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Base sizes for simple groups and a conjecture of Cameron

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Abstract

Let G be a permutation group on a finite set Ω. A base for G is a subset $B \subseteq \Omega$ with pointwise stabilizer in G that is trivial; we write $b(G)$ for the smallest size of a base for G. In this paper we prove that $b(G) \leqslant 6$ if G is an almost simple group of exceptional Lie type and Ω is a primitive faithful G-set. An important consequence of this result, when combined with other recent work, is that $b(G) \leqslant 7$ for any almost simple group G in a non-standard action, proving a conjecture of Cameron. The proof is probabilistic and uses bounds on fixed point ratios.

1. *Introduction*

Let G be a permutation group on a set Ω. A *base* for G is a subset $B \subseteq \Omega$ with pointwise stabilizer in G that is trivial. We write $b(G) = b(G, \Omega)$ for the smallest size of a base for G. Bases have been of interest since the early days of group theory in the nineteenth century. For example, a classical result of Bochert [**3**] states that if G is a primitive permutation group of degree n not containing A_n, then $b(G) \leqslant n/2$. In more recent years, bases have been used extensively in the computational study of finite permutation groups. In this respect, small bases are particularly significant, and so it is important to establish accurate bounds on the minimal base size.

In this paper we study base sizes for finite almost simple primitive groups. More precisely, we are interested in the so-called *non-standard* actions which we define as follows. A primitive action of a finite almost simple group G is said to be *standard* if either G has socle A_n and the action is on subsets or partitions of $\{1, \ldots, n\}$, or G is a classical group acting on an orbit of subspaces (or pairs of subspaces of complementary dimension) of the natural module. Non-standard actions are defined accordingly. (For a precise list of standard actions see Definitions 1.1 and 2.1 in [**7**].)

A well-known conjecture of Cameron and Kantor [**12**, **14**] asserts that there exists an absolute constant c such that $b(G) \leqslant c$ for all finite almost simple groups G in faithful primitive non-standard actions. In general, it is easy to see that $b(G)$ can be arbitrarily large for standard actions.

The Cameron–Kantor conjecture was settled in the affirmative by Liebeck and Shalev in [**48**]. However, this is strictly an existence result and the proof of [**48**, Theorem 1.3] does not yield an explicit value for c. Recently, a number of papers have appeared where more explicit base size results are obtained. For example, in the forthcoming paper by T. C. Burness, R. M. Guralnick and J. Saxl, 'Base sizes for actions of simple groups', it is shown that if G has socle A_n and $n > 12$ then $b(G) = 2$ for all non-standard actions; it quickly follows that $b(G) \leqslant 3$ for all n. Minimal base sizes for standard actions of alternating and symmetric groups are determined by James in [**29**], while precise results for primitive actions of sporadic groups will appear in the forthcoming paper [**11**]. Non-standard actions of finite classical groups are

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considered in [7] where it is shown that either $b(G) \leqslant 4$, or $G = U_6(2).2$, $G_\omega = U_4(3).2^2$ and $b(G) = 5$. Precise base size results for classical groups have been determined in specific cases; see [27, 30] for example.

In [13], referring to the constant c in the statement of the Cameron–Kantor conjecture, Cameron writes, '*Probably this constant is* 7*, and the extreme case is the Mathieu group* M_{24}' (see [13, p. 122]). In this paper we prove Cameron's conjecture for groups of exceptional Lie type. For such groups, this is the first paper to give explicit bounds on the minimal base size; a concise version of our main result is Theorem 1 below. We refer the reader to Theorems 3 and 4 for more comprehensive results.

Theorem 1. *Let G be a finite almost simple group of exceptional Lie type, and let Ω be a primitive faithful G-set. Then $b(G) \leqslant 6$.*

Now, the main theorem in [11] states that if G is an almost simple primitive group with sporadic socle then $b(G) \leqslant 7$, with equality if and only if $G = M_{24}$ acting on 24 points. Therefore, in view of the results discussed above for alternating and classical groups, we see that Theorem 1 completes the proof of Cameron's conjecture in full generality.

Corollary 1. *Let G be a finite almost simple group in a primitive faithful non-standard action. Then $b(G) \leqslant 7$, with equality if and only if G is the Mathieu group M_{24} in its natural action of degree* 24.

Remark 1. The bound in Theorem 1 is best possible. Indeed, with the aid of the computer package Magma (see [4]) we calculate that $b(G) = 6$ if $G = E_6(2)$ and G_ω is the maximal parabolic subgroup P_1 (or P_6). It would be interesting to know if there are only finitely many examples with $b(G) = 6$, although it is easy to see that there are infinitely many with $b(G) = 5$. For example, if $G = E_8(q)$, and G_ω is the maximal parabolic subgroup P_8 then $b(G) = 5$ for any q (see Theorem 4).

