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UNRAMIFIED REPRESENTATIONS OF REDUCTIVE GROUPS OVER FINITE RINGS

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Abstract. Lusztig has given a construction of certain representations of reductive groups over finite local principal ideal rings of characteristic p, extending the construction of Deligne and Lusztig of representations of reductive groups over finite fields. We generalize Lusztig's results to reductive groups over arbitrary finite local rings. This generalization uses the Greenberg functor and the theory of group schemes over Artinian local rings.

Introduction

In [12] Lusztig gave a construction of certain representations of a reductive group over a finite ring coming from the ring of integers in a local field of characteristic $p > 0$, modulo some power of its maximal ideal. Such rings can equivalently be characterized as finite local principal ideal rings of characteristic p. The construction, which is cohomological in nature and is a generalization of the construction of Deligne and Lusztig [2], attaches irreducible representations to certain characters of "maximal tori". It was stated in [12] that the restriction on the ring is not essential and that a similar method applies in the case when the ring is a finite quotient of the ring of integers of an arbitrary non-archimedean local field. Thus the first natural problem is to realize the construction for arbitrary finite local principal ideal rings. Unlike the characteristic p case, it turns out that for arbitrary rings of this type it is no longer possible to stay in the realm of algebraic groups over fields, and instead the proper setting is that of group schemes over Artinian local rings and the theory of the Greenberg functor. Now this general setting makes it clear that the construction does not have to be restricted to principal ideal rings, but can in fact be carried out uniformly for reductive groups over arbitrary finite local rings. Since any finite (commutative) ring R can be decomposed as a direct sum of finite local rings $R \cong \bigoplus_i R_i$, it follows that if $\mathbf{G}$ is an affine group scheme over R, then the group of points $\mathbf{G}(R)$ can be written as a direct sum $\mathbf{G}(R) \cong \bigoplus_i \mathbf{G}(R_i)$. In the study of representations of $\mathbf{G}(R)$ it is therefore enough to consider representations of the points of $\mathbf{G}$ over finite local rings.

In this paper we generalize Lusztig's construction to reductive group schemes over arbitrary finite local rings. In particular, we thus go beyond the original conception of rings of integers in local fields. Throughout the paper we shall work with a fixed Artinian local ring A with residue field $\overline{\mathbb{F}}_q$. We write $\mathfrak{m}$ for the maximal ideal of A and denote by r the smallest positive integer such that $\mathfrak{m}^r = 0$. Let $\mathbf{G}$

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be a reductive group scheme over A. The representations we construct depend in a sense only on the structure of the reductive group $\mathbf{G} \times_A \overline{\mathbb{F}}_q$ and are essentially independent of the arithmetic of the ring in question. This is a reason why we call these *unramified* representations. As Lusztig remarks in [12], it seems likely that the representations we construct (in the principal ideal ring case with $r \geq 2$ and $\mathbf{G}$ split) coincide with those constructed by Gérardin [5] by a non-cohomological method. The latter are closely related to unramified maximal tori and the unramified discrete series representations of p-adic groups, and this is another reason for our choice of terminology. There is also some overlap with the representations constructed by Hill [8, 9] in the case $\mathbf{G} = \mathrm{GL}_n$, again by a non-cohomological method.

After we have set up the framework of group schemes over local Artinian rings and their associated algebraic groups, and proved several auxiliary results, the proofs of the main theorems follow closely those of Lusztig. We shall therefore give a detailed comparison between the present paper and the contents of [12].

The first section sets up some basic notation and introduces reductive group schemes over Artinian local rings with residue field $\overline{\mathbb{F}}_q$, the Greenberg functor, and the corresponding algebraic groups. For the theory of group schemes we shall frequently refer to SGA 3 [13], which seems to be the only complete reference covering what we need. We sometimes also refer to Demazure's thesis [3], which is a convenient summary of many of the results we need from SGA 3. For definitions and results concerning the Greenberg functor, we refer to the original papers of Greenberg [6, 7]. In [12], group schemes could be bypassed altogether because the base ring $\overline{\mathbb{F}}_q[[\varepsilon]]/(\varepsilon^r)$ is an $\overline{\mathbb{F}}_q$-algebra, and so one can start with an affine algebraic group G_1 over $\overline{\mathbb{F}}_q$ and consider the group of points $G = G_1(\overline{\mathbb{F}}_q[[\varepsilon]]/(\varepsilon^r))$. By using elementary considerations rather than the general formalism of the Greenberg functor, one can then show that G carries a structure of an affine algebraic group over $\overline{\mathbb{F}}_q$. Moreover, there is a natural inclusion $G_1 \subseteq G$, so the whole subgroup structure of G_1 can be easily transferred to G. In the general situation considered in the present paper, we are forced instead to work directly with the structure of $\mathbf{G}$ over A. We then simply write G for the $\overline{\mathbb{F}}_q$-points of the algebraic group associated to $\mathbf{G}$ via the Greenberg functor. Thus G is an affine algebraic group over $\overline{\mathbb{F}}_q$ such that $G \cong \mathbf{G}(A)$. One reason why this approach is possible is that the theory of affine smooth group schemes over strictly Henselian Artinian rings in many ways resembles the classical theory of algebraic groups over algebraically closed fields.

In the second section, Lemmas 2.4, 2.5, 2.6, and 2.10 are due to Lusztig, although of course stated here in our wider generality. In order to clarify some steps of the proof of Lemma 2.5, we have given a proof using the new Lemma 2.3. Several results in [12] were stated under the hypothesis $r \geq 2$, and since the results were already known for $r = 1$, this is no loss of generality. Nevertheless, we have removed this hypothesis in order to emphasize that the proofs in fact work uniformly for all $r \geq 1$, except for the proof of Theorem 3.1 where one has to separate the case $r = 1$ (where regularity of characters is not required) from the case $r \geq 2$ (where regularity is used and the proof becomes longer). This also affects Proposition 3.3 and Corollary 3.4, which we have stated in a form that includes the case $r = 1$.

The proofs of Lemmas 2.2, 2.3, 2.8 and 2.9 are new, although the results have to some degree been known earlier either implicitly in unpublished form or in various special cases. In particular, Lemma 2.2 provides a commutator relation and Iwahori decomposition for (certain) group schemes over local rings. These results are well-known for certain classes of groups and rings, but our results hold quite generally and are proved by arguments which are more geometric than the classical group theoretic approaches. Lemmas 2.8 and 2.9 were stated in [12] without proof, and since our proofs are not obvious, we have included them here. The proof of Lemma 2.9(c) is especially long and is an example of the extra complications that appear in our general setting compared to the case of rings of characteristic p, where the proof can be reduced to the case of SL_2.

In the final section we have collected all the main results, including our version of Lusztig's Lemma 1.9 which we have given the status of Theorem since its proof is the longest and most difficult in the entire paper and its consequences include the most important results. The ideas of the proofs in this section are due to Lusztig, and our presentation follows [12], except with regards to the use of the elements $\hat{w}$ (see below). We have also added some references to various results used in the proofs, and some clarifying remarks. We have included these reworkings of Lusztig's proofs in order to get a complete and coherent exposition, and we believe this to be a more satisfying solution (both logically and from the reader's point of view) than if we had simply stated the generalized main results and referred to proofs appearing in a more special context.

If $\mathbf{T}$ and $\mathbf{T}'$ are two maximal tori in $\mathbf{G}$, we shall denote the corresponding closed subgroups of G by T and T', respectively. Reducing modulo $\mathfrak{m}$ we get the maximal tori $T_1 = (\mathbf{T} \times_A \overline{\mathbb{F}}_q)(\overline{\mathbb{F}}_q)$ and $T_1' = (\mathbf{T}' \times_A \overline{\mathbb{F}}_q)(\overline{\mathbb{F}}_q)$ in $G_1 = (\mathbf{G} \times_A \overline{\mathbb{F}}_q)(\overline{\mathbb{F}}_q)$. A remark applicable to both of the last two sections is that unlike the case where the ring A has characteristic p, in general we cannot directly transfer elements of the transporter $N(T_1, T_1') = \{n \in G_1 \mid n^{-1}T_1 n = T_1'\}$ to elements of $N(T, T') = \{n \in G \mid n^{-1}Tn = T'\}$. Instead we apply results from SGA 3 showing that the transporter (or normalizer) group schemes of maximal tori are smooth, and using this we are able to conclude that the natural map $N(T, T') \to N(T_1, T_1)$ is surjective. For any element $\dot{w} \in N(T_1, T_1')$ we can thus work with a lift $\hat{w} \in N(T, T')$ under this map. It turns out that the ambiguity in the choice of lifts does not affect the results, and this provides a sense in which the results only depend on structures over the residue field.

1. Notation

Throughout this paper a ring will always refer to a (unital, associative) commutative ring. Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and perfect residue field k. Let r denote the smallest positive integer such that $\mathfrak{m}^r = 0$. Let X be a scheme of finite type over A (as usual, we shall speak of a scheme over the ring A rather than over the scheme $\operatorname{Spec} A$). Greenberg [6, 7] has defined a functor $\mathcal{F}_A$ from the category of schemes of finite type over A to the category of schemes of finite type over k, such that there exists a canonical isomorphism $X(A) \cong (\mathcal{F}_A X)(k)$. It is shown in loc. cit. that the functor $\mathcal{F}_A$ preserves affine and separated schemes, respectively. Furthermore, it maps group schemes over A to group schemes over k, schemes smooth over A to schemes smooth over k, and preserves subschemes (of any kind). If X is smooth over A and $X \times_A k$ is reduced and irreducible, then $\mathcal{F}_A X$ is reduced and irreducible ([7], 2, Corollary 2).

Suppose that $\mathbf{G}$ is an affine smooth group scheme over A. Thus it is in particular of finite type over A. We take the residue field k to be an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ of characteristic p. For any integer r' such that $r \geq r' \geq 0$, we define

$$G_{r'} = \mathcal{F}_A(\mathbf{G} \times_A A/\mathfrak{m}^{r'})(k).$$

Note that for $r' = 0$ the ring $A/\mathfrak{m}^{r'}$ is the trivial ring $\{0 = 1\}$, and so G_0 consists of exactly one point. In particular, for $r' = r$ we write G for the group $G_r = (\mathcal{F}_A\mathbf{G})(k) \cong \mathbf{G}(A)$. In general, we shall write group schemes over A in boldface type and the corresponding algebraic group over k using the same letter in normal type. By the results of Greenberg, each group $G_{r'}$ is the k-points of an affine smooth group scheme over k. It is thus an affine algebraic group over k, connected if $\mathbf{G} \times k$ is connected. Since $\mathbf{G}$ is smooth it follows that the reduction map $A \to A/\mathfrak{m}^{r'}$ induces a surjective homomorphism $\varphi_{r,r'} = \varphi_{r'} : G \to G_{r'}$. The kernel of $\varphi_{r'}$ is denoted by $G^{r'}$. We have

$$\{1\} = G^r \subseteq G^{r-1} \subseteq \cdots \subseteq G^1 \subseteq G^0 = G.$$

Let $G^{r',*} = G^{r'} - G^{r'+1}$, for $r' < r$. We thus have a partition

$$G = G^{0,*} \sqcup G^{1,*} \sqcup \cdots \sqcup G^{r-1,*} \sqcup \{1\}.$$

From now on, let $\mathbf{G}$ be a reductive group scheme over A (cf. [3], 2.1 or [13], XIX 2.7). This means that $\mathbf{G}$ is an affine smooth group scheme over A such that its fibre $\mathbf{G} \times k$ is a connected reductive group in the classical sense. We shall be interested in the situation where G is endowed with a Frobenius endomorphism $F : G \to G$, which in the most general sense is just a surjective endomorphism with finite fixed point group G^F.

