + *\mathrm{u}^2\right), \tag{2.13}$$

while the energy–momentum tensor takes the form:

$$\mathrm{T} = \frac{1}{2k^2}\left((\varepsilon+\chi)(\mathrm{u} + *\mathrm{u})^2 + (\varepsilon-\chi)(\mathrm{u} - *\mathrm{u})^2\right) + \frac{1}{k^2}(p - \varepsilon + \tau) * \mathrm{u}^2. \tag{2.14}$$

In holographic systems, the boundary enjoys remarkable conformal properties as it defines a conformal class, rather than a specific metric. Under Weyl transformations

$$\mathrm{d}s^2 \to \frac{\mathrm{d}s^2}{\mathcal{B}^2}, \tag{2.15}$$

the velocity form components u_μ are traded for $u_\mu/\mathcal{B}$, the energy and heat densities have

[6]The hodge-dual of a scalar is a two-form and would spell with a suffix star. Instead, Θ^* is just another scalar.

[7]For any vector v and a function f, $\mathrm{v}(f)$ stands for $v^\mu\partial_\mu f$. We remind the following identities: $\mathrm{d}^\dagger\mathrm{d}f = -\Box f$ with $\mathrm{d}^\dagger \mathrm{w} = *\mathrm{d} * \mathrm{w} = -\nabla^\mu w_\mu$ and $\mathrm{d}f = \frac{1}{k^2}\left(*\mathrm{u}(f) * \mathrm{u} - \mathrm{u}(f)\mathrm{u}\right)$, $*\mathrm{d}f = \frac{1}{k^2}\left(*\mathrm{u}(f)\mathrm{u} - \mathrm{u}(f) * \mathrm{u}\right)$.

weight 2, and the local-equilibrium equation of state is conformal

$$\varepsilon = p, \tag{2.16}$$

which is accompanied by Stefan's law (σ is the Stefan–Boltzmann constant):

$$\varepsilon = \sigma T^2. \tag{2.17}$$

Hence, the trace of the energy–momentum tensor is τ. In the absence of anomalies it vanishes and $T_{\mu\nu}$ is invariant under (2.15). If τ is non-vanishing, the fluid is not conformal and τ is an anomalous weight-2 quantity.

Covariantization with respect to rescalings requires to introduce a Weyl connection one-form:[8]

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \Theta \mathrm{u}\right) = \frac{1}{k^2}\left(\Theta^* * \mathrm{u} - \Theta \mathrm{u}\right), \tag{2.18}$$

which transforms as $\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathcal{B}$. Ordinary covariant derivatives ∇ are thus traded for Weyl covariant ones $\mathscr{D} = \nabla + w\,\mathrm{A}$, w being the conformal weight of the tensor under consideration. We provide for concreteness the Weyl covariant derivative of a form v_μ and of a scalar function Φ, both of weight w:

$$\begin{aligned} \mathscr{D}_\nu v_\mu &= \nabla_\nu v_\mu + (w+1) A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu} A^\rho v_\rho, \\ \mathscr{D}_\nu \Phi &= \partial_\nu \Phi + w A_\nu \Phi. \end{aligned} \tag{2.19}$$

The Weyl covariant derivative is metric-compatible with effective torsion:

$$\mathscr{D}_\rho g_{\mu\nu} = 0, \tag{2.20}$$

$$\left(\mathscr{D}_\mu \mathscr{D}_\nu - \mathscr{D}_\nu \mathscr{D}_\mu\right) f = w f F_{\mu\nu}, \tag{2.21}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.22}$$

is the Weyl-invariant field strength. Its dual

$$F = *\mathrm{dA} = \eta^{\mu\nu}\partial_\mu A_\nu = \frac{1}{k^2}\left(*\mathrm{u}(\Theta) - \mathrm{u}(\Theta^*)\right) \tag{2.23}$$

is a weight-2 scalar.

Commuting the Weyl-covariant derivatives acting on vectors, one defines the Weyl covariant Riemann tensor

$$\left(\mathscr{D}_\mu \mathscr{D}_\nu - \mathscr{D}_\nu \mathscr{D}_\mu\right) V^\rho = \mathscr{R}^\rho{}_{\sigma\mu\nu} V^\sigma + w F_{\mu\nu} V^\rho \tag{2.24}$$

[8] The explicit form of A is obtained by demanding $\mathscr{D}_\mu u^\mu = 0$ and $u^\lambda \mathscr{D}_\lambda u_\mu = 0$.

(V^ρ are weight-w) and the usual subsequent quantities. In two spacetime dimensions, the covariant Ricci tensor (weight-0) and the scalar (weight-2) curvatures read:

$$\mathscr{R}_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda - F_{\mu\nu}, \tag{2.25}$$
$$\mathscr{R} = R + 2\nabla_\mu A^\mu. \tag{2.26}$$

It turns out that $R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda$ vanishes identically. Hence

$$\mathscr{R} = 0 \Leftrightarrow R = 2\mathrm{d}^\dagger \mathrm{A} \quad \text{and} \quad \mathscr{R}_{\mu\nu} = -F_{\mu\nu}. \tag{2.27}$$

The ordinary scalar curvature has a weight-2 anomalous transformation

$$R \to \mathcal{B}^2 \left(R + 2\Box \ln \mathcal{B}\right) \tag{2.28}$$

(the box operator is here referring to the metric before the Weyl transformation).

Hydrodynamic equations and the hydrodynamic-frame covariance

Using the above tools as well as the identity

$$\nabla^\mu T_{\mu\nu} = \mathscr{D}^\mu T_{\mu\nu} - A_\nu T^\mu{}_\mu, \tag{2.29}$$

(based on Eqs. (2.19) and Leibniz rule, for a weight-0, rank-2 symmetric tensor), the general fluid equations (2.1) with $\varepsilon = p$, projected on the light-cone directions $\mathrm{u} \pm *\mathrm{u}$ read:[9]

$$\begin{cases} (u^\mu + *u^\mu)\,\mathscr{D}_\mu\,(\varepsilon + \chi) + (u^\mu - *u^\mu)\,f_\mu = -\Theta\tau - \Theta^*\tau - *\mathrm{u}(\tau), \\ (u^\mu - *u^\mu)\,\mathscr{D}_\mu\,(\varepsilon - \chi) + (u^\mu + *u^\mu)\,f_\mu = -\Theta\tau + \Theta^*\tau + *\mathrm{u}(\tau). \end{cases} \tag{2.30}$$

Equivalently, these equations are expressed as

$$\begin{cases} \mathrm{d}\left(\sqrt{\varepsilon + \chi + \tau/2}\,(\mathrm{u} + *\mathrm{u})\right) + \dfrac{1}{2\sqrt{\varepsilon + \chi + \tau/2}}(\mathrm{u} - *\mathrm{u}) \wedge *\left(\mathrm{f} - \tfrac{1}{2}\mathrm{d}\tau\right) = 0, \\ \mathrm{d}\left(\sqrt{\varepsilon - \chi + \tau/2}\,(\mathrm{u} - *\mathrm{u})\right) - \dfrac{1}{2\sqrt{\varepsilon - \chi + \tau/2}}(\mathrm{u} + *\mathrm{u}) \wedge *\left(\mathrm{f} - \tfrac{1}{2}\mathrm{d}\tau\right) = 0. \end{cases} \tag{2.31}$$

Changing hydrodynamic frame, *i.e.* the fluid velocity field, amounts to perform an arbitrary local Lorentz transformation on the Cartan mobile frame

$$\begin{pmatrix} \mathrm{u}' \\ *\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\psi(x) & \sinh\psi(x) \\ \sinh\psi(x) & \cosh\psi(x) \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ *\mathrm{u} \end{pmatrix}, \tag{2.32}$$

[9] Notice that any congruence with $w = -1$ in two dimensions obeys $\mathscr{D}_\mu u_\nu = \nabla_\mu u_\nu + \frac{1}{k^2} u_\mu a_\nu - \Theta h_{\mu\nu} = 0$ due to the absence of shear and vorticity, and similarly $\mathscr{D}_\mu * u_\nu = 0$.

or for the null directions $\mathrm{u}' \pm *\mathrm{u}' = (\mathrm{u} \pm *\mathrm{u})\, \mathrm{e}^{\pm\psi}$. This affects the Weyl connection and Weyl curvature scalar as follows

$$\mathrm{A}' = \mathrm{A} - *\mathrm{d}\psi \tag{2.33}$$
$$F' = F + \Box\psi. \tag{2.34}$$

The transformation (2.32) keeps the energy–momentum tensor invariant provided the energy density and the heat density transform appropriately. Imposing that in the new frame (2.16) holds, *i.e.* $\varepsilon' = p'$, we conclude that

$$\begin{pmatrix} \varepsilon' \\ \chi' \end{pmatrix} = \begin{pmatrix} \cosh 2\psi(x) & -\sinh 2\psi(x) \\ -\sinh 2\psi(x) & \cosh 2\psi(x) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \chi \end{pmatrix} + \tau \sinh\psi(x) \begin{pmatrix} \sinh\psi(x) \\ -\cosh\psi(x) \end{pmatrix}, \tag{2.35}$$

while, due to the invariance of the trace,

$$\tau' = \tau. \tag{2.36}$$

Equivalently one can use $\sqrt{\left(\varepsilon' \pm \chi' + \frac{\tau'}{2}\right)} = \sqrt{\left(\varepsilon \pm \chi + \frac{\tau}{2}\right)}\, \mathrm{e}^{\mp\psi}$.

The energy–momentum tensor can be diagonalized with a specific local Lorentz transformation. By definition, the corresponding hydrodynamic frame is the Landau–Lifshitz frame, where the heat current χ_{LL} is vanishing. We find

$$\mathrm{T} = \frac{\varepsilon_{\mathrm{LL}}}{k^2} \mathrm{u}_{\mathrm{LL}}^2 + \frac{\varepsilon_{\mathrm{LL}} + \tau}{k^2} * \mathrm{u}_{\mathrm{LL}}^2 \tag{2.37}$$

since $\tau_{\mathrm{LL}} = \tau$ and $\chi_{\mathrm{LL}} = 0$. The latter condition allows to find the local boost towards the Landau–Lifshitz frame

$$\mathrm{e}^{4\psi_{\mathrm{LL}}} = \frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}. \tag{2.38}$$

With this, the eigenvalues are easily computed. One finds the Landau–Lifshitz energy density

$$\varepsilon_{\mathrm{LL}} = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} - \frac{\tau}{2}. \tag{2.39}$$

It exhibits an upper bound for χ^2, $\chi^2_{\max} = (\varepsilon + \tau/2)^2$, which translates causality and unitarity properties of the underlying microscopic field theory. The eigenvalue[10] $\varepsilon_{\mathrm{LL}}$ is supported by the time-like eigenvector

$$\mathrm{u}_{\mathrm{LL}} = \frac{1}{2}\left(\left(\frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}\right)^{1/4} (\mathrm{u} + *\mathrm{u}) + \left(\frac{\varepsilon - \chi + \tau/2}{\varepsilon + \chi + \tau/2}\right)^{1/4} (\mathrm{u} - *\mathrm{u})\right), \tag{2.40}$$

[10] We make for simplicity the implicit assumption that the energy density is positive. This needs not be true, however, and the holographic fluid dual to pure AdS_3 has indeed negative energy.

whereas

$$\varepsilon^*_{\mathrm{LL}} = \varepsilon_{\mathrm{LL}} + \tau = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} + \frac{\tau}{2} \tag{2.41}$$

is the eigenvalue along the space-like eigenvector $*\mathrm{u}_{\mathrm{LL}}$. Using the above expressions in the Landau–Lifshitz frame, the fluid equations (2.31) are recast as follows

$$\begin{cases} 2\sqrt{\varepsilon_{\mathrm{LL}}}\,\mathrm{d}^\dagger\left(\sqrt{\varepsilon_{\mathrm{LL}}}\,\mathrm{u}_{\mathrm{LL}}\right) - \mathrm{u}_{\mathrm{LL}}\cdot \mathrm{f} - \Theta_{\mathrm{LL}}\tau = 0\,, \\ 2\sqrt{\varepsilon^*_{\mathrm{LL}}}\,\mathrm{d}^\dagger\left(\sqrt{\varepsilon^*_{\mathrm{LL}}} * \mathrm{u}_{\mathrm{LL}}\right) + *\mathrm{u}_{\mathrm{LL}}\cdot \mathrm{f} + \Theta^*_{\mathrm{LL}}\tau = 0\,. \end{cases} \tag{2.42}$$

A non-anomalous conformal fluid in two dimensions is defined through the relations (2.16), (2.17) and

$$\tau = 0. \tag{2.43}$$

Under these assumptions, the last term of (2.14) drops, whereas following the fluid equations (2.31) at zero external force ($\mathrm{f} = f_\mu \mathrm{d}x^\mu = 0$), the forms $\sqrt{\varepsilon \pm \chi}(\mathrm{u} \pm *\mathrm{u})$ are *closed*, and can be used to define a privileged light-cone coordinate system, adapted to the fluid configuration. In this specific case, the on-shell Weyl scalar curvature reads

$$F = -\frac{1}{2}\Box \ln \sqrt{\frac{\varepsilon + \chi}{\varepsilon - \chi}}. \tag{2.44}$$

For conformal fluids, the hydrodynamic-frame transformation (2.32) acts on the energy and heat densities as a spin-two electric–magnetic boost, the energy being electric and the heat magnetic.

The entropy current

We would like to close this overview on two-dimensional conformal fluids with the entropy current. The entropy appears in Gibbs–Duhem equation

$$Ts = p + \varepsilon, \tag{2.45}$$

and is easily computed for conformal fluids in terms of the energy density, using Eq. (2.16) and Stefan's law (2.17):

$$s = 2\sqrt{\sigma\varepsilon}. \tag{2.46}$$

The entropy current is an involved concept. In arbitrary dimension, there is no generic and closed expression in terms of the dissipative tensors for this current, which is generally constructed order by order as a derivative expansion (see [27]). Whether this expansion can be hydrodynamic-frame invariant, and at the same time compatible with the underlying already quoted microscopic laws (unitarity and causality) as well as with the second law of thermodynamics is not known in full generality, although this is in principle part of the rationale behind frame invariance.

In two dimensions, the ingredients for building a hydrodynamic-frame-invariant entropy current are the time-like invariant vector u_{LL} (given in (2.40)) and its space-like dual $*\mathrm{u}_{\mathrm{LL}}$, plus the invariant scalars $\varepsilon_{\mathrm{LL}}$ and $\varepsilon^*_{\mathrm{LL}}$ (or any combination, see (2.39) and (2.41)). The entropy current should have non-negative divergence, vanishing for a free (*i.e.* at zero external force) perfect fluid. In the case at hand, a perfect fluid is necessarily conformal since it must have vanishing τ.

A good candidate for a hydrodynamic-frame-invariant entropy current is

$$\mathrm{S}_0 = s_{\mathrm{LL}}\mathrm{u}_{\mathrm{LL}} = 2\sqrt{\sigma\varepsilon_{\mathrm{LL}}}\,\mathrm{u}_{\mathrm{LL}}, \tag{2.47}$$

which can be expressed in any frame using Eqs. (2.39) and (2.40). This is usually adopted as the entropy current of a perfect fluid, and in that case it is divergence-free when external forces vanish. Here, it obeys (see (2.42))

$$\nabla\cdot\mathrm{S}_0 = -\sqrt{\frac{\sigma}{\varepsilon_{\mathrm{LL}}}}\left(\Theta_{\mathrm{LL}}\tau+\mathrm{u}_{\mathrm{LL}}\cdot\mathrm{f}\right) = -\frac{1}{T_{\mathrm{LL}}}\left(\Theta_{\mathrm{LL}}\tau+\mathrm{u}_{\mathrm{LL}}\cdot\mathrm{f}\right), \tag{2.48}$$

which can be recast in terms of arbitrary-frame data using the already quoted (2.39), (2.40) and the divergence of the latter. Expanding this result up to first order for $\chi,\tau\ll\varepsilon$, we find for a free fluid

$$\nabla\cdot\mathrm{S}_{0(1)} = -\frac{1}{T}\Theta\tau = \frac{\zeta}{T}\Theta^2, \tag{2.49}$$

where we have used in the last equality the first-order derivative expansion of τ, given in (2.11). For this to be positive one finds the usual requirement $\zeta>0$. From this perspective, the current S_0 seems fine.

The expansion of S_0 up to second order in $\chi,\tau\ll\varepsilon$,

$$\mathrm{S}_0 = 2\sqrt{\sigma\varepsilon}\,\mathrm{u}+\chi\sqrt{\frac{\sigma}{\varepsilon}}*\mathrm{u}-\frac{\chi^2}{4\varepsilon}\sqrt{\frac{\sigma}{\varepsilon}}\mathrm{u}-\frac{\tau\chi}{2\varepsilon}\sqrt{\frac{\sigma}{\varepsilon}}*\mathrm{u}+\cdots = s\mathrm{u}+\frac{\mathrm{q}}{T}-\frac{\chi^2}{4\varepsilon T}\mathrm{u}-\frac{\tau}{2\varepsilon T}\mathrm{q}+\cdots, \tag{2.50}$$

is in agreement with the usual expectations dictated by *extended irreversible thermodynamics* (completing the first-order *classical irreversible thermodynamics*) [27]. These can be summarized as follows, the order referring to the dissipative expansion:

1. free perfect limit: $\mathrm{S}|_{\chi=\tau=0} = \mathrm{S}_{(0)} = s\mathrm{u} = 2\sqrt{\sigma\varepsilon}\,\mathrm{u}$;
2. stability $\left.\frac{\partial\mathrm{S}\cdot\mathrm{u}}{\partial\tau}\right|_{\chi=\tau=0} = 0$;
3. first-order (CIT) correction: $\mathrm{S}_{(1)} = \frac{\mathrm{q}}{T}$;
4. second-order (EIT) corrections: $\mathrm{S}_{(2)}$ might contain $\frac{\tau^2}{\varepsilon T}\mathrm{u}$, $\frac{\chi^2}{\varepsilon T}\mathrm{u}$ and $\frac{\tau}{\varepsilon T}\mathrm{q}$;
5. second law: $\nabla\cdot\mathrm{S}\geqslant 0$.

Other invariant terms may be considered in the definition of S as long as the above requirements are satisfied. In the absence of a concrete proposal for selecting other terms, we will not pursue the argument any further. Related discussions can be found in [28–31].[11]

2.1.1 Light-cone versus Randers–Papapetrou frames

Light-cone frame Every two-dimensional metric is amenable by diffeomorphisms to a conformally flat form. This suggests to use:[12]

$$\mathrm{d}s^2 = \mathrm{e}^{-2\omega}\mathrm{d}x^+\mathrm{d}x^- \tag{2.51}$$

(with usual time and space coordinates defined as $x^\pm = x \pm kt$), where ω is an arbitrary function of x^+ and x^-.

Any normalized congruence has the following form:

$$\mathrm{u} = u_+\mathrm{d}x^+ + u_-\mathrm{d}x^- \quad \Leftrightarrow \quad *\mathrm{u} = -u_+\mathrm{d}x^+ + u_-\mathrm{d}x^-, \tag{2.52}$$

where $u_\pm$, functions of x^+ and x^-, are related by the normalization condition

$$u_+u_- = -\frac{k^2}{4}\mathrm{e}^{-2\omega}. \tag{2.53}$$

We can parameterize the velocity field as

$$u_+ = -\frac{k}{2}\mathrm{e}^{-\omega}\sqrt{\xi}, \quad u_- = \frac{k}{2}\mathrm{e}^{-\omega}\frac{1}{\sqrt{\xi}}, \tag{2.54}$$

where $\xi = \xi(x^+, x^-)$ is defined as the ratio

$$\xi = -\frac{u_+}{u_-}. \tag{2.55}$$

The choice $\xi = 1$ corresponds to a comoving fluid because in this case $\mathrm{u} = -k^2\mathrm{e}^{-\omega}\mathrm{d}t$.

For the congruence at hand

$$\Theta \pm \Theta^* = \pm 2k\mathrm{e}^{2\omega}\partial_\pm \mathrm{e}^{-\left(\omega \pm \ln\sqrt{\xi}\right)}. \tag{2.56}$$

We can also determine the Weyl connection and field strength:

$$\mathrm{A} = -\mathrm{d}\omega + *\mathrm{d}\ln\sqrt{\xi} \quad \text{and} \quad F = -\Box \ln\sqrt{\xi} = -2\mathrm{e}^{2\omega}\partial_+\partial_-\ln\xi, \tag{2.57}$$

[11] It should be quoted that S as defined in (2.47) does not coincide with the entropy current proposed in Ref. [31]. Hydrodynamic-frame invariance and CIT/EIT arguments were not part of the agenda in this work, based essentially on the second law of thermodynamics.

[12] With this choice, $g_{+-} = 1/2\,\mathrm{e}^{-2\omega}$, $\eta_{+-} = 1/2\,\mathrm{e}^{-2\omega}$, $\eta^{+-} = -2\mathrm{e}^{2\omega}$, $\eta_+{}^+ = 1$, $\eta_-{}^- = -1$. Notice also that $*\left(\mathrm{d}x^+ \wedge \mathrm{d}x^-\right) = \eta^{+-} = -2\mathrm{e}^{2\omega}$.

whereas the ordinary (non Weyl-covariant) scalar curvature reads (see (2.27))

$$R = 2\Box\omega = 8\mathrm{e}^{2\omega}\partial_+\partial_-\omega. \tag{2.58}$$

In the present light-cone frame $\{\mathrm{d}x^+, \mathrm{d}x^-\}$, the components of a general energy–momentum tensor, with $\epsilon = p$, are

$$\begin{gathered} T_{++} = \frac{\xi}{2}\left(\varepsilon - \chi + \frac{\tau}{2}\right)\mathrm{e}^{-2\omega}, \quad T_{--} = \frac{1}{2\xi}\left(\varepsilon + \chi + \frac{\tau}{2}\right)\mathrm{e}^{-2\omega}, \\ T_{+-} = T_{-+} = \frac{\tau}{4}\mathrm{e}^{-2\omega}. \end{gathered} \tag{2.59}$$

For a conformal fluid Eqs. (2.43) lead to $T_{+-} = T_{-+} = 0$ and

$$(\varepsilon + \chi)(\varepsilon - \chi) = 4\mathrm{e}^{4\omega}T_{++}T_{--}, \quad \frac{\varepsilon + \chi}{\varepsilon - \chi} = \frac{T_{--}}{T_{++}}\xi^2. \tag{2.60}$$

In the latter case, and in the absence of external forces, the forms (2.31) are closed, which in light-cone coordinates implies that $(\varepsilon - \chi)\mathrm{e}^{-2\omega}\xi$ is locally a function of x^+, and $(\varepsilon + \chi)\frac{\mathrm{e}^{-2\omega}}{\xi}$ a function of x^-. Observe that in the Landau–Lifshitz frame ($\chi_{\mathrm{LL}} = 0$)

$$\xi^2_{\mathrm{LL}} = \frac{T_{++}}{T_{--}}, \quad \varepsilon^2_{\mathrm{LL}} = 4\mathrm{e}^{4\omega}T_{++}T_{--}. \tag{2.61}$$

In this frame, on-shell, F vanishes. Moving from a given hydrodynamic frame to another by a local Lorentz boost, amounts to perform the following transformation on the function ξ

$$\xi(x^+, x^-) \to \xi'(x^+, x^-) = \mathrm{e}^{-2\psi(x^+, x^-)}\xi(x^+, x^-). \tag{2.62}$$

Randers–Papapetrou frame The light-cone frame is not well suited for the Carrollian limit, which is the ultra-relativistic limit reached at vanishing k, and emerging at the null-infinity conformal boundary of a flat spacetime (subject of next section). As discussed in [18], Carrollian fluid dynamics is elegantly reached in the Randers–Papapetrou frame, where

$$\mathrm{d}s^2 = -k^2\left(\Omega\mathrm{d}t - b_x\mathrm{d}x\right)^2 + a\mathrm{d}x^2 \tag{2.63}$$

with all three functions of the coordinates t and x.