In [14], Cameron and Kantor formulate a stronger base size conjecture. More precisely, they assert that there is an absolute constant c' such that the probability that a random c'-element subset of Ω forms a base for G tends to 1 as the order of G tends to infinity. Here G is any finite almost simple group, and Ω is a faithful primitive non-standard G-set. Now, if the socle of G is an alternating group then an elementary argument of Cameron and Kantor [14] establishes the conjecture with a best possible constant $c' = 2$. The general case was finally settled by Liebeck and Shalev [48, Theorem 1.3], although their probabilistic proof does not yield an explicit value for c'.

From the proof of Theorem 1, it is easy to see that the conjecture holds with the constant $c' = 6$ for groups of exceptional Lie type. If G is a classical group with natural module of dimension greater than 15 then a theorem of Liebeck and Shalev [49, Theorem 1.11] establishes the conjecture with a best possible constant $c' = 3$. By considering the remaining classical groups of small rank we prove the following theorem.

Theorem 2. *Let G be a finite almost simple group, and let Ω be a primitive faithful non-standard G-set. Then the probability that a random* 6*-tuple in Ω is a base for G tends to* 1 *as* $|G| \rightarrow \infty$.

Our proof of Theorem 1 is probabilistic and uses bounds on fixed point ratios. This is very similar to the approach taken in [7] for classical groups, originating in [48]. Recall that if G acts on a set Ω then the *fixed point ratio of* x, which we denote by $\mathrm{fpr}(x)$, is the proportion of points in Ω which are fixed by x. It is easy to see that if G acts transitively on Ω then

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}, \tag{1.1}$$

where $H = G_\omega$ for some $\omega \in \Omega$. As observed in the proof of [48, Theorem 1.3], the connection between fixed point ratios and base sizes arises as follows. Let $Q(G,c)$ be the probability that a randomly chosen c-tuple of points in Ω is not a base for G, so G admits a base of size c if and only if $Q(G,c) < 1$. Of course, a c-tuple in Ω fails to be a base if and only if it is fixed by an element $x \in G$ of prime order, and we note that the probability that a random c-tuple is fixed by x is at most $\mathrm{fpr}(x)^c$. Let $\mathscr{P}$ be the set of elements of prime order in G, and let $x_1, \ldots, x_k$ be a set of representatives for the G-classes of elements in $\mathscr{P}$. Since G is transitive, fixed point ratios are constant on conjugacy classes (see (1.1)) and it follows that

$$Q(G,c) \leqslant \sum_{x \in \mathscr{P}} \mathrm{fpr}(x)^c = \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i)^c =: \widehat{Q}(G,c). \tag{1.2}$$

In particular, we can apply upper bounds on fixed point ratios to bound $\widehat{Q}(G,c)$ from above. Detailed information on fixed point ratios for primitive actions of finite exceptional groups of Lie type can be found in [38] and we make extensive use of the results and methods therein.

Let us now state a more detailed version of Theorem 1. We record our results for parabolic and non-parabolic actions in Theorems 3 and 4, respectively.

In the statement of Theorem 3, we write P_I for the standard parabolic subgroup of G which corresponds to deleting the nodes in $I \subseteq \{1, \ldots, r\}$ from the associated Dynkin diagram of G, where r is the (untwisted) Lie rank of G. We follow [5, p. 250] in labelling Dynkin diagrams. In addition, γ is an involutory graph automorphism of $E_6^\epsilon(q)$, while ψ denotes an involutory graph-field automorphism (if one exists) of $F_4(q)$ $(p=2)$ and $G_2(q)$ $(p=3)$, where $q = p^a$.

THEOREM 3. *Let G be a finite almost simple group of exceptional Lie type over $\mathbb{F}_q$ with socle G_0, where $q = p^a$ with p a prime. Let H be a maximal parabolic subgroup of G, and let Ω be the set of right cosets of H in G. Then $b(G) \leqslant c$, where c is defined as follows. Here an asterisk indicates that $b(G) = c$ for all values of q.*

(i) *If $G_0 = {}^3D_4(q)$, ${}^2F_4(q)'$, ${}^2G_2(q)$ or ${}^2B_2(q)$ then either $c = 3^*$, or $G_0 = {}^3D_4(q)$, $H = P_2$ and $c = 4^*$.*

(ii) *In all other cases, the values of c are as follows.*

	$H = P_1$	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$G_0 = E_8(q)$	4^*	3^*	3^*	3^*	3^*	3^*	4	5^*
$E_7(q)$	5	4^*	4	3^*	3^*	4^*	6	
$E_6(q)$	6	5	4	4	4	6		
$F_4(q)$	5	4	4	5				
$G_2(q)$	4	4						