Remark. We show how a situation as above typically arises. Let A_0 be an arbitrary finite local ring. Then A_0 is obviously Artinian with residue field $\mathbb{F}_q$, for some q. Let $\mathbf{G}_0$ be a reductive group scheme over A_0. Then by results of Greenberg, A_0 is an algebra over the ring of Witt vectors $W_n(\mathbb{F}_q)$, where $\operatorname{char} A_0 = p^{n+1}$. Let

$$A = A_0 \otimes_{W_n(\mathbb{F}_q)} W_n(\overline{\mathbb{F}}_q).$$

Then by [6], 1, Prop. 4, A is a local Artinian ring with residue field $\overline{\mathbb{F}}_q$. The algebra A carries an endomorphism F induced by the Frobenius map of $W_n(\overline{\mathbb{F}}_q)$. If we now let $\mathbf{G} = \mathbf{G}_0 \times_{A_0} A$, then G inherits a Frobenius endomorphism from the endomorphism F on $\mathbf{G}(A)$ such that $G^F \cong \mathbf{G}(A)^F$. Note however that not all Frobenius endomorphisms of G are of this form; there are also those that give rise to twisted groups.

Assume henceforth that G is an algebraic group over $\overline{\mathbb{F}}_q$ obtained from a reductive group scheme $\mathbf{G}$ as above, and provided with a Frobenius endomorphism $F : G \to G$. Since our base A is a local Artinian ring with algebraically closed residue field, it is strictly Henselian, and thus maximal tori and Borel subgroups exist in $\mathbf{G}$ (cf. [13], XXVI 7.15). If $\mathbf{T}$ is a maximal torus contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, we have a semidirect product $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ (cf. [13], XXII 5.11.4). We then have the respective associated algebraic subgroups T, B, U of G, and a semidirect product $B = TU$. Note that for $r \geq 2$, T will not be a maximal torus of G in the sense of algebraic groups; nor will B be a Borel subgroup of G.

Throughout this paper we fix a prime $l \neq p$. If X is an algebraic variety over $\overline{\mathbb{F}}_q$ we write $H_c^j(X)$ instead of $H_c^j(X, \overline{\mathbb{Q}}_l)$. For a finite group Γ, we write $\widehat{\Gamma}$ for the group of linear characters $\operatorname{Hom}(\Gamma, \overline{\mathbb{Q}}_l^\times)$.

2. Lemmas

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, and let $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ be the set of roots of $\mathbf{G}$, relative to $\mathbf{T}$ (cf. [13], XIX 3.6). Thus Φ consists of elements in $\mathrm{Hom}_{A\text{-gr}}(\mathbf{T}, (\mathbb{G}_m)_A)$ whose image in $\mathrm{Hom}_{k\text{-gr}}(\mathbf{T} \times k, (\mathbb{G}_m)_k)$ is a root in the usual sense. We write the groups of characters additively, and denote by $\mathbf{U}_\alpha$ the root subgroup of $\mathbf{G}$ corresponding to the root α.

We shall consider the notion of a *splitting* (déploiement) of a reductive group $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$ (cf. [13], XXII 1.13, or [3], 3.1), and we call $\mathbf{G}$ *split* (déployé) with respect to $\mathbf{T}$ if such a splitting exists. A torus over a general base $\mathbf{S}$ is called *trivial* if it is diagonalizable, that is, if it is isomorphic to some $(\mathbb{G}_m^n)_{\mathbf{S}}$.

Lemma 2.1. *Let A be a strictly Henselian ring and let $\mathbf{G}$ be a reductive group scheme over A. Then there exists a maximal torus in $\mathbf{G}$, and $\mathbf{G}$ is split with respect to any of its maximal tori.*

Proof. By [13], XXVI 7.15 (or XIV 3.20) a maximal torus exists in $\mathbf{G}$. By [13], X 4.6, a group of multiplicative type and of finite type (in particular a maximal torus) over a Henselian base is isotrivial, that is, it is trivial after a finite surjective étale extension. Since $\operatorname{Spec} A$ does not have any non-trivial finite étale extensions, it follows that any maximal torus of $\mathbf{G}$ is trivial. Since A is local, the lemma now follows from [13], XXII 2.2. □

The existence of splitting implies that the root data of $\mathbf{G}$ relative to $\mathbf{T}$ is canonically isomorphic to the root data of $\mathbf{G} \times k$ relative to $\mathbf{T} \times k$ (cf. [13] XXII 1.15 b)). In particular, the map $\mathrm{Hom}_{A\text{-gr}}(\mathbf{T}, (\mathbb{G}_m)_A) \to \mathrm{Hom}_{k\text{-gr}}(\mathbf{T} \times k, (\mathbb{G}_m)_k)$ is a bijection on the roots. As for algebraic groups over fields, a choice of Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$ defines a set of positive roots Φ^+, and the splitting of $\mathbf{G}$ with respect to $\mathbf{T}$ implies that for some fixed but arbitrary ordering of Φ^+ we have

$$\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$$

(see [3], 3.3.3). On the level of groups of points this yields $U = \prod_{\alpha \in \Phi^+} U_\alpha$, where an element of U is expressed uniquely as a product of elements of the U_α.

From now on, let $\mathbf{T}$ and $\mathbf{T}'$ be two maximal tori of $\mathbf{G}$ such that the corresponding subgroups T and T' of G are F-stable. Let $\mathbf{U}$ (resp. $\mathbf{U}'$) be the unipotent radical of a Borel subgroup of $\mathbf{G}$ that contains $\mathbf{T}$ (resp. $\mathbf{T}'$), and let U and U' be the corresponding subgroups of G. Note that U and U' are not necessarily F-stable.

Let $N(T_1, T_1') = \{g \in G_1 \mid g^{-1}T_1 g = T_1'\}$. Then T_1 acts on $N(T_1, T_1')$ by left multiplication and T_1' acts on $N(T_1, T_1')$ by right multiplication. The orbits of T_1 are in natural bijection with the orbits of T_1'. We set $W(T_1, T_1') = T_1 \backslash N(T_1, T_1') \cong N(T_1, T_1')/T_1'$; this is a finite set because if $a \in G_1$ is an element such that ${}^aT_1 = T_1'$, then $g \mapsto ga$ gives a bijection between $N(T_1, T_1')$ and the normalizer $N_{G_1}(T_1) = N(T_1, T_1)$, and this induces a bijection between $W(T_1, T_1')$ and the Weyl group $W(T_1) = W(T_1, T_1)$. For each $w \in W(T_1, T_1')$ we choose a representative $\dot{w} \in N(T_1, T_1')$. Since the normalizer $N_{\mathbf{G}}(\mathbf{T})$ is smooth over A (cf. [3], 1.5.1), the map φ_1 induces a surjection

$$N_{\mathbf{G}}(\mathbf{T})(A) \longrightarrow N_{\mathbf{G}}(\mathbf{T})(k).$$

By the definition of the normalizer group scheme and the fact that k is algebraically closed, we have $N_{\mathbf{G}}(\mathbf{T})(A) \subseteq N_{\mathbf{G}(A)}(\mathbf{T}(A))$ and $N_{\mathbf{G}}(\mathbf{T})(k) = N_{\mathbf{G}(k)}(\mathbf{T}(k))$ (cf. [10], I 2.6). Thus φ_1 also induces a surjection

$$N_G(T) \cong N_{\mathbf{G}(A)}(\mathbf{T}(A)) \longrightarrow N_{\mathbf{G}(k)}(\mathbf{T}(k)) = N_{G_1}(T_1).$$

Let $N(T,T') = \{g \in G \mid g^{-1}Tg = T'\}$. It follows from the conjugacy of maximal tori ([3], 1.5.3) that T and T' are conjugate in G by an element whose image in G_1 conjugates T_1 to T_1'. Thus we have in the same way as above a bijection between $N(T,T')$ and $N_G(T)$, and hence a surjection $N(T,T') \to N(T_1,T_1')$ (this also follows from the smoothness of transporters [13], XXII 5.3.9). For each $\dot{w} \in N(T_1,T_1')$ we can therefore choose a lift $\hat{w} \in N(T,T')$, and throughout this paper we shall work with a fixed set of lifts $\hat{w}$. As we shall see, the main results are independent of the choice of these lifts.

Define the variety

$$\Sigma = \{(x,x',y) \in F(U) \times F(U') \times G \mid xF(y) = yx'\}.$$

The Bruhat decomposition in G_1 implies that there is a bijection between double B_1-B_1 cosets indexed by $W(T_1,T_1')$, and double B_1-B_1 cosets indexed by $W(T_1)$. Indeed, if $w \in W(T_1,T_1')$ and if $a \in G_1$ is such that ${}^aT_1 = T_1'$, then the map $B_1wB_1 \mapsto B_1waB_1$ is injective since if $B_1w'B_1$ has the same image as B_1wB_1, then $B_1waB_1 = B_1w'aB_1$. Therefore, by Bruhat decomposition, $\dot{w}a$ and $\dot{w}'a$ have the same image in $W(T_1)$; that is, $\dot{w}$ and $\dot{w}'$ have the same image in $W(T_1,T_1')$. We thus have $G_1 = \bigsqcup_{w\in W(T_1,T_1')} G_{1,w}$, where $G_{1,w} = U_1T_1\dot{w}U_1' = U_1\dot{w}T_1'U_1'$. Let G_w be the inverse image of $G_{1,w}$ under $\varphi_1 : G \to G_1$ and let

$$\Sigma_w = \{(x,x',y) \in \Sigma \mid y \in G_w\}.$$

This defines a partition of Σ. The group $T^F \times T'^F$ acts on Σ by $(t,t') : (x,x',y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. This restricts to an action of $T^F \times T'^F$ on Σ_w for any $w \in W(T_1,T_1')$.

If $\theta \in \widehat{T^F}$, $\theta' \in \widehat{T'^F}$, and M is a $T^F \times T'^F$-module, we shall write $M_{\theta^{-1},\theta}$ for the subspace of M on which $T^F \times T'^F$ acts according to $\theta^{-1} \boxtimes \theta'$; that is,

$$M_{\theta^{-1},\theta'} = \{m \in M \mid (t,t')m = \theta^{-1}(t)\theta'(t')m,\ \forall\,(t,t') \in T^F \times T'^F\}.$$

Lemma 2.2. *Let $\mathbf{G}$ be an affine group scheme over a local ring A with maximal ideal $\mathfrak{m}$. For $i \geq 0$, write $G = \mathbf{G}(A)$, $G_i = \mathbf{G}(A/\mathfrak{m}^i)$, and $G^i = \mathrm{Ker}(G \to G_i)$. Then the following holds:*

(a) *For any integers $i,j \geq 0$ we have the commutator relation $[G^i,G^j] \subseteq G^{i+j}$.*

(b) *(Iwahori decomposition) Assume in addition that $\mathbf{G}$ is reductive and split over A, with respect to a maximal torus $\mathbf{T}$. Let $\mathbf{T}$ be contained in a Borel subgroup with unipotent radical $\mathbf{U}$, and let $\mathbf{U}^-$ be the unipotent radical of a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, such that $\mathbf{U} \cap \mathbf{U}^- = \{1\}$. Let T, U, and U^- be the respective groups of A-points, and let T^1, U^1, and $(U^-)^1$ be the respective kernels. Then we have*

$$G^1 = (U^-)^1T^1U^1,$$

and each element $g \in G^1$ decomposes uniquely as $g = u^-tu$, where $u^- \in (U^-)^1$, $t \in T^1$, and $u \in U^1$.

