To transform the chromatic number research to the chromatic index one in cubic graph

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To transform the chromatic number search to the chromatic index one in cubic graph

Summary: We present a personal transformation of the chromatic number research in any graph into chromatic index research in bridgeless cubic graph.

Keywords: Chromatic number, chromatic index, edge coloring, vertices coloring, graph theory

Résumé : Nous présentons une manière personnelle de transformer la recherche d'un nombre chromatique d'un graphe quelconque en la recherche de l'indice chromatique d'un graphe 3-régulier dit cubique, sans isthme.

Mots Clefs : Nombre chromatique, indice chromatique, coloration d'arêtes, coloration de sommets, théorie de graphes.

1. Introduction

We first remind some notions and naming convention and particularly the Cartesian sum of graph.

Knowing an algorithm for edges colouring of a 3-regular graph called cubic, without isthmus, we know how to find the chromatic number of any graph. Indeed, we explain how to reduce it to a problem SAT, then 3-SAT and following the article of Ian Holyer [3], we reduce it to a problem of chromatic index research of a cubic graph without isthmus.

2. Reminders and conventions

- **G = (X, A)** is the graph which has X as set of vertices and A as set of edges.
- **|Y|** is the cardinal of a set Y.
- We note $\mathbf{d_G(x)}$ the degree of x vertex on graph G.
- A vertex clique is a set of vertices linked together in pairs by edges. We note ω the maximum number of a clique.
- **N** is the vertices number of G (or |X|) and **M** the edges number (or |A|).
- We name χ', the chromatic index of G corresponding to the minimum number of colours to strongly colour the edges. In the books of Claude Berge [1] [2], this number is noted q(G).
- χ is the chromatic number of G corresponding to the minimum number of colours to strongly colour the vertices. In the books of Claude Berge [1] [2], this number is noted γ(G).
- $\mathbf{K_q}$ represents the order clique q.
- $\overline{G}$ is noted as the complement of G.
- G1 + G2 denotes the Cartesian sum of two graphs (cf. [1] [2]).

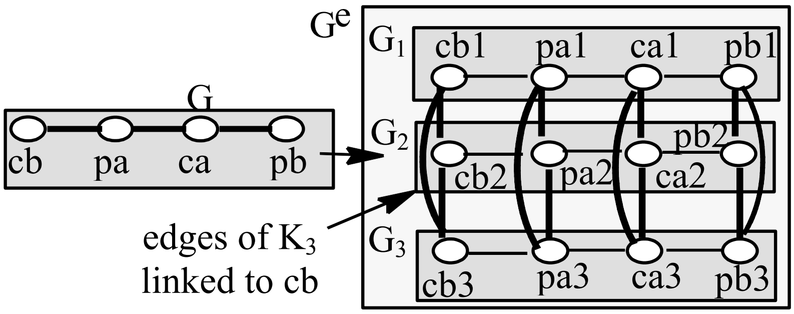
Reminder: *Cartesian sum of graph*. A Cartesian sum G^e of a graph G with another graph R is a graph consisting of the union of k duplications of the graph G, where k is the cardinal of X_R and edges of R connecting the vertices of the different duplicates (Gi). We can see graphically a duplication Gi in a horizontal axis and a duplication Rj in the vertical plane.

Here we will mainly use the Cartesian sum G + K_q.

Example: Let a graph G with X = {pa, pb, ca, cb} and A= {(pa, ca), (pa, cb), (pb, ca)},

G^e = G + K_3 = {(pa1, pa2), (pa1, pa3), (pa2, pa3), (pb1, pb2), (pb1, pb3), (pb2, pb3), (ca1, ca2), (ca1, ca3), (ca2, ca3), (cb1, cb2), (cb1, cb3), (cb2, cb3)}).

It is the Cartesian sum, G + K_3 which gives graphically:



Resulting graph

The edges set {(cb1, cb2), (cb1, cb3), (cb2, cb3)} will be named $\mathbf{K_3(cb)}$ linked to the vertex cb of G.

Remark: for the chromatic number research χ, we will associate the colour *i* to the ith duplication.

3. From chromatic number to chromatic index

This part takes up the graph sum Cartesian G + K_q and is based on the work of Ian Holyer seen in [3]. We will show that looking for the chromatic number of G is like developing G in several steps: first in a graph G+K_q. Once done, the second step is to create a problem of SAT who will be in third turned into a problem 3-SAT. The fourth step resulting from the work of Ian Holyer [3] will apply a transformation of the problem 3-SAT into a cubic graph where it will only remain to apply chromatic index research.

Remark: Garey et Johnson [4][5] have already shown how to reduce a vertex coloring (chromatic number) problem into a 3-SAT. The following steps present a personal version.