	$P_{1,6}$	P_2	$P_{3,5}$	P_4
${}^2E_6(q)$	4^*	5	3^*	4
$E_6(q).\langle\gamma\rangle$	6	5	4	4
${}^2E_6(q).\langle\gamma\rangle$	4^*	5	3^*	4

	$P_{1,4}$	$P_{2,3}$
$F_4(q).\langle\psi\rangle$	5	3^*

	$P_{1,2}$
$G_2(q).\langle\psi\rangle$	4

Furthermore, the probability that a random c-tuple in Ω forms a base for G tends to 1 as $|G| \to \infty$.

REMARK 2. Several of the non-asterisked bounds on $b(G)$ in Theorem 3 are in fact sharp, provided that we exclude a few values of q. For example, Theorem 3 states that $b(G) \leqslant 5$ if $G_0 = E_7(q)$ and $H = P_1$. In this case, Proposition 2.4 implies that $b(G) = 5$ for all $q > 3$. Similarly, we deduce that $b(G) = 4$ if $G = E_6(q)$, $H = P_3$ (or $H = P_5$) and $q > 2$.

The next theorem is our main result on non-parabolic actions.

THEOREM 4. *Let G be a finite almost simple group of exceptional Lie type over $\mathbb{F}_q$ with socle G_0. Let H be a maximal non-parabolic subgroup of G, and let Ω be the set of right cosets of H in G. Then $b(G) \leqslant c$, where c is defined as in Table 1.*

TABLE 1. *Non-parabolic actions.*

G_0	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$	$G_2(q)'$	${}^2F_4(q)'$	${}^2G_2(q)$	${}^2B_2(q)$	${}^3D_4(q)$
c	5	6	6	6	5	3	3	2	5

Furthermore, the probability that a random c-tuple in Ω forms a base for G tends to 1 as $|G| \to \infty$.

It is worth noting that in some specific cases we obtain a better bound on $b(G)$ than that presented in the statement of Theorem 4 (see Lemmas 4.16, 4.20 and 4.27, for example).

For some small rank groups defined over small fields we can use MAGMA to determine $b(G)$.

PROPOSITION 1. *Let G be a finite almost simple group of exceptional Lie type over $\mathbb{F}_q$ with socle G_0, where*

$$G_0 \in \{ {}^2B_2(8),\ {}^2B_2(32),\ {}^2G_2(27),\ G_2(3), G_2(4), G_2(5),\ {}^3D_4(2),\ {}^2F_4(2)' \}.$$

Then for each faithful primitive action of G, the precise value of $b(G)$ is recorded in Tables 11 and 12 in Section 6.

This paper is organized as follows. In Section 2 we record various preliminary results which we will need in the proof of Theorem 1. In particular, we present some results from Lawther's forthcoming paper [**35**] on the fusion of unipotent classes in maximal subgroups of exceptional algebraic groups. In Section 3 we consider parabolic actions and we prove Theorem 3; the remaining non-parabolic actions are dealt with in Section 4. In Section 5 we give a short proof of Theorem 2, and in the final section we present some miscellaneous results which we refer to in the proof of Theorem 1. For example, we record some useful information on the conjugacy classes of semisimple elements of prime order in the groups $E_6(2)$, ${}^2E_6(2).3$ and $F_4(2)$. Here one can also find the precise base size results referred to in the statement of Proposition 1.

NOTATION. Our notation for groups of Lie type is standard (see [**33**], for example). We write T_i for an i-dimensional torus. In addition, (a, b) denotes the highest common factor of the integers a and b, while $\delta_{i,j}$ is the familiar Kronecker delta. If X is a subset of a group then we write $i_m(X)$ for the number of elements of order m in X. Also, if H and G are groups then $H.G$ denotes an extension of H by G, and we write $H : G$ if this extension is split.

2. *Preliminaries*

We begin with some additional notational remarks which apply for the remainder of the paper.

NOTATION. Let G_0 be a finite simple group of exceptional Lie type over $\mathbb{F}_q$, where $q = p^a$ for a prime p. Let $\bar{G}$ be a simple adjoint exceptional algebraic group over the algebraic closure $K = \bar{\mathbb{F}}_q$ which admits a Frobenius morphism σ such that $\bar{G}_\sigma := \{x \in \bar{G} : x^\sigma = x\}$ has socle G_0.

The following result is an easy consequence of the order formulae for exceptional groups.

PROPOSITION 2.1. *We have* $\frac{1}{2}q^{\dim \bar{G}} < |\bar{G}_\sigma| < q^{\dim \bar{G}}$.