Proof. We prove (a) using a Hopf algebra approach. Let $A[\mathbf{G}]$ be the affine algebra of $\mathbf{G}$; thus $A[\mathbf{G}]$ is a commutative Hopf algebra over A. Let $\Delta : A[\mathbf{G}] \to A[\mathbf{G}]\otimes A[\mathbf{G}]$ and $\varepsilon : A[\mathbf{G}] \to A$ denote its coproduct and counit, respectively. Let $I = \mathrm{Ker}\,\varepsilon$ be the augmentation ideal. If $\alpha : A \to R$ is an A-algebra, then the identity element of the group $G(R) = \mathrm{Hom}(A[\mathbf{G}], R)$ is given by $\alpha \circ \varepsilon$. For any $i \geq 0$, the reduction map $\varphi_i : G = \mathrm{Hom}(A[\mathbf{G}], A) \to \mathrm{Hom}(A[\mathbf{G}], A/\mathfrak{m}^i) = G_i$ sends any $g \in G$ to $\varphi_i \circ g$. Now let $g \in G^i$ and $h \in G^j$, for some integers $i, j \geq 0$. Then

$$\varphi_i \circ g = \varphi_i \circ \varepsilon \quad \text{and} \quad \varphi_j \circ h = \varphi_j \circ \varepsilon,$$

respectively (recall that φ_i denotes both the map $G \to G_i$ and $A \to A/\mathfrak{m}^i$). Thus $\varphi_i(g(I)) = 0$; that is, $g(I) \subseteq \mathfrak{m}^i$. Similarly, we have $h(I) \subseteq \mathfrak{m}^j$. Since $a \mapsto a \cdot 1 : A \to A[\mathbf{G}]$ is a section of ε, we have $A[\mathbf{G}] = A \cdot 1 \oplus I$, as A-modules. This implies that $A[\mathbf{G}] \otimes A[\mathbf{G}] = A(1 \otimes 1) \oplus (A \otimes I) \oplus (I \otimes A) \oplus (I \otimes I)$. Let $x \in I$ and write $\Delta(x) = a_1(1 \otimes 1) + a_2 \otimes y_1 + y_2 \otimes a_3 + y_3 \otimes y_4$, where $a_k \in A$, $y_k \in I$. The Hopf algebra axiom $(\varepsilon \circ \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \circ \varepsilon) \circ \Delta$ implies that $a_1 + a_2 y_1 = a_1 + y_2 a_3 = x \in I$, and so $a_1 \in I$; that is, $a_1 = 0$ and $x = a_2 y_1 = y_2 a_3$. Hence $\Delta(x) \in a_2 \otimes y_1 + y_2 \otimes a_3 + I \otimes I = 1 \otimes a_2 y_1 + y_2 a_3 \otimes 1 + I \otimes I$, and so we have

$$\Delta(x) \in x \otimes 1 + 1 \otimes x + I \otimes I, \qquad \text{for all } x \in I.$$

The product $gh \in G$ is given by the element $(g \otimes h) \circ \Delta \in \mathrm{Hom}(A[\mathbf{G}], A)$. Hence

$$gh(x) \in g(x) + h(x) + g(I)h(I) \subseteq g(x) + h(x) + \mathfrak{m}^{i+j}, \qquad \text{for all } x \in I,$$

and so $(\varphi_{i+j} \circ (gh - g - h))(I) = 0$. Thus the map $\varphi_{i+j} \circ (gh - g - h)$ factors through ε, and since φ_{i+j} is the unique A-algebra map $A \to A/\mathfrak{m}^{i+j}$, we must have $\varphi_{i+j} \circ (gh - g - h) = \varphi_{i+j} \circ \varepsilon$. This means exactly that the element $gh - g - h \in \mathrm{Hom}(A[\mathbf{G}], A)$ lies in the kernel G^{i+j}. We thus see that $gh = g + h = h + g = hg$, modulo G^{i+j}, and the result follows.

We now prove (b). Let W be the group generated by simple reflections corresponding to the root system of $\mathbf{G}$ relative to $\mathbf{T}$ (cf. [13], XXI 1.1.8). By [13], XXII 3.3, respectively, 3.8, we have a natural inclusion $W \subseteq N_{\mathbf{G}}(\mathbf{T})(A)/\mathbf{T}(A)$, respectively, surjection $N_{\mathbf{G}}(\mathbf{T})(A) \to (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(A)$. For any $w \in W$ we can thus choose a lift $n_w \in N_{\mathbf{G}}(\mathbf{T})(A)$. Since A is local, we have

$$G = \bigcup_{w \in W} n_w U^- T U$$

(cf. [13], XXII 5.7.4 (ii) and also 5.7.5 (ii)). In particular, we may take $n_1 = 1$ as a representative for the trivial element $1 \in W$.

Now, if $\varphi_1(n_w u^- t u) = 1$ for some $w \in W$, $u^- \in U^-$, $t \in T$, $u \in U$, then $B_1 \varphi_1(n_w) U_1^- \subseteq B_1 U_1^-$, and the Bruhat decomposition in G_1 with respect to the Borel subgroups B_1 and B_1^- implies that $\varphi_1(n_w) \in T_1$. Hence $\varphi_1(u^-) = \varphi_1(n_w t) = \varphi_1(u) = 1$. Let k be the residue field of A. The morphism $N_{\mathbf{G}}(\mathbf{T}) \to N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ yields a commutative diagram

$$\begin{tikzcd} N_{\mathbf{G}}(\mathbf{T})(A) \arrow[r] \arrow[d, "\varphi_1"'] & (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(A) \arrow[d, "\varphi_1"] \\ N_{\mathbf{G}}(\mathbf{T})(k) \arrow[r] & (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(k). \end{tikzcd}$$

Since $\mathbf{G}$ is split reductive, its root datum is canonically isomorphic to the root datum of $\mathbf{G} \times k$ ([13], XXII 1.15 b)), and hence the map $\varphi_1 : (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(A) \to (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(k)$ restricts to an isomorphism between W and the Weyl group of the root datum of $\mathbf{G} \times k$ considered as a subgroup of $(N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(k)$. The image of $\varphi_1(n_w) \in T_1 \subseteq N_{\mathbf{G}}(\mathbf{T})(k)$ in $(N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})(k)$ is trivial, and by the commutativity of the above diagram, the image of n_w in $W \subseteq N_{\mathbf{G}}(\mathbf{T})(A)/\mathbf{T}(A)$ is thus the trivial element. It follows from our choice of representatives that $n_w = n_1 = 1$, whence $G^1 = (U^-)^1T^1U^1$. Uniqueness follows immediately, since if $u^-tu = u_1^-t_1u_1$, then $(u^-)^{-1}u_1^- \in U^- \cap B = \{1\}$, and similarly, $t^{-1}t_1 = u^{-1}u_1 = 1$. □

Remark. For basic facts about Hopf algebras we have followed [10], 2.4. In the present paper we apply Iwahori decomposition only in the case of a reductive group scheme over a strictly Henselian base, and such groups are split by Lemma 2.1.

We return to our situation where $\mathbf{G}$ is a reductive group scheme over the Artinian local ring A, with residue field k. Using the isomorphism $G = (\mathcal{F}_A\mathbf{G})(k) \cong \mathbf{G}(A)$ together with Lemma 2.2 we get corresponding commutator relations and Iwahori decomposition in G.

We now prove a result which is a form of Bruhat decomposition for G and is both a strengthened and a generalized form of a result of Hill (cf. [8], 2.6). Let $\mathbf{U}^-$ (resp. $\mathbf{U}'^-$) be the unipotent radical of a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ (resp. $\mathbf{T}'$) such that $\mathbf{U} \cap \mathbf{U}^- = \{1\}$ (resp. $\mathbf{U}' \cap \mathbf{U}'^- = \{1\}$). Let $U^- = (\mathcal{F}_A\mathbf{U}^-)(k)$ and $U'^- = (\mathcal{F}_A\mathbf{U}'^-)(k)$ be the corresponding subgroups of G.

Lemma 2.3. *Let U, U', U^-, and U'^- be as above. Then G decomposes as*

$$G = \bigsqcup_{w \in W(T_1, T_1')} (U \cap \hat{w}U'^-\hat{w}^{-1})\hat{w}T'((U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w})U',$$

and every element $g \in G$ can be written uniquely in the form $g = u\hat{w}t'ku'$, where $u \in U \cap \hat{w}U'^-\hat{w}^{-1}$, $t' \in T'$, $k \in (U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w}$, and $u' \in U'$.

Proof. In the case $r = 1$ the result is a well-known consequence of Bruhat's lemma. Using the surjection φ_r we lift the decomposition to G, and so we may write

$$G = \bigsqcup_{w \in W(T_1, T_1')} (U \cap \hat{w}U'^-\hat{w}^{-1})\hat{w}G^1T'U'.$$

Now, by Iwahori decomposition we have $G^1 = (U'^-)^1T^1(U')^1$, so

$$(U \cap \hat{w}U'^-\hat{w}^{-1})\hat{w}G^1T'U' = (U \cap \hat{w}U'^-\hat{w}^{-1})\hat{w}(U'^-)^1T'U'.$$

The formula $U = \prod_{\alpha\in\Phi^+} U_\alpha$ implies that we may write

$$(U'^-)^1 = ((U'^-)^1 \cap \hat{w}^{-1}U\hat{w})((U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w}),$$

and since $\hat{w}((U'^-)^1 \cap \hat{w}^{-1}U\hat{w})\hat{w}^{-1} \in (U \cap \hat{w}U'^-\hat{w}^{-1})$, we have

$$G = \bigsqcup_{w \in W(T_1, T_1')} (U \cap \hat{w}U'^-\hat{w}^{-1})\hat{w}((U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w})T'U'.$$

Since T' normalizes $(U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w}$, we get the desired decomposition. Now let $u\hat{w}t'ku' = u_1\hat{w}t_1'k_1u_1'$ for $u, u_1 \in U \cap \hat{w}U'^-\hat{w}^{-1}$, $t', t_1' \in T'$, $k, k_1 \in (U'^-)^1 \cap \hat{w}^{-1}U^-\hat{w}$, and $u', u_1' \in U'$. Then $u'u_1'^{-1} = (u\hat{w}t'k)^{-1}u_1\hat{w}t_1'k_1$, and since $u'u_1'^{-1} \in U'$ and $(u\hat{w}t'k)^{-1}u_1\hat{w}t_1'k_1 \in T'U'^-$, we must have $u' = u_1'$ and $u\hat{w}t'k = u_1\hat{w}t_1'k_1$, or equivalently $t'kk_1^{-1}t_1'^{-1} = \hat{w}^{-1}u^{-1}u_1\hat{w}$. Since $t'kk_1^{-1}t_1'^{-1} \in T'\hat{w}^{-1}U^-\hat{w} = \hat{w}^{-1}TU^-\hat{w}$ and $\hat{w}^{-1}u^{-1}u_1\hat{w} \in \hat{w}^{-1}U\hat{w}$, we conclude that $t'k = t_1'k_1$ and $u = u_1$. Thus also $t' = t_1'$ and $k = k_1$, and the lemma is proved. □

If $\mathcal{T}$ is a commutative algebraic group over $\overline{\mathbb{F}}_q$ with fixed $\mathbb{F}_q$-structure and with Frobenius map $F : \mathcal{T} \to \mathcal{T}$, then for any integer $n \geq 1$ we have a norm map

$$N_F^{F^n} : \mathcal{T}^{F^n} \longrightarrow \mathcal{T}^F, \qquad t \longmapsto tF(t)F^2(t)\cdots F^{n-1}(t).$$

Lemma 2.4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two commutative connected algebraic groups over $\overline{\mathbb{F}}_q$ with fixed $\mathbb{F}_q$-rational structures with Frobenius maps $F:\mathcal{T}\to\mathcal{T}$ and $F:\mathcal{T}'\to\mathcal{T}'$. Let $f:\mathcal{T}\xrightarrow{\sim}\mathcal{T}'$ be an isomorphism of algebraic groups over $\overline{\mathbb{F}}_q$. Let $n\geq 1$ be such that $F^nf=fF^n:\mathcal{T}\to\mathcal{T}'$; thus $f:\mathcal{T}^{F^n}\xrightarrow{\sim}\mathcal{T}'^{F^n}$. Let*

$$H=\{(t,t')\in\mathcal{T}\times\mathcal{T}'\mid f(F(t)^{-1}t)=F(t')^{-1}t'\}$$

(a subgroup of $\mathcal{T}\times\mathcal{T}'$ containing $\mathcal{T}^F\times\mathcal{T}'^F$). Let $\theta\in\widehat{\mathcal{T}^F}$ and $\theta'\in\widehat{\mathcal{T}'^F}$ be such that $\theta^{-1}\boxtimes\theta'$ is trivial on $(\mathcal{T}^F\times\mathcal{T}'^F)\cap H^0$. Then $\theta N_F^{F^n}=\theta' N_F^{F^n}f\in\widehat{\mathcal{T}^{F^n}}$.