A generic velocity vector field u reads:

$$\mathrm{u} = \gamma\left(\partial_t + v^x\partial_x\right). \tag{2.64}$$

It is convenient to parametrize the velocity v^x (see [18]) as[13]

$$v^x = \frac{k^2\Omega\beta^x}{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{b}} \Leftrightarrow \beta^x = \frac{v^x}{k^2\Omega\left(1-\frac{v^x b_x}{\Omega}\right)} \tag{2.65}$$

with Lorentz factor

$$\gamma = \frac{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{b}}{\Omega\sqrt{1-k^2\boldsymbol{\beta}^2}}. \tag{2.66}$$

The velocity form and its Hodge-dual read:

$$\mathrm{u} = -\frac{k^2}{\sqrt{1-k^2\boldsymbol{\beta}^2}}\left(\Omega \mathrm{d}t - (b_x+\beta_x)\,\mathrm{d}x\right), \quad *\mathrm{u} = k\sqrt{a}\,\Omega\gamma\left(\mathrm{d}x - v^x\mathrm{d}t\right), \tag{2.67}$$

while the corresponding vector is

$$*\,\mathrm{u} = \frac{k}{\sqrt{a}\sqrt{1-k^2\boldsymbol{\beta}^2}}\left(\frac{b_x+\beta_x}{\Omega}\partial_t + \partial_x\right). \tag{2.68}$$

We can determine the form of the heat current q, which must be proportional to $*\mathrm{u}$, in terms of a single component q_x. We find

$$\chi = \frac{q_x}{k\sqrt{a}\,\Omega\gamma} = \frac{q^x\sqrt{a}\sqrt{1-k^2\boldsymbol{\beta}^2}}{k}. \tag{2.69}$$

Similarly, for the viscous stress tensor

$$\tau = \frac{\tau_{xx}}{a\Omega^2\gamma^2} = \tau^{xx}a\left(1-k^2\boldsymbol{\beta}^2\right). \tag{2.70}$$

Performing a local Lorentz boost (2.32) on the hydrodynamic frame does not affect the geometric objects Ω, b_x or a, and is thus entirely captured by the transformation of the vector $\boldsymbol{\beta}$. Parameterizing the boost in terms of a Carrollian vector $\boldsymbol{B} = B^x\partial_x$ as

$$\cosh\psi = \Gamma = \frac{1}{\sqrt{1-k^2\boldsymbol{B}^2}}, \quad \sinh\psi = \Gamma k\sqrt{a}\,B^x = \frac{k\sqrt{a}\,B^x}{\sqrt{1-k^2\boldsymbol{B}^2}}, \tag{2.71}$$

we get:

$$\boldsymbol{\beta}' = \frac{\boldsymbol{\beta}+\boldsymbol{B}}{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{B}}, \tag{2.72}$$

as expected from the velocity rule composition in special relativity. Using (2.35), we also

[13] With these definitions, β^x transforms as the component of a genuine Carrollian vector $\boldsymbol{\beta} = \beta^x\partial_x$, when considering the flat limit of the bulk spacetime. Notice that $\beta_x + b_x = -\frac{\Omega u_x}{k u_0}$. We define as usual $b^x = a^{xx}b_x$, $\beta_x = a_{xx}\beta^x$, $v_x = a_{xx}v^x$ with $a_{xx} = 1/a^{xx} = a$, $\boldsymbol{b}^2 = b_x b^x$, $\boldsymbol{\beta}^2 = \boldsymbol{\beta}\cdot\boldsymbol{\beta} = \beta_x\beta^x$ and $\boldsymbol{b}\cdot\boldsymbol{\beta} = b_x\beta^x$.

obtain

$$\varepsilon' = \frac{1}{1-k^2\boldsymbol{B}^2}\left(\left(1+k^2\boldsymbol{B}^2\right)\varepsilon - k\sqrt{a}B^x 2\chi + k^2\boldsymbol{B}^2\tau\right), \tag{2.73}$$

$$\chi' = \frac{1}{1-k^2\boldsymbol{B}^2}\left(\left(1+k^2\boldsymbol{B}^2\right)\chi - k\sqrt{a}B^x(2\varepsilon+\tau)\right), \tag{2.74}$$

accompanying (2.36). Together with (2.69) and (2.70), we finally reach:

$$\frac{q'_x}{\sqrt{a}} = \left(\left(1+k^2\boldsymbol{B}^2\right)\chi - k\sqrt{a}B^x(2\varepsilon+\tau)\right)k\frac{\left(1+k^2\left(\boldsymbol{\beta}\cdot\boldsymbol{B}+(\boldsymbol{\beta}+\boldsymbol{B})\cdot\boldsymbol{b}\right)\right)}{\left(1-k^2\boldsymbol{\beta}^2\right)^{1/2}\left(1-k^2\boldsymbol{B}^2\right)^{3/2}}, \tag{2.75}$$

$$\frac{\tau'_{xx}}{a} = \tau\frac{\left(1+k^2\left(\boldsymbol{\beta}\cdot\boldsymbol{B}+(\boldsymbol{\beta}+\boldsymbol{B})\cdot\boldsymbol{b}\right)\right)^2}{\left(1-k^2\boldsymbol{\beta}^2\right)\left(1-k^2\boldsymbol{B}^2\right)}. \tag{2.76}$$

2.2 Carrollian fluids

The Carrollian geometry

The Carrollian geometry $\mathbb{R}\times\mathscr{S}$ is obtained as the vanishing-k limit of the two-dimensional pseudo-Riemannian geometry $\mathscr{M}$ equipped with metric (2.63). In this limit, the line $\mathscr{S}$ inherits a metric[14]

$$\mathrm{d}\ell^2 = a\,\mathrm{d}x^2, \tag{2.77}$$

and $t\in\mathbb{R}$ is the Carrollian time. Much like a Galilean space is observed from a spatial frame moving with respect to a local inertial frame with velocity $\mathbf{w}$, a Carrollian frame is described by a form $\boldsymbol{b} = b_x(t,x)\,\mathrm{d}x$. The latter is *not* a velocity because in Carrollian spacetimes motion is forbidden. It is rather an inverse velocity, describing a "temporal frame" and plays a dual role. A scalar $\Omega(t,x)$ also remains in the $k\to 0$ limit (as in the Galilean case, see [18] – this reference will be useful along the present section).

We define the Carrollian diffeomorphisms as

$$t' = t'(t,x) \quad \text{and} \quad x' = x'(x). \tag{2.78}$$

The ordinary exterior derivative of a scalar function does not transform as a form. To overcome this issue, it is desirable to introduce a Carrollian derivative as

$$\hat{\partial}_x = \partial_x + \frac{b_x}{\Omega}\partial_t, \tag{2.79}$$

transforming as a form. With this derivative we can proceed and define a Carrollian covariant derivative $\hat{\nabla}_x$, based on Levi–Civita–Carroll connection

$$\hat{\gamma}^x_{xx} = \hat{\partial}_x \ln\sqrt{a}. \tag{2.80}$$

[14]This metric lowers all x indices.

As we will see in 3.2, in the framework of flat holography, the spatial surface $\mathscr{S}$ emerges as the null infinity $\mathscr{I}^+$ of the Ricci-flat geometry. The geometry of $\mathscr{I}^+$ is equipped with a conformal class of metrics rather than with a metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to study conformal Carrollian geometry as opposed to plain Carrollian geometry (see [32]).

The action of Weyl transformations on the elements of the Carrollian geometry on a surface $\mathscr{S}$ is inherited from (2.15)

$$a \to \frac{a}{\mathcal{B}^2}, \quad b_x \to \frac{b_x}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \quad \beta_x \to \frac{\beta_x}{\mathcal{B}}, \tag{2.81}$$

where $\mathcal{B} = \mathcal{B}(t,x)$ is an arbitrary function. However, the Levi–Civita–Carroll covariant derivatives are not covariant under (2.81). Following [18], they must be replaced with Weyl–Carroll covariant spatial and time metric-compatible derivatives built on the Carrollian acceleration φ_x and the Carrollian expansion θ,

$$\varphi_x = \frac{1}{\Omega}(\partial_t b_x + \partial_x \Omega) = \partial_t \frac{b_x}{\Omega} + \hat{\partial}_x \ln \Omega, \tag{2.82}$$

$$\theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \tag{2.83}$$

which transform as connections:

$$\varphi_x \to \varphi_x - \hat{\partial}_x \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{1}{\Omega}\partial_t \mathcal{B}. \tag{2.84}$$

In particular, these can be combined in[15]

$$\alpha_x = \varphi_x - \theta b_x, \tag{2.85}$$

transforming under Weyl rescaling as

$$\alpha_x \to \alpha_x - \partial_x \ln \mathcal{B}. \tag{2.86}$$

The spatial Weyl–Carrol derivative is

$$\hat{\mathscr{D}}_x \Phi = \hat{\partial}_x \Phi + w \varphi_x \Phi, \tag{2.87}$$

for a weight-w scalar function Φ, and

$$\hat{\mathscr{D}}_x V^x = \hat{\nabla}_x V^x + (w-1)\varphi_x V^x, \tag{2.88}$$

[15]Contrary to φ_x, α_x is not a Carrollian one-form, *i.e.* it does not transform covariantly under Carrollian diffeomorphisms (2.78).

for a vector with weight-w component V^x. It does not alter the conformal weight, and is generalized to any tensor by Leibniz rule.

Similarly we define the temporal Weyl–Carroll derivative by its action on a weight-w function Φ

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi = \frac{1}{\Omega}\partial_t\Phi + w\theta\Phi, \tag{2.89}$$

which is a scalar of weight $w+1$ under (2.81). Accordingly, the action of the Weyl–Carroll time derivative on a weight-w vector is

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^x = \frac{1}{\Omega}\partial_t V^x + w\theta V^x. \tag{2.90}$$

This is the component of a genuine Carrollian vector of weight $w+1$, and Leibinz rule allows to generalize this action to any tensor.

The Weyl–Carroll connections have curvature. Here, the only non-vanishing piece is the curvature one-form resulting from the commutation of $\hat{\mathscr{D}}_x$ and $\frac{1}{\Omega}\hat{\mathscr{D}}_t$, which has weight 1:

$$\mathscr{R}_x = \frac{1}{\Omega}\left(\partial_t\alpha_x - \partial_x(\theta\Omega)\right) = \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{2.91}$$

Carrollian fluid observables

A relativistic fluid satisfying Eq. (2.1) will obey Carrollian dynamics at vanishing k. The original relativistic fluid is not at rest, but has a velocity parametrized with $\boldsymbol{\beta} = \beta_x \mathrm{d}x$ (see (2.65)), which remains in the Carrollian limit as the kinematical "inverse-velocity" variable. We will keep calling it abusively "velocity". This variable transforms as a Carrollian vector and allows to define further kinematical objects.

- We introduce the acceleration $\boldsymbol{\gamma} = \gamma_x \mathrm{d}x$

$$\gamma_x = \frac{1}{\Omega}\partial_t\beta_x. \tag{2.92}$$

  This is not Weyl-covariant, as opposed to

$$\delta_x = \gamma_x - \theta\beta_x = \frac{\sqrt{a}}{\Omega}\partial_t\frac{\beta_x}{\sqrt{a}}, \tag{2.93}$$

  which has weight 0.

- The suracceleration is the weight-1 conformal Carrollian one-form

$$\mathscr{A}_x = \frac{1}{\Omega}\hat{\mathscr{D}}_t\frac{1}{\Omega}\hat{\mathscr{D}}_t\beta_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right). \tag{2.94}$$

It can be combined with the curvature (2.91), which has equal weight,

$$s_x = \mathscr{A}_x + \mathscr{R}_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right) + \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{2.95}$$

This appears as a conformal Carrollian total (*i.e.* kinematical plus geometric) suracceleration, and enables us to define a weight-2 conformal Carrollian scalar:

$$s = \frac{s_x}{\sqrt{a}}. \tag{2.96}$$

The latter originates from the Weyl curvature F of the pseudo-Riemannian ascendent manifold $\mathscr{M}$:

$$s = -\lim_{k\to 0} kF. \tag{2.97}$$

Notice that the ordinary scalar curvature of $\mathscr{M}$ given in (2.27) is not Weyl-covariant (see (2.28)) and can be expressed in terms of Carrollian non-Weyl-covariant scalars of $\mathbb{R}\times\mathscr{S}$:

$$R = \frac{2}{k^2}\left(\theta^2 + \frac{1}{\Omega}\partial_t\theta\right) - 2\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x. \tag{2.98}$$

Besides the inverse velocity, acceleration and suracceleration, other physical data describe a Carrollian fluid.

- The energy density ε and the pressure p, related here through $\varepsilon = p$. The Carrollian energy and pressure are the zero-k limits of the corresponding relativistic quantities, and have weight 2. It is implicit that they are finite, and in order to avoid inflation of symbols, we have kept the same notation.

- The heat current $\boldsymbol{\pi} = \pi_x(t,x)\mathrm{d}x$ of conformal weight 1, inherited from the relativistic heat current (see (2.2)) as follows:[16]

$$q^x = k^2\pi^x + \mathrm{O}\left(k^4\right). \tag{2.99}$$

 This translates the expected (see (2.69)) small-k behaviour of χ:

$$\chi = \chi_\pi k + \mathrm{O}\left(k^3\right), \tag{2.100}$$

[16]In arbitrary dimensions one generally admits $q^x = Q^x + k^2\pi^x + \mathrm{O}\left(k^4\right)$ (see [18]), which amounts assuming $\chi = \frac{\chi_Q}{k} + \chi_\pi k + \mathrm{O}\left(k^3\right)$. This is actually more natural because vanishing χ_Q is not a hydrodynamic-frame-invariant feature in the presence of friction. Keeping $\chi_Q \neq 0$, however, is not viable holographically in two boundary dimensions because it would create a $1/k^2$ divergence inside the derivative expansion. Since the Carrollian limit destroys anyway the hydrodynamic-frame invariance, our choice is consistent from every respect. Ultimately these behaviours should be justified within a microscopic quantum/statistical approach, missing at present.

leading to

$$\pi^x = \frac{\chi_\pi}{\sqrt{a}}. \tag{2.101}$$

- The weight-0 viscous stress tensors $\mathbf{\Sigma} = \Sigma_{xx}\mathrm{d}x^2$ and $\mathbf{\Xi} = \Xi_{xx}\mathrm{d}x^2$, obtained from the relativistic viscous stress tensor $\frac{\tau}{k^2} * \mathrm{u} * \mathrm{u}$ as

$$\tau^{xx} = -\frac{\Sigma^{xx}}{k^2} - \Xi^{xx} + \mathrm{O}\left(k^2\right). \tag{2.102}$$

  For this to hold, following (2.70), we expect

$$\tau = \frac{\tau_\Sigma}{k^2} + \tau_\Xi + \mathrm{O}\left(k^2\right), \tag{2.103}$$

  and find (in the Carrollian geometry, indices are lowered with $a_{xx} = a$):

$$\Sigma^x{}_x = -\tau_\Sigma, \quad \Xi^x{}_x = -\tau_\Xi - \boldsymbol{\beta}^2\tau_\Sigma. \tag{2.104}$$

  As we will see later, this is in agreement with the form of τ for the relativistic systems at hand (see Eqs. (2.98) and (3.2)).

- Finally, we assume that the components of the external force density behave as follows, providing further Carrollian power and tension:

$$\begin{cases} \frac{k}{\Omega} f_0 = \frac{f}{k^2} + e + \mathrm{O}\left(k^2\right), \\ f^x = \frac{h^x}{k^2} + g^x + \mathrm{O}\left(k^2\right). \end{cases} \tag{2.105}$$

Hydrodynamic equations

The hydrodynamic equations for a Carrollian fluid are obtained as the zero-k limit of the relativistic equations (see [18]):

$$-\left(\frac{1}{\Omega}\partial_t + 2\theta\right)\left(\varepsilon - \boldsymbol{\beta}^2\Sigma^x{}_x\right) + \left(\hat{\nabla}^x + 2\varphi^x\right)\left(\beta_x\Sigma^x{}_x\right) + \theta\left(\Xi^x{}_x - \boldsymbol{\beta}^2\Sigma^x{}_x\right) = e, \tag{2.106}$$

$$\theta\Sigma^x{}_x = f, \tag{2.107}$$

$$\left(\hat{\nabla}_x + \varphi_x\right)\left(\varepsilon - \Xi^x{}_x\right) + \varphi_x\left(\varepsilon - \boldsymbol{\beta}^2\Sigma^x{}_x\right) + \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\pi_x + \beta_x\left(2\varepsilon - \Xi^x{}_x\right)\right) = g_x, \tag{2.108}$$

$$-\left(\hat{\nabla}_x + \varphi_x\right)\Sigma^x{}_x - \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\beta_x\Sigma^x{}_x\right) = h_x. \tag{2.109}$$

Generically, the above equations are not invariant under Carrollian local boosts, acting as

$$\beta'_x = \beta_x + B_x \tag{2.110}$$

(vanishing-k limit of (2.72)). This should not come as a surprise. Such an invariance is exclusive to the relativistic case for obvious physical reasons, and is also known to be absent from Galilean fluid equations, which are not invariant under local Galilean boosts. Nevertheless, as we will see in Sec. 4, in specific situations a residual invariance persists.

3 Three-dimensional bulk reconstruction

3.1 Anti-de Sitter

Three-dimensional Einstein spacetimes are peculiar because the usual derivative expansion terminates at finite order. This happens also for the Fefferman–Graham expansion (see *e.g.* [17]). The reason is that most geometric and fluid tensors vanish (like the shear or the vorticity), reducing the number of available terms compatible with conformal invariance. As opposed to higher dimension, the heat current can nevertheless enter directly. We obtain:

$$\boxed{\mathrm{d}s^2_{\text{Einstein}} = 2\frac{\mathrm{u}}{k^2}\left(\mathrm{d}r + r\mathrm{A}\right) + r^2\mathrm{d}s^2 + \frac{8\pi G}{k^4}\mathrm{u}\left(\varepsilon\mathrm{u} + \chi * \mathrm{u}\right),} \tag{3.1}$$

where A is displayed in (2.18), ε and χ being the energy and heat densities of the fluid. These enter the fluid energy–momentum tensor (2.14) together with τ, which carries the anomaly:

$$\tau = \frac{R}{8\pi G} = \frac{1}{4\pi G k^2}\left(\Theta^2 - \Theta^{*2} + \mathrm{u}(\Theta) - *\mathrm{u}(\Theta^*)\right) \tag{3.2}$$

(we keep the conformal state equation $\varepsilon = p$). For a flat boundary this anomaly is absent, but Weyl transformations bring it back.

The metric (3.1) provides an *exact* Einstein, asymptotically AdS spacetime with $R = 6\Lambda = -6k^2$, under the necessary and sufficient condition that the non-conformal fluid energy–momentum tensor (2.14) obeys

$$\nabla^\mu\left(T_{\mu\nu} + D_{\mu\nu}\right) = 0, \tag{3.3}$$

where $D_{\mu\nu}$ is a symmetric and traceless tensor which reads:

$$D_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{1}{8\pi G k^4}\left(\left(\mathrm{u}(\Theta) + *\mathrm{u}(\Theta^*) - \frac{k^2}{2}R\right)\left(\mathrm{u}^2 + *\mathrm{u}^2\right) - 4 * \mathrm{u}(\Theta)\mathrm{u} * \mathrm{u}\right). \tag{3.4}$$

On the one hand, the holographic energy–momentum tensor is the sum $T_{\mu\nu} + D_{\mu\nu}$, and this can be shown following the Balasubramanian–Kraus method [33].[17] On the other hand, the holographic fluid is subject to an external force with density

$$f_\nu = -\nabla^\mu D_{\mu\nu}. \tag{3.5}$$

[17] For this computation we used the conventions of [34].

Its longitudinal and transverse components are

$$\begin{cases} u^\mu f_\mu = -\frac{1}{4\pi G}\left(*\mathrm{u}(F) + 2\Theta^* F + \frac{1}{2}\Theta R\right), \\ *u^\mu f_\mu = \frac{1}{8\pi G}\left(*\mathrm{u}(R) + \Theta^* R\right). \end{cases} \tag{3.6}$$

Combining (2.30), (3.2) and (3.6) we find the following equations:

$$\begin{cases} (u^\mu + *u^\mu)\mathscr{D}_\mu(\varepsilon + \chi) = \frac{1}{4\pi G} * u^\mu \mathscr{D}_\mu F, \\ (u^\mu - *u^\mu)\mathscr{D}_\mu(\varepsilon - \chi) = \frac{1}{4\pi G} * u^\mu \mathscr{D}_\mu F. \end{cases} \tag{3.7}$$

Notice that eventually these equations are Weyl-covariant (weight-3) despite the conformal anomaly.

An important remark is in order regarding the holographic fluid. Rather than $T_{\mu\nu}$, we could have adopted $T_{\mu\nu} + D_{\mu\nu}$ as its energy–momentum tensor. The latter would have been decomposed as in (2.2), with $\tilde{\varepsilon} = \tilde{p}$ and $\tilde{\chi}$ though ($\tilde{\tau} = \tau$ since $D_{\mu\nu}$ has vanishing trace):

$$\tilde{\varepsilon} = \varepsilon + \frac{1}{8\pi G k^2}\left(\mathrm{u}(\Theta) + *\mathrm{u}(\Theta^*)\right) - \frac{R}{16\pi G}, \tag{3.8}$$

$$\tilde{\chi} = \chi - \frac{1}{4\pi G k^2} * \mathrm{u}(\Theta). \tag{3.9}$$

We did not make this choice for two reasons: (i) in the formula (3.1) we used ε and χ rather than $\tilde{\varepsilon}$ and $\tilde{\chi}$ for reconstructing the bulk; (ii) ε and χ/k are finite in the limit of vanishing k, whereas $\tilde{\varepsilon}$ and $\tilde{\chi}/k$ are not. This last fact is not an obstruction, but it would require to reconsider the Carrollian hydrodynamic equations developed in Ref. [18] and applied here.

Expression (3.1) is the most general locally AdS spacetime in Eddington–Finkelstein coordinates. The corresponding gauge includes but does not always coincide with BMS.[18] From that perspective, this result is new although it may not contain any new solutions compared to Bañados', all captured either in BMS or in Fefferman–Graham gauge (see [19]). The bonus is the hydrodynamical interpretation. Here the corresponding fluid is defined on a generally curved boundary and has an arbitrary velocity field. This should be contrasted with the treatment of three-dimensional fluid/gravity correspondence worked out in Refs. [2, 3], where the host geometry was flat, avoiding the issue of conformal anomaly. Furthermore the fluid was assumed perfect by hydrodynamic-frame choice, which permits a subclass of Bañados solutions only, as we will see in Sec. 4 by computing the conserved charges.

For practical purposes, we can work in light-cone coordinates, introduced in Eq. (2.51). Using the expression (2.54) for the congruence u, and solving the fluid equations (3.7), we

[18]There is no definition of Eddington–Finkelstein gauge. Within the three-dimensional derivative expansion, one can nevertheless refer to it as a gauge because the r-dependence is fixed. This does not exhaust all freedom, but allows comparison with BMS. Actually, fluid/gravity approach is not meant to lock completely the coordinates for describing the most general solution in terms of a minimal set of functions.

obtain the fluid densities ε and χ in terms of two arbitrary chiral functions

$$\varepsilon = \frac{e^{2\omega}}{4\pi G}\left(\frac{\ell_+}{\xi}+\xi\ell_- - \frac{3\left(\partial_+\xi\right)^2}{4\xi^3}+\frac{\partial_+^2\xi}{2\xi^2}+\frac{\left(\partial_-\xi\right)^2}{4\xi}-\frac{\partial_-^2\xi}{2}\right), \tag{3.10}$$

$$\chi = \frac{e^{2\omega}}{4\pi G}\left(-\frac{\ell_+}{\xi}+\xi\ell_- + \frac{3\left(\partial_+\xi\right)^2}{4\xi^3}-\frac{\partial_+^2\xi}{2\xi^2}+\frac{\left(\partial_-\xi\right)^2}{4\xi}-\frac{\partial_-^2\xi}{2}+\frac{\partial_+\xi\partial_-\xi}{\xi^2}-\frac{\partial_+\partial_-\xi}{\xi}\right). \tag{3.11}$$

Gathering these data inside (3.1) provides, in the gauge at hand, the general class of locally AdS three-dimensional spacetime with curved conformal boundary. The conformal factor $\exp 2\omega$ plays actually no role because, as one readily sees from the above expressions, it can be reabsorbed with the redefinition of r into $r\exp\omega$, bringing (3.1) to its flat-boundary form.[19] As we will shortly see, the arbitrary function $\xi(x^+,x^-)$ is more insidious regarding the charges.

We could proceed and display similar expressions in the Randers–Papapetrou boundary frame, describing the general locally anti-de Sitter spacetimes in terms of the three geometric data $\Omega(t,x)$, $b_x(t,x)$ and $a_{xx}=a(t,x)$, and whatever integration functions would appear in the process of solving the hydrodynamic equations (3.7). Usually, this resolution cannot be conducted explicitly as it happens in light-cone coordinates, and we end up with an implicit description of the bulk metric. We should quote here that a specific example of curved boundary[20] was investigated in Ref. [35], outside of the fluid/gravity framework, and the output agrees with our general results. We should also stress, following the discussion of footnote 18, that the Randers–Papapetrou boundary frame produces in (3.1) order-r $\mathrm{d}t\mathrm{d}x$ components absent in the BMS gauge.