Stage 1. Let us take again the original graph G. We transform it into a graph G+K_q with Gi = (Xi, Ai) the graph of ith iteration where vertices xi will mean « x is of colour i » and with K_q(x) the clique of cardinality q and linked to the vertices x, linking then all vertices xi, i=1 to q. We can initialize by default q as being the maximum degree.

Stage 2. We will transform this resulting graph into a problem to satisfy (SAT in the sense of Garey and Johnson [4][5]) with clauses et literals. These problems have boolean variables *xi* used in literals (*xi* or not *xi* noted $\neg$*xi*). These literals are linked by the operator « *or* » noted $\vee$ to create clauses which are linked between them by the operator « *and* » noted $\wedge$. Thus the transformation takes each vertex xi in litteral and puts three types of clauses which are:

- On a same Gi=(Xi, Ai), if (*xi*, *yi*) belongs to Ai, then the following clause is: (not *xi* or not *yi*) or ($\neg$*xi* $\vee$ $\neg$*yi)*
  - This indicates that two vertices connected by an edge cannot have the same colour *i* but cannot be coloured by this colour *i* (both variables are 'false')
- On a same K_q(x), for all *xi* of Gi and all *xj* of Gj, (not *xi* or not *xj*) or ($\neg$*xi* $\vee$ $\neg$*xj)*
  - This indicates that the same vertex cannot cumulatively have two colours i and j
- On a same K_q(x) of cardinality q, (*x1* or *x2* or … or *xq*) or (*x1* $\vee$ *x2* $\vee$ … $\vee$ *xq*)
  - This indicates that there is at least one colour between 1 and q for the same vertex x

Stage 3. These clauses are then retransformed into 3-SAT: clauses with two literals have one more literal which is always false and the clause with n literals is transformed by the intermediate literal method (a or b or c or d or e) = (a or b or int1) and (no int1 or c or int2) and (not int2 or d or e). Note this transformation 3-SAT(G).

Stage 4. We apply the Ian Holyer [3] method. We note 3Sat-3Reg (problem 3-SAT) the method of creating a 3-regular graph for the search of its chromatic index from a problem 3-SAT. If the graph of degree 3 resulting from this method is 4 colourable then the problem 3-SAT is not satisfied. If it is 3-colourable, then the 3-SAT problem is satisfied giving the colouring of the graph G.

Iteration on the four stages. We loop from i = q to 1, starting from graphs G + Ki starting with a 'q' close to an upper bound (max degree for example). If the 3-SAT problem coming from the graph G+K_i is satisfied but the graph G + K_{i-1} is not satisfied then the chromatic is equal to *i*.

4. Synthesis

Here is a summary of the main results with G=(X,A) and $G+K_q = \bigcup_{i=1}^{q} Gi \cup \bigcup_{x \in X} Kq(x)$ where $G_i = (X_i, A_i)$ i^{th} copy of G and $K_q(x)$ copy of K_q linked to a vertex x of X :

- χ *to* χ' ; Let SAT_i=

  {

  $$\bigwedge_{x \in X} \Big(\bigwedge_{\substack{xi \in Gi \subset X+Kq \\ xj \in Gj \subset X+Kq \\ xi,xj \in Kq(x)}} (\neg xi \vee \neg xj) \Big) \wedge$$

  $$\bigwedge_{Ai \subset G+Kq} \Big(\bigwedge_{(xi,yi) \in Ai} (\neg xi \vee \neg yi) \Big) \wedge$$
  $$\bigwedge_{Kq(x) \subset X+Kq} (x1 \vee \ldots \vee xq)$$

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With the two transformations, 3Sat-3Reg to transform a 3-SAT problem to a cubic graph and 3-SAT the transformation of a NSAT problem into a 3SAT one, we have:

$\exists$ i $1 \le i \le$ upper bound of χ (max degree for example), such that $\chi'(3Sat\text{-}3Reg(3\text{-}SAT(SAT_i)))=3$ and $\chi'(3Sat\text{-}3Reg(3\text{-}SAT(SAT_{i-1})))=4$ and $\chi(G)=i$.

5. Conclusion

We describe how one can reduce the search for the chromatic number of any graph to the search of the chromatic index of a particular cubic graph by way of transformations into problems SAT.

We will show in further research how to carry out this search for the chromatic index of a cubic graph.

We will show in future article the use of these algorithms and a particular construction of graph to solve some problems of operational research as scheduling of tasks and even of allocation of resources. This article will explain how to set at the same time constraints of order, disjunction and even cumulative "a minima" or "a maxima" in order to optimize these resources.

6. References

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