Proof. See [12], 1.1. ☐

From now on, let $\mathcal{T}=T^{r-1}$ and $\mathcal{T}'=T'^{r-1}$. Note that in the case $r=1$ we have $\mathcal{T}=T$ and $\mathcal{T}'=T'$.

Lemma 2.5. *Let $w\in W(T_1,T_1')$, and let $\theta\in\widehat{T^F}$, $\theta'\in\widehat{T'^F}$. Assume that $H_c^j(\Sigma_w)_{\theta^{-1},\theta'}\neq 0$ for some $j\in\mathbb{Z}$. Let $g=F(\hat{w})^{-1}$ and let $n\geq 1$ be such that $g\in G^{F^n}$. Then $\mathrm{Ad}(g)$ (conjugation by g) carries $\mathcal{T}^{F^n}$ onto $\mathcal{T}'^{F^n}$ and $\theta|_{\mathcal{T}^F}\circ N_F^{F^n}\in\widehat{\mathcal{T}^{F^n}}$ to $\theta'|_{\mathcal{T}'^F}\circ N_F^{F^n}\in\widehat{\mathcal{T}'^{F^n}}$.*

Proof. Put $U_{\hat{w}}=U\cap\hat{w}U'^{-}\hat{w}^{-1}$ and $K=(U'^{-})^1\cap\hat{w}^{-1}U^{-}\hat{w}$. By Lemma 2.3, we then have an isomorphism

$$\begin{aligned}\tilde{\Sigma}_{\hat{w}}:=\{&(x,x',u,u',k,\nu)\in F(U)\times F(U')\times U_{\hat{w}}\times U'\times K\times\hat{w}T'\mid\\ &xF(u)F(\nu)F(k)F(u')=u\nu ku'x'\}\xrightarrow{\sim}\Sigma_w,\end{aligned}$$

given by $(x,x',u,u',k,\nu)\mapsto(x,x',u\nu ku')$. This isomorphism is compatible with the $T^F\times T'^F$-actions, where $T^F\times T'^F$ acts on $\tilde{\Sigma}_{\hat{w}}$ by

(a) $\quad (t,t'):(x,x',u,u',k,\nu)\longmapsto(txt^{-1},t'x't'^{-1},tut^{-1},t'u't'^{-1},t'kt'^{-1},t\nu t'^{-1}).$

Hence we have $H_c^j(\tilde{\Sigma}_{\hat{w}})_{\theta^{-1},\theta'}\neq 0$. By the substitution $xF(u)\mapsto x$, $x'F(u')^{-1}\mapsto x'$, the variety $\tilde{\Sigma}_{\hat{w}}$ is rewritten as

(b) $$\begin{aligned}\{&(x,x',u,u',k,\nu)\in F(U)\times F(U')\times U_{\hat{w}}\times U'\times K\times\hat{w}T'\mid\\ &xF(k)F(\nu)=u\nu ku'x'\};\end{aligned}$$

in these coordinates, the action of $T^F\times T'^F$ is still given by (a). Let

$$H=\{(t,t')\in\mathcal{T}\times\mathcal{T}'\mid t'F(t')^{-1}=F(\hat{w})^{-1}tF(t)^{-1}F(\hat{w})\}$$

(a closed subgroup of $T\times T'$). It acts on the variety (b) by the same formula as in (a) (we use Lemma 2.2 to show that $\mathcal{T}$ and $\mathcal{T}'$ centralize G^1). By [2], 6.5, the induced action of H on $H_c^j(\tilde{\Sigma}_{\hat{w}})$ is trivial when restricted to the connected component H^0. In particular, the intersection $(T^F\times T'^F)\cap H^0$ acts trivially on $H_c^j(\tilde{\Sigma}_{\hat{w}})$. Since $H_c^j(\tilde{\Sigma}_{\hat{w}})_{\theta^{-1},\theta'}\neq 0$, it follows that $\theta^{-1}\boxtimes\theta'$ is trivial on $(T^F\times T'^F)\cap H^0$. Let $g=F(\hat{w})^{-1}$ and let $n\geq 1$ be such that $g\in G^{F^n}$. Then $\mathrm{Ad}(g)$ carries $\mathcal{T}^{F^n}$ onto $\mathcal{T}'^{F^n}$ and (by Lemma 2.4 with $f=\mathrm{Ad}(g)$) it carries $\theta|_{\mathcal{T}^F}\circ N_F^{F^n}$ to $\theta'|_{\mathcal{T}'^F}\circ N_F^{F^n}$. ☐

With the above lemma proved for each Σ_w we can deduce a similar statement for the whole variety Σ. This will be used later (in Proposition 3.2) to prove a result which is a generalization of Theorem 6.2 of Deligne and Lusztig [2].

Lemma 2.6. *Let $\theta \in \widehat{T^F}$, $\theta' \in \widehat{T'^F}$ be such that*

$$H_c^j(\Sigma)_{\theta^{-1},\theta'} \neq 0 \tag{a}$$

for some $j \in \mathbb{Z}$. Then there exists $n \geq 1$ and $g \in N(T', T)^{F^n}$ such that $\mathrm{Ad}(g)$ *carries $\theta|_{\mathcal{T}^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}'^{F^n}}$.*

Proof. It is well-known that the subvarieties $G_{1,w}$ of G_1 have the following property: for some ordering $\leq$ of $W(T_1, T_1')$, the unions $\bigcup_{w' \leq w} G_{1,w'}$ are closed in G_1. It follows that the unions $\bigcup_{w' \leq w} G_{w'}$ are closed in G and the unions $\bigcup_{w' \leq w} \Sigma_{w'}$ are closed in Σ. The spectral sequence associated to the filtration of Σ by these unions, together with (a), shows that there exists $w \in W(T_1, T_1')$ and $j' \in \mathbb{Z}$ such that $H_c^{j'}(\Sigma_w)_{\theta^{-1},\theta'} \neq 0$. We can therefore apply Lemma 2.5 to get an element $g = F(\hat{w})^{-1} \in N(T', T)^{F^n}$, for some $n \geq 1$, satisfying the conclusion. □

For each root $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$ we have a unique coroot $\check{\alpha} \in \mathrm{Hom}_{A\text{-gr}}((\mathbb{G}_m)_A, \mathbf{T})$. Let $\mathbf{T}^\alpha$ denote the image of $\check{\alpha}$ in $\mathbf{T}$, so that $\mathbf{T}^\alpha$ is a 1-dimensional torus in $\mathbf{T}$ (cf. [13], XX 3). Keeping with our notational conventions, we let $U_\alpha = (\mathcal{F}_A \mathbf{U}_\alpha)(k)$ and $T^\alpha = (\mathcal{F}_A \mathbf{T}^\alpha)(k)$. We also write $\mathcal{T}^\alpha = (T^\alpha)^{r-1}$ (a 1-dimensional subgroup of $\mathcal{T} = T^{r-1}$; cf. [7], 3).

The following definition was introduced in [12], 1.5.

Definition 2.7. Let $\chi \in \widehat{\mathcal{T}^F}$. We say that χ is *regular* if for any $\alpha \in \Phi$ and any $n \geq 1$ such that $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, the restriction of $\chi \circ N_F^{F^n} : \mathcal{T}^{F^n} \to \overline{\mathbb{Q}}_l^\times$ to $(\mathcal{T}^\alpha)^{F^n}$ is non-trivial. If $\theta \in \widehat{T^F}$, we say that θ is *regular* if $\theta|_{\mathcal{T}^F}$ is regular.

Lemma 2.8. *Let $\chi \in \widehat{\mathcal{T}^F}$, and suppose that there exists an $n \geq 1$ such that for all $\alpha \in \Phi$, $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$ and the restriction of $\chi \circ N_F^{F^n}$ to $(\mathcal{T}^\alpha)^{F^n}$ is non-trivial. Then χ is regular.*

Proof. We first show some properties of the norm map. Let $\mathcal{T}$ be a commutative algebraic group defined over $\mathbb{F}_q$ with Frobenius F. Let a and b be two positive integers such that $b = ka$, for some integer k. Then clearly $\mathcal{T}^{F^a} \subseteq \mathcal{T}^{F^b}$. We extend the definition of the norm map by defining the map $N_{F^a}^{F^b} : \mathcal{T}^{F^b} \to \mathcal{T}^{F^a}$, $x \mapsto xF^a(x)F^{2a}(x) \cdots F^{(k-1)a}(x)$. We then have

$$N_F^{F^a} N_{F^a}^{F^b}(x) = \prod_{j=0}^{a-1} F^j \left(\prod_{i=0}^{k-1} F^{ia}(x) \right) = \prod_{j=0}^{a-1} \prod_{i=0}^{k-1} F^{j+ia}(x) = \prod_{i=0}^{b-1} F^i(x) = N_F^{F^b}(x),$$

so $N_F^{F^b} = N_F^{F^a} \circ N_{F^a}^{F^b}$. Now suppose that $\mathcal{H}$ is a closed connected subgroup of $\mathcal{T}$ which is stable under F^a and F^b. The map $N_{F^a}^{F^b}$ restricts to a map $N_{F^a}^{F^b} : \mathcal{H}^{F^b} \to \mathcal{H}^{F^a}$, which we claim is surjective. Indeed, if $x \in \mathcal{H}^{F^a}$, then by the Lang-Steinberg theorem there exists some $y \in \mathcal{H}$ such that $y^{-1}F^b(y) = x$. Now, $F^a(x) = x$ implies that $F^a(y^{-1}F^b(y)) = y^{-1}F^b(y)$, and so $F^b(y^{-1}F^a(y)) = y^{-1}F^a(y)$. Thus $y^{-1}F^a(y) \in \mathcal{H}^{F^b}$ and $N_{F^a}^{F^b}(y^{-1}F^a(y)) = y^{-1}F^b(y) = x$.

Now let m be the minimal positive integer such that $F^m(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, for all $\alpha \in \Phi$. Write $n = gm + h$ with integers $g \geq 1$ and $0 \leq h < m$. Then $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha \, \forall \alpha$ implies that $F^h(\mathcal{T}^\alpha) = \mathcal{T}^\alpha \, \forall \alpha$, so the minimality of m forces $h = 0$. If for some α we have $\chi \circ N_F^{F^m}((\mathcal{T}^\alpha)^{F^m}) = 1$, then $N_F^{F^n} = N_F^{F^m} \circ N_{F^m}^{F^n}$ implies $N_F^{F^n}((\mathcal{T}^\alpha)^{F^n}) \subseteq N_F^{F^m}((\mathcal{T}^\alpha)^{F^m})$, so $\chi \circ N_F^{F^n}((\mathcal{T}^\alpha)^{F^n}) = 1$, which contradicts the hypothesis. Thus m is such that the restriction of $\chi \circ N_F^{F^m}$ to $(\mathcal{T}^\alpha)^{F^m}$ is non-trivial, for all α.