3.2 Ricci-flat

Our starting point is the finite derivative expansion of an asymptotically AdS_3 spacetime, Eq. (3.1). The fundamental question is whether the latter admits a smooth zero-k limit.

We have implicitly assumed that the Randers–Papapetrou data of the two-dimensional pseudo-Riemannian conformal boundary $\mathscr{I}$ associated with the original Einstein spacetime, a, b and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new spatial one-dimensional boundary $\mathscr{S}$ emerging at $\mathscr{I}^+$. Following again the detailed analysis performed in [18], we can match the various two-dimensional Riemannian quantities with the corresponding one-dimensional Carrollian ones:

$$\mathrm{u} = -k^2\left(\Omega\mathrm{d}t-\left(b_x+\beta_x\right)\mathrm{d}x\right)+\mathrm{O}\left(k^4\right),\quad *\mathrm{u} = k\sqrt{a}\,\mathrm{d}x+\mathrm{O}\left(k^3\right) \tag{3.12}$$

[19] This should be contrasted with the more intricate situation regarding this conformal factor inside the analogous formula in Fefferman–Graham gauge, Eq. (2.21) of Ref. [19].

[20] In that case $\Omega=\exp 2\beta$, $b_x=0$, $a=1$ and, in our language, the fluid velocity would have been $\mathrm{u}=-k^2 e^{2\beta}\mathrm{d}t$, *i.e.* comoving.

and

$$\begin{aligned} \Theta &= \theta + \mathrm{O}\left(k^2\right), \\ \mathrm{a} &= k^2\left(\varphi_x + \gamma_x\right)\mathrm{d}x + \mathrm{O}\left(k^4\right), \\ \mathrm{A} &= \theta\Omega\mathrm{d}t + \left(\alpha_x + \delta_x\right)\mathrm{d}x + \mathrm{O}\left(k^2\right), \end{aligned} \tag{3.13}$$

where the left-hand-side quantities are Riemannian, and the right-hand-side ones Carrollian (see (2.82), (2.83), (2.85), (2.92), (2.93)).

The closed form (3.1) is smooth at zero k. In this limit the metric reads:

$$\boxed{\begin{aligned} \mathrm{d}s^2_{\text{flat}} = &-2\left(\Omega\mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\left(\mathrm{d}r + r\left(\boldsymbol{\varphi} + \boldsymbol{\gamma} + \theta\left(\Omega\mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\right)\right) \\ &+ r^2\mathrm{d}\ell^2 + 8\pi G\left(\Omega\mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\left(\varepsilon\left(\Omega\mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right) - \boldsymbol{\pi}\right), \end{aligned}} \tag{3.14}$$

Here $\mathrm{d}\ell^2$, Ω, $\boldsymbol{b} = b_x\mathrm{d}x$, $\boldsymbol{\varphi} = \varphi_x\mathrm{d}x$ and θ are the Carrollian geometric objects introduced earlier. The bulk Ricci-flat spacetime is now dual to a Carrollian fluid with kinematics captured in $\boldsymbol{\beta} = \beta_x\mathrm{d}x$ and $\boldsymbol{\gamma} = \gamma_x\mathrm{d}x$, energy density ε (zero-k limit of the corresponding relativistic function), and heat current $\boldsymbol{\pi} = \pi_x\mathrm{d}x$ (obtained in Eqs.(2.99), (2.100) and (2.101)).

For the fluid under consideration, there is also a pair of Carrollian stress tensors originating from the anomaly (3.2). Using expressions (2.98) and (2.103), we can determine τ_Σ and τ_Ξ, and Eqs. (2.104) provide in turn the Carrollian stress:

$$\Sigma^x{}_x = -\frac{1}{4\pi G}\left(\theta^2 + \frac{\partial_t\theta}{\Omega}\right), \quad \Xi^x{}_x = \frac{1}{4\pi G}\left(\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x - \boldsymbol{\beta}^2\left(\theta^2 + \frac{\partial_t\theta}{\Omega}\right)\right). \tag{3.15}$$

This is the Carrollian emanation of the relativistic conformal anomaly.

Expression (3.14) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (3.1) was Einstein are fulfilled in the zero-k limit. These are the set of Carrollian hydrodynamic equations (2.106), (2.107), (2.108) and (2.109), with Carrollian power and force densities e, f, g_x, h_x obtained using their definition (2.105) and the expressions of f_μ displayed in (3.6). Equations (2.107) and (2.109) are automatically satisfied, whereas (2.106) and (2.108) lead to[21]

$$\begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{4\pi G}\left(\frac{2s_x}{\Omega}\hat{\mathscr{D}}_t\beta^x + \frac{\beta_x}{\Omega}\hat{\mathscr{D}}_t s^x + \hat{\mathscr{D}}^x s_x\right) = 0, \\ \hat{\mathscr{D}}_x\varepsilon - \frac{\beta_x}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{\Omega}\hat{\mathscr{D}}_t\left(\pi_x + 2\varepsilon\beta_x\right) = 0 \end{cases} \tag{3.16}$$

with s_x given in (2.95). The unknown functions, which bear the fluid configuration, are $\varepsilon(t,x)$, $\pi_x(t,x)$ and $\beta_x(t,x)$. These cannot be all determined by the two equations at hand. Hence, there is some redundancy, originating from the relativistic fluid frame invariance – responsible *e.g.* for the arbitrariness of $\xi(x^+,x^-)$ in the description of AdS spacetimes using

[21] We remind that Weyl–Carroll covariant derivatives are defined in Eqs. (2.87), (2.88), (2.89) and (2.90). Here ε, β^x, π_x and s^x have weights 2, 1, 1 and 3. For example $\hat{\mathscr{D}}_x s^x = \hat{\nabla}_x s^x + 2\varphi_x s^x = \frac{1}{\sqrt{a}}\hat{\partial}_x\left(\sqrt{a}s^x\right) + 2\varphi_x s^x$.

the light-cone boundary frame. More will be said about this in Sec. 4.2.

Equations (3.16) are Carroll–Weyl covariant. The Ricci-flat line element (3.14) inherits Weyl invariance from its relativistic ancestor. The set of transformations (2.81), (2.84) and (2.86), supplemented with $\varepsilon \to \mathcal{B}^2\varepsilon$ and $\pi_x \to \mathcal{B}\pi_x$, can indeed be absorbed by setting $r \to \mathcal{B}r$, resulting thus in the invariance of (3.14). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this is rooted to the location of the one-dimensional spatial boundary $\mathscr{S}$ at null infinity $\mathscr{I}^+$.

We would like to close this chapter with a specific but general enough situation to encompass all Barnich–Troessaert Ricci-flat three-dimensional spacetimes. The Carrollian geometric data are $b_x = 0$, $\Omega = 1$ and $a = \exp 2\Phi(t,x)$, and the kinematic variable of the Carrollian dual fluid β_x is left free. Hence (3.14) reads:

$$\begin{aligned} \mathrm{d}s^2_{\text{flat}} &= -2\left(\mathrm{d}t - \beta_x \mathrm{d}x\right)\left(\mathrm{d}r + r\left(\partial_t\Phi \mathrm{d}t + \left(\partial_t - \partial_t\Phi\right)\beta_x \mathrm{d}x\right)\right) \\ &\quad + r^2 \mathrm{e}^{2\Phi}\mathrm{d}x^2 + 8\pi G\left(\mathrm{d}t - \beta_x\mathrm{d}x\right)\left(\varepsilon\mathrm{d}t - \left(\pi_x + \varepsilon\beta_x\right)\mathrm{d}x\right), \end{aligned} \tag{3.17}$$

where $\varepsilon(t,x)$ and $\pi(t,x)$ obey Eqs. (3.16) in the form

$$\begin{cases} \left(\partial_t + 2\partial_t\Phi\right)\varepsilon + \dfrac{1}{4\pi G}\left(2s_x\left(\partial_t + \partial_t\Phi\right)\beta^x + \beta_x\left(\partial_t + 3\partial_t\Phi\right)s^x + \left(\partial_x + \partial_x\Phi\right)s^x\right) = 0, \\ \partial_x\varepsilon + \left(\partial_t + \partial_t\Phi\right)\pi_x + 2\varepsilon\partial_t\beta_x + \beta_x\partial_t\varepsilon = 0. \end{cases} \tag{3.18}$$

Here, s_x takes the simple form

$$s_x = \partial_t^2\beta_x - \partial_t\left(\beta_x\partial_t\Phi\right) - \partial_t\partial_x\Phi. \tag{3.19}$$

For vanishing β_x, the results (3.17) and (3.18) coincide precisely with those obtained in [19] by demanding Ricci-flatness in the BMS gauge. Here, they are reached from purely Carrollian-fluid considerations, and for generic $\beta_x(t,x)$, the metric (3.17) lays outside the BMS gauge.

4 Two-dimensional flat boundary and conserved charges

We will now restrict the previous analysis to non-anomalous and Weyl-flat boundaries, both in AdS and Ricci-flat spacetimes. This enables us to compute the conserved charges, and analyze the role of the velocity and the heat current of the boundary fluid.

4.1 Charges in AdS spacetimes

The flatness requirements are equivalent to setting $R = 0$ and $F = 0$. In the light-cone frame (2.51), this amounts to (see (2.57) and (2.58))

$$\omega = 0 \quad \text{and} \quad \xi(x^+, x^-) = -\frac{\xi^-(x^-)}{\xi^+(x^+)}, \tag{4.1}$$

where the minus sign is conventional.

Using the general solutions (3.10) and (3.11) in the bulk expression (3.1), and trading the chiral functions $\ell_\pm$ for $L_\pm$ defined as

$$\ell_\pm = \frac{1}{(\xi^\pm)^2}\left(L_\pm - \frac{(\xi^{\pm\prime})^2 - 2\xi^\pm\xi^{\pm\prime\prime}}{4}\right), \tag{4.2}$$

we obtain the following metric:

$$\begin{aligned} \mathrm{d}s^2_{\text{Einstein}} \;=\; & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}\mathrm{d}x^+ - \sqrt{-\frac{\xi^+}{\xi^-}}\mathrm{d}x^-\right)\mathrm{d}r \\ & + \left(\frac{L_+}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{+\prime}\right)\left(\frac{\mathrm{d}x^+}{\xi^+}\right)^2 + \left(\frac{L_-}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{-\prime}\right)\left(\frac{\mathrm{d}x^-}{\xi^-}\right)^2 \\ & + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}(\xi^{+\prime} + \xi^{-\prime}) + \frac{L_+ + L_-}{k^2\xi^+\xi^-}\right)\mathrm{d}x^+\mathrm{d}x^-. \end{aligned} \tag{4.3}$$

This metric depends on four arbitrary functions: $\xi^+(x^+)$ and $\xi^-(x^-)$ carrying information about the holographic fluid velocity (see (2.54)), and $L_+(x^+)$, $L_-(x^-)$, which together with $\xi^+(x^+)$ and $\xi^-(x^-)$ shape the energy–momentum tensor – here traceless due to the absence of anomaly for flat boundaries. Indeed we have

$$\varepsilon = -\frac{1}{4\pi G}\frac{L_+ + L_-}{\xi^+\xi^-}, \quad \chi = \frac{1}{4\pi G}\frac{L_+ - L_-}{\xi^+\xi^-}, \tag{4.4}$$

and in turn

$$T_{\pm\pm} = \frac{L_\pm}{4\pi G(\xi^\pm)^2}. \tag{4.5}$$

In three dimensions, any Einstein spacetime is locally anti-de Sitter. Hence, there exists always a coordinate transformation that can be used to bring it into a canonical AdS_3 form. This is a large gauge transformation whenever the original Einstein spacetime has non-trivial conserved charges. The determination of the latter is therefore crucial for a faithful identification of the solution under consideration. It allows to evaluate the precise role played by the above arbitrary functions.

The charge computation requires a complete family of asymptotic Killing vectors. Those are determined according to the gauge, *i.e.* to the fall-off behaviour at large-r. The family (4.3) does not fit BMS gauge, unless $\xi^\pm$ are constant. This is equivalent to saying that the fluid has a uniform velocity, and can therefore be set at rest by an innocuous global Lorentz boost tuning $\xi^+ = 1$ and $\xi^- = -1$.[22] We will first focus on this case, where the asymptotic Killing vectors are known, and move next to the other extreme, demanding the fluid be perfect, *i.e.* in Landau–Lifshitz hydrodynamic frame. In the latter instance we will have to determine this family of vectors beforehand, as the gauge will no longer be BMS. Investigating the general situation captured by (4.3) is not relevant for our argument, which is meant to show that fluid/gravity holographic reconstruction is hydrodynamic-frame dependent.

Dissipative static fluid As anticipated, this class of solutions is reached by demanding $\xi^\pm = \pm 1$, while keeping $L^\pm$ arbitrary. We obtain

$$\mathrm{d}s^2_{\text{Einstein}} = -\frac{1}{k}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)\mathrm{d}r + r^2\mathrm{d}x^+\mathrm{d}x^- + \frac{1}{k^2}\left(L_+\mathrm{d}x^+ - L_-\mathrm{d}x^-\right)\left(\mathrm{d}x^+ - \mathrm{d}x^-\right), \quad (4.6)$$

which is the canonical expression of Bañados solutions in BMS gauge. Following (4.4), the boundary fluid energy and heat densities are $\varepsilon = {}^{1}\!/_{4\pi G}\left(L_+ + L_-\right)$ and $\chi = -{}^{1}\!/_{4\pi G}\left(L_+ - L_-\right)$. Therefore the heat current is not vanishing, and in the present hydrodynamic frame the fluid is at rest and dissipative.

The class of metrics (4.6) are form-invariant under

$$\zeta = \zeta^r\partial_r + \zeta^+\partial_+ + \zeta^-\partial_- \quad (4.7)$$

with

$$\begin{aligned}\zeta^r &= -\frac{r}{2}\left(Y^{+\prime} + Y^{-\prime}\right) + \frac{1}{2k}\left(Y^{+\prime\prime} - Y^{-\prime\prime}\right)\\ &\quad - \frac{1}{2k^2 r}\left(L_+ - L_-\right)\left(Y^{+\prime} - Y^{-\prime}\right), \quad (4.8)\\ \zeta^\pm &= Y^\pm - \frac{1}{2kr}\left(Y^{+\prime} - Y^{-\prime}\right), \quad (4.9)\end{aligned}$$

for arbitrary chiral functions $Y^+(x^+)$ and $Y^-(x^-)$. These vector fields generate diffeomorphisms, which alter the functions appearing in (4.6) according to

$$-\mathscr{L}_\zeta g_{MN} = \delta_\zeta g_{MN} = \frac{\partial g_{MN}}{\partial L_+}\delta_\zeta L_+ + \frac{\partial g_{MN}}{\partial L_-}\delta_\zeta L_- \quad (4.10)$$

with

$$\delta_\zeta L_\pm = -Y^\pm L_\pm' - 2L_\pm Y^{\pm\prime} + \frac{1}{2}Y^{\pm\prime\prime\prime}. \quad (4.11)$$

[22] Observe that one may reabsorb ξ^+ and ξ^- by redefining $\mathrm{d}x^\pm \to \xi^\pm\mathrm{d}x^\pm$ and $r \to {}^{r}\!/\sqrt{-\xi^+\xi^-}$ inside (4.3). This does not prove, however, that $\xi^\pm$ play no role, and this is why we treat them separately.

The last term in this expression is responsible for the emergence of a central charge in the surface-charge algebra. These vectors obey an algebra for the modified Lie bracket (see e.g. [19]):

$$\zeta_3 = [\zeta_1, \zeta_2]_{\mathrm{M}} = [\zeta_1, \zeta_2] - \delta_{\zeta_2}\zeta_1 + \delta_{\zeta_1}\zeta_2 \tag{4.12}$$

with[23] $\zeta_a = \zeta(Y_a^+, Y_a^-)$ and

$$Y_3^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm. \tag{4.13}$$

The surface charges are computed for an arbitrary metric g of the type (4.6) with empty AdS_3 as reference background. The latter has metric $\bar{g}$ with $L_+ = L_- = -1/4$ *i.e.* $\varepsilon = -1/8\pi G$ and $\chi = 0$. The final integral is performed over the compact spatial boundary coordinate $x \in [0, 2\pi]$:

$$Q_Y[g - \bar{g}, \bar{g}] = \frac{1}{8\pi k G}\int_0^{2\pi} \mathrm{d}x \left(Y^+\left(L_+ + \frac{1}{4}\right) + Y^-\left(L_- + \frac{1}{4}\right)\right). \tag{4.14}$$

These charges are in agreement with the quoted literature, and their algebra is determined as usual:

$$\{Q_{Y_1}, Q_{Y_2}\} = \delta_{\zeta_1} Q_{Y_2} = -\delta_{\zeta_2} Q_{Y_1}. \tag{4.15}$$

Introducing the modes

$$L_m^\pm = \frac{1}{8\pi k G}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}m x^\pm}\left(L_\pm + \frac{1}{4}\right) \tag{4.16}$$

the algebra reads:

$$\mathrm{i}\left\{L_m^\pm, L_n^\pm\right\} = (m-n) L_{m+n}^\pm + \frac{c}{12} m\left(m^2 - 1\right)\delta_{m+n,0}, \quad \left\{L_m^\pm, L_n^\mp\right\} = 0. \tag{4.17}$$

This double realization of Virasoro algebra with Brown–Henneaux central charge $c = 3/2kG$ was expected for Bañados solutions (4.6).

Perfect fluid with arbitrary velocity In Landau–Lifshitz frame the heat current vanishes ($\chi = 0$) and the boundary conformal fluid is perfect. Equation (4.4) requires for this

$$L_+ = L_- = \frac{M}{2}, \tag{4.18}$$

[23] Here $\delta_{\zeta_2}\zeta_1$ stands for the variation produced on ζ_1 by ζ_2, and this is not vanishing because ζ_1 depends explicitly on $L_\pm$: $\delta_{\zeta_2}\zeta_1 = \left(\frac{\partial \zeta_1^N}{\partial L_+}\delta_{\zeta_2} L_+ + \frac{\partial \zeta_1^N}{\partial L_-}\delta_{\zeta_2} L_-\right)\partial_N$.

with M constant, while it gives for energy density $\varepsilon = -M/4\pi G\xi^+\xi^-$. As for the general case, the reconstructed bulk family of metrics

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Einstein}} \;=\; & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}\mathrm{d}x^+ - \sqrt{-\frac{\xi^+}{\xi^-}}\mathrm{d}x^-\right)\mathrm{d}r \\
& + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{+\prime}\right)\left(\frac{\mathrm{d}x^+}{\xi^+}\right)^2 + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{-\prime}\right)\left(\frac{\mathrm{d}x^-}{\xi^-}\right)^2 \\
& + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime} + \xi^{-\prime}\right) + \frac{M}{k^2\xi^+\xi^-}\right)\mathrm{d}x^+\mathrm{d}x^-
\end{aligned}
\tag{4.19}
$$

is not in BMS gauge, unless $\xi^\pm$ are constant. Again this latter subset is entirely captured by $\xi^\pm = \pm 1$, and the resulting solution is BTZ together with all non-spinning zero-modes of Bañados family:

$$
\mathrm{d}s^2_{\text{Einstein}} = -\frac{1}{k}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)\mathrm{d}r + r^2\mathrm{d}x^+\mathrm{d}x^- + \frac{M}{2k^2}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)^2. \tag{4.20}
$$

The asymptotic structure rising in (4.19) is now respected by the following family of asymptotic Killing vectors

$$
\eta = \eta^r\partial_r + \eta^+\partial_+ + \eta^-\partial_-, \tag{4.21}
$$

expressed in terms of two arbitrary chiral functions $\epsilon^\pm(x^\pm)$

$$
\eta^r = -\frac{r}{2}\left(\epsilon^{+\prime} + \epsilon^{-\prime}\right), \quad \eta^\pm = \epsilon^\pm. \tag{4.22}
$$

These vectors, slightly different from those found for the dissipative boundary fluids (4.7), (4.8), (4.9), appear as the result of an exhaustive analysis of (4.19). They do not support subleading terms, and since they do not depend on the the functions $\xi^\pm$, they form an algebra for the Lie bracket:

$$
[\eta_1, \eta_2] = \eta_3 \tag{4.23}
$$

with $\eta_a = \eta\left(\epsilon^+_a, \epsilon^-_a\right)$ and

$$
\epsilon^\pm_3 = \epsilon^\pm_1\epsilon^{\pm\prime}_2 - \epsilon^\pm_2\epsilon^{\pm\prime}_1. \tag{4.24}
$$

They induce the exact transformation

$$
-\mathscr{L}_\eta g_{MN} = \delta_\eta g_{MN} = \frac{\partial g_{MN}}{\partial\xi^+}\delta_\eta\xi^+ + \frac{\partial g_{MN}}{\partial\xi^{+\prime}}\delta_\eta\xi^{+\prime} + \frac{\partial g_{MN}}{\partial\xi^-}\delta_\eta\xi^- + \frac{\partial g_{MN}}{\partial\xi^{-\prime}}\delta_\eta\xi^{-\prime} \tag{4.25}
$$

with

$$
\delta_\eta\xi^\pm = \epsilon^\pm\xi^{\pm\prime} - \xi^\pm\epsilon^{\pm\prime}. \tag{4.26}
$$

Following the customary pattern, we can determine the conserved charges, with AdS_3 as reference background, now reached with $\xi^{\pm} = \pm 1$ and $M = {}^{-1}\!/_{2}$ (again $\varepsilon = {}^{-1}\!/_{8\pi G}$ and $\chi = 0$):

$$Q_{\epsilon}\left[g - \bar{g}, \bar{g}\right] = \frac{1}{16\pi kG}\int_0^{2\pi} \mathrm{d}x \left(\epsilon^+ \left(\frac{1}{\xi^{+2}} - 1\right) + \epsilon^- \left(\frac{1}{\xi^{-2}} - 1\right)\right), \tag{4.27}$$

as well as their algebra:

$$\left\{Q_{\epsilon_1}, Q_{\epsilon_2}\right\} = \delta_{\eta_1} Q_{\epsilon_2} = -\delta_{\eta_2} Q_{\epsilon_1}. \tag{4.28}$$

Defining now

$$L_m^{\pm} = \frac{1}{16\pi kG}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx^{\pm}} \left(\frac{1}{\xi^{\pm 2}} - 1\right) \tag{4.29}$$

we find

$$\left\{L_m^{\pm}, L_n^{\pm}\right\} = \mathrm{i}(m-n)L_{m+n}^{\pm} + \frac{im}{4kG}\delta_{m+n,0}, \quad \left\{L_m^{\pm}, L_n^{\mp}\right\} = 0. \tag{4.30}$$

The central extension of this algebra is trivial. Indeed, it can be reabsorbed in the following redefinition of the modes $L_m^{\pm}$

$$\tilde{L}_m^{\pm} = L_m^{\pm} + \frac{1}{8kG}\delta_{m,0}. \tag{4.31}$$

Therefore, (4.30) becomes

$$\left\{\tilde{L}_m^{\pm}, \tilde{L}_n^{\pm}\right\} = \mathrm{i}(m-n)\tilde{L}_{m+n}^{\pm}, \quad \left\{\tilde{L}_m^{\pm}, \tilde{L}_n^{\mp}\right\} = 0. \tag{4.32}$$

The algebra at hand (4.32) is de Witt rather than Virasoro, and this outcome demonstrates the already advertised result: the family of locally anti-de Sitter spacetimes obtained holographically from two-dimensional fluids in the Landau–Lifshitz frame overlap only partially the space of Bañados solutions. This overlap encompasses the non-spinning BTZ and excess or defects geometries provided in (4.20).

4.2 Charges in Ricci-flat spacetimes

The absence of anomaly in the Carrollian framework is equivalent to setting $\Sigma^x{}_x = \Xi^x{}_x = 0$ (see (3.15)), whereas the Weyl–Carroll flatness requires $s = 0$ (see (2.96)). This amounts to taking $\Omega = a = 1$ and $b_x = 0$,[24] and with those data $s = 0$ reads

$$\partial_t^2 \beta_x = 0. \tag{4.33}$$

[24] Actually the absence of anomaly requires rather $\Omega = \Omega(t)$, $a = a(x)$ and $b_x = b_x(x)$, which can be reabsorbed trivially with Carrollian diffeomorphisms.