Finally, suppose that m' is an arbitrary positive integer such that $F^{m'}(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, for all α. Then as we have seen, $m \mid m'$. By the surjectivity and transitivity of the norm map, we get $N_F^{F^{m'}}((\mathcal{T}^\alpha)^{F^{m'}}) = N_F^{F^m} \circ N_{F^m}^{F^{m'}}((\mathcal{T}^\alpha)^{F^{m'}}) = N_F^{F^m}((\mathcal{T}^\alpha)^{F^m})$. Thus $\chi \circ N_F^{F^{m'}}((\mathcal{T}^\alpha)^{F^{m'}}) = \chi \circ N_F^{F^m}((\mathcal{T}^\alpha)^{F^m}) \neq 1$ for all α, and so χ is regular. $\square$

As before, $\mathbf{U}$ is the unipotent radical of a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Let $\mathbf{V}$ be the unipotent radical of another such Borel subgroup, and let $\mathbf{U}^-$ (resp. $\mathbf{V}^-$) be the unipotent radical of a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $\mathbf{U} \cap \mathbf{U}^- = \{1\}$ (resp. $\mathbf{V} \cap \mathbf{V}^- = \{1\}$). The corresponding subgroups of G are denoted by U, U^-, V, V^-, respectively. Let

$$\Phi^+ = \{\alpha \in \Phi \mid \mathbf{U}_\alpha \subseteq \mathbf{V}\}, \qquad \Phi^- = \{\alpha \in \Phi \mid \mathbf{U}_\alpha \subseteq \mathbf{V}^-\}$$

be the positive, respectively negative, roots corresponding to the choice of $\mathbf{V}$ and $\mathbf{V}^-$. Then $\Phi = \Phi^+ \sqcup \Phi^-$ and $\Phi^- = \{-\alpha \mid \alpha \in \Phi^+\}$. For $\alpha \in \Phi^+$ let $\mathrm{ht}(\alpha)$ be the largest integer $n \geq 1$ such that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $\alpha_i \in \Phi^+$. In the following, for two elements x, y of a group, we shall write $[x, y] = xyx^{-1}y^{-1}$ for their commutator.

The following result was given without proof in [12], 1.6, where it is an easy consequence of well-known facts. In our present context, the last part requires a different and longer proof.

Lemma 2.9. *Let $x \in (U_\alpha)^b$ and $x' \in (U_{\alpha'})^c$, where $\alpha, \alpha' \in \Phi$ and $0 \leq b, c \leq r$. Then the following holds:*

(a) *If $b + c \geq r$, then $xx' = x'x$.*

(b) *If $b + c \leq r$ and $\alpha \neq -\alpha'$, then*

$$[x, x'] = \prod_{\substack{i,i' \geq 1 \\ i\alpha + i'\alpha' \in \Phi}} u_{i,i'},$$

where $u_{i,i'} \in (U_{i\alpha+i'\alpha'})^{b+c}$. (The factors in the product are written in a fixed but arbitrary order.)

(c) *If $b + c \geq r - 1$, $b + 2c \geq r$, and $\alpha = -\alpha'$, then $[x, x'] = \tau_{x,x'}u$, where $\tau_{x,x'} \in \mathcal{T}^\alpha$ and $u \in (U_\alpha)^{r-1}$ are uniquely determined.*

Proof. Part (a) follows immediately from Lemma 2.2(a). Part (b) is Chevalley's commutator formula (cf. [3], 3.3.4) applied to the various subgroups $\mathbf{U}_{i\alpha+i'\alpha'}(A)$ of $\mathbf{G}(A)$. For each α, choose corresponding isomorphisms $p_\alpha : (\mathbb{G}_a)_A \to \mathbf{U}_\alpha$ as in [13], XX 1.20. Functorial properties then imply that

$$p_\alpha(\mathrm{Ker}(\mathbb{G}_a(A) \to \mathbb{G}_a(A/\mathfrak{m}^j))) = \mathrm{Ker}(\mathbf{U}_\alpha(A) \to \mathbf{U}_\alpha(A/\mathfrak{m}^j)) \cong (U_\alpha)^j,$$

for any $0 \leq j \leq r$ (note that we abuse notation since p_α is really a map of group functors rather than groups), and the formula follows. Finally, we prove (c). Let $\tilde{x}, \tilde{x}' \in \mathbb{G}_a(A)$ be such that $p_\alpha(\tilde{x}) = x$ and $p_{-\alpha}(\tilde{x}') = x'$. Then $\tilde{x} \in \mathrm{Ker}(\mathbb{G}_a(A) \to \mathbb{G}_a(A/\mathfrak{m}^b))$ and $\tilde{x}' \in \mathrm{Ker}(\mathbb{G}_a(A) \to \mathbb{G}_a(A/\mathfrak{m}^c))$, respectively; thus $1 + a\tilde{x}\tilde{x}' \in \mathbb{G}_m(A)$, for any $a \in \mathbb{G}_m(A)$. The hypotheses $b + c \geq r - 1$ and $b + 2c \geq r$ imply that $\tilde{x}\tilde{x}' \in \mathrm{Ker}(\mathbb{G}_a(A) \to \mathbb{G}_a(A/\mathfrak{m}^{r-1}))$ and $\tilde{x}\tilde{x}'^2 = 0$. By [13], XX 2.2 we have, for some $a \in \mathbb{G}_m(A)$,

$$p_\alpha(\tilde{x})p_{-\alpha}(\tilde{x}') = p_{-\alpha}(\frac{\tilde{x}'}{1 + a\tilde{x}\tilde{x}'})\check{\alpha}(1 + a\tilde{x}\tilde{x}')p_\alpha(\frac{\tilde{x}}{1 + a\tilde{x}\tilde{x}'}).$$

From this formula we get

$$
\begin{aligned}
[x,x'] &= p_\alpha(\tilde{x})p_{-\alpha}(\tilde{x}')p_\alpha(-\tilde{x})p_{-\alpha}(-\tilde{x}') \\
&= p_{-\alpha}(\frac{\tilde{x}'}{1+a\tilde{x}\tilde{x}'})\check{\alpha}(1+a\tilde{x}\tilde{x}')p_\alpha(\frac{\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&\quad\times p_{-\alpha}(\frac{-\tilde{x}'}{1+a\tilde{x}\tilde{x}'})\check{\alpha}(1+a\tilde{x}\tilde{x}')p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2 p_{-\alpha}((1+a\tilde{x}\tilde{x}')^4\frac{\tilde{x}'}{1+a\tilde{x}\tilde{x}'})p_\alpha((1+a\tilde{x}\tilde{x}')^2\frac{\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&\quad\times p_{-\alpha}((1+a\tilde{x}\tilde{x}')^2\frac{-\tilde{x}'}{1+a\tilde{x}\tilde{x}'})p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&\quad\text{(repeatedly using the facts that } \forall\, t\in\mathbf{T}(A), z\in\mathbb{G}_a(A), \alpha\in\Phi, \text{ we have} \\
&\quad\ tp_\alpha(z)t^{-1} = p_\alpha(\alpha(t)z) \text{ and } \alpha\check{\alpha}(z)=z^2) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2 p_{-\alpha}(\tilde{x}'(1+a\tilde{x}\tilde{x}')^3)p_\alpha(\tilde{x}(1+a\tilde{x}\tilde{x}')) \\
&\quad\times p_{-\alpha}(-\tilde{x}'(1+a\tilde{x}\tilde{x}'))p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2 p_{-\alpha}(\tilde{x}')p_\alpha(\tilde{x}(1+a\tilde{x}\tilde{x}'))p_{-\alpha}(-\tilde{x}')p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&\quad\text{(using } \tilde{x}\tilde{x}'^2=0) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2 p_{-\alpha}(\tilde{x}')p_{-\alpha}(\frac{-\tilde{x}'}{1-a\tilde{x}'\tilde{x}(1+\tilde{x}\tilde{x}')}) \\
&\quad\times \check{\alpha}(1-a\tilde{x}'\tilde{x}(1+\tilde{x}\tilde{x}'))p_\alpha(\frac{\tilde{x}(1+\tilde{x}\tilde{x}')}{1-a\tilde{x}'\tilde{x}(1+\tilde{x}\tilde{x}')})p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2 p_{-\alpha}(\tilde{x}')p_{-\alpha}(\frac{-\tilde{x}'}{1-a\tilde{x}\tilde{x}'}) \\
&\quad\times \check{\alpha}(1-a\tilde{x}\tilde{x}')p_\alpha(\frac{\tilde{x}(1+a\tilde{x}\tilde{x}')}{1-a\tilde{x}\tilde{x}'})p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')^2\check{\alpha}(1-a\tilde{x}\tilde{x}')p_{-\alpha}(\tilde{x}'(1-a\tilde{x}\tilde{x}')^2)p_{-\alpha}(\frac{-\tilde{x}'}{1-a\tilde{x}\tilde{x}'}(1-a\tilde{x}\tilde{x}')^2) \\
&\quad\times p_\alpha(\frac{\tilde{x}(1+a\tilde{x}\tilde{x}')}{1-a\tilde{x}\tilde{x}'})p_\alpha(\frac{-\tilde{x}}{1+a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}((1+a\tilde{x}\tilde{x}')^2(1-a\tilde{x}\tilde{x}'))p_{-\alpha}(0)p_\alpha(\frac{\tilde{x}\tilde{x}'}{1-a\tilde{x}\tilde{x}'}) \\
&= \check{\alpha}(1+a\tilde{x}\tilde{x}')p_\alpha(\frac{\tilde{x}\tilde{x}'}{1-a\tilde{x}\tilde{x}'}).
\end{aligned}
$$

Now

$$\check{\alpha}(1+a\tilde{x}\tilde{x}') \in \mathrm{Ker}(\mathbf{T}^\alpha(A)\to\mathbf{T}^\alpha(A/\mathfrak{m}^{r-1})) \cong \mathcal{T}^\alpha$$

and

$$p_\alpha(\frac{\tilde{x}\tilde{x}'}{1-a\tilde{x}\tilde{x}'}) \in \mathrm{Ker}(\mathbf{U}_\alpha(A)\to\mathbf{U}_\alpha(A/\mathfrak{m}^{r-1})) \cong (U_\alpha)^{r-1}.$$

Using the canonical isomorphism $G \cong \mathbf{G}(A)$, we conclude that for elements $x \in (U_\alpha)^b$ and $x' \in (U_{-\alpha})^c$ we have $[x,x'] \in \mathcal{T}^\alpha(U_\alpha)^{r-1}$. Finally, because of the semidirect product TU in G, the decomposition of $[x,x']$ as an element of $\mathcal{T}^\alpha(U_\alpha)^{r-1}$ is unique. □

The following result and its proof appear in [12], 1.7. Note that the proof uses Lemma 2.9(c).

Lemma 2.10. *We fix an order on Φ^+. For any $z \in V$ and $\beta \in \Phi^+$, define elements $x^z_\beta \in U_\beta$ via the decomposition $z = \prod_{\beta\in\Phi^+} x^z_\beta$ (factors written using the given order on Φ^+). Let $\alpha \in \Phi^-$ and a be an integer such that $1 \leq a \leq r-1$. Suppose that $z \in V^a$ is an element such that $x^z_\beta \in (U_\beta)^{a+1}$, for all $\beta \in \Phi^+$ with $\mathrm{ht}(\beta) > \mathrm{ht}(-\alpha)$. Then for any $\xi \in (U_\alpha)^{r-a-1}$, we have*

$$[\xi, z] = \tau_{\xi,z}\omega_{\xi,z}, \quad \textit{where } \tau_{\xi,z} \in \mathcal{T}^\alpha \textit{ and } \omega_{\xi,z} \in (V^-)^{r-1}.$$

Proof. We argue by induction on $N_z = \#\{\beta \in \Phi^+ \mid x^z_\beta \neq 1\}$. If $N_z = 0$ the result is clear. Now assume that $N_z = 1$ so that $z \in U_\beta$ with $\beta \in \Phi^+$. If $\alpha = -\beta$ the result follows from Lemma 2.9(c). If $\alpha \neq -\beta$ and $\mathrm{ht}(\beta) > \mathrm{ht}(-\alpha)$, then $z \in (U_\alpha)^{a+1}$ and $\xi z = z\xi$ by Lemma 2.9(b). If $\alpha \neq -\beta$ and $\mathrm{ht}(\beta) \leq \mathrm{ht}(-\alpha)$, then by Lemma 2.9(b) we have $[\xi, z] = \prod_{i,i'\geq 1, i\alpha+i'\beta\in\Phi} u_{i,i'}$ with $u_{i,i'} \in (U_{i\alpha+i'\beta})^{r-1}$, and it is enough to show that if $i, i' \geq 1$, we cannot have $i\alpha + i'\beta \in \Phi^+$. Now if we had $i\alpha + i'\beta \in \Phi^+$ for some $i, i' \geq 1$, then general properties of root systems imply that $\alpha + \beta \in \Phi^+$, and hence we would have $\mathrm{ht}(\beta) > \mathrm{ht}(-\alpha)$; a contradiction.

Now assume that $N_z \geq 2$ and that the assertion is true for all z' such that $N_{z'} \leq N_z$. We can write $z = z'z''$, where $z', z'' \in V^a$, $N_{z'} < N_z$, $N_{z''} < N_z$. Using the induction hypothesis, we have

$$\xi z = \xi z'z'' = \tau_{\xi,z'}\omega_{\xi,z'}z'\xi z'' = \tau_{\xi,z'}\omega_{\xi,z'}z'\tau_{\xi,z''}\omega_{\xi,z''}z''\xi,$$

where $\tau_{\xi,z'} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z'} \in (V^-)^{r-1}$. Since $a+r-1 \geq r$, we have $z'\tau_{\xi,z''} = \tau_{\xi,z''}z'$ and $z'\omega_{\xi,z''} = \omega_{\xi,z''}z'$, by Lemma 2.2. Hence

$$\xi z = \tau_{\xi,z'}\omega_{\xi,z'}z'\tau_{\xi,z''}\omega_{\xi,z''}z''\xi = \tau_{\xi,z'}\omega_{\xi,z'}\tau_{\xi,z''}\omega_{\xi,z''}z'z''\xi = \tau_{\xi,z'}\tau_{\xi,z''}\omega_{\xi,z'}\omega_{\xi,z''}z\xi,$$

and so

$$[\xi, z] = \tau_{\xi,z}\omega_{\xi,z},$$

where $\tau_{\xi,z} = \tau_{\xi,z'}\tau_{\xi,z''}$ and $\omega_{\xi,z} = \omega_{\xi,z'}\omega_{\xi,z''}$. $\square$

Let $\mathbf{Z} = \mathbf{U}^- \cap \mathbf{V}$, $Z = (\mathcal{F}_A\mathbf{Z})(k) = U^- \cap V$, and $\Phi' = \{\beta \in \Phi \mid \mathbf{U}_\beta \subseteq \mathbf{Z}\}$. We obviously have $\Phi' \subseteq \Phi^+$. Let $\mathcal{X}$ be the set of all subsets $I \subseteq \Phi'$ such that $I \neq \emptyset$, and $\mathrm{ht} : \Phi^+ \to \mathbb{N}$ is constant on I.

To any $z \in Z^1 - \{1\}$ we associate a pair (a, I_z), where a is an integer $1 \leq a \leq r-1$, and $I_z \in \mathcal{X}$, as follows. We define a by the condition that $z \in Z^{a,*}$. If $x^z_\beta \in U_\beta$ are defined as in Lemma 2.10 in terms of a fixed order on Φ^+, then $x^z_\beta \in (U_\beta)^a$ for all $\beta \in \Phi'$ and $x^z_\beta = 1$ for all $\beta \in \Phi^+ - \Phi'$ (this is a consequence of the formula $Z^a = \prod_{\beta\in\Phi'} U^a_\beta$). We define the set I_z as

$$I_z = \{\alpha' \in \Phi' \mid x^z_{\alpha'} \in (U_{\alpha'})^{a,*} \text{ and } x^z_\beta \in (U_\beta)^{a+1}\ \forall\, \beta \in \Phi^+ \text{ s.t. } \mathrm{ht}(\beta) > \mathrm{ht}(\alpha')\}.$$

The definition of I_z does not depend on the choice of order on Φ^+. For any integer $1 \leq a \leq r-1$ and $I \in \mathcal{X}$, let $Z^{a,*,I}$ be the set of all $z \in Z^1 - \{1\}$ such that $z \in Z^{a,*}$ and $I = I_z$. Thus we have a partition

$$(*) \qquad Z^1 - \{1\} = \bigsqcup_{\substack{1\leq a\leq r-1\\ I\in\mathcal{X}}} Z^{a,*,I}.$$

3. The main results

Recall the definitions of the groups $T, T', U, U', \mathcal{T}, \mathcal{T}'$ and the variety Σ from Section 2. After having set up the general framework, we are now ready to give results generalizing those in [12], with the structures of the proofs remaining more or less the same. All the ideas of the proofs in this section are due to Lusztig. The only thing that requires a comment here is the use of the elements $\hat{w}$. In [12], the inclusion $G_1 \subseteq G$ (in our notation) allows one to view the elements of $N(T_1, T_1')$ as elements of $N(T, T')$. However, in the general case which we consider here there is no such inclusion, and instead we have to use lifts $\hat{w} \in N(T, T')$ of the elements $\dot{w} \in N(T_1, T_1')$. The following theorem does not depend on the choice of lift $\hat{w}$ for each $\dot{w}$. This can be seen from the proof, because we show that

$$\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\Sigma_w)_{\theta^{-1},\theta'} = \sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}_{\hat{w}})_{\theta^{-1},\theta'},$$

where $\hat{\Sigma}_{\hat{w}}$ is the variety defined below, $\hat{w}$ is an arbitrary lift of $\dot{w}$, and the latter sum is equal to 1 if $F(w) = w$ and $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ', and equals 0 otherwise. Thus, if $\hat{w}'$ is another lift of $\dot{w}$, then $\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}_{\hat{w}})_{\theta^{-1},\theta'} = \sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}_{\hat{w}'})_{\theta^{-1},\theta'}$, and so whenever $F(w) = w$, we see that $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ' if and only if $\mathrm{Ad}(\hat{w}') : T'^F \to T^F$ carries θ to θ'.

Theorem 3.1. *Let $\theta \in \widehat{T^F}$ and $\theta' \in \widehat{T'^F}$. If $r \geq 2$, assume that θ' is regular. Then $\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\Sigma)_{\theta^{-1},\theta'}$ is equal to the number of $w \in W(T_1, T_1')^F$ such that $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ'.*

Proof. Using the partition $\Sigma = \bigsqcup_w \Sigma_w$ and the additivity of Lefschetz numbers (cf. [4], 10.7), we see that it is enough to prove that $\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\Sigma_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ', and equals 0 otherwise. We now fix $w \in W(T_1, T_1')$. We have

$$\Sigma_w = \{(x, x', y) \in F(U)\times F(U')\times G \mid xF(y) = yx',\ y \in UG^1\hat{w}T'U' = UZ^1\hat{w}T'U'\},$$

where $Z^1 = (U^-)^1 \cap \hat{w}(U'^-)^1\hat{w}^{-1}$ (the equality $UG^1\hat{w}T'U' = UZ^1\hat{w}T'U'$ follows from Lemma 2.3). Let

$$\begin{aligned}\hat{\Sigma}_{\hat{w}} = \{&(x, x', u, u', z, \tau') \in F(U) \times F(U') \times U \times U' \times Z^1 \times T' \mid \\ &xF(u)F(z)F(\hat{w})F(\tau')F(u') = uz\hat{w}\tau'u'x'\}.\end{aligned}$$

The map $\hat{\Sigma}_{\hat{w}} \to \Sigma_w$ given by $(x, x', u, u', z, \tau') \mapsto (x, x', uz\hat{w}\tau'u')$ is a locally trivial fibration with all fibres isomorphic to a fixed affine space. This map is compatible with the $T^F \times T'^F$-actions where $T^F \times T'^F$ acts on $\hat{\Sigma}_{\hat{w}}$ by

(a)
$$\begin{aligned}&(t, t') : (x, x', u, u', z, \tau') \\ &\qquad\longmapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tzt^{-1}, \hat{w}^{-1}t\hat{w}\tau't'^{-1}).\end{aligned}$$

Hence, by [11], 1.9 it is enough to show that $\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}_{\hat{w}})_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ', and equals 0 otherwise.

By the change of variables $xF(u) \mapsto x$, $x'F(u')^{-1} \mapsto x'$ we may rewrite $\hat{\Sigma}_{\hat{w}}$ as

$$\begin{aligned}\hat{\Sigma}_{\hat{w}} = \{&(x, x', u, u', z, \tau') \in F(U) \times F(U') \times U \times U' \times Z^1 \times T' \mid \\ &xF(z)F(\hat{w})F(\tau') = uz\hat{w}\tau'u'x'\},\end{aligned}$$

with the $T^F \times T'^F$-action still given by (a). We have a partition $\hat{\Sigma}_{\hat{w}} = \hat{\Sigma}'_{\hat{w}} \sqcup \hat{\Sigma}''_{\hat{w}}$, where

$$\begin{aligned}\hat{\Sigma}'_{\hat{w}} &= \{(x, x', u, u', z, \tau') \in F(U) \times F(U') \times U \times U' \times (Z^1 - \{1\}) \times T' \mid \\ &\quad xF(z)F(\hat{w})F(\tau') = uz\hat{w}\tau'u'x'\},\\ \hat{\Sigma}''_{\hat{w}} &= \{(x, x', u, u', 1, \tau') \in F(U) \times F(U') \times U \times U' \times \{1\} \times T' \mid \\ &\quad xF(\hat{w})F(\tau') = u\hat{w}\tau'u'x'\}\end{aligned}$$

are stable under the $T^F \times T'^F$-action. It is then enough to show that

(b) $\displaystyle\sum_{j\in\mathbb{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}''_{\hat{w}})_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $\mathrm{Ad}(\hat{w}) : T'^F \to T^F$ carries θ to θ', and equals 0 otherwise.

(c) $H_c^j(\hat{\Sigma}'_{\hat{w}})_{\theta^{-1},\theta'} = 0$, for all j.

We first prove (c). For $r = 1$ we have $\hat{\Sigma}'_{\hat{w}} = \emptyset$, so in this case (c) is clear. Suppose now that $r \geq 2$. If M is a $\mathcal{T}'^F$-module we shall write $M_{(\chi)}$ for the subspace of M on which $\mathcal{T}'^F$ acts according to χ; that is, $M_{(\chi)} = \{m \in M \mid t'm = \chi(t')m,\ \forall t' \in \mathcal{T}'^F\}$. Now $\mathcal{T}'^F$ acts on $\hat{\Sigma}'_{\hat{w}}$ by

$$t' : (x, x', u, u', z, \tau') \longmapsto (x, t'x't'^{-1}, u, t'u't'^{-1}, z, \tau't'^{-1}).$$

Hence $H_c^j(\hat{\Sigma}'_{\hat{w}})$ becomes a $\mathcal{T}'^F$-module. Since $H_c^j(\hat{\Sigma}'_{\hat{w}}) = \bigoplus_\chi H_c^j(\hat{\Sigma}'_{\hat{w}})_{(\chi)}$, it is enough to show that $H_c^j(\hat{\Sigma}'_{\hat{w}})_{(\chi)} = 0$. We shall use the definitions and results in Lemmas 2.9, 2.10, and the partition $(*)$ at the end of Section 2, relative to U, U^-, V, V^-, where we take U, U^- as above, and $V = \hat{w}(U')^-\hat{w}^{-1}$, $V^- = \hat{w}U'\hat{w}^{-1}$. The partition $(*)$ gives rise to a partition

$$\hat{\Sigma}'_{\hat{w}} = \bigsqcup_{\substack{1\leq a\leq r-1\\ I\in\mathcal{X}}} \hat{\Sigma}^{a,I}_{\hat{w}}, \text{ where } \hat{\Sigma}^{a,I}_{\hat{w}} = \{(x, x', u, u', z, \tau') \in \hat{\Sigma}'_{\hat{w}} \mid z \in Z^{a,*,I}\}.$$

It is easy to see that there is a total order on the set of indices (a, I) such that the union of the $\hat{\Sigma}^{a,I}_{\hat{w}}$ for (a, I) less than or equal to some given (a^0, I^0) is closed in $\hat{\Sigma}'_{\hat{w}}$. Since the subsets $\hat{\Sigma}^{a,I}_{\hat{w}}$ are stable under the action of $\mathcal{T}'^F$, we see that in order to prove (c) it is enough to show that

(d) $H_c^j(\hat{\Sigma}^{a,I}_{\hat{w}})_{(\chi)} = 0$, for any fixed (a, I).