In the Carrollian spacetime at hand, the fluid equations of motion (3.16) are

$$\begin{cases} \partial_t \varepsilon = 0, \\ \partial_x \varepsilon + \partial_t (\pi_x + 2\varepsilon\beta_x) = 0. \end{cases} \tag{4.34}$$

Equations (4.33) and (4.34) can be integrated in terms of four arbitrary functions of x: $\varepsilon(x)$, $\varpi(x)$, $\lambda(x)$ and $\mu(x)$. We find

$$\pi_x(t,x) = -2\varepsilon(x)\beta_x(t,x) + \varpi(x) - t\varepsilon'(x), \tag{4.35}$$

$$\beta_x(t,x) = \frac{\lambda(x)}{2\varepsilon(x)} - \frac{t\mu'(x)}{2\mu(x)} \tag{4.36}$$

(this parameterization of β_x will be appreciated later). The Ricci-flat (even locally flat) holographically reconstructed spacetime from these Carrollian fluid data is obtained from the general expression (3.14):

$$\begin{aligned} \mathrm{d}s^2_{\text{flat}} = & -2\left(\mathrm{d}t - \beta_x \mathrm{d}x\right)\left(\mathrm{d}r + r\partial_t\beta_x \mathrm{d}x\right) + r^2\mathrm{d}x^2 \\ & + 8\pi G\left(\varepsilon(\mathrm{d}t - \beta_x\mathrm{d}x)^2 - \pi_x \mathrm{d}x(\mathrm{d}t - \beta_x \mathrm{d}x)\right), \end{aligned} \tag{4.37}$$

where β_x and π_x are meant to be as in (4.35) and (4.36).

On the one hand, the arbitrary functions $\varepsilon(x)$ and $\varpi(x)$ are reminiscent of the functions $L_\pm(x^\pm)$ (or $\varepsilon(t,x)$ and $\chi(t,x)$) present in the AdS solutions. A vanishing-k limit was indeed used in Ref. [26] to obtain $\varepsilon(x)$ and $\varpi(x)$ from $L_\pm(x^\pm)$. On the other hand, $\lambda(x)$ and $\mu(x)$ remind $\zeta^\pm(x^\pm)$, and are indeed a manifestation of a residual hydrodynamic frame invariance, which survives the Carrollian limit. Considering indeed the Carrollian hydrodynamic-frame transformations (2.110)

$$\beta'_x = \beta_x + B_x, \tag{4.38}$$

in the present framework ($\Sigma^x{}_x = \Xi^x{}_x = 0$), and using Eqs. (2.73), (2.74), (2.75), (2.76), (2.99), (2.100), (2.101), we obtain the transformations:

$$\varepsilon' = \varepsilon, \quad \pi'_x = \pi_x - 2\varepsilon B_x, \tag{4.39}$$

which leave the Carrollian fluid equations (4.34) invariant. The new velocity field β'_x is compatible with the Weyl–Carroll flatness (4.33) provided the transformation function B_x is linear in time, hence parameterized in terms of two arbitrary functions of x. This is how $\lambda(x)$ and $\mu(x)$ emerge.

Observe also that the residual Carrollian hydrodynamic frame invariance enables us to define here a Carrollian Landau–Lifshitz hydrodynamic frame. Indeed, combining (4.35) and (4.36) we obtain

$$\pi_x(t,x) = -\lambda(x) + \varpi(x) + t\varepsilon(x)\partial_x \ln\frac{\mu(x)}{\varepsilon(x)}. \tag{4.40}$$

Adjusting the velocity field β_x such that

$$\lambda(x) = \varpi(x) \quad \text{and} \quad \frac{\mu(x)}{\varepsilon(x)} = \frac{1}{\varepsilon_0} \tag{4.41}$$

with ε_0 a constant, makes the Carrollian fluid perfect: $\pi_x = 0$.

In complete analogy with the AdS analysis, we will first compute the charges for vanishing velocity $\beta_x = 0$ (which is given by $\lambda(x) = 0$ and $\mu(x) = 1$) in terms of $\varepsilon(x)$ and $\varpi(x)$, and next perform the similar computation for perfect fluids with velocity β_x parameterized with two arbitrary functions $\lambda(x)$ and $\mu(x)$. Here empty Minkowski bulk is realized with $\mu = 1$, $\lambda = 0$, $\varpi = 0$ and $\varepsilon_0 = -1/8\pi G$.

As for the AdS instance discussed in Sec. 4.1, the class (4.37) is not in the BMS gauge, unless β_x is constant, which can then be reabsorbed by a global Carrollian boost (constant B_x).[25] We will first discuss this situation, where the asymptotic Killings are the canonical generators of $\mathfrak{bms}_3$. Outside the BMS, we will perform the determination of the asymptotic isometry for metrics reconstructed from perfect fluids, and proceed with the surface charges and their algebra. Our conclusion is here that asymptotically flat fluid/gravity correspondence is sensitive to the residual hydrodynamic-frame invariance.

Dissipative static fluid The metric (4.37) for vanishing β_x takes the simple form

$$\mathrm{d}s^2_{\text{flat}} = -2\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}x^2 + 8\pi G\left(\varepsilon\mathrm{d}t - \left(\varpi - t\varepsilon'\right)\mathrm{d}x\right)\mathrm{d}t, \tag{4.42}$$

compatible with BMS gauge with asymptotic Killing vectors

$$\zeta = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^x\partial_x, \tag{4.43}$$

where

$$\zeta^r = -rY' + H'' + tY''' + \frac{4\pi G}{r}\left(\varpi - t\varepsilon'\right)\left(H' + tY''\right), \tag{4.44}$$

$$\zeta^t = H + tY', \tag{4.45}$$

$$\zeta^x = Y - \frac{1}{r}\left(H' + tY''\right). \tag{4.46}$$

[25] The functions $\lambda(x)$ and $\mu(x)$ entering (4.37) via (4.35) and (4.36) can be reabsorbed in any case by performing the coordinate transformation $\mathrm{d}x \to \frac{\mathrm{d}x}{\sqrt{\mu(x)}}$, $\mathrm{d}t \to \frac{1}{\sqrt{\mu(x)}}(\mathrm{d}t + \beta_x\mathrm{d}x)$ and $r \to r\sqrt{\mu(x)}$. This leads to the same form as the one reached by setting $\mu = 1$ and $\lambda = 0$, *i.e* (4.42).

Here H and Y are functions of x only. Vectors (4.44), (4.45), (4.46) are the vanishing-k limit of (4.7), (4.8), (4.9), reached by trading light-cone frame as $x^{\pm} = x \pm kt$, and setting $Y^{\pm}(x^{\pm}) = Y(x) \pm k(H(x) + tY'(x))$.

This family of vectors produces the following variation on the metric fields:

$$-\mathscr{L}_{\zeta} g_{MN} = \delta_{\zeta} g_{MN} = \frac{\partial g_{MN}}{\partial \varepsilon}\delta_{\zeta}\varepsilon + \frac{\partial g_{MN}}{\partial \varepsilon'}\delta_{\zeta}\varepsilon' + \frac{\partial g_{MN}}{\partial \varpi}\delta_{\zeta}\varpi, \tag{4.47}$$

with

$$\delta_{\zeta}\varepsilon = -2\varepsilon Y' - Y\varepsilon' + \frac{Y'''}{4\pi G}, \tag{4.48}$$

$$\delta_{\zeta}\varpi = -\frac{H'''}{4\pi G} + \frac{1}{H}\left(\varepsilon H^2\right)' - \frac{1}{Y}\left(\varpi Y^2\right)'. \tag{4.49}$$

Their algebra closes for the same modified Lie bracket (4.12) with $\zeta_a = \zeta(H_a, Y_a)$ and

$$Y_3 = Y_1 Y_2' - Y_2 Y_1' \quad H_3 = Y_1 H_2' + H_1 Y_2' - Y_2 H_1' - H_2 Y_1'. \tag{4.50}$$

We can compute the charges of g in (4.42), using Minkowski as reference background $\bar{g}$. They read:

$$Q_{H,Y}[g - \bar{g}, \bar{g}] = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x \left[H\left(\varepsilon + \frac{1}{8\pi G}\right) - Y\varpi\right]. \tag{4.51}$$

With a basis of functions $\exp imx$ for H and Y, we find the standard collection of charges

$$P_m = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{imx}\left(\varepsilon + \frac{1}{8\pi G}\right), \quad J_m = -\frac{1}{2}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{imx}\varpi, \tag{4.52}$$

which coincide with the computation performed *e.g.* in [26]. Using

$$\{Q_{H_1,Y_1}, Q_{H_2,Y_2}\} = \delta_{\zeta_1} Q_{H_2,Y_2} = -\delta_{\zeta_2} Q_{H_1,Y_1}, \tag{4.53}$$

we obtain the following surface-charge algebra:

$$\mathrm{i}\{J_m, P_n\} = (m-n)P_{m+n} + \frac{c}{12}m\left(m^2 - 1\right)\delta_{m+n,0}, \quad \mathrm{i}\{J_m, J_n\} = (m-n)J_{m+n}, \quad \{P_m, P_n\} = 0 \tag{4.54}$$

with $c = {}^{3}/_{G}$. This is the $\mathfrak{bms}_3$ algebra, and this analysis demonstrates that a non-perfect Carrollian fluid, even with $\beta_x = 0$, is sufficient for generating holographically all Barnich–Troessaert flat three-dimensional spacetimes. This goes along with the analogue conclusion reached in AdS for Bañados spacetimes.

Perfect fluid with velocity Consider now the resummed metric (4.37) assuming (4.41). We obtain

$$\mathrm{d}s^2_{\text{flat}} = -2\left(\mathrm{d}t - \beta_x \mathrm{d}x\right)\left(\mathrm{d}r - \frac{r\mu'}{2\mu}\mathrm{d}x\right) + r^2\mathrm{d}x^2 + 8\pi G\varepsilon_0\mu\left(\mathrm{d}t - \beta_x\mathrm{d}x\right)^2 \tag{4.55}$$

with β_x given by

$$\beta_x = \frac{1}{2\mu}\left(\frac{\lambda}{\varepsilon_0} - t\mu'\right). \tag{4.56}$$

Unless β_x is constant, the metrics (4.55) are not in BMS gauge. The BMS subset is entirely captured by $\mu = 1$, $\lambda = 0$ with resulting solutions plain Minkowski ($\varepsilon_0 = {-1}/{8\pi G}$) and the non-spinning zero-modes of Barnich–Troessaert family:

$$\mathrm{d}s^2_{\text{flat}} = -2\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}x^2 + 8\pi G\varepsilon_0\mathrm{d}t^2. \tag{4.57}$$

The asymptotic isometries of (4.55) are now generated by[26]

$$\eta = \eta^r\partial_r + \eta^t\partial_t + \eta^x\partial_x, \tag{4.58}$$

expressed in terms of two arbitrary functions $h(x)$ and $\rho(x)$

$$\eta^r = -r\rho', \quad \eta^t = h + t\rho', \quad \eta^x = \rho. \tag{4.59}$$

The algebra of asymptotic Killing vectors closes for the ordinary Lie bracket

$$[\eta_1, \eta_2] = \eta_3 \tag{4.60}$$

with $\eta_a = \eta\left(h_a, \rho_a\right)$ and

$$\rho_3 = \rho_1'\rho_2 - \rho_2\rho_1', \quad h_3 = \rho_1 h_2' + h_1\rho_2' - \rho_2 h_1' - h_2\rho_1'. \tag{4.61}$$

It respects the form of the metric

$$-\mathscr{L}_\eta g_{MN} = \delta_\eta g_{MN} = \frac{\partial g_{MN}}{\partial\mu}\delta_\eta\mu + \frac{\partial g_{MN}}{\partial\mu'}\delta_\eta\mu' + \frac{\partial g_{MN}}{\partial\lambda}\delta_\eta\lambda \tag{4.62}$$

with

$$\delta_\eta\lambda = -2\lambda\rho' - \rho\lambda' + \varepsilon_0\left(2\mu h' + h\mu'\right), \tag{4.63}$$

$$\delta_\eta\mu = -2\mu\rho' - \rho\mu'. \tag{4.64}$$

The charges of g in (4.55) are computed as usual with Minkowski as reference back-

[26] Again the fields (4.58), (4.59) are alternatively obtained by an appropriate zero-k limit of (4.21) and (4.22).

ground $\bar{g}$. They read:

$$Q_{h,\rho}[g-\bar{g},\bar{g}] = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x \left[h\left(\varepsilon_0\mu + \frac{1}{8\pi G}\right) - \rho\lambda\right]. \tag{4.65}$$

With a basis of unimodular exponentials for h and ρ, we find again

$$P_m = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{imx}\left(\varepsilon_0\mu + \frac{1}{8\pi G}\right), \quad J_m = -\frac{1}{2}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{imx}\lambda, \tag{4.66}$$

and

$$\{Q_{h_1,\rho_1}, Q_{h_2,\rho_2}\} = \delta_{\eta_1} Q_{h_2,\rho_2} = -\delta_{\eta_2} Q_{h_1,\rho_1} \tag{4.67}$$

provide the surface-charge algebra:

$$\mathrm{i}\{J_m, P_n\} = (m-n)P_{m+n} - \frac{m}{4G}\delta_{m+n,0}, \quad \mathrm{i}\{J_m, J_n\} = (m-n)J_{m+n}, \quad \{P_m, P_n\} = 0. \tag{4.68}$$

As for the anti-de Sitter case, the central extension of this algebra is trivial. By translating the modes

$$\tilde{P}_m = P_m - \frac{1}{8G}\delta_{m,0}, \tag{4.69}$$

we obtain

$$\mathrm{i}\{J_m, \tilde{P}_n\} = (m-n)\tilde{P}_{m+n}, \quad \mathrm{i}\{J_m, J_n\} = (m-n)J_{m+n}, \quad \{\tilde{P}_m, \tilde{P}_n\} = 0. \tag{4.70}$$

This algebra (that could have been obtained from (4.32) in the zero-k limit) has no central charge. Therefore, our computation shows unquestionably that holographic locally flat spacetimes based on perfect Carrollian fluids – fluids in Carrollian Landau–Lifshitz frame – cover only in some measure the family on Barnich–Troessaert solutions. Among those one finds (4.57).

5 Conclusion

We can now summarize our achievements. The motivations of the present work have been twofold: (i) reconstruct asymptotically anti-de Sitter and flat three-dimensional spacetimes using fluid/gravity holographic correspondence in a unified framework; (ii) investigate the emergence of hydrodynamic-frame invariance and its potential holographic breakdown.

Solutions to three-dimensional vacuum Einstein's equations have been searched systematically since the seminal work of BTZ, and their asymptotic symmetries as well as the corresponding conserved charges are thoroughly understood. In parallel, many aspects of their boundary properties in the anti-de Sitter case were discussed before the advent of the holographic correspondence, and lately for the flat case in relation with the BMS asymptotic symmetries. However, setting up a precise correspondence between a general two-dimensional relativistic fluid defined on an arbitrary background and a three-dimensional anti-de Sitter spacetime was only superficially analyzed, whereas the possible relationship among flat spacetimes and Carrolian fluid dynamics had never been considered. This has been the core of our inquiry.

Because relativistic fluid dynamics in two spacetime dimensions is rather simple, it allows to perform an exhaustive and exact study of the equations of motion, and of their form invariance under hydrodynamic-frame transformations – local Lorentz boosts. We have assumed for commodity a conformal equation of state, keeping the fluid non-conformal though (*i.e.* with non-zero viscous bulk pressure). Hence, the relativistic fluid is described by an arbitrary velocity field, the energy and heat densities, and the viscous pressure, all transforming appropriately under local Lorentz boosts so as to keep the energy–momentum tensor invariant. The extreme situation corresponds to the Landau–Lifshitz frame, where the heat current vanishes and the energy–momentum tensor is diagonal.

Three-dimensional Einstein spacetime reconstruction is then achieved with the derivative expansion, following the usual pattern of higher dimensions. Here it is not an expansion but a finite sum, involving all boundary data. Holographic fluids have an anomalous viscous pressure proportional to the curvature of the host geometry. Owing to this fact, the holographic fluid does not move freely, but is subject to a force, entirely determined by its kinematical configuration and by the geometry. Using light-cone coordinates and conformally flat boundary makes it easy to obtain the general fluid configuration, and a general and closed expression for locally anti-de Sitter spacetimes, in a gauge which is less restrictive than BMS.

With this general result, it is possible to address the question of whether a boundary fluid configuration observed from different hydrodynamic frames gives rise to distinct bulk geometries. This is discussed in the simpler (but sufficient for the argument) case of flat boundaries with vanishing Weyl curvature, for which the fluid is conformal (no anomaly). The reconstructed bulk geometries are then described in terms of two pairs of chiral functions, $\xi^{\pm}$ and $L_{\pm}$. The former parameterize the velocity of the fluid, while the latter its energy and heat densities. With these data two extreme configurations emerge: (i) a fluid at rest with heat current; (ii) a fluid with arbitrary velocity and vanishing heat current (hence perfect since the viscous pressure is also zero) *i.e.* in the Landau–Lifshitz frame. For both cases one determines the bulk asymptotic Killing vectors together with the algebra of conserved surface charges. In the first instance, the left and right Virasoro algebras appear with their canonical central charges. In the second, the central charges vanish, demonstrating thereby that the bulk-metric derivative expansion is sensitive to the boundary-fluid hydrodynamic frame. In particular, the Landau–Lifshitz frame fails to reproduce faithfully all Banãdos' solutions, contrary to the common expectation.

The above pattern has been resumed for the Ricci-flat spacetimes. The conformal boundary is now at null infinity, and is endowed with a Carrollian $1+1$-dimensional structure. Boundary dynamics is carried by a Carrollian fluid, obeying a set of hydrodynamic equations for energy and heat densities, two viscous stress scalars as well as a kinematic variable referred to as "inverse-velocity". Generically, these equations do not exhibit any sort of hydrodynamic-frame invariance.

The reconstruction of three-dimensional Ricci-flat spacetimes is achieved by considering the vanishing-k limit of the anti-de Sitter derivative expansion, which is finite. Information is supplied in this Ricci-flat derivative expansion by the Carrollian fluid defined at null infinity. In particular, the original conformal anomaly is carefully identified as a source of Carrollian stress.

As for Einstein spacetimes, we do not consider the most general situation, but impose equivalent restrictions: absence of anomaly and zero Weyl–Carroll curvature. The derivative-expansion gauge is slightly less restrained than BMS, and a residual hydrodynamic-frame-like invariance emerges, which allows to treat the same Carrollian dynamics from two equivalent perspectives: (i) a Carrollian fluid with vanishing inverse velocity and non-zero heat current; (ii) a Carrollian fluid with inverse velocity and vanishing heat current (*i.e.* a sort of Carrollian Landau–Lifshitz frame). Although equivalent from the Carrollian-fluid perspective, these two patterns lead to Ricci-flat spacetimes with different surface charge algebras. The former family fits in BMS gauge and reproduces all Barnich–Troessaert spacetimes with the appropriate charges. The algebra is $\mathfrak{bms}_3$ with central charge. The set of Ricci-flat metrics obtained with a Carrollian perfect fluids exhibit an algebra without central charge.

The above is the core of our work. Our findings raise several immediate questions. The most important concerns the systematic analysis of asymptotic Killing vectors and conserved charges under the general fall-off behaviours suggested by the derivative expansion. This question is valid in both anti-de Sitter and flat spacetime. The latter case calls for a deeper Hamiltonian understanding of the charges within the appropriate intrinsic Carrollian setup recently developed in [36]. All this also concerns fluid/gravity holographic correspondence irrespective of the dimension. Even though the possible breakdown of the Landau–Lifshitz-frame paradigm has been quoted for three-dimensional holographic boundary fluids [37], no concrete result is available at present.

Aside from the interplay between gravity and fluids, a purely hydrodynamic issue was also discussed: the entropy current. In relativistic systems, this current is expected to be hydrodynamic-frame invariant – by essence of this invariance. It is also physically restricted, to comply with fundamental laws. No closed expression exists and this object is usually constructed order-by-order in the derivative expansion. In two dimensions, we have the possibility to implement frame invariance exactly and we proposed a closed expression, which however is not unique and deserves further investigation. At the first place, one should understand whether and why this choice is the natural one. It could also wondered if it is useful for systems of dimension higher than two. Eventually, in the spirit of considering its Carrollian limit, one should try to give a meaning to entropy in ultrarelativistic systems.

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References

[1] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear fluid dynamics from gravity*, JHEP **0802** (2008) 045, arXiv:0712.2456 [hep-th].

[2] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*, JHEP **0810** (2008) 063, arXiv:0806.4602 [hep-th].

[3] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, *Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions*, JHEP **0812** (2008) 116, arXiv:0809.4272 [hep-th].

[4] V. E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, arXiv:1107.5780 [hep-th].

[5] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, J. Phys. **A45** (2012) 473001, arXiv:1205.5040 [hep-th].

[6] P. Romatschke, *New developments in relativistic viscous hydrodynamics*, Int. J. Mod. Phys. **E19** (2010) 1, arXiv:0902.3663 [hep-th].

[7] C. Fefferman and C. R. Graham, *Conformal invariants*, in *Elie Cartan et les mathématiques d'aujourd'hui,* Astérisque, 1985, numéro hors série Soc. Math. France, Paris, 95.

[8] C. Fefferman and C. R. Graham, *The ambient metric*, arXiv:0710.0919 [math.DG].

[9] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Flat holography and Carrollian fluids*, JHEP **07** (2018) 165, arXiv:1802.06809 [hep-th].

[10] L. D. Landau et E. M. Lifchitz, *Physique Théorique*, Vol. 6 *Mécanique des fluides*, Editions Mir, Moscou, 1969.

[11] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli and K. Siampos, *Vorticity in holographic fluids*, Proc. of Science **CORFU11** (2012) 076, arXiv:1206.4351 [hep-th].

[12] A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli and K. Siampos, *Holographic perfect fluidity, Cotton energy–momentum duality and transport properties*, JHEP **1404** (2014) 136, arXiv:1309.2310 [hep-th].

[13] P. M. Petropoulos, *Gravitational duality, topologically massive gravity and holographic fluids*, Lect. Notes Phys. **892** (2015) 331, arXiv:1406.2328 [hep-th].

[14] J. Gath, A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Petrov Classification and holographic reconstruction of spacetime*, JHEP **1509** (2015) 005, arXiv:1506.04813 [hep-th].

[15] P. M. Petropoulos and K. Siampos, *Integrability, Einstein spaces and holographic fluids*, proceedings of the workshop *About various kinds of interactions* in honour of the 65th birthday of Professor Philippe Spindel, N. Boulanger and S. Detournay, eds., Mons 2017, arXiv:1510.06456 [hep-th].

[16] M. Bañados, *Three-dimensional quantum geometry and black holes*, AIP Conf. Proc. **484** (1999) no.1, 147, hep-th/9901148.

[17] K. Skenderis and S. N. Solodukhin, *Quantum effective action from the AdS/CFT correspondence*, Phys. Lett. **B472** (2000) 316, arXiv:9910023 [hep-th].

[18] L. Ciambelli, C. Marteau, A.C. Petkou, P. M. Petropoulos and K. Siampos, *Covariant Galiliean versus Carrollian hydrodynamics from relativistic fluids*, Class. Quant. Grav. **35**, no. 16, 165001 (2018), arXiv:1802.05286 [hep-th].

[19] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, JHEP **1005** (2010) 062, arXiv:1001.1541 [hep-th].

[20] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, Nucl. Phys. **B633** (2002) 3, arXiv:0111246 [hep-th].

[21] G. Barnich and G. Compère, *Surface charge algebra in gauge theories and thermodynamic integrability*, J. Math. Phys. **49** (2008) 042901, arXiv:0708.2378 [gr-qc].

[22] G. Compère, package *SurfaceCharges*, SurfaceCharges

[23] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space–time*, Phys. Rev. Lett. **69** (1992) 1849, arXiv:9204099 [hep-th].

[24] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. **D48** (1993) 1506, Erratum: Phys. Rev. **D88** (2013) 069902, arXiv:9302012 [gr-qc].

[25] M. Bañados, *Global charges in Chern-Simons field theory and the (2+1) black hole*, Phys. Rev. **D52** (1996) 5816, arXiv:9405171 [hep-th].

[26] G. Barnich, A. Gomberoff and H. A. Gonzalez, *The flat limit of three dimensional asymptotically anti-de Sitter spacetimes,* Phys. Rev. **D86** (2012) 024020, arXiv:1204.3288 [gr-qc].

[27] L. Rezzolla and O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, Oxford UK, 2013.

[28] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, *A theory of first-order dissipative superfluid dynamics*, JHEP **1405** (2014) 147, arXiv:1105.3733 [hep-th].

[29] C. Eling, Y. Oz, S. Theisen and S. Yankielowicz, *Conformal anomalies in hydrodynamics*, JHEP **1305** (2013) 037, arXiv:1301.3170 [hep-th].