We choose $\alpha' \in I$, and let $\alpha = -\alpha'$. Then $U_\alpha \subseteq U \cap V^- = U \cap \hat{w}U'\hat{w}^{-1}$.

For any $z \in Z^{a,*}$ and $\xi \in (U_\alpha)^{r-a-1}$ we have

$$[\xi, z] = \tau_{\xi,z}\omega_{\xi,z},$$

where $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in \hat{w}(U')^{r-1}\hat{w}^{-1}$ are uniquely determined (cf. Lemma 2.10). Moreover, the map $(U_\alpha)^{r-a-1} \to \mathcal{T}^\alpha$, $\xi \mapsto \tau_{\xi,z}$ factors through an isomorphism

$$\lambda_z : (U_\alpha)^{r-a-1}/(U_\alpha)^{r-a} \xrightarrow{\sim} \mathcal{T}^\alpha.$$

Let $\pi : (U_\alpha)^{r-a-1} \to (U_\alpha)^{r-a-1}/(U_\alpha)^{r-a}$ be the canonical homomorphism. Since $\mathbf{U}_\alpha$ is an affine space, there exists a morphism of algebraic varieties

$$\psi : (U_\alpha)^{r-a-1}/(U_\alpha)^{r-a} \longrightarrow (U_\alpha)^{r-a-1}$$

such that $\pi \circ \psi = \mathrm{Id}$ and $\psi(1) = 1$. Let

$$\mathcal{H}' = \{t' \in \mathcal{T}' \mid t'^{-1}F(t') \in \hat{w}^{-1}\mathcal{T}^{\alpha}\hat{w}\}.$$

This is a closed subgroup of $\mathcal{T}'$. For any $t' \in \mathcal{H}'$ we define $f_{t'} : \hat{\Sigma}_{\hat{w}}^{a,I} \to \hat{\Sigma}_{\hat{w}}^{a,I}$ by

$$f_{t'}(x, x', u, u', z, \tau') = (xF(\xi), \hat{x}', u, F(t')^{-1}u'F(t'), z, \tau'F(t')),$$

where

$$\xi = (\psi\lambda_{z^{-1}}^{-1}(\hat{w}F(t')^{-1}t'\hat{w}^{-1}))^{-1} \in (U_\alpha)^{r-a-1} \subseteq U \cap \hat{w}U'\hat{w}^{-1},$$

and $\hat{x}' \in G$ is determined by the condition that defines the variety $\hat{\Sigma}_{\hat{w}}^{a,I}$; that is,

$$xF(\xi)F(z)F(\hat{w})F(\tau'F(t')) = uz\hat{w}\tau'F(t')F(t')^{-1}u'F(t')\hat{x}'.$$

In order for this to be well defined we must check that $\hat{x}' \in F(U')$. Thus we must show that

$$xF(\xi)F(z)F(\hat{w})F(\tau'F(t')) \in uz\hat{w}\tau'u'F(t')F(U').$$

By Lemma 2.10 we have

$$\xi z = (z^{-1}\xi^{-1})^{-1} = (\omega_{\xi^{-1},z^{-1}}^{-1}\tau_{\xi^{-1},z^{-1}}^{-1}\xi^{-1}z^{-1})^{-1} = z\xi\tau_{\xi^{-1},z^{-1}}\omega_{\xi^{-1},z^{-1}}.$$

Thus the above condition is equivalent to

$$xF(z)F(\xi)F(\tau_{\xi^{-1},z^{-1}})F(\omega_{\xi^{-1},z^{-1}})F(\hat{w})F(\tau'F(t')) \in uz\hat{w}\tau'u'F(t')F(U').$$

Since $xF(z) = uz\hat{w}\tau'u'x'F(\tau')^{-1}F(\hat{w})^{-1}$, it is enough to show that

$$\begin{aligned} uz\hat{w}\tau'u'x'F(\tau')^{-1}F(\hat{w})^{-1}F(\xi)F(\tau_{\xi^{-1},z^{-1}})F(\omega_{\xi^{-1},z^{-1}})F(\hat{w})F(\tau'F(t')) \\ \in uz\hat{w}\tau'u'F(t')F(U') \end{aligned}$$

or that

$$x'F(\tau')^{-1}F(\hat{w})^{-1}F(\xi)F(\tau_{\xi^{-1},z^{-1}})F(\omega_{\xi^{-1},z^{-1}})F(\hat{w})F(\tau'F(t')) \in F(t')F(U').$$

Since $x' \in F(U')$ and $F(\hat{w})^{-1}F(\omega_{\xi^{-1},z^{-1}})F(\hat{w}) \in F(U')$ it is enough to check that

$$F(\tau')^{-1}F(\hat{w})^{-1}F(\xi)F(\tau_{\xi^{-1},z^{-1}})F(\hat{w})F(\tau'F(t')) \in F(t')F(U').$$

Since $F(\hat{w}^{-1})F(\xi)F(\hat{w}) \in F(U')$ it is enough to check that

$$F(\tau')^{-1}F(\hat{w})^{-1}F(\tau_{\xi^{-1},z^{-1}})F(\hat{w})F(\tau'F(t')) \in F(t')F(U')$$

or that

$$F(\hat{w})^{-1}F(\tau_{\xi^{-1},z^{-1}})F(\hat{w})F(F(t')) \in F(t')F(\tau')F(U')F(\tau')^{-1} = F(t')F(U'),$$

which is equivalent to

$$F(\hat{w})^{-1}F(\tau_{\xi^{-1},z^{-1}})F(\hat{w})F(F(t')) = F(t').$$

That is, $\hat{w}^{-1}\tau_{\xi^{-1},z^{-1}}\hat{w} = F(t')^{-1}t'$ or $\lambda_{z^{-1}}(\pi(\xi^{-1})) = \tau_{\xi^{-1},z^{-1}}$, which holds because of the definitions of the element ξ and the map $\lambda_{z^{-1}}$.

Thus, $f_{t'} : \hat{\Sigma}_{\hat{w}}^{a,I} \to \hat{\Sigma}_{\hat{w}}^{a,I}$ is well defined and has an obvious inverse, so it is clearly an isomorphism for any $t' \in \mathcal{H}'$. Note however that this does not define an action of the group $\mathcal{H}'$ on $\hat{\Sigma}_{\hat{w}}^{a,I}$, since $f_{t_1't_2'} \neq f_{t_1'} \circ f_{t_2'}$ in general. Nevertheless, $f_{t'}$ is in particular a well-defined isomorphism for any $t' \in \mathcal{H}'^0$, where $\mathcal{H}'^0$ is the connected component of $\mathcal{H}'$, and by general principles (cf. the proof of Proposition 6.4 in [2]) the induced map $f_{t'}^* : H_c^j(\hat{\Sigma}_{\hat{w}}^{a,I}) \to H_c^j(\hat{\Sigma}_{\hat{w}}^{a,I})$ is constant when t' varies in $\mathcal{H}'^0$. In particular, it is constant when t' varies in $\mathcal{T}' \cap \mathcal{H}'^0$. Now $\mathcal{T}'^F \subseteq \mathcal{H}'$ and for $t' \in \mathcal{T}'^F$, the map $f_{t'}$ coincides with the action of t'^{-1} in the $\mathcal{T}'^F$-action on $\hat{\Sigma}_{\hat{w}}^{a,I}$ (we use the fact that $\psi(1)=1$). We see that the induced action of $\mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_{\hat{w}}^{a,I})$ is trivial when restricted to $\mathcal{T}'\cap\mathcal{H}'^0$.

Now let $n\geq 1$ be an integer such that $F^n(\hat{w}^{-1}\mathcal{T}^\alpha\hat{w})=\hat{w}^{-1}\mathcal{T}^\alpha\hat{w}$. Then

$$t'\longmapsto t'F(t')F^2(t')\cdots F^{n-1}(t')$$

is a well-defined morphism $\hat{w}^{-1}\mathcal{T}^\alpha\hat{w}\to\mathcal{H}'$. Its image is a connected subgroup of $\mathcal{H}'$, hence contained in $\mathcal{H}'^0$. If $t'\in(\hat{w}^{-1}\mathcal{T}^\alpha\hat{w})^{F^n}$, then $N_F^{F^n}(t')\in\mathcal{T}'^F$; thus $N_F^{F^n}(t')\in\mathcal{T}'^F\cap\mathcal{H}'^0$. We see that the action of $N_F^{F^n}(t')\in\mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_{\hat{w}}^{a,I})$ is trivial for any $t'\in(\hat{w}^{-1}\mathcal{T}^\alpha\hat{w})^{F^n}$.

If we assume that $H_c^j(\hat{\Sigma}_{\hat{w}}^{a,I})_{(\chi)}\neq 0$, it follows that $t'\mapsto\chi(N_F^{F^n}(t'))$ is the trivial character of $(\hat{w}^{-1}\mathcal{T}^\alpha\hat{w})^{F^n}$. This contradicts our assumption that χ is regular. Thus (d) holds, and hence (c) holds.

We now prove (b). Let

$$\tilde{H}=\{(t,t')\in T\times T'\mid tF(t)^{-1}=F(\hat{w})t'F(t')^{-1}F(\hat{w})^{-1}\}.$$

This is a closed subgroup of $T\times T'$ containing $T^F\times T'^F$. Now the action of $T^F\times T'^F$ on $\hat{\Sigma}''_{\hat{w}}$ extends to an action of $\tilde{H}$ given by the same formula. To see this, consider $(t,t')\in\tilde{H}$ and $(x,x',u,u',1,\tau')\in\hat{\Sigma}''_{\hat{w}}$. We must show that

$$(txt^{-1},t'x't'^{-1},tut^{-1},t'u't'^{-1},1,\hat{w}^{-1}t\hat{w}\tau't'^{-1})\in\hat{\Sigma}''_{\hat{w}}.$$

That is,

$$txt^{-1}F(\hat{w})F(\hat{w}^{-1})F(t)F(\hat{w})F(\tau')F(t'^{-1})=tut^{-1}\hat{w}\hat{w}^{-1}t\hat{w}\tau't'u't'^{-1}t'x't'^{-1}$$

or that

$$xt^{-1}F(t)F(\hat{w})F(\tau')F(t'^{-1})=u\hat{w}\tau'u'x't'^{-1}$$

or that

$$xt^{-1}F(t)F(\hat{w})F(\tau')F(t'^{-1})=xF(\hat{w})F(\tau')t'^{-1}$$

or that $t^{-1}F(t)F(\hat{w})F(t'^{-1})=F(\hat{w})t'^{-1}$, which is clear. Let T_* (resp. T'_*) be the reductive part of T (resp. T') (thus T_* is a torus isomorphic to T). Let $\tilde{H}_*=\tilde{H}\cap(T_*\times T'_*)$. Then $\tilde{H}_*^0$ is a torus acting on $\hat{\Sigma}''_{\hat{w}}$ by restriction of the $\tilde{H}$-action. The fixed point set $(\hat{\Sigma}''_{\hat{w}})^{\tilde{H}_*^0}$ is stable under the action of $T^F\times T'^F$, and by [4], 4.5 (compare 11.2) and 10.15 we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}&(-1)^j\dim H_c^j(\hat{\Sigma}''_{\hat{w}})_{\theta^{-1},\theta'}\\&=|T^F\times T'^F|^{-1}\sum_{(t,t')\in T^F\times T'^F}\mathcal{L}((t,t'),\hat{\Sigma}''_{\hat{w}})\theta(t)\theta'(t')^{-1}\\&=|T^F\times T'^F|^{-1}\sum_{(t,t')\in T^F\times T'^F}\mathcal{L}((t,t'),(\hat{\Sigma}''_{\hat{w}})^{\tilde{H}_*^0})\theta(t)\theta'(t')^{-1}\\&=\sum_{j\in\mathbb{Z}}(-1)^j\dim H_c^j((\hat{\Sigma}''_{\hat{w}})^{\tilde{H}_*^0})_{\theta^{-1},\theta'}.\end{aligned}$$

It is then enough to show that

(e) $\sum_{j\in\mathbb{Z}}(-1)^j\dim H_c^j((\hat{\Sigma}''_{\hat{w}})^{\tilde{H}_*^0})_{\theta^{-1},\theta'}$ is equal to 1 if $F(w)=w$ and $\mathrm{Ad}(\hat{w}):T'^F\to T^F$ carries θ to θ', and equals 0 otherwise.