[30] R. Banerjee, *Exact results in two-dimensional chiral hydrodynamics with gravitational anomalies*, Eur. Phys. J. **C74** (2014) 2824, arXiv:1303.5593 [gr-qc].

[31] R. Banerjee, S. Dey and B. R. Majhi, *Entropy current in two dimensional anomalous hydrodynamics and a bound on the sum of the response parameters*, Phys. Rev. **D92** (2015) 044019, arXiv:1412.5878 [hep-th].

[32] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,* Class. Quant. Grav. **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[33] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, Commun. Math. Phys. **208** (1999) 413, arXiv:9902121 [het-th]

[34] M. M. Caldarelli, O. J. C. Dias, R. Monteiro and J. E. Santos, *Black funnels and droplets in thermal equilibrium*, JHEP **1105** (2011) 116, arXiv:1102.4337 [hep-th].

[35] R. Fareghbal, A. Naseh and S. Rouhani, *Aspects of ultra-relativistic field theories via flat-space holography*, Phys. Lett. **B771** (2017) 189, arXiv:1511.01774 [hep-th].

[36] L. Ciambelli and C. Marteau, *Carrollian conservation laws and Ricci-flat gravity*, arXiv:1810.11037 [hep-th]

[37] L. Ciambelli, A. C. Petkou, P. M. Petropoulos and K. Siampos, *The Robinson–Trautman spacetime and its holographic fluid*, PoS **CORFU2016** (2016) 076, arXiv:1707.02995 [hep--th].

Weyl Connections and their Role in Holography

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Abstract

It is a well known property of holographic theories that diffeomorphism invariance in the bulk spacetime implies Weyl invariance of the dual holographic field theory in the sense that the field theory couples to a conformal class of background metrics. The usual Fefferman-Graham formalism, which provides us with a holographic dictionary between the two theories, breaks explicitly this symmetry by choosing a specific boundary metric and a corresponding specific metric ansatz in the bulk. In this paper, we show that a simple extension of the Fefferman-Graham formalism allows us to sidestep this explicit breaking; one finds that the geometry of the boundary includes an induced metric and an induced connection on the tangent bundle of the boundary that is a *Weyl connection* (rather than the more familiar Levi-Civita connection uniquely determined by the induced metric). Properly invoking this boundary geometry has far-reaching consequences: the holographic dictionary extends and naturally encodes Weyl-covariant geometrical data, and, most importantly, the Weyl anomaly gains a clearer geometrical interpretation, cohomologically relating two Weyl-transformed volumes. The boundary theory is enhanced due to the presence of the Weyl current, which participates with the stress tensor in the boundary Ward identity.

1 Introduction

The basic principle of general relativity is invariance under diffeomorphisms with, as it is usually formulated, a metric playing the role of the dynamical degrees of freedom. Nonetheless, we usually make use of specific choices of coordinates and parametrizations of the metric, since we are often interested in particular subregions of the space-time manifold. These parametrizations are not harmless in that they break (or gauge fix) some subset of the diffeomorphisms, and one has a restricted class of diffeomorphisms which explicitly preserves the form of a given parametrization. It is most clarifying to choose a parametrization such that the unbroken symmetries act geometrically on the subregion of spacetime. This is particularly important, for example, for hypersurfaces of any type and co-dimension, but even more generally, for sub-bundles (distributions) of the tangent bundle.

Fefferman and Graham in their seminal works [1, 2] found a bulk gauge (FG gauge) preserving the structure of time-like hypersurfaces in AdS_{d+1} spacetimes. This is useful to discuss the time-like conformal boundary; which in suitable coordinates is located at $z = 0$, z being the holographic coordinate such that $z = const$ hypersurfaces are time-like. The FG gauge induces on the boundary a metric and its Levi-Civita connection. Although everything is consistent, there exists some leftover freedom in choosing the boundary metric. This comes about because the induced metric on the $z = 0$ hypersurface is defined, because of certain bulk diffeomorphisms, up to a rescaling by a non-trivial function of the boundary coordinates. We therefore often refer to the boundary as possessing a conformal class of metrics and say that the boundary enjoys Weyl symmetry. The latter is however often ignored in physical applications, for we usually fix the boundary metric and thus break this symmetry.

In an attempt to bring electromagnetism and gravity into a unified framework [3], Weyl introduced the concept of Weyl transformation, which encapsulates the possibility of rescaling the metric with an arbitrary scalar function. Weyl symmetry is not considered in many physical systems, but it is a key feature of holography. For instance, it is a very powerful tool in the fluid/gravity correspondence [4–7], where it is exploited in organizing the boundary theory.

The main observation that we focus on here is that the Levi-Civita connection is not Weyl-covariant, the metricity condition being the source of this non-covariance. This problem can be sidestepped by introducing the notion of a Weyl connection and more generally of Weyl geometry [8, 9]. These concepts have been mentioned in the literature from time to time with reference to a variety of proposed physical applications, mostly in conformal gravitational theory, but also in cosmology and in particle physics, see e.g. [10–22].[1] In the present paper, we will show that Weyl connections play a role in the holographic correspondence, on the field theory side of the duality. Indeed, our first result will be to show that, by slightly generalizing the FG ansatz to what we call the Weyl-Fefferman-Graham gauge (WFG), the *Weyl diffeomorphism* responsible for the rescaling of the boundary metric becomes a geometric symmetry. The consequences of this modification are simple: this bulk geometry induces on the boundary a metric and a Weyl connection, instead of its Levi-Civita counterpart. In the dual quantum field theory, these objects act as backgrounds and sources for current operators. Thus, Weyl geometry makes an appearance in holography, not through a modification of the bulk gravitational theory, but in the organization of the dual field theory.

To establish these results, it is important to employ the notion of a (possibly non-integrable) distribution (i.e., a sub-bundle of the tangent bundle), replacing the less general notion of hypersurfaces and foliations. Since this may be unfamiliar to the casual reader, we take some time to review the mathematics, which is informed by theorems of Frobenius. In this way of thinking, the more relevant object is a tangent space, rather than a space itself.

The FG gauge admits an expansion of the metric from the boundary to the bulk in powers of the holographic coordinate z. Solving Einstein equations allows the extraction of the different terms of the

[1] For a review on applications of Weyl geometry in physics, see [23] and references therein.

expansion, all being determined by two terms in the expansion: the leading order, which defines the boundary conformal class of metrics and the term at order z^{d-2} which gives the vacuum expectation value of the energy-momentum tensor operator of the dual field theory, as originally discussed in [24–26]. It is a theorem that, given these two quantities, one can reconstruct, at least order by order, a bulk AdS spacetime in FG gauge — with some caveats due to the Weyl anomaly, which we will discuss shortly. The resolution of Einstein equations order by order for the WFG gauge on the other hand leads to a modification of the subleading terms in this expansion. In fact, we will demonstrate that the modifications are such that each term is Weyl-covariant; in the FG gauge, the subleading terms transform under Weyl transformations in a very complicated non-linear fashion (which, as we discuss, comes about because they are determined by non-Weyl-covariant Levi-Civita boundary curvature tensors).

It is a familiar aspect of the FG formalism that the on-shell bulk action diverges as one approaches the boundary. Traditionally, this is dealt with by including local counterterms which are functionals of the induced geometry, in a solution-independent way [25, 27–29]. There remains one physical subtlety, which is the appearance of a simple pole in $d - 2k$, with k integer. This effect is more appropriately thought of as an anomaly in the Weyl Ward identity, a basic feature of renormalization theory [30]. This anomaly can be traced back to the fact that holographic renormalization breaks Weyl covariance by fixing a $z = \epsilon$ hypersurface to regulate the theory. No Weyl-covariant renormalization procedures exist. Consequently, a Weyl anomaly is present, and contributes in any even-dimensional boundary theory. There is of course a huge literature on this subject, but an interesting historical account on the Weyl anomaly is [31], with a useful list of relevant references therein, as e.g. [32–34]. Notice also that a more field-theoretical approach to the anomaly, inspired by string theory and based on the non-invariance of path integral measure under Weyl transformations can be found in [35, 36]. The Weyl anomaly is an integral over geometrical tensors, the form of which depends on dimension. These tensors have been classified in [37]. We will unravel a different packaging of the Weyl anomaly, through the use of the WFG gauge — the Weyl anomaly will in fact become an integral over Weyl-covariant geometrical tensors. This result reorganizes the theory in a much simpler fashion and opens the door to a relevant direction of investigation, which is the determination of this coefficient in any even dimension. Inspired by [38], we will moreover present a simple cohomological interpretation of the Weyl anomaly, based on the difference of two Weyl-related bulk top forms.

The presence of the anomaly is usually encoded in the fact that the boundary energy-momentum tensor acquires an anomalous trace [39–41]. Indeed in FG gauge, it is found that it must be *a priori* traceless. This boundary Ward identity is obtained by considering the boundary background as dictated by the induced metric only. It is thus natural that there is only one sourced current. However, one finds that one must typically *improve* the energy-momentum tensor, as originally found in [42]. We advocate in this paper a different interpretation, corroborated by the WFG extension. Specifically, we interpret the boundary theory as defined on a background metric (again given by the induced-from-the-bulk metric) and a background Weyl connection, given by the leading order of the bulk dual. In this respect, we are now really sourcing two different currents, which can and indeed do both participate in the boundary Ward identity. From this perspective we are gauging the Weyl symmetry in the boundary [43–46], although more properly, we should view it as a local *background* symmetry. Actually, it is the WFG in the bulk that is promoting this Weyl connection to a background configuration in the boundary. We will in particular show that the holographic dictionary furnishes directly this boundary Ward identity relating the trace of the energy-momentum with the divergence of the Weyl current. This will be elegantly verified directly from the boundary action, without invoking holography. Consequently, our setup is useful also to analyze the profound relationship between Weyl invariance and conformal invariance, a subject which has been discussed for instance in [47, 48].

This paper is organized as follows. Section 2 introduces the Weyl connection, its metricity and torsion properties and its curvature tensors and associated identities. Emphasis is given to its relationship with the ordinary Levi-Civita connection. We then analyze in Section 3 the FG gauge and define the Weyl-Fefferman-Graham gauge. We show that the WFG gauge is form-invariant under the Weyl diffeomorphism.

We then discuss the important result that we are indeed inducing a Weyl connection on the boundary. The latter makes the (tangent bundle of the) boundary a (generally non-integrable) distribution. Section 4 describes the improved holographic dictionary: the boundary Ward identity is derived and it is shown that every term in the bulk-to-boundary expansion is by construction Weyl-covariant. These results are supported by Appendix A, to which we delegate useful details for the computation of Einstein equations order by order. The next part of this section is devoted to a through analysis of the Weyl anomaly, and its cohomological derivation. In Section 5, we present some relevant field theoretical results: we re-derive the Ward identity intrinsically and present examples of simple Weyl-invariant actions. We then conclude and offer some final remarks in Section 6.

2 Weyl Connections and Weyl Manifolds

Recall that given a manifold M with metric g and connection ∇ (on the tangent bundle TM), we define the metricity ∇g and torsion T via

$$\begin{aligned} \nabla_{\underline{X}} g(\underline{Y}, \underline{Z}) &= \nabla_{\underline{X}}(g(\underline{Y}, \underline{Z})) - g(\nabla_{\underline{X}}\underline{Y}, \underline{Z}) - g(\underline{Y}, \nabla_{\underline{X}}\underline{Z}), && (1)\\ T(\underline{X}, \underline{Y}) &= \nabla_{\underline{X}}\underline{Y} - \nabla_{\underline{Y}}\underline{X} - [\underline{X}, \underline{Y}]\,, && (2)\end{aligned}$$

where $\underline{X}, ...$ are arbitrary vector fields and $[\underline{X}, \underline{Y}]$ denotes the Lie bracket. Suppose we have a basis $\{\underline{e}_a\}$ of vector fields, and define the connection coefficients via

$$\nabla_{\underline{e}_a}\underline{e}_b = \Gamma^c_{ab}\underline{e}_c. \tag{3}$$

It is a familiar theorem that requiring both the metricity and torsion of the connection to vanish leads to a uniquely determined set of connection coefficients, those of the Levi-Civita (LC) connection. Indeed, further defining the rotation coefficients

$$[\underline{e}_a, \underline{e}_b] = C_{ab}{}^c\underline{e}_c, \tag{4}$$

we find the general result

$$\mathring{\Gamma}^d_{ac} = \tfrac{1}{2}g^{db}\Big(\underline{e}_a(g_{bc}) + \underline{e}_c(g_{ab}) - \underline{e}_b(g_{ca})\Big) - \tfrac{1}{2}g^{db}\Big(C_{ab}{}^f g_{fc} + C_{ca}{}^f g_{fb} - C_{bc}{}^f g_{fa}\Big), \tag{5}$$

where $g_{ab} \equiv g(\underline{e}_a, \underline{e}_b)$ and we use the circle notation to refer to the LC quantities. This reduces with the choice of coordinate basis $\underline{e}_a = \underline{\partial}_a$ to the familiar Christoffel symbols.

The vanishing of metricity and torsion are certainly invariant under diffeomorphisms. Therefore, all the geometrical objects built using the LC connection transform nicely under diffeomorphisms. We note though that metricity is not invariant under Weyl transformations[2] $g \to g/\mathcal{B}^2$, instead transforming as

$$\nabla g \to \nabla g - 2d\ln\mathcal{B} \otimes g. \tag{8}$$

Consequently, if we wish to consider geometric theories in which Weyl transformations play a role, it is inconvenient to choose the usual LC connection. Instead, one attains a connection that is covariant with

[2] The Weyl transformation should not be confused with a conformal transformation, which is a diffeomorphism. They do look similar in their actions on the components of the metric,

$$\begin{aligned} Weyl: &\quad g_{ab}(x) \mapsto g_{ab}(x)/\mathcal{B}(x)^2, && (6)\\ conformal: &\quad g_{ab}(x) \mapsto g'_{ab}(x') = g_{ab}(x)/\omega(x)^2. && (7)\end{aligned}$$

Here though, $\mathcal{B}(x)$ is an arbitrary function, while $\omega(x)$ is a specific function, associated with a special diffeomorphism that is a conformal isometry.

respect to both Weyl transformations and diffeomorphisms by introducing a *Weyl connection* A which transforms non-linearly under a Weyl transformation

$$g \to g/\mathcal{B}^2, \qquad A \to A - d\ln\mathcal{B}. \tag{9}$$

By design then, the *Weyl metricity* is covariant[3]

$$(\nabla g - 2A\otimes g) \to (\nabla g - 2A\otimes g)/\mathcal{B}^2, \tag{10}$$

and it makes sense to set it to zero if one wishes. Fortunately, there is a theorem which states that there is a unique connection (also generally referred to as a Weyl connection) that has zero torsion and Weyl metricity. In this case, the connection coefficients are given by the formula

$$\begin{aligned}\Gamma^d_{ac} \;=\;& \tfrac{1}{2}g^{db}\Big(\underline{e}_a(g_{bc}) + \underline{e}_c(g_{ab}) - \underline{e}_b(g_{ca})\Big) - \tfrac{1}{2}g^{db}\Big(C_{ab}{}^f g_{fc} + C_{ca}{}^f g_{fb} - C_{bc}{}^f g_{fa}\Big)\\ &-\Big(A_a\delta^d{}_c + A_c\delta^d{}_a - g^{db}A_b g_{ca}\Big).\end{aligned} \tag{11}$$

We note that these connection coefficients are in fact *invariant* under Weyl transformations. Consequently, the curvature of the Weyl connection has components[4]

$$R^a{}_{bcd} = \underline{e}_c(\Gamma^a_{db}) - \underline{e}_d(\Gamma^a_{cb}) + \Gamma^f_{db}\Gamma^a_{cf} - \Gamma^f_{cb}\Gamma^a_{df} - C_{cd}{}^f\Gamma^a_{fb} \tag{13}$$

that are themselves Weyl invariant. This Weyl-Riemann tensor possesses less symmetries than its Levi-Civita counterpart, and indeed the degrees of freedom contained within are in one-to-one correspondence with the Levi-Civita Riemann tensor, plus a 2-form F, which is the field strength $F = dA$. To see this, we can write the Weyl curvature components in terms of the LC curvature components,

$$R^a{}_{bcd} \;=\; \mathring{R}^a{}_{bcd} + \mathring{\nabla}_d A_b\delta^a{}_c - \mathring{\nabla}_c A_b\delta^a{}_d + (\mathring{\nabla}_d A_c - \mathring{\nabla}_c A_d)\delta^a{}_b + \mathring{\nabla}_c A^a g_{bd} - \mathring{\nabla}_d A^a g_{bc} \tag{14}$$

$$+A_b(A_d\delta^a{}_c - A_c\delta^a{}_d) + A^a(g_{bd}A_c - g_{bc}A_d) + A^2(g_{bc}\delta^a{}_d - g_{bd}\delta^a{}_c). \tag{15}$$

The corresponding Weyl-Ricci tensor, which we define as $Ric_{ab} = R^c{}_{acb}$, is given by

$$Ric_{ab} = \mathring{R}ic_{ab} - \tfrac{d}{2}F_{ab} + (d-2)\Big(\mathring{\nabla}_{(a}A_{b)} + A_aA_b\Big) + \Big(\mathring{\nabla}\cdot A - (d-2)A^2\Big)g_{ab} \tag{16}$$

in space-time dimension d. We then read off that the Weyl-Ricci tensor has an antisymmetric part

$$Ric_{[ab]} = -\tfrac{d}{2}F_{ab}, \tag{17}$$

while the symmetric part differs from the LC Ricci tensor,

$$Ric_{(ab)} = \mathring{R}ic_{ab} + (d-2)\Big(\mathring{\nabla}_{(a}A_{b)} + A_aA_b\Big) + \Big(\mathring{\nabla}\cdot A - (d-2)A^2\Big)g_{ab}. \tag{18}$$

The corresponding Weyl-Ricci scalar is the trace,

$$R = \mathring{R} + 2(d-1)\mathring{\nabla}\cdot A - (d-1)(d-2)A^2. \tag{19}$$

[3]To be more specific, what we mean by this notation is

$$(\nabla g - 2A\otimes g)(\underline{X},\underline{Y},\underline{Z}) = \nabla_{\underline{X}}g(\underline{Y},\underline{Z}) - 2A(\underline{X})g(\underline{Y},\underline{Z})$$

The notation $A(\underline{X})$ used here and throughout the paper refers to the contraction of a 1-form with a vector, $A(\underline{X}) \equiv i_{\underline{X}}A \equiv A_aX^a$.

[4]Here we are using the convention

$$R^a{}_{bcd}\underline{e}_a \equiv R(\underline{e}_b,\underline{e}_c,\underline{e}_d) \equiv \nabla_{\underline{e}_c}\nabla_{\underline{e}_d}\underline{e}_b - \nabla_{\underline{e}_d}\nabla_{\underline{e}_c}\underline{e}_b - \nabla_{[\underline{e}_c,\underline{e}_d]}\underline{e}_b \tag{12}$$

Under a Weyl transformation, $R \to R\mathcal{B}^2$, so we see that the LC Ricci scalar must transform very non-trivially under Weyl,

$$\mathring{R} \to \mathcal{B}^2\Big(\mathring{R} + 2(d-1)\mathring{\nabla}^2 \ln\mathcal{B} - 2(d-1)(d-2)A\cdot d\ln\mathcal{B} + (d-1)(d-2)(d\ln\mathcal{B})^2\Big) \tag{20}$$

in order to cancel the transformation of the non-Weyl-invariant expression involving the Weyl connection A. We thus see the important role played by the Weyl connection. Organize the theory with respect to the latter is a more natural prescription, whenever this theory includes Weyl transformations.

Given a Weyl connection, we can organize tensors in such a way that they have a specific Weyl weight and we use the notation

$$\hat{\nabla}_{\underline{X}} t = \nabla_{\underline{X}} t + w_t A(\underline{X})\ t. \tag{21}$$

whereby

$$t \mapsto \mathcal{B}^{w_t} t, \qquad \hat{\nabla} t \mapsto \mathcal{B}^{w_t}\hat{\nabla} t. \tag{22}$$

For the specific case of a scalar field ϕ, we would then write $\hat{\nabla}_a\phi = \underline{e}_a(\phi) + w_\phi A_a \phi$. The condition that Weyl metricity vanishes is translated in this notation as $\hat{\nabla} g = 0$.

Finally we remark that the Bianchi identity for the Weyl-Riemann tensor is

$$\nabla_a R^e{}_{bcd} + \nabla_c R^e{}_{bda} + \nabla_d R^e{}_{bac} = 0 \tag{23}$$

Contracting the e, c indices, we get the once-contracted Bianchi identity

$$\nabla_a Ric_{bd} - \nabla_d Ric_{ba} + \nabla_c R^c{}_{bda} = 0. \tag{24}$$

which given that the Weyl-Riemann and Weyl-Ricci tensors are Weyl invariant, can also be written as

$$\hat{\nabla}_a Ric_{bd} - \hat{\nabla}_d Ric_{ba} + \hat{\nabla}_c R^c{}_{bda} = 0. \tag{25}$$

If we multiply by g^{ab}, we find

$$g^{ab}\hat{\nabla}_a Ric_{bd} - \hat{\nabla}_d R + \hat{\nabla}_c(g^{ab} R^c{}_{bda}) = 0. \tag{26}$$

This can be simplified further by noting that

$$g^{ab} R^c{}_{bda} = g^{cb}\Big(Ric_{bd} + 2F_{bd}\Big) \tag{27}$$

and hence the twice contracted Bianchi identity can be simplified to

$$g^{ab}\hat{\nabla}_a(G_{bc} + F_{bc}) = 0 \tag{28}$$

where $G_{ab} = Ric_{ab} - \frac{1}{2}Rg_{ab}$ is the Weyl-Einstein tensor. Since G and F have Weyl weight zero, this can also be written as

$$g^{ab}\nabla_a(G_{bc} + F_{bc}) = 0 \tag{29}$$

This is the analogue of the familiar result in Riemannian geometry, $\mathring{\nabla}^a\mathring{G}_{ac} = 0$.

3 Weyl Invariance and Holography

The Fefferman-Graham theorem says that the metric of a locally asymptotically AdS_{d+1} (LaAdS) geometry can be always put in the form

$$ds^2 = L^2\frac{dz^2}{z^2} + h_{\mu\nu}(z;x)dx^\mu dx^\nu \tag{30}$$

The conformal boundary is a constant-z hypersurface at $z = 0$ in these coordinates. To obtain this form, one has used up all of the diffeomorphism invariance, apart from residual transformations of the $x^\mu \to x'^\mu(x)$, which of course would change the components of $h_{\mu\nu}$ in general.

Near $z = 0$, $h_{\mu\nu}(z;x)$ may be expanded

$$h_{\mu\nu}(z;x) = \frac{L^2}{z^2}\left[\gamma^{(0)}_{\mu\nu}(x) + \frac{z^2}{L^2}\gamma^{(2)}_{\mu\nu}(x) + \frac{z^4}{L^4}\gamma^{(4)}_{\mu\nu}(x) + ...\right] + \frac{z^{d-2}}{L^{d-2}}\left[\pi^{(0)}_{\mu\nu}(x) + \frac{z^2}{L^2}\pi^{(2)}_{\mu\nu}(x) + ...\right]. \tag{31}$$

Here, we are regarding the boundary dimension d as variable[5] (in fact, we will regard $d \in \mathbb{C}$ formally as needed. This is discussed further later in the paper). $\gamma^{(0)}_{\mu\nu}(x)$ has an interpretation as an induced boundary metric:

$$\frac{z^2}{L^2}ds^2 \underset{z\to 0}{\longrightarrow} \gamma^{(0)}_{\mu\nu}(x)dx^\mu dx^\nu = ds^2_{bdy}. \tag{32}$$

It is this object that sources the stress energy tensor in the dual field theory, with $\pi^{(0)}_{\mu\nu}(x)$ its vev. All of the other terms in the series are determined in terms of $\gamma^{(0)}_{\mu\nu}(x), \pi^{(0)}_{\mu\nu}(x)$ by the bulk classical equations of motion.

Equation (32) defines the induced boundary metric up to a Weyl transformation. We see indeed that there is an ambiguity in the construction of this metric which amounts in defining the latter up to a scalar function of the boundary coordinates. Although it is often stated, this ambiguity is usually disregarded.

The following bulk diffeomorphism (which we refer to as the *Weyl diffeomorphism*)

$$z \to z' = z/\mathcal{B}(x), \qquad x^\mu \to x'^\mu = x^\mu \tag{33}$$

plays an important role. It has the effect of inducing a Weyl transformation of the boundary metric: using (32) with now holographic coordinate z' we obtain

$$ds^2_{bdy} = \frac{\gamma^{(0)}_{\mu\nu}(x)}{\mathcal{B}(x)^2}dx^\mu dx^\nu. \tag{34}$$

However, this diffeomorphism does not leave the bulk metric in the Fefferman-Graham gauge, but rather transforms it to

$$ds^2 = L^2\left(\frac{dz'}{z'} + \partial_\mu \ln\mathcal{B}(x)\ dx^\mu\right)^2 + h_{\mu\nu}(z'\mathcal{B}(x);x)dx^\mu dx^\nu \tag{35}$$

where

$$h_{\mu\nu}(z'\mathcal{B}(x);x) = \frac{L^2}{z'^2}\left[\frac{\gamma^{(0)}_{\mu\nu}(x)}{\mathcal{B}(x)^2} + \frac{z'^2}{L^2}\gamma^{(2)}_{\mu\nu}(x) + \frac{z'^4}{L^4}\mathcal{B}(x)^2\gamma^{(4)}_{\mu\nu}(x) + ...\right] \tag{36}$$

$$+\frac{z'^{d-2}}{L^{d-2}}\left[\mathcal{B}(x)^{d-2}\pi^{(0)}_{\mu\nu}(x) + \frac{z'^2}{L^2}\mathcal{B}(x)^d\pi^{(2)}_{\mu\nu}(x) + ...\right]. \tag{37}$$

[5]This avoids the necessary introduction of logarithms that occur when d is an even integer.

Thus, this diffeomorphism takes us out of FG gauge (as it is one of the diffs that was fixed in going to that gauge), and acts on the boundary tensors $\gamma^{(k)}_{\mu\nu}(x)$ and $\pi^{(k)}_{\mu\nu}(x)$ by a local Weyl rescaling with specific k-dependent weights.

The standard way to deal with the fact that we have been taken out of FG gauge is to employ an additional diffeomorphism acting on the $x^\mu \to x^\mu + \xi^\mu(z;x)$ which becomes trivial at the conformal boundary in such a way that $\gamma^{(0)}_{\mu\nu}(x)$ is left unchanged, but the cross term in (35) is cancelled. However, this diffeomorphism unfortunately has a complicated effect on all of the subleading terms in the metric — they no longer transform linearly as in (36), but instead transform non-linearly under the combined transformations and, we claim, this obscures the geometric significance of the sub-leading terms. There is nothing inconsistent here: indeed, in FG gauge, the subleading terms are given on-shell by expressions involving the Levi-Civita curvature of the induced metric, which themselves transform non-linearly under Weyl transformations.

We will instead consider here a revised ansatz, which we refer to as Weyl-Fefferman-Graham (WFG) gauge, defined as[6]

$$ds^2 = L^2 \left(\frac{dz}{z} - a_\mu(z;x)dx^\mu\right)^2 + h_{\mu\nu}(z;x)dx^\mu dx^\nu. \tag{38}$$

The constant-z hypersurface Σ at $z = 0$ remains the conformal boundary with induced metric $\gamma^{(0)}$, as

$$\frac{z^2}{L^2}ds^2 \underset{z\to 0}{\longrightarrow} \gamma^{(0)}_{\mu\nu}(x)dx^\mu dx^\nu. \tag{39}$$

Thus the presence of a_μ in the ansatz does not modify the induced metric at $z = 0$. However, the metric is no longer diagonal in the z, x^μ coordinates, and so we must take greater care in interpreting how we approach the conformal boundary.

It is natural, given the metric ansatz (38), to introduce the 1-form

$$e \equiv \Omega(z;x)^{-1}\left(\frac{dz}{z} - a_\mu(z;x)dx^\mu\right) \tag{40}$$

This form defines a distribution $C_e \subset TM$ defined as

$$C_e = ker(e) = span\left\{\underline{X} \in \Gamma(TM)\middle| i_{\underline{X}}e = 0\right\}. \tag{41}$$

Note that there is an ambiguity in multiplying e (or equivalently the $\underline{X}$'s) by a function on M, and we have represented this ambiguity by introducing the function Ω.

We remark that if a_μ were zero, then C_e is the span of the vectors $\underline{\partial}_\mu$ and can be thought of as related to constant-z hypersurfaces. More generally, it is convenient to introduce a basis for C_e as the set of vectors

$$\underline{D}_\mu \equiv \underline{\partial}_\mu + a_\mu(z;x)z\underline{\partial}_z. \tag{42}$$

This implies that we can regard a_μ as providing a lift[7] from $T\Sigma$ (with basis $\{\underline{\partial}_\mu\}$) to C_e, that is, it can be thought of as an Ehresmann connection. By the Frobenius theorem, C_e is an integrable distribution if

$$[\underline{D}_\mu, \underline{D}_\nu] \in C_e. \tag{43}$$

[6]It is also possible to generalize the ansatz by the inclusion of a scalar function in front of the first term, essentially a radial lapse function. We will discuss this further in the following.

[7]Here, we are regarding Σ as an isolated hypersurface in M. We can thus regard M as a fibre bundle $\pi : M \to \Sigma$. An Ehresmann connection provides a splitting of the tangent bundle $TM = H \oplus V$, and the $\underline{D}_\mu$ vectors form a basis of H, identified with C_e, at the point (z, x^μ).

To understand this condition, it is convenient to introduce a vector dual to e,

$$\underline{e} \equiv \Omega(z;x) z \underline{\partial}_z \tag{44}$$

which has been normalized to $e(\underline{e}) = 1$, and we regard $\{\underline{e}, \underline{D}_\mu\}$ as a basis for $T_{(z;x)}M$. We then compute

$$\left[\underline{D}_\mu, \underline{D}_\nu\right] = \Omega(z;x)^{-1} f_{\mu\nu} \underline{e}, \qquad f_{\mu\nu} \equiv D_\mu a_\nu - D_\nu a_\mu \tag{45}$$

So we find that integrability is the condition $f_{\mu\nu} = 0$, and thus by Frobenius, the distribution C_e would define under that circumstance a foliation of M by co-dimension one hypersurfaces.

By taking $\underline{e}$ in the form (44), we have fixed some of the diffeomorphism invariance;[8] the diffeomorphisms that preserve the form of $\underline{e}$ are given by $z' = z'(z;x), x'^\mu = x'^\mu(x)$. Given the interpretation of holography in terms of renormalization, we expect that these diffeomorphisms correspond to generic *local* (in x) coarse grainings. These residual diffeomorphisms act on the form e as

$$\frac{\partial x'^\nu(x)}{\partial x^\mu} a'_\nu(z';x') = \frac{\partial \ln z'(z;x)}{\partial \ln z} a_\mu(z;x) + \frac{\partial \ln z'(z;x)}{\partial x^\mu}, \qquad \Omega'(z';x') = \frac{\partial \ln z'(z;x)}{\partial \ln z} \Omega(z;x). \tag{47}$$

The first equation is consistent with the interpretation of a as an Ehresmann connection. The second equation implies that the inherent ambiguity in the definition of the distribution C_e represented by $\Omega(z;x)$ can be thought of as the (local) reparametrization invariance of z. We can for example use this reparametrization invariance to set $\Omega(z;x) \to L^{-1}$ if we wish. The residual diffeomorphisms that preserve this choice (or, more generally preserve any specific $\Omega(z;x)$) are of the form $z' = z/\mathcal{B}(x), x'^\mu = x'^\mu(x)$, which are the Weyl diffeomorphisms. These give

$$\frac{\partial x'^\nu(x)}{\partial x^\mu} a'_\nu(z';x') = a_\mu(z;x) - \partial_\mu \ln \mathcal{B}(x), \tag{48}$$

and so we are to interpret the $a_\mu(z;x)$ as a connection for the Weyl diffeomorphisms (33). Given this result, it will not come as a surprise that there will be an induced Weyl connection on the conformal boundary. To recap, using $\Omega = L^{-1}$, we have the following setup

$$\{\underline{e}, \underline{D}_\mu\} = \left\{L^{-1} z \underline{\partial}_z, \underline{\partial}_\mu + a_\mu z \underline{\partial}_z\right\}, \qquad \left[\underline{D}_\mu, \underline{D}_\nu\right] = L f_{\mu\nu} \underline{e}. \tag{49}$$

To proceed further, we Fourier analyze $a_\mu(z;x)$ and $h_{\mu\nu}(z;x)$ in the sense that we will expand them in eigenfunctions of $\underline{e}$. Such eigenfunctions are of course just the monomials in $z \in \mathbb{R}^+$. For $h_{\mu\nu}(z;x)$ we obtain then the same expansion as before, eq. (31), and for $a_\mu(z;x)$ we write

$$a_\mu(z;x) = \left[a_\mu^{(0)}(x) + \frac{z^2}{L^2} a_\mu^{(2)}(x) + \ldots\right] + \frac{z^{d-2}}{L^{d-2}} \left[p_\mu^{(0)}(x) + \frac{z^2}{L^2} p_\mu^{(2)}(x) + \ldots\right], \tag{50}$$

which is of the same form as the expansion of a massless gauge field in Fefferman-Graham. Given these expressions, we observe that $a_\mu^{(0)}$ is not part of the boundary metric, although as we will show, it is part of the induced boundary connection and thus should be regarded as part of the boundary geometry.

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[8] Indeed, the vector field $\underline{e}$ could more generally be of the form

$$\underline{e} \to \underline{e}' = \underline{e} + \theta^\mu(z;x) \underline{D}_\mu \tag{46}$$

which satisfies $e(\underline{e}) = 1$ for any θ^μ. (In the language of footnote 7 (see page 8), the $\underline{e}$ of (44) is special in that $\underline{e} \in V$.) In the general case, we have $\left[\underline{D}_\mu, \underline{D}_\nu\right] = f_{\mu\nu}\underline{e}' - f_{\mu\nu}\theta^\lambda \underline{D}_\lambda$ and thus integrability remains the condition $f_{\mu\nu} = 0$. The second diffeomorphism, discussed earlier, that returns the metric to the FG ansatz after a boundary Weyl transformation corresponds on the contrary to setting $a_\mu \to 0$ at the expense of keeping $\theta^\mu \neq 0$.

More precisely, what we will show is that for the WFG ansatz, the induced connection is *not* the Levi-Civita connection of the induced metric, but instead a Weyl connection. Given the expansions (31,50), we see that the Weyl diffeomorphism (33) acts as

$$\gamma^{(k)}_{\mu\nu}(x) \to \gamma^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{k-2}, \qquad \pi^{(k)}_{\mu\nu}(x) \to \pi^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{d-2+k} \tag{51}$$

$$a^{(k)}_{\mu}(x) \to a^{(k)}_{\mu}(x)\mathcal{B}(x)^{k} - \delta_{k,0}\partial_\mu \ln\mathcal{B}(x), \qquad p^{(k)}_{\mu}(x) \to p^{(k)}_{\mu}(x)\mathcal{B}(x)^{d-2+k} \tag{52}$$

and so in particular

$$\gamma^{(0)}_{\mu\nu}(x) \to \gamma^{(0)}_{\mu\nu}(x)/\mathcal{B}(x)^2, \qquad a^{(0)}_{\mu}(x) \to a^{(0)}_{\mu}(x) - \partial_\mu \ln\mathcal{B}(x) \tag{53}$$

and thus we may anticipate that $a^{(0)}_\mu$ will play the role of a boundary Weyl connection. All of the other subleading functions in the expansions (31,50) are interpreted to have, à la (51–52), definite Weyl weights, that is they are Weyl tensors. It is then natural to anticipate that they will be determined in terms of the Weyl curvature, which we discussed in the last section.

We introduced the concept of the distribution C_e precisely in order to properly discuss the notion of an induced connection, as C_e is a sub-bundle of TM. That is, given a connection ∇ on TM (which we will take to be the LC connection), we can apply it to vectors in C_e, which will be of the general form

$$\nabla_{\underline{D}_\mu}\underline{D}_\nu = \Gamma^\lambda_{\mu\nu}\underline{D}_\lambda + \Gamma^e_{\mu\nu}\underline{e} \tag{54}$$

The coefficients of the induced connection on C_e are by definition the $\Gamma^\lambda_{\mu\nu}$ appearing in (54). Notice that these connection coefficients should not be confused with the usual Christoffel symbols, which are associated with coordinate bases. By direct computation, we find

$$\Gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu} \equiv \tfrac{1}{2}h^{\lambda\rho}\Big(D_\mu h_{\rho\nu} + D_\nu h_{\mu\rho} - D_\rho h_{\nu\mu}\Big) \tag{55}$$

and furthermore if we evaluate this expression at $z=0$, we find

$$\gamma^{(0)\lambda}{}_{\mu\nu} = \tfrac{1}{2}\gamma^{\lambda\rho}_{(0)}\Big((\partial_\mu - 2a^{(0)}_\mu)\gamma^{(0)}_{\nu\rho} + (\partial_\nu - 2a^{(0)}_\nu)\gamma^{(0)}_{\mu\rho} - (\partial_\rho - 2a^{(0)}_\rho)\gamma^{(0)}_{\mu\nu}\Big) \tag{56}$$

This result can be compared to (11), from which we conclude that the induced connection on the boundary is in fact a Weyl connection, with the role of the geometric data g_{ab} and A_a in (11) being played here by $\gamma^{(0)}_{\mu\nu}$ and $a^{(0)}_\mu$. In comparing, we make use of the fact that here the intrinsic rotation coefficients are $C_{\mu\nu}{}^{\lambda} = 0$, as in (45). We will use the notation $\nabla^{(0)}$ for the corresponding Weyl connection (whose Weyl-Christoffel symbols are given by (56)), and the curvature as $R^{(0)\lambda}{}_{\mu\rho\nu}$. A tensor with components $t_{\mu_1\ldots\mu_n}(x)$ that has Weyl weight w_t transforms as $t_{\mu_1\ldots\mu_n}(x) \mapsto \mathcal{B}(x)^{w_t}t_{\mu_1\ldots\mu_n}(x)$, while $\hat{\nabla}^{(0)}_\nu t_{\mu_1\ldots\mu_n}(x) \equiv \nabla^{(0)}_\nu t_{\mu_1\ldots\mu_n}(x) + w_t a^{(0)}_\nu t_{\mu_1\ldots\mu_n}(x)$ transforms covariantly with the same weight. As noted above, all of the component fields aside form $a^{(0)}_\mu$ transform covariantly with respect to arbitrary Weyl transformations, and the Weyl weights of the various component fields are given above in (51). In the next section, we will briefly study some aspects of the holographic dictionary, and we will find that every equation is covariant with respect to arbitrary Weyl transformations — it is a *bona fide* (background) symmetry of the dual field theory. In particular, we will find that the appearance of $a^{(0)}_\mu(x)$, since it transforms non-linearly under Weyl transformations, is through Weyl-covariant derivatives of other fields, or through expressions involving the Weyl-invariant field strength $f^{(0)}_{\mu\nu}$.

4 The Holographic Dictionary and the Weyl Anomaly

In this section, we will explore some details of the holographic dictionary corresponding to the WFG ansatz. The LC connection in the bulk has the form

$$\begin{aligned} \nabla_{\underline{D}_\mu}\underline{D}_\nu &= \gamma^\lambda_{\mu\nu}\underline{D}_\lambda - h_{\nu\lambda}\psi^\lambda{}_\mu\underline{e} && (57)\\ \nabla_{\underline{D}_\mu}\underline{e} &= \psi^\lambda{}_\mu\underline{D}_\lambda && (58)\\ \nabla_{\underline{e}}\underline{D}_\mu &= \psi^\lambda{}_\mu\underline{D}_\lambda + L\varphi_\mu\underline{e} && (59)\\ \nabla_{\underline{e}}\underline{e} &= -Lh^{\lambda\rho}\varphi_\rho\underline{D}_\lambda && (60)\end{aligned}$$

where

$$\psi^\mu{}_\nu = \rho^\mu{}_\nu + \frac{L}{2}h^{\mu\lambda}f_{\lambda\nu}, \qquad \rho^\mu{}_\nu = \tfrac{1}{2}h^{\mu\lambda}\underline{e}(h_{\lambda\nu}), \qquad \varphi_\mu = \underline{e}(a_\mu), \qquad f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu \tag{61}$$

and we note that φ_μ is proportional to the rotation coefficient $C_{e\mu}{}^e$, i.e., $[\underline{e}, \underline{D}_\mu] = L\varphi_\mu\underline{e}$. In addition, we will use the notation[9] $\theta = tr\rho = \underline{e}(\ln\sqrt{-\det h})$ and $\zeta^\mu{}_\nu = \rho^\mu{}_\nu - \frac{1}{d}\theta\delta^\mu{}_\nu$. In Appendix A, we record some additional details, including the Weyl-Riemann curvature components.

As we have detailed above, the WFG metric ansatz has two bulk fields $h_{\mu\nu}$ and a_μ, and $\gamma^{(0)}_{\mu\nu}(x)$ and $a^{(0)}_\mu(x)$ appear as sources (and/or backgrounds), while $\pi^{(0)}_{\mu\nu}(x)$ and $p^{(0)}_\mu(x)$ appear as the corresponding vevs. The corresponding operators in the dual field theory are *Weyl-covariant currents* $\hat{T}_{\mu\nu}(x)$ and $\hat{J}_\mu(x)$, each of Weyl weight $d-2$. We will discuss these operators more fully in Section 5.

As usual, one finds that the bulk equations of motion determine the subleading component fields in terms of $\gamma^{(0)}_{\mu\nu}(x), a^{(0)}_\mu(x), \pi^{(0)}_{\mu\nu}(x)$ and $p^{(0)}_\mu(x)$. Here we will assume that we have a vacuum solution that is asymptotically locally anti-de Sitter. For example, the ee-component of the vacuum Einstein equations is

$$0 = G_{ee} + \Lambda g_{ee} = -\tfrac{1}{2}tr(\rho\rho) - \frac{3L^2}{8}tr(ff) - \tfrac{1}{2}\bar{R} + \tfrac{1}{2}\theta^2 \tag{62}$$

where $\Lambda = -\frac{d(d-1)}{2L^2}$ is the cosmological constant of AdS_{d+1} and we define for the sake of brevity

$$\bar{R}^\lambda{}_{\mu\rho\nu} = D_\rho\gamma^\lambda_{\nu\mu} - D_\nu\gamma^\lambda_{\rho\mu} + \gamma^\delta_{\nu\mu}\gamma^\lambda_{\rho\delta} - \gamma^\delta_{\rho\mu}\gamma^\lambda_{\nu\delta} \tag{63}$$

with $\bar{R} = h^{\mu\nu}\bar{R}^\rho{}_{\mu\rho\nu}$ the corresponding Ricci scalar. Expanding (62) we find

$$0 = \left[\Lambda + \frac{d(d-1)}{2L^2}\right] - \frac{1}{2}\frac{z^2}{L^2}\left[2(d-1)L^{-2}X^{(1)} + R^{(0)}\right] + ... - (d-1)\frac{z^d}{L^d}\left[\frac{d}{2}L^{-2}Y^{(1)} + \hat{\nabla}\cdot p_{(0)}\right] + ... \tag{64}$$

where $R^{(0)}$ is the boundary Weyl-Ricci scalar and

$$X^{(1)} = \gamma^{\mu\nu}_{(0)}\gamma^{(2)}_{\mu\nu}, \qquad Y^{(1)} = \gamma^{\mu\nu}_{(0)}\pi^{(0)}_{\mu\nu}. \tag{65}$$

In (64), the order one equation is trivially satisfied while the z^2 contribution gives $X^{(1)}$ entirely in terms of the Weyl-Ricci scalar curvature:

$$X^{(1)} = -\frac{L^2}{2(d-1)}R^{(0)}. \tag{66}$$

As in the FG story, we must be careful with the $O(z^d)$ terms here because of divergences in the evaluation of the on-shell action — those divergences are responsible for the Weyl anomaly in the dual field theory,

[9] The notation used here can be interpreted in terms of expansion (θ), shear (ζ), vorticity (f) and acceleration (φ) of the radial congruence $\underline{e}$.

the structure of which we will discuss in detail below. Nevertheless, we may read off the 'left-hand-side' of the Weyl Ward identity from this,

$$Y^{(1)} + \frac{2L^2}{d}\hat{\nabla} \cdot p_{(0)}. \tag{67}$$

We will see later that this is the expected form given the interpretation of $\pi^{(0)}_{\mu\nu}$ and $p^{(0)}_{\mu}$ as vevs of currents in the dual field theory. We will also study the form of the anomalous right-hand-side later.

Similarly, one finds that the leading $O(z^2)$ term in $G_{e\mu}$ is proportional to

$$\gamma^{\lambda\nu}_{(0)}\nabla^{(0)}_{\nu}\Big(G^{(0)}_{\lambda\mu} + f^{(0)}_{\lambda\mu}\Big) = 0, \tag{68}$$

the vanishing of which is the twice-contracted Bianchi identity of the Weyl connection, as was discussed above (see eq. (29)).

The leading non-trivial terms in the $\mu\nu$-components of the Einstein equations determine

$$\gamma^{(2)}_{\mu\nu} = -\frac{L^2}{d-2}\Big(Ric^{(0)}_{(\mu\nu)} - \frac{1}{2(d-1)}R^{(0)}\gamma^{(0)}_{\mu\nu}\Big) = -\frac{L^2}{d-2}L^{(0)}_{(\mu\nu)}, \tag{69}$$

where $L^{(0)}$ is the Weyl-Schouten tensor. Its trace (65) correctly reproduces (66). We take each of these results as representative of the fact that the subleading terms in the expansion of the metric are determined by the Weyl curvature, analogous to what happens in the usual FG gauge in which they are determined by the LC curvature of the induced metric. As we mentioned previously, the difference is that now all of the subleading terms in the bulk fields are Weyl-covariant. One expects that the same is true for a_μ as well, along with the transversality of such solutions. For example, the $O(z^4)$ term in the $e\mu$-component of the bulk Einstein equation involves $a^{(2)}_\mu$ in the form $Max(a^{(2)})_\mu$ where Max refers to the Weyl-Maxwell differential operator

$$Max(a^{(2)})_\mu \equiv \hat{\nabla}^{(0)} \cdot \hat{\nabla}^{(0)} a^{(2)}_\mu - \hat{\nabla}^{(0)}_\mu(\hat{\nabla}^{(0)} \cdot a^{(2)}) + (Ric^{(0)}_{\nu\mu} + 4f^{(0)}_{\nu\mu})\gamma^{\nu\lambda}_{(0)}a^{(2)}_\lambda. \tag{70}$$

The appearance of the Maxwell operator here is the analogue of the appearance of the tranverse tensor $\Pi^{\mu\nu}$ in the bulk solutions for a massless gauge field, when the boundary is Minkowski space-time. Note that both the Weyl-Ricci tensor and $f^{(0)}_{\mu\nu}$ appear in the Laplacian because $a^{(2)}$ is a vector field that has non-zero Weyl charge (weight).

The holographic dictionary for WFG will be taken to be the obvious generalization of the usual relationship, i.e.,

$$Z_{bulk}[g;\gamma^{(0)}, a^{(0)}] = exp(-S_{o.s.}[h, a; \gamma^{(0)}, a^{(0)}]) = Z_{FT}[\gamma^{(0)}, a^{(0)}] \tag{71}$$

where on the left we have the on-shell action of the bulk classical theory whose metric is given by h, a with asymptotic configurations $\gamma^{(0)}, a^{(0)}$, while the right-hand-side is the generating functional of correlation functions of operators sourced by $\gamma^{(0)}, a^{(0)}$. Although this is expressed in terms of the "bare" sources, it is implicit that a regularization scheme for the left-hand-side is employed and that the boundary counter-terms are introduced to absorb power divergences that arise in the evaluation of the on-shell action. Here, we will organize the discussion by taking the space-time dimension d to be formally complex; the on-shell action is convergent for sufficiently small d, and as we move d up along the real axis, we encounter additional divergences as d approaches an even integer. It is well-known in the context of Fefferman-Graham that as a byproduct this divergence induces the Weyl anomaly of the dual field theory, and is associated with the appearance of logarithms in the field expansions when d is precisely an even integer. Here we will review this bit of physics, as the existence of the Weyl connection, as we will see, organizes the Weyl anomaly in a much more symmetric fashion than is usually described.