Let $(x, x', u, u', 1, \tau') \in (\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*}$. By the Lang-Steinberg theorem the first projection $\tilde{H}_* \to T_*$ is surjective. It follows that the first projection $\tilde{H}^0_* \to T_*$ is surjective. Similarly the second projection $\tilde{H}^0_* \to T'_*$ is as well. Hence for any $t \in T_*$, $t' \in T_*$ we have

$$txt^{-1} = x,\ t'x't'^{-1} = x',\ tut^{-1} = u,\ t'u't'^{-1} = u',$$

and hence $x = x' = u = u' = 1$. Thus $(\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*}$ is contained in

$$\{(1,1,1,1,1,\tau') \mid \tau' \in T', F(\hat{w}\tau') = \hat{w}\tau'\}. \tag{f}$$

The set (f) is clearly contained in the fixed point set of $\tilde{H}$. Note that (f) is empty unless $F(w) = w$, by Bruhat decomposition in G_1. If (f) is empty, then $\sum_{j\in\mathbb{Z}}(-1)^j \dim H^j_c((\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*})_{\theta^{-1},\theta'} = 0$. We can therefore assume that $F(w) = w$. Now (f) is stable under the action of $\tilde{H}$. Indeed, if $\tau' \in T'$ is such that $F(\hat{w}\tau') = \hat{w}\tau'$ and $(t, t') \in \tilde{H}$, then

$$\begin{aligned} F(\hat{w}\hat{w}^{-1}t\hat{w}\tau't'^{-1}) &= F(\hat{w})F(t')t'^{-1}F(\hat{w})^{-1}tF(\hat{w})F(\tau')F(t'^{-1}) \\ &= tF(\hat{w})F(t')t'^{-1}F(\tau')F(t'^{-1}) = tF(\hat{w})F(\tau')t'^{-1} = \hat{w}\hat{w}^{-1}t\hat{w}\tau't'^{-1}. \end{aligned}$$

Thus in particular, (f) is stable under $\tilde{H}^0_*$. Since (f) is a finite set and $\tilde{H}^0_*$ is connected, we see that $\tilde{H}^0_*$ must act trivially on (f). Thus, (f) is exactly the fixed point set of $\tilde{H}^0_*$, and hence $(\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*} \cong (\hat{w}T')^F$. Since

$$\begin{aligned} \#((\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*})^{(t,t')} &= \#\{\hat{w}\tau' \in (\hat{w}T')^F \mid \hat{w}^{-1}t\hat{w}\tau't'^{-1} = \tau'\} \\ &= \#\{\hat{w}\tau' \in (\hat{w}T')^F \mid \hat{w}^{-1}t\hat{w} = t'\} = \begin{cases} |(\hat{w}T')^F| = |T'^F| & \text{if } \hat{w}^{-1}t\hat{w} = t', \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

it follows from the facts quoted above together with [11], 1.10 that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}(-1)^j \dim H^j_c((\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*})_{\theta^{-1},\theta'} \\ &\quad = |T^F \times T'^F|^{-1} \sum_{(t,t')\in T^F\times T'^F} \mathcal{L}((t,t'), (\hat{\Sigma}''_{\hat{w}})^{\tilde{H}^0_*})\theta(t)\theta'(t')^{-1} \\ &\quad = |T^F \times T'^F|^{-1} \sum_{t\in T^F} |T'^F|\theta(t)\theta'(\hat{w}^{-1}t\hat{w})^{-1} \\ &\quad = \langle \theta, {}^{\hat{w}^{-1}}\theta' \rangle_{T^F} = \begin{cases} 1 & \text{if } {}^{\hat{w}}\theta = \theta', \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we have established (e), and so the theorem is proved. □

We finish by giving some important consequences of the preceding results. Let $\mathcal{R}(G^F)$ be the group of virtual representations of G^F over $\overline{\mathbb{Q}}_l$. Let $\langle\cdot,\cdot\rangle$ be the standard inner product $\mathcal{R}(G^F) \times \mathcal{R}(G^F) \to \mathbb{Z}$. Let

$$S_{T,U} = \{g \in G \mid g^{-1}F(g) \in F(U)\}.$$

The finite group $G^F \times T^F$ acts on $S_{T,U}$ by $(g_1, t) : g \mapsto g_1gt^{-1}$. For any $i \in \mathbb{Z}$ we have an induced action of $G^F \times T^F$ on $H^i_c(S_{T,U})$. For $\theta \in \widehat{T^F}$ we denote by

$H_c^i(S_{T,U})_\theta$ the subspace of $H_c^i(S_{T,U})$ on which T^F acts according to θ. This is a G^F-submodule of $H_c^i(S_{T,U})$. Let

$$R_{T,U}^\theta = \sum_{i\in\mathbb{Z}}(-1)^i H_c^i(S_{T,U})_\theta \in \mathcal{R}(G^F).$$

Note that the definition of $R_{T,U}^\theta$ is formally identical to that of [12], 2.1, even though the objects involved (such as the groups G and U, and the variety $S_{T,U}$) are in general not isomorphic to their analogues in [12].

The following result appears in [12], 2.2.

Proposition 3.2. *Let the notation be as before. Then the following holds:*

(a) *Assume that there exist integers i and i' and an irreducible G^F-module that appears in the G^F-module $(H_c^i(S_{T,U})_{\theta^{-1}})^*$ (dual of $H_c^i(S_{T,U})_{\theta^{-1}}$) and in the G^F-module $H_c^{i'}(S_{T',U'})_{\theta'}$. Then there exists $n\geq 1$ and $g\in N(T',T)^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta\circ N_F^{F^n}|_{\mathcal{T}^{F^n}}\in\widehat{\mathcal{T}^{F^n}}$ to $\theta'\circ N_F^{F^n}|_{\mathcal{T}'^{F^n}}\in\widehat{\mathcal{T}'^{F^n}}$.*

(b) *Assume that there exists an irreducible G^F-module that appears in the virtual G^F-module $R_{T,U}^\theta$ and in the virtual G^F-module $R_{T',U'}^{\theta'}$. Then there exists $n\geq 1$ and $g\in N(T,T')^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta\circ N_F^{F^n}|_{\mathcal{T}^{F^n}}\in\widehat{\mathcal{T}^{F^n}}$ to $\theta'\circ N_F^{F^n}|_{\mathcal{T}'^{F^n}}\in\widehat{\mathcal{T}'^{F^n}}$.*

Proof. We prove (a). Consider the free G^F-action on $S_{T,U}\times S_{T',U'}$ given by $g_1:(g,g')\mapsto(g_1g,g_1g')$. The map

$$(g,g')\longmapsto(x,x',y),\quad x=g^{-1}F(g), x'=g'^{-1}F(g'), y=g^{-1}g'$$

defines an isomorphism from $G^F\backslash(S_{T,U}\times S_{T',U'})$ to Σ (the fact that it is an isomorphism and not merely a bijective homomorphism is proved in [1], pp. 221–222 in the situation where $r=1$; the same argument works in general). The action of $T^F\times T'^F$ on $S_{T,U}\times S_{T',U'}$ given by right multiplication by t^{-1} on the first factor and by t'^{-1} on the second factor, becomes an action of $T^F\times T'^F$ on Σ given by $(x,x',y)\mapsto(txt^{-1},t'x't'^{-1},tyt'^{-1})$. Our assumption implies that the G^F-module $H_c^i(S_{T,U})_{\theta^{-1}}\otimes H_c^{i'}(S_{T',U'})_{\theta'}$ contains the trivial representation with non-zero multiplicity; that is, $(H_c^i(S_{T,U})_{\theta^{-1}}\otimes H_c^{i'}(S_{T',U'})_{\theta'})^{G^F}\neq 0$. By [4], 10.9 and 10.10(i) we have an inclusion

$$(H_c^i(S_{T,U})_{\theta^{-1}}\otimes H_c^{i'}(S_{T',U'})_{\theta'})^{G^F}\hookrightarrow H_c^{i+i'}(G^F\backslash(S_{T,U}\times S_{T',U'}))_{\theta^{-1},\theta'},$$

and so $H_c^{i+i'}(G^F\backslash(S_{T,U}\times S_{T',U'}))_{\theta^{-1},\theta'}\neq 0$. By the above isomorphism we thus have $H_c^{i+i}(\Sigma)_{\theta^{-1},\theta'}\neq 0$. We now use Lemma 2.6, and (a) follows.

We prove (b). By [4], 11.4 we have

$$\sum_i(-1)^i(H_c^i(S_{T,U})_{\theta^{-1}})^* = \sum_i(-1)^i H_c^i(S_{T,U})_\theta.$$

Hence the assumption of (b) implies that the assumption of (a) holds. Hence the conclusion of (a) holds. The proposition is proved. □

The following result and its corollary correspond to 2.3 and 2.4 in [12].

Proposition 3.3. *Assume that θ or θ' is regular (see Definition 2.7). Then*

$$\langle R_{T,U}^\theta, R_{T',U'}^{\theta'}\rangle = \#\{w\in W(T_1,T_1')^F \mid {}^{\hat{w}}\theta=\theta'\}.$$

Proof. We may assume that θ' is regular. We have

$$\begin{aligned}\langle R^{\theta}_{T,U}, R^{\theta'}_{T',U'}\rangle &= \sum_{i,i'\in\mathbb{Z}} (-1)^{i+i'} \dim(H^i_c(S_{T,U})_{\theta^{-1}} \otimes H^{i'}_c(S_{T',U'})_{\theta'})^{G^F} \\ &= \sum_{j\in\mathbb{Z}}(-1)^j \dim H^j_c(G^F\backslash(S_{T,U}\times S_{T',U'}))_{\theta^{-1},\theta'} \\ &= \sum_{j\in\mathbb{Z}}(-1)^j \dim H^j_c(\Sigma)_{\theta^{-1},\theta'}.\end{aligned}$$

It remains to use Theorem 3.1. □

Corollary 3.4. *Assume that $\theta \in \widehat{T^F}$ is regular. Then*

(a) *$R^{\theta}_{T,U}$ is independent of the choice of U.*

(b) *Assume in addition that there is no non-trivial element $w \in W(T_1)^F$ such that $\hat{w}$ fixes θ. Then $\pm R^{\theta}_{T,U}$ is an irreducible G^F-module.*

Proof. We prove (a). Let V be the subgroup of G associated with the unipotent radical $\mathbf{V}$ of another Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. By Proposition 3.3 we have

$$\langle R^{\theta}_{T,U}, R^{\theta}_{T,U}\rangle = \langle R^{\theta}_{T,U}, R^{\theta}_{T,U'}\rangle = \langle R^{\theta}_{T,U'}, R^{\theta}_{T,U}\rangle = \langle R^{\theta}_{T,U'}, R^{\theta}_{T,U'}\rangle.$$

Hence $\langle R^{\theta}_{T,U} - R^{\theta}_{T,U'}, R^{\theta}_{T,U} - R^{\theta}_{T,U'}\rangle = 0$, and so $R^{\theta}_{T,U} = R^{\theta}_{T,U'}$. This proves (a). In the setup of (b), we see from Proposition 3.3 that $\langle R^{\theta}_{T,U}, R^{\theta}_{T,U}\rangle = 1$, which proves (b). □

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