It is taken for granted that Z_{bulk} is diffeomorphism invariant. Under the holographic map this implies, among other things, that the dual field theory can be regulated in a diffeomorphism-invariant fashion. However, the bulk calculation is classical, and thus, in principle, is a functional of the bulk metric g as well as the boundary values. We therefore suppose that

$$\frac{Z_{bulk}\left[g'; \gamma'_{(0)}, a'_{(0)}, ...\middle| z', x'\right]}{Z_{bulk}\left[g; \gamma_{(0)}, a_{(0)}, ...\middle| z, x\right]} = 1, \tag{72}$$

where the notation refers to the fact that we are computing the partition function in different coordinate systems. Here of course we are particularly interested in the Weyl diffeomorphism $(z', x') = (z/\mathcal{B}(x), x)$ which relates the boundary values $\gamma'_{(0)} = \gamma_{(0)}/\mathcal{B}^2$, $a'_{(0)} = a_{(0)} - d\ln\mathcal{B}$. Z_{bulk} is given in the classical limit by evaluating the (renormalized) on-shell action, $Z_{bulk} = e^{-S_{o.s.}[g;\gamma_{(0)},a_{(0)},...|z,x]}$. We then ask, is it also true that this cleanly induces a Weyl transformation on the boundary? That is, is it true that

$$\frac{Z_{bdy}[x; \gamma'_{(0)}, a'_{(0)}, ...]}{Z_{bdy}[x; \gamma_{(0)}, a_{(0)}, ...]} \stackrel{?}{=} 1, \tag{73}$$

where Z_{bdy} is the generating functional in the given background. As is well-established, what happens is that there is an anomaly

$$\frac{Z_{bulk}\left[g'; \gamma'_{(0)}, a'_{(0)}, ...\middle| z', x\right]}{Z_{bulk}\left[g; \gamma_{(0)}, a_{(0)}, ...\middle| z, x\right]} = e^{\mathcal{A}_k}\frac{Z_{bdy}[x; \gamma'_{(0)}, a'_{(0)}, ...]}{Z_{bdy}[x; \gamma_{(0)}, a_{(0)}, ...]} \tag{74}$$

in dimension $d = 2k$. Recall that we are employing the specific Weyl diffeomorphism, which is inducing a Weyl transformation on the boundary, but no boundary diffeomorphism. If we take the log of these expressions, the result is that

$$0 = S_{bulk}[g'; \gamma'_{(0)}, ...|z', x] - S_{bulk}[g; \gamma_{(0)}, ...|z, x] = S_{bdy}[x; \gamma'_{(0)}, a'_{(0)}, ...] - S_{bdy}[x; \gamma_{(0)}, a_{(0)}, ...] + \mathcal{A}_k. \tag{75}$$

That is, when we compare the evaluation of the bulk on-shell action in different coordinate systems, the result appears as the difference of boundary actions in Weyl-equivalent backgrounds, *up to an anomalous term, which is not the difference of two such actions*. The only source for such a term is a pole at $d = 2k$ in the evaluation of the bulk action, which arises because the on-shell action is not a boundary term, but contains pieces that must be integrated over z. The bulk action is generally given by ($vol_S = \sqrt{-\det h}d^dx$)

$$S_{bulk}[g; \gamma_{(0)}, ...|z, x] = \frac{1}{16\pi G}\int_M e \wedge vol_S(R - 2\Lambda). \tag{76}$$

On shell, it evaluates to

$$S_{bulk}[g; \gamma_{(0)}, ...|z, x] = -\frac{d}{8\pi GL^2}\int_M e \wedge vol_S = -\frac{d}{8\pi GL}\int_M \frac{dz}{z} \wedge d^dx\sqrt{-\det h}, \tag{77}$$

where we remind that d is the boundary dimension. We then expand $\sqrt{-\det h}$ in powers of z:

$$S_{bulk}[g; \gamma_{(0)}, ...|z, x] = -\frac{d}{8\pi GL}\int_M dz \wedge d^dx\Big(\frac{L}{z}\Big)^{d+1}\sqrt{-\det\gamma^{(0)}}\Big(1 + \frac{z^2}{L^2}\frac{X^{(1)}}{2} + \dots\Big). \tag{78}$$

Consider now the difference of Weyl-transformed bulk actions as in (75) and define $vol_\Sigma = \sqrt{-\det\gamma^{(0)}}d^dx$. The idea is to start with $S_{bulk}[g';\gamma'_{(0)},...|z',x]$, use the explicit Weyl transformation of the different quantities in the expansion (see (51)) and then change the name of the integration variable from z' to z.[10] We will demonstrate this for the first pole, which occurs at $d=2$. We then obtain

$$\begin{aligned} 0 &= \frac{d}{8\pi G}\int_M d\left(\frac{\mathcal{B}^{-d}}{d}\left(\frac{L}{z}\right)^d\right)\wedge vol_\Sigma - \frac{d}{8\pi G}\int_M d\left(\frac{1}{d}\left(\frac{L}{z}\right)^d\right)\wedge vol_\Sigma \\ &\quad + \frac{d}{16\pi G}\int_M d\left(\frac{\mathcal{B}^{-(d-2)}}{d-2}\left(\frac{L}{z}\right)^{d-2}\right)\wedge \mathcal{G}_\Sigma - \frac{d}{16\pi G}\int_M d\left(\frac{1}{d-2}\left(\frac{L}{z}\right)^{d-2}\right)\wedge\mathcal{G}_\Sigma + \ldots, \end{aligned} \tag{79}$$

with $\mathcal{G}_\Sigma = X^{(1)}vol_\Sigma$ (Weyl weight $-(d-2)$). We observe that the offending term in $d\to 2^-$ is

$$\frac{d}{16\pi G}\int_M d\left(\frac{\mathcal{B}^{-(d-2)}}{d-2}\left(\frac{L}{z}\right)^{d-2}\right)\wedge\mathcal{G}_\Sigma - \frac{d}{16\pi G}\int_M d\left(\frac{1}{d-2}\left(\frac{L}{z}\right)^{d-2}\right)\wedge\mathcal{G}_\Sigma = -\frac{1}{8\pi GL}\int_\Sigma \ln\mathcal{B}\;\mathcal{G}_\Sigma. \tag{80}$$

The equality in this equation is obtained expanding $\mathcal{B}$ around 1 and eventually imposing $d=2$. For concreteness we expand this final result using the holographic value of $X^{(1)}$, (66). Then, we read from (75):

$$\mathcal{A}_1 = \frac{1}{8\pi GL}\int_\Sigma \ln\mathcal{B}\;\mathcal{G}_\Sigma = -\frac{L}{16\pi G}\int_\Sigma \ln\mathcal{B}\;R^{(0)}vol_\Sigma. \tag{81}$$

This numerical coefficient is the correct one that leads to the central charge $c=\dfrac{3L}{2G}$. We will shortly comment on the implications, but notice already that $R^{(0)}$ is not the Levi-Civita curvature, as usually found, but rather the Weyl curvature. As such, it is a Weyl-covariant scalar.

The Weyl anomaly in $d=2$ then is best expressed cohomologically as the difference:

$$(e\wedge\mathcal{G}_\Sigma)' - (e\wedge\mathcal{G}_\Sigma) = d(\ln\mathcal{B}\;\mathcal{A}_1\;vol_\Sigma), \tag{82}$$

with $\mathcal{A}_1$ proportional to $X^{(1)}$. Each term on the left is expected to be closed (because they are top forms in the bulk!) but the difference is in general exact, with its strength determining the Weyl anomaly of the boundary theory.

Some comments are in order here. Firstly, we have obtained a very powerful new result: the Weyl anomaly $\mathcal{A}_1$ is now dictated in 2d by the Weyl-covariant scalar curvature $R^{(0)}$. This is not the case if we start with the FG gauge in the bulk, for which the Levi-Civita scalar curvature appears. The Weyl covariance of all the subleading terms in the WFG gauge implies that the anomaly in every even boundary dimension will have Weyl-covariant curvature coefficients in our framework. Secondly, we expect the cohomological derivation of the anomaly to be a general feature, not restricted to the 2-dimensional case. In fact, recalling that the metric determinant is expanded as (cf (A.22))

$$\sqrt{-\det h(z;x)} \;=\; \left(\frac{L}{z}\right)^d\sqrt{-\det\gamma^{(0)}(x)}\left[1+\frac{1}{2}\frac{z^2}{L^2}X^{(1)}+\frac{1}{2}\frac{z^4}{L^4}X^{(2)}+\ldots+\frac{1}{2}\frac{z^d}{L^d}Y^{(1)}+\ldots\right], \tag{83}$$

we deduce that a similar derivation as for the 2-dimensional case holds in any even dimension, with $\mathcal{G}_\Sigma$ generally replaced by

$$\mathcal{G}^{(k)}_\Sigma = X^{(k)}vol_\Sigma. \tag{84}$$

[10] To evaluate these expressions, a regulator is required. The last step of renaming the integration variable has a corresponding effect on the cutoff and thus is not innocuous in the renormalization procedure. Such a regulator is not Weyl-covariant, which is consistent with the fact that an anomaly arises. Most of the details of the renormalization occur in expressions that are the difference of two Weyl-equivalent actions, whereas the anomaly is not and has been cleanly extracted.

We therefore claim that in any even dimension $d = 2k$,

$$\left(e \wedge \mathcal{G}_{\Sigma}^{(k)}\right)' - \left(e \wedge \mathcal{G}_{\Sigma}^{(k)}\right) = d(\ln \mathcal{B}\ \mathcal{A}_k\ vol_\Sigma), \tag{85}$$

the $\mathcal{A}_k$ term on the right-hand side being proportional to $X_{(k)}$. Looking for a universal form of $X_{(k)}$ as a function of the Weyl curvature tensors of the boundary is an appealing future direction of investigation.

5 Field Theory Aspects

In this section, we will make some preliminary remarks about the dual field theory. The holographic analysis implies that we should now consider a field theory coupled to a background metric and Weyl connection, with action $S[\gamma^{(0)}, a^{(0)}; \Phi]$ where Φ denotes some collection of dynamical fields to which we will assign some definite Weyl weights. As we will explain, this is perfectly natural from the field theory perspective as well, but constitutes a new organization of such field theories (which in the usual formulation are coupled only to a background metric). The quantum theory possesses a partition function $Z[\gamma^{(0)}, a^{(0)}]$ that depends on the background, both through explicit dependence in the action and in the definition of the functional integral measure. A background Ward identity is generated by changing integration variables $\Phi(x) \mapsto \mathcal{B}(x)^{w_\Phi}\Phi(x)$ giving

$$Z[\gamma^{(0)}, a^{(0)}] = e^{\mathcal{A}[\mathcal{B}]} Z[\mathcal{B}(x)^{-2}\gamma^{(0)}, a^{(0)} - d\ln \mathcal{B}(x)] \tag{86}$$

with $\mathcal{A}$ a possible anomalous contribution. Thus the Weyl Ward identity is a relationship between *different* theories, that is, field theories in different backgrounds and so, more properly, we refer to the above equation as a background Ward identity. Strictly speaking, this argument applies to free theories, whereby (if Φ is a scalar) $w_\Phi = \frac{1}{2}(d-2)$ is the engineering dimension. An example of an action in this context is

$$S[\gamma^{(0)}, a^{(0)}; \Phi] = -\tfrac{1}{2}\int d^dx \sqrt{-\det \gamma^{(0)}}\ \gamma^{ab}_{(0)} \hat{\nabla}_a \Phi \hat{\nabla}_b \Phi \tag{87}$$

where $\hat{\nabla}_a \Phi = \partial_a \Phi + w_\Phi a_a^{(0)} \Phi$ is Weyl-covariant.[11] Notice that the stress tensor of this theory has the form

$$\mathrm{T}_{ab}^{\gamma^{(0)},a^{(0)}}(x) = \frac{2}{\sqrt{-\det \gamma^{(0)}}} \frac{\delta S[\gamma^{(0)}, a^{(0)}; \Phi]}{\delta \gamma_{ab}^{(0)}(x)} = \hat{\nabla}_a\Phi(x)\hat{\nabla}_b\Phi(x) - \tfrac{1}{2}\gamma_{ab}^{(0)}(x)\gamma^{(0)cd}(x)\hat{\nabla}_c\Phi(x)\hat{\nabla}_d\Phi(x) \tag{88}$$

Here we have used pedantic notation to emphasize that the definition of the operator depends on the background fields. This operator is Weyl-covariant, by which we mean

$$\mathrm{T}_{ab}^{\mathcal{B}(x)^{-2}\gamma^{(0)}, a^{(0)} - d\ln\mathcal{B}(x)}(x) = \mathcal{B}(x)^{d-2}\mathrm{T}_{ab}^{\gamma^{(0)},a^{(0)}}(x) \tag{89}$$

That is, if we compare correlation functions of the stress tensor in two Weyl-related backgrounds, there will be a relative factor of $\mathcal{B}(x)^{d-2}$ for each instance of the stress tensor; for brevity, we refer to this as the stress tensor (with two lower indices) having Weyl weight $w_T = d-2$. Similarly, we have the Weyl current

$$\mathrm{J}_a^{a^{(0)}}(x) = \frac{1}{\sqrt{-\det \gamma^{(0)}}} \frac{\delta S[\gamma^{(0)}, a^{(0)}; \Phi]}{\delta a_a^{(0)}(x)} = w_\Phi \Phi(x)\hat{\nabla}_a\Phi(x) \tag{90}$$

This operator is also Weyl-covariant in the same sense as the stress tensor and is of weight $d-2$. Thus $\hat{\mathrm{T}}_{ab}$ and $\hat{\mathrm{J}}_a$ have the properties of the operators sourced in the holographic WFG theory. In a holographic

[11] An independently Weyl invariant action term is $\int d^dx\sqrt{-\det\gamma^{(0)}}\ R^{(0)}\Phi^2$. It is well-known that using the LC connection, only a specific linear combination of the kinetic term and such a curvature term is Weyl invariant, at least up to a total derivative.

theory, we would not have the free field discussion given here, but we can still discuss sourcing these operators (in a given background).

Earlier, we saw that the classical Weyl Ward identity involved a linear combination of the trace of the stress tensor and the divergence of the Weyl current. This is in fact easily established in general terms. Here we will use classical language, but the argument easily extends to the quantum case by making use of (86). Indeed, suppose that the classical action satisfies

$$S[\gamma^{(0)}, a^{(0)}; \mathcal{B}^{w_\Phi}\Phi] = S[\gamma^{(0)}/\mathcal{B}^2, a^{(0)} - d\ln\mathcal{B}; \Phi] \tag{91}$$

As mentioned above, this is what we mean by Weyl being a background symmetry. By expanding both sides for small $\ln\mathcal{B}$ and going on-shell, we find

$$0 = \int d^dx \frac{\delta S}{\delta a^{(0)}_\mu(x)}\partial_\mu \ln\mathcal{B}(x) + \int d^dx \frac{\delta S}{\delta\gamma^{(0)}_{\mu\nu}(x)}\Big(-2\ln\mathcal{B}(x)\gamma^{(0)}_{\mu\nu}(x)\Big) \tag{92}$$

We recognize that this may be written as

$$0 = \int d^dx\sqrt{-\det\gamma^{(0)}}\ \mathrm{J}^\mu(x)\partial_\mu\ln\mathcal{B}(x) + \int d^dx\sqrt{-\det\gamma^{(0)}}\ \mathrm{T}^{\mu\nu}(x)\Big(-\ln\mathcal{B}(x)\gamma^{(0)}_{\mu\nu}(x)\Big) \tag{93}$$

and, by integrating by parts, we have

$$0 = -\int d^dx\sqrt{-\det\gamma^{(0)}}\Big(\hat{\nabla}_\mu\mathrm{J}^\mu(x) + \mathrm{T}^{\mu\nu}(x)\gamma^{(0)}_{\mu\nu}(x)\Big)\ln\mathcal{B}(x) \tag{94}$$

This result serves to identify the relative normalization of $\pi^{(0)}_{\mu\nu}$ and $p^{(0)}_\mu$ and their relation with the currents defined here. Incidentally, the Weyl-covariant derivative appears in (94) precisely because the current J^μ (with raised index) has Weyl weight d.

We remark that typical discussions of related topics are rife with 'improvements' to operators such as the stress tensor, including mixing with a so-called 'virial current'. The operators that we have defined here have the advantage of transforming linearly, and in particular do not mix with each other, under Weyl transformations. Note also that the Weyl current in the free theory is in fact a total derivative. Thus, at least in the absence of edges or boundaries [49, 50], one might suppose that this operator is in a sense trivial. We will explain elsewhere the symmetry structure of these operators.

6 Conclusions

In this work, we have discussed the consequences of bringing a Weyl connection into the formulation of holography. In order to address this, we first intrinsically analyzed such connections and their associated geometrical tensors. The need for a Weyl connection arises in theories that, in addition to diffeomorphisms, admit a local rescaling of the metric by an arbitrary local function. The vanishing of the metricity required for the familiar Levi-Civita connection is indeed not maintained under such rescalings, and the Weyl connection is defined as the unique torsionless connection with vanishing Weyl metricity, a Weyl-covariant statement. Although richer than its Levi-Civita counterparts, the geometrical tensors built out of this connection turn out to be tractable.

It has long been understood that holographic field theories possess a Weyl invariance, in the sense that they couple not to a metric, but to a conformal class of metrics. The introduction of a (background) Weyl connection in holographic field theories is a suitable reformulation in which local Weyl transformations relate such theories in different backgrounds. In our account, the bulk gravitational theory is unmodified, but the gauge-fixing is relaxed (to what we called Weyl-Fefferman-Graham gauge) in such a way that the Weyl diffeomorphisms act geometrically on tensors parametrizing the bulk metric. The Weyl diffeomorphisms correspond to rescaling the holographic coordinate by functions of the transverse coordinates while leaving the latter unchanged. While the FG expansion induces the LC connection associated to the induced boundary metric, we have proven that the WFG expansion induces on the boundary a Weyl connection. This result indicates that the WFG gauge is the proper bulk parametrization that leaves the bulk diffeomorphisms corresponding to the boundary Weyl transformations unfixed. This leads to the interpretation of the Weyl connection in the boundary as a background field together with the boundary metric; essentially, the pair $(\gamma^{(0)}, a^{(0)})$ replace $[\gamma^{(0)}]$. An interesting consequence of the WFG gauge is that the boundary hypersurface is generally not part of a foliation, the distribution that is involved being generally non-integrable. We expect that the details of holographic renormalization require a slightly more sophisticated regulator than is usually employed, but the results of this paper do not rely on such details.

The WFG gauge involves an expansion in powers of the holographic coordinate in which every coefficient is Weyl-covariant by construction. This result is a powerful reorganization of the holographic dictionary. The Weyl connection sources a Weyl current which explicitly appears in the subleading expansion of the bulk geometry. Subleading orders of the bulk Einstein equations unravel the boundary Weyl geometrical tensors and relationships between boundary expectation values of the sourced operators. In particular we find the boundary Ward identity relating the trace of the energy-momentum tensor with the divergence of the Weyl current, and in the last section have shown that this is the expected result.

We then scrutinized the implications of our setup for the Weyl anomaly. Not surprisingly, we found the latter to be given now in terms of Weyl-covariant geometrical objects, instead of the corresponding Levi-Civita objects. We expect that this outcome will have implications for the study and characterization of the anomaly in higher even boundary dimensions. The presence of Weyl geometrical tensors allowed for a cohomological description of the anomaly as a difference of Weyl-related bulk volumes, which offers a clear geometrical interpretation of the anomaly.

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A Details of Bulk Expansions

We recapitulate here our geometrical setup both in the bulk and in the boundary, and compute the leading orders of the expansion toward $z = 0$ of the main quantities involved. These are useful to evaluate Einstein equations order by order, and hence solve for the various geometrical objects. Concretely, we work in the non-coordinate basis

$$ds^2 = e \otimes e + h_{\mu\nu} dx^\mu \otimes dx^\nu, \qquad e = L\Big(\frac{dz}{z} - a_\mu dx^\mu\Big). \tag{A.1}$$

The dual vectors are

$$\underline{e} = L^{-1} z \underline{\partial}_z, \qquad \underline{D}_\mu = \underline{\partial}_\mu + z a_\mu \underline{\partial}_z, \tag{A.2}$$

and they form an orthonormal basis

$$e(\underline{e}) = 1, \qquad e(\underline{D}_\mu) = 0, \qquad dx^\mu(\underline{D}_\nu) = \delta^\mu_\nu, \qquad dx^\mu(\underline{e}) = 0. \tag{A.3}$$

The vector commutators give

$$[\underline{e}, \underline{D}_\mu] = L\underline{e}(a_\mu)\underline{e} = L\varphi_\mu \underline{e}, \qquad [\underline{D}_\mu, \underline{D}_\nu] = L\Big(\underline{D}_\mu a_\nu - \underline{D}_\nu a_\mu\Big)\underline{e} = L f_{\mu\nu}\underline{e}, \tag{A.4}$$

from which we read

$$C_{e\mu}{}^e = L\varphi_\mu, \qquad C_{\mu\nu}{}^e = L f_{\mu\nu}, \qquad C_{\mu\nu}{}^\alpha = 0. \tag{A.5}$$

Throughout this Appendix, we refer for brevity to generalized bulk indices as $M = (e, \mu)$ and thus vectors $\underline{e}_M = (\underline{e}, \underline{D}_\mu)$, the most general non-coordinatized Levi-Civita connection is then

$$\Gamma^P_{MN} = \tfrac{1}{2} g^{PQ}\Big(\underline{e}_M(g_{NQ}) + \underline{e}_N(g_{QM}) - \underline{e}_Q(g_{MN})\Big) - \tfrac{1}{2} g^{PQ}\Big(C_{MQ}{}^R g_{RN} + C_{NM}{}^R g_{RQ} - C_{QN}{}^R g_{RM}\Big). \tag{A.6}$$

The metric and its inverse are given in components by

$$g_{\mu\nu} = h_{\mu\nu}, \quad g_{e\mu} = 0, \quad g_{ee} = 1, \quad g^{\mu\nu} = h^{\mu\nu}, \quad g^{\mu e} = 0, \quad g^{ee} = 1. \tag{A.7}$$

Then, calling $\theta = tr\rho$ with $\rho^\mu_\nu = \frac{1}{2} h^{\mu\alpha}\underline{e}(h_{\alpha\nu})$, the Christoffel symbols evaluate to

$$\Gamma^e_{ee} = 0 \tag{A.8}$$
$$\Gamma^e_{e\mu} = C_{e\mu}{}^e = L\varphi_\mu \tag{A.9}$$
$$\Gamma^e_{\mu e} = 0 \tag{A.10}$$
$$\Gamma^e_{\mu\nu} = -\tfrac{1}{2}\underline{e}(h_{\mu\nu}) + \frac{L}{2} f_{\mu\nu} \tag{A.11}$$
$$\Gamma^\mu_{ee} = h^{\mu\nu} C_{\nu e}{}^e = -L h^{\mu\nu}\varphi_\nu \tag{A.12}$$
$$\Gamma^\mu_{e\nu} = \rho^\mu_\nu + \frac{L}{2} f^\mu{}_\nu \tag{A.13}$$
$$\Gamma^\mu_{\nu e} = \rho^\mu_\nu + \frac{L}{2} f^\mu{}_\nu \tag{A.14}$$
$$\Gamma^\mu_{\mu e} = \theta \tag{A.15}$$
$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} h^{\mu\nu}\Big(\underline{D}_\alpha h_{\beta\nu} + \underline{D}_\beta h_{\alpha\nu} - \underline{D}_\nu h_{\alpha\beta}\Big) \equiv \gamma^\mu_{\alpha\beta}. \tag{A.16}$$

These connections are explicitly reported in (57), (58), (59) and (60). We additionally define

$$m_{(k)}{}^\mu{}_\nu \equiv (\gamma^{-1}_{(0)}\gamma_{(k)})^\mu{}_\nu, \qquad n_{(k)}{}^\mu{}_\nu \equiv (\gamma^{-1}_{(0)}\pi_{(k)})^\mu{}_\nu, \tag{A.17}$$

and the scalars

$$X^{(1)} = tr(m_{(2)}), \tag{A.18}$$
$$X^{(2)} = tr(m_{(4)}) - \tfrac{1}{2} tr(m^2_{(2)}) + \tfrac{1}{4}\Big(tr(m_{(2)})\Big)^2, \tag{A.19}$$
$$Y^{(1)} = tr(n_{(0)}). \tag{A.20}$$

Starting from the metric (31) and the Weyl connection (50) expansions, we compute the inverse metric, the determinant and the various connection components appearing in (61). We expand the two series enough to be able to capture the two leading orders. The result is:

$$h^{\mu\lambda}(z;x) = \frac{z^2}{L^2}\left[\gamma_{(0)}^{-1} - \frac{z^2}{L^2}m_{(2)}\gamma_{(0)}^{-1} - \frac{z^4}{L^4}(m_{(4)} - m_{(2)}^2)\gamma_{(0)}^{-1} + ...\right]^{\mu\lambda} - \frac{z^{d+2}}{L^{d+2}}\left[n_{(0)}\gamma_{(0)}^{-1} + ...\right]^{\mu\lambda} \quad (A.21)$$

$$\sqrt{-\det h(z;x)} = \left(\frac{L}{z}\right)^d \sqrt{-\det \gamma^{(0)}(x)}\left[1 + \frac{1}{2}\frac{z^2}{L^2}X^{(1)} + \frac{1}{2}\frac{z^4}{L^4}X^{(2)} + ... + \frac{1}{2}\frac{z^d}{L^d}Y^{(1)} + ...\right] \quad (A.22)$$

$$\rho^\mu{}_\nu(z;x) = L^{-1}\left[-\delta^\mu{}_\nu + \frac{z^2}{L^2}m_{(2)}{}^\mu{}_\nu + \frac{z^4}{L^4}(2m_{(4)} - m_{(2)}^2)^\mu{}_\nu + ... + \frac{d}{2}\frac{z^d}{L^d}n_{(0)}{}^\mu{}_\nu + ...\right] \quad (A.23)$$

$$\theta(z;x) = L^{-1}\left[-d + \frac{z^2}{L^2}X^{(1)} + \frac{z^4}{L^4}2(X^{(2)} - \frac{1}{4}(X^{(1)})^2) + ... + \frac{d}{2}\frac{z^d}{L^d}Y^{(1)} + ...\right] \quad (A.24)$$

$$\varphi_\mu(z;x) = L^{-1}\left[\frac{z^2}{L^2}2a_\mu^{(2)} + ... + \frac{z^{d-2}}{L^{d-2}}(d-2)p_\mu^{(0)} + ...\right] \quad (A.25)$$

$$f_{\mu\nu}(z;x) = f_{\mu\nu}^{(0)}(x) + \frac{z^2}{L^2}(\hat\nabla_\mu^{(0)}a_\nu^{(2)} - \hat\nabla_\nu^{(0)}a_\mu^{(2)}) + ... + \frac{z^{d-2}}{L^{d-2}}(\hat\nabla_\mu^{(0)}p_\nu^{(0)} - \hat\nabla_\nu^{(0)}p_\mu^{(0)}) + ... \quad (A.26)$$

with $f_{\mu\nu}^{(0)} = \partial_\mu a_\nu^{(0)} - \partial_\nu a_\mu^{(0)}$. In the expression for $f_{\mu\nu}$ we used the boundary derivative introduced in (21), which is the Weyl derivative shifted with the Weyl weight of the object it acts upon. For instance, looking at (52), $a_\mu^{(2)}$ and $p_\mu^{(0)}$ are Weyl-covariant with weights 2 and $d-2$ respectively and therefore:

$$\hat\nabla_\mu^{(0)}a_\nu^{(2)} = \nabla_\mu^{(0)}a_\nu^{(2)} + 2a_\mu^{(0)}a_\nu^{(2)}, \quad (A.27)$$

$$\hat\nabla_\mu^{(0)}p_\nu^{(0)} = \nabla_\mu^{(0)}p_\nu^{(0)} + (d-2)a_\mu^{(0)}p_\nu^{(0)}, \quad (A.28)$$

with $\nabla^{(0)}$ the boundary Weyl connection (its connection coefficients are explicitly given in (56)).

The expansion of the geometrical objects constructed from (63) is also reported

$$\begin{aligned}\gamma^\lambda_{\mu\nu} = {}& \gamma^{(0)\lambda}_{\mu\nu} + \frac{z^2}{L^2}\Big[\tfrac{1}{2}\gamma_{(0)}^{\lambda\xi}\Big(\hat\nabla_\nu^{(0)}\gamma_{\mu\xi}^{(2)} + \hat\nabla_\mu^{(0)}\gamma_{\xi\nu}^{(2)} - \hat\nabla_\xi^{(0)}\gamma_{\mu\nu}^{(2)}\Big) - \Big(a_\mu^{(2)}\delta^\lambda{}_\nu + a_\nu^{(2)}\delta^\lambda{}_\mu - a_\xi^{(2)}\gamma_{(0)}^{\lambda\xi}\gamma_{\mu\nu}^{(0)}\Big)\Big] + ...\\ &-\frac{z^{d-2}}{L^{d-2}}\Big[p_\mu^{(0)}\delta^\lambda{}_\nu + p_\nu^{(0)}\delta^\lambda{}_\mu - p_\rho^{(0)}\gamma_{(0)}^{\lambda\rho}\gamma_{\mu\nu}^{(0)}\Big] + ...\end{aligned} \quad (A.29)$$

$$\begin{aligned}\bar{Ric}_{\mu\nu} = {}& Ric_{\mu\nu}^{(0)} + \frac{z^2}{L^2}\Big[\tfrac{1}{2}\hat\nabla_\lambda^{(0)}\Big(\gamma_{(0)}^{\lambda\xi}\Big(\hat\nabla_\nu^{(0)}\gamma_{\mu\xi}^{(2)} + \hat\nabla_\mu^{(0)}\gamma_{\xi\nu}^{(2)} - \hat\nabla_\xi^{(0)}\gamma_{\mu\nu}^{(2)}\Big)\Big)\\ &+(d-1)\hat\nabla_\nu^{(0)}a_\mu^{(2)} - \hat\nabla_\mu^{(0)}a_\nu^{(2)} + \gamma_{\mu\nu}^{(0)}\hat\nabla^{(0)}\cdot a^{(2)} - \tfrac{1}{2}\hat\nabla_\nu^{(0)}\hat\nabla_\mu^{(0)}X^{(1)}\Big]\\ &+... + \frac{z^{d-2}}{L^{d-2}}\Big[(d-1)\hat\nabla_\nu p_\mu^{(0)} - \hat\nabla_\mu^{(0)}p_\nu^{(0)} + \gamma_{\mu\nu}^{(0)}\hat\nabla^{(0)}\cdot p^{(0)}\Big] + ...\end{aligned} \quad (A.30)$$

$$\begin{aligned}\bar R = {}& \frac{z^2}{L^2}R^{(0)} + \frac{z^4}{L^4}\Big[\gamma_{(0)}^{\lambda\nu}\hat\nabla_\lambda^{(0)}\hat\nabla_\mu\Big(m_{(2)}{}^\mu{}_\nu - tr(m_{(2)})\delta^\mu{}_\nu\Big) + 2(d-1)\hat\nabla\cdot a^{(2)} - tr(m_{(2)}\gamma_{(0)}^{-1}Ric^{(0)})\Big]\\ &+... + 2(d-1)\frac{z^d}{L^d}\hat\nabla\cdot p^{(0)} + ...\end{aligned} \quad (A.31)$$

$$\begin{aligned}\bar G_{\mu\nu} = {}& G_{\mu\nu}^{(0)} + \frac{z^2}{L^2}\Big[\tfrac{1}{2}\hat\nabla_\lambda\Big(\gamma_{(0)}^{\lambda\xi}\Big(\hat\nabla_\nu\gamma_{\xi\mu}^{(2)} + \hat\nabla_\mu\gamma_{\xi\nu}^{(2)} - \hat\nabla_\xi\gamma_{\mu\nu}^{(2)}\Big)\Big) + (d-1)\hat\nabla_\nu a_\mu^{(2)} - \hat\nabla_\mu a_\nu^{(2)} - (d-2)\gamma_{\mu\nu}^{(0)}\hat\nabla\cdot a^{(2)}\\ &-\tfrac{1}{2}\hat\nabla_\nu\hat\nabla_\mu X^{(1)} - \tfrac{1}{2}\gamma_{\mu\nu}^{(2)}R^{(0)} - \tfrac{1}{2}\gamma_{\mu\nu}^{(0)}\hat\nabla_\lambda\hat\nabla_\phi\Big((\gamma_{(0)}^{-1}\gamma^{(2)}\gamma_{(0)}^{-1})^{\phi\lambda} - X^{(1)}\gamma_{(0)}^{\phi\lambda}\Big)\Big] + ...\\ &+\frac{z^{d-2}}{L^{d-2}}\Big[(d-1)\hat\nabla_\nu p_\mu^{(0)} - \hat\nabla_\mu p_\nu^{(0)} - (d-2)\hat\nabla\cdot p^{(0)}\gamma_{\mu\nu}^{(0)}\Big] + ...\end{aligned} \quad (A.32)$$

These quantities appear explicitly in the Einstein tensor. We then compute the bulk Ricci tensor:

$$Ric_{MN} = R^P{}_{MPN} = \underline{e}_P(\Gamma^P_{NM}) - \underline{e}_N(\Gamma^P_{PM}) + \Gamma^Q_{NM}\Gamma^P_{PQ} - \Gamma^Q_{PM}\Gamma^P_{NQ} - C_{PN}{}^Q\Gamma^P_{QM}, \quad (A.33)$$

and so

$$
\begin{aligned}
Ric_{ee} &= -L\nabla_\mu\varphi^\mu - L^2\varphi^2 - \underline{e}(\theta) - tr(\rho\rho) - \frac{L^2}{4}tr(ff) && \text{(A.34)}\\
Ric_{e\mu} &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.35)}\\
Ric_{\mu e} &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.36)}\\
Ric_{\mu\nu} &= \bar{Ric}_{\mu\nu} - L\nabla_\nu\varphi_\mu - (\underline{e}+\theta)\Big(\rho_{\mu\nu} + \frac{L}{2}f_{\mu\nu}\Big) - L^2\varphi_\mu\varphi_\nu + 2\rho_\mu^\alpha\rho_{\alpha\nu} + \frac{L^2}{2}f_{\nu\alpha}f^\alpha{}_\mu. && \text{(A.37)}
\end{aligned}
$$

Notice that $Ric_{e\mu} = Ric_{\mu e}$. The trace of the Ricci tensor gives the scalar curvature

$$R = g^{MN}\Big(\underline{e}_P(\Gamma^P_{NM}) - \underline{e}_N(\Gamma^P_{PM}) + \Gamma^Q_{NM}\Gamma^P_{PQ} - \Gamma^Q_{PM}\Gamma^P_{NQ} - C_{PN}{}^Q\Gamma^P_{QM}\Big). \tag{A.38}$$

It evaluates to

$$R = -2\underline{e}(\theta) + \frac{L^2}{4}tr(ff) - tr(\rho\rho) - 2Lh^{\mu\nu}\nabla_\mu\varphi_\nu + \bar{R} - \theta^2 - 2L^2\varphi_\mu\varphi_\nu h^{\mu\nu}. \tag{A.39}$$

Therefore the various components of the Einstein tensor read

$$
\begin{aligned}
G_{ee} &= -\tfrac{1}{2}tr(\rho\rho) - \frac{3L^2}{8}tr(ff) - \tfrac{1}{2}\bar{R} + \tfrac{1}{2}\theta^2 && \text{(A.40)}\\
G_{e\mu} &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.41)}\\
G_{\mu e} &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.42)}\\
G_{\mu\nu} &= \bar{G}_{\mu\nu} - L\nabla_\nu\varphi_\mu - (\underline{e}+\theta)\Big(\rho_{\mu\nu} + \frac{L}{2}f_{\mu\nu}\Big) - L^2\varphi_\mu\varphi_\nu + 2\rho_\mu^\alpha\rho_{\alpha\nu} + \frac{L^2}{2}f_{\nu\alpha}f^\alpha{}_\mu && \text{(A.43)}\\
&\quad + h_{\mu\nu}\Big(\underline{e}(\theta) - \frac{L^2}{8}tr(ff) + \tfrac{1}{2}tr(\rho\rho) + L\nabla_\alpha\varphi^\alpha + \tfrac{1}{2}\theta^2 + L^2\varphi^2\Big). && \text{(A.44)}
\end{aligned}
$$

Finally, vacuum Einstein equations are given by

$$G_{MN} + \Lambda g_{MN} = 0. \tag{A.45}$$

They become

$$
\begin{aligned}
0 &= -\tfrac{1}{2}tr(\rho\rho) - \frac{3L^2}{8}tr(ff) - \tfrac{1}{2}\bar{R} + \tfrac{1}{2}\theta^2 + \Lambda && \text{(A.46)}\\
0 &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.47)}\\
0 &= \nabla_\alpha\Big(\rho_\mu^\alpha + \frac{L}{2}f^\alpha{}_\mu\Big) - \underline{D}_\mu\theta + L^2\varphi^\alpha f_{\alpha\mu} && \text{(A.48)}\\
0 &= \bar{G}_{\mu\nu} - L\nabla_\nu\varphi_\mu - (\underline{e}+\theta)\Big(\rho_{\mu\nu} + \frac{L}{2}f_{\mu\nu}\Big) - L^2\varphi_\mu\varphi_\nu + 2\rho_\mu^\alpha\rho_{\alpha\nu} + \frac{L^2}{2}f_{\nu\alpha}f^\alpha{}_\mu && \text{(A.49)}\\
&\quad + h_{\mu\nu}\Big(\underline{e}(\theta) - \frac{L^2}{8}tr(ff) + \tfrac{1}{2}tr(\rho\rho) + L\nabla_\alpha\varphi^\alpha + \tfrac{1}{2}\theta^2 + L^2\varphi^2 + \Lambda\Big). && \text{(A.50)}
\end{aligned}
$$

We can obtain relationships among all the various terms in the expansion of $h_{\mu\nu}$ and a_μ by solving these equations order by order in z. For instance, (A.46) is expanded in (64), the expansion of (A.47) gives (68) and (70). Eventually, expanding (A.49) we obtain at first non-trivial order $\gamma^{(2)}_{\mu\nu}$ as written in (69).

References

[1] C. Fefferman and C. R. Graham, "Conformal invariants," *in Elie Cartan et les mathématiques d'aujourd'hui, Astérisque 1985, numéro hors série Soc. Math. France, Paris, 95* .

[2] C. Fefferman and C. R. Graham, "The ambient metric," *Ann. Math. Stud.* **178** (2011) 1–128, arXiv:0710.0919 [math.DG].

[3] H. Weyl, "Gravitation and electricity," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1918** (1918) 465. [,24(1918)].

[4] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," *JHEP* **02** (2008) 045, arXiv:0712.2456 [hep-th].

[5] R. Loganayagam, "Entropy Current in Conformal Hydrodynamics," *JHEP* **05** (2008) 087, arXiv:0801.3701 [hep-th].

[6] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," *JHEP* **12** (2008) 116, arXiv:0809.4272 [hep-th].

[7] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, "Flat holography and Carrollian fluids," arXiv:1802.06809 [hep-th].

[8] G. B. Folland, "Weyl manifolds," *J. Diff. Geom.* **4** (1970) 145–153.

[9] G. Hall, "Weyl manifolds and connections," *J Math Phys* **33** (1992) 2633.

[10] T. Fulton, F. Rohrlich, and L. Witten, "Conformal invariance in physics," *Rev. Mod. Phys.* **34** (1962) 442–457.

[11] N. Matsuo, "Einstein Gravity as Spontaneously Broken Weyl Gravity," *Gen. Rel. Grav.* **22** (1990) 561–593.

[12] J. M. Salim and S. L. Sautu, "Gravitational theory in Weyl integrable space-time," *Class. Quant. Grav.* **13** (1996) 353–360.

[13] W. Drechsler and H. Tann, "Broken Weyl invariance and the origin of mass," *Found. Phys.* **29** (1999) 1023–1064, arXiv:gr-qc/9802044 [gr-qc].

[14] I. Quiros, "The Weyl anomaly and the nature of the background geometry," arXiv:gr-qc/0011056 [gr-qc].

[15] C. Romero, J. B. Fonseca-Neto, and M. L. Pucheu, "General Relativity and Weyl Geometry," *Class. Quant. Grav.* **29** (2012) 155015, arXiv:1201.1469 [gr-qc].

[16] I. P. Lobo, A. B. Barreto, and C. Romero, "Space-time singularities in Weyl manifolds," *Eur. Phys. J.* **C75** (2015) no. 9, 448, arXiv:1506.02180 [gr-qc].

[17] E. Mottola, "Scalar Gravitational Waves in the Effective Theory of Gravity," *JHEP* **07** (2017) 043, arXiv:1606.09220 [gr-qc]. [erratum: JHEP09,107(2017)].

[18] S. Dengiz, "A noncompact Weyl–Einstein–Yang–Mills model: A semiclassical quantum gravity," *Annals Phys.* **383** (2017) 560–578, arXiv:1609.02475 [hep-th].

[19] M. de Cesare, J. W. Moffat, and M. Sakellariadou, "Local conformal symmetry in non-Riemannian geometry and the origin of physical scales," *Eur. Phys. J.* **C77** (2017) no. 9, 605, arXiv:1612.08066 [hep-th].

[20] I. Oda, "Planck and Electroweak Scales Emerging from Conformal Gravity," *Eur. Phys. J.* **C78** (2018) no. 10, 798, arXiv:1806.03420 [hep-th].

[21] I. Quiros and R. De Arcia, "On local scale invariance and the questionable theoretical basis of the conformal transformations' issue," arXiv:1811.02458 [gr-qc].

[22] D. M. Ghilencea, "Spontaneous breaking of Weyl quadratic gravity to Einstein action and Higgs potential," *JHEP* **03** (2019) 049, arXiv:1812.08613 [hep-th].

[23] E. Scholz, "The unexpected resurgence of Weyl geometry in late 20-th century physics," *Einstein Stud.* **14** (2018) 261–360, arXiv:1703.03187 [math.HO].

[24] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[25] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, arXiv:hep-th/9902121 [hep-th].

[26] K. Skenderis, "Asymptotically Anti-de Sitter space-times and their stress energy tensor," *Int. J. Mod. Phys.* **A16** (2001) 740–749, arXiv:hep-th/0010138 [hep-th]. [,394(2000)].

[27] R. Emparan, C. V. Johnson, and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," *Phys. Rev.* **D60** (1999) 104001, arXiv:hep-th/9903238 [hep-th].

[28] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595–622, arXiv:hep-th/0002230 [hep-th].

[29] K. Skenderis, "Lecture notes on holographic renormalization," *Class. Quant. Grav.* **19** (2002) 5849–5876, arXiv:hep-th/0209067 [hep-th].

[30] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," *JHEP* **07** (1998) 023, arXiv:hep-th/9806087 [hep-th].

[31] M. J. Duff, "Twenty years of the Weyl anomaly," *Class. Quant. Grav.* **11** (1994) 1387–1404, arXiv:hep-th/9308075 [hep-th].

[32] D. M. Capper and M. J. Duff, "Trace anomalies in dimensional regularization," *Nuovo Cim.* **A23** (1974) 173–183.

[33] S. Deser, M. J. Duff, and C. J. Isham, "Nonlocal Conformal Anomalies," *Nucl. Phys.* **B111** (1976) 45–55.

[34] M. J. Duff, "Observations on Conformal Anomalies," *Nucl. Phys.* **B125** (1977) 334–348.

[35] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," *Phys. Lett.* **B103** (1981) 207–210. [,598(1981)].

[36] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," *Rev. Mod. Phys.* **60** (1988) 917.

[37] S. Deser and A. Schwimmer, "Geometric classification of conformal anomalies in arbitrary dimensions," *Phys. Lett.* **B309** (1993) 279–284, arXiv:hep-th/9302047 [hep-th].

[38] P. O. Mazur and E. Mottola, "Weyl cohomology and the effective action for conformal anomalies," *Phys. Rev.* **D64** (2001) 104022, arXiv:hep-th/0106151 [hep-th].

[39] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories," *Nucl. Phys.* **B363** (1991) 486–526.

[40] D. R. Karakhanian, R. P. Manvelyan, and R. L. Mkrtchian, "Trace anomalies and cocycles of Weyl and diffeomorphism groups," *Mod. Phys. Lett.* **A11** (1996) 409–422, arXiv:hep-th/9411068 [hep-th].

[41] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997. http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997.

[42] C. G. Callan, Jr., S. R. Coleman, and R. Jackiw, "A New improved energy - momentum tensor," *Annals Phys.* **59** (1970) 42–73.

[43] A. Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger, "Weyl gauging and conformal invariance," *Nucl. Phys.* **B495** (1997) 433–450, arXiv:hep-th/9607110 [hep-th].

[44] S. Dengiz, E. Kilicarslan, and B. Tekin, "Weyl-gauging of Topologically Massive Gravity," *Phys. Rev.* **D86** (2012) 104014, arXiv:1209.1251 [hep-th].

[45] C. Coriano, L. Delle Rose, C. Marzo, and M. Serino, "The dilaton Wess–Zumino action in six dimensions from Weyl gauging: local anomalies and trace relations," *Class. Quant. Grav.* **31** (2014) 105009, arXiv:1311.1804 [hep-th].

[46] G. K. Karananas and A. Monin, "Weyl and Ricci gauging from the coset construction," *Phys. Rev.* **D93** (2016) no. 6, 064013, arXiv:1510.07589 [hep-th].

[47] K. Farnsworth, M. A. Luty, and V. Prilepina, "Weyl versus Conformal Invariance in Quantum Field Theory," *JHEP* **10** (2017) 170, arXiv:1702.07079 [hep-th].

[48] F. Wu, "Note on Weyl versus Conformal Invariance in Field Theory," *Eur. Phys. J.* **C77** (2017) no. 12, 886, arXiv:1704.05210 [hep-th].

[49] W. Donnelly and L. Freidel, "Local subsystems in gauge theory and gravity," *JHEP* **09** (2016) 102, arXiv:1601.04744 [hep-th].

[50] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," arXiv:1703.05448 [hep-th].

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ÉCOLE DOCTORALE
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Titre: La voie hydrodynamique vers l'holographie plate.

Mots clés: AdS/CFT, symétrie de Weyl, correspondance holographique, physique carrollienne.

Résumé: L'objectif de cette thèse est l'étude de la correspondance fluide/gravité, réalisation macroscopique de la dualité AdS/CFT dans la limite où la constante cosmologique tend vers zéro (limite plate). La jauge de Fefferman-Graham, utilisée dans le dictionnaire holographique, est singulière dans cette limite. Ici, en passant par la formulation hydrodynamique de la théorie vivant au bord, nous construirons une jauge appelée jauge du développement en série dérivative où cette limite est bien définie. Sur la géométrie du bord, elle correspond à faire tendre la vitesse de la lumière vers zéro, situation connue sous le nom de limite carrollienne. Un fluide relativiste admet une telle limite qui donne lieu à l'hydrodynamique carrollienne que l'on étudie ici en dimension arbitraire, parallèlement à son homologue galiléen. Ensuite, nous montrerons en dimensions quatre et trois qu'il est possible de construire des solutions asymptotiquement plates des équations d'Einstein en partant de systèmes hydrodynamiques conformes carrolliens du bord. En quatre dimensions, nous introduirons des conditions d'intégrabilité permettant de resommer la série dérivative. En trois dimensions, toute configuration fluide du bord aboutit à une solution des équations d'Einstein. Les solutions de Bañados sont un sous-ensemble des solutions obtenues et identifiées au moyen de leurs charges de surface. Nous accorderons une attention particulière au rôle du repère hydrodynamique, souvent ignoré en holographie. Pour terminer, nous nous concentrerons sur la formulation de la correspondance AdS/CFT dans laquelle la symétrie de Weyl est explicite. Bien que cette symétrie soit un ingrédient incontournable de la correspondance fluide/gravité, elle n'est pas codée dans la formulation habituelle de l'holographie. Nous introduirons une nouvelle jauge et analyserons ses conséquences.

Title: Paving the Fluid Road to Flat Holography

Keywords: AdS/CFT, Weyl symmetry, holographic correspondence, Carrollian physics.

Abstract: In this thesis we discuss the limit of vanishing cosmological constant (flat limit) of the fluid/gravity correspondence, which is a macroscopic realization of the AdS/CFT. The holographic dictionary is usually implemented in a gauge (Fefferman-Graham), which does not admit a flat limit. In the hydrodynamic formulation of the boundary theory, we introduce a gauge, called derivative expansion, where such a limit turns out to be smooth. In the boundary we show that this corresponds to a Carrollian limit, i.e. a limit where the speed of light vanishes. We present Carrollian hydrodynamics, together with its dual Galilean counterpart. Then, for four and three bulk dimensions, we exhibit a resummed line element, which provides an asymptotically flat bulk solution of Einstein equations starting only from boundary (i.e. null infinity) conformal Carrollian hydrodynamic data. In four dimensions we exploit specific integrability conditions, which restrict the achievable class of solutions in the bulk. In three dimensions every boundary fluid configuration leads to an exact solution of Einstein equations. Bañados solutions are a subset of the solutions reached in this way. They are rigorously identified with their surface charges and the corresponding algebra. We emphasize the choice of hydrodynamic frame, often sidestepped in holography. Finally, we focus on the formulation of AdS/CFT to encompass Weyl symmetry. This symmetry is a key ingredient of fluid/gravity but it is not naturally encoded in the usual formulation of holography. We introduce an appropriate gauge for realizing it, and analyze its far-reaching consequences.

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