The next result is a well-known theorem of Steinberg (see [**15**, Theorem 6.6.1], for example).

PROPOSITION 2.2. *The group* $\bar{G}_\sigma$ *contains precisely* $q^{\dim \bar{G} - r}$ *unipotent elements, where* r *is the rank of* $\bar{G}$.

In this paper we adopt the terminology of [**28**] for describing the various automorphisms of G_0 (see [**28**, Definition 2.5.13] in particular). Another familiar theorem of Steinberg [**69**, Theorem 30] states that $\mathrm{Aut}(G_0)$ is generated by inner, diagonal, field and graph automorphisms. We refer the reader to [**38**, Proposition 1.1] for a convenient list of the various possibilities for the centralizer $C_{G_0}(x)$ when x is a graph automorphism of prime order. Also, we note that $\bar{G}_\sigma$ is the subgroup of $\mathrm{Aut}(G_0)$ generated by inner and diagonal automorphisms of G_0.

The following elementary result plays an important role in the proof of Theorem 1.

PROPOSITION 2.3. *Let* G *be a transitive permutation group on a finite set* Ω *and write* $H = G_\omega$ *for some* $\omega \in \Omega$. *Suppose that* $x_1, \ldots, x_m$ *represent distinct* G*-classes such that* $\sum_i |x_i^G \cap H| \leqslant A$ *and* $|x_i^G| \geqslant B$ *for all* $1 \leqslant i \leqslant m$. *Then*

$$\sum_{i=1}^{m} |x_i^G| \cdot \mathrm{fpr}(x_i)^c \leqslant B(A/B)^c$$

for all $c \in \mathbb{N}$.

Proof. Write $a_i = |x_i^G \cap H|$ and $b_i = |x_i^G|$, so that $\sum a_i \leqslant A$ and $b_i \geqslant B$. Since $fpr(x_i) = a_i/b_i$, the left hand side of the required inequality is

$$\sum a_i^c/b_i^{c-1} \leqslant \sum a_i^c/B^{c-1} \leqslant (\sum a_i)^c/B^{c-1} \leqslant A^c/B^{c-1},$$

as required. □

By definition, if $B \subseteq \Omega$ is a base for G then the elements of G are uniquely determined by their action on B. This trivial observation yields the following useful lower bound for $b(G)$.

PROPOSITION 2.4. *If* G *is a permutation group on a finite set* Ω *then* $b(G) \geqslant \log_{|\Omega|} |G|$.

To conclude this short preliminary section we present some results from Lawther's forthcoming paper [**35**] on the fusion of unipotent classes in maximal non-parabolic subgroups of exceptional algebraic groups. To obtain these results, one first derives expressions for root elements of the given maximal subgroup $\bar{M}$ of $\bar{G}$ and then uses them to form representatives of the unipotent classes in $\bar{M}$. Then one determines their Jordan block structure, typically on the Lie algebra of $\bar{G}$, and finally concludes by inspecting the relevant tables in [36].

Notation. In Tables 2–6 we denote the class of a unipotent element x in a classical algebraic group $\bar{G}$ by the partition of $\dim V$ which encodes the Jordan form of x on the natural $\bar{G}$-module V. However, if $p = 2$ and $\bar{G}$ is a symplectic or orthogonal group then we adopt the standard Aschbacher–Seitz [1] notation for involution classes. It is well–known that if $p \neq 2$ and $\bar{G}$ is classical then either each unipotent class in $\bar{G}$ is uniquely determined by its corresponding Jordan form, or $\bar{G}$ is an even-dimensional orthogonal group and two distinct unipotent classes correspond to the same partition λ if and only if λ has no odd parts. In this latter case, we use

Table 2. $D_8 < E_8$.

$p > 2$		$p = 2$	
D_8-class	E_8-class	D_8-class	E_8-class
(15,1)	$E_8(a_4)$	c_8	$4A_1$
(13,3)	$E_8(a_5)$	c_6	$4A_1$
(11,5)	$E_8(a_6)$	a_8'	$4A_1$
$(11,2^2,1)$	$E_7(a_3)$	a_8''	$3A_1$
(9,7)	$E_8(b_6)$	a_6	$3A_1$
$(9,3,2^2)$	$E_7(a_4)$	c_4	$3A_1$
$(8^2)'$	$E_6(a_1)$	a_4	$2A_1$
(7,5,3,1)	$E_8(a_7)$	c_2	$2A_1$
$(7,4^2,1)$	$D_6(a_2)$	a_2	A_1
$(7,2^4,1)$	$D_5(a_1)$		
$(6^2,3,1)$	$E_6(a_3)+A_1$		
$(6^2,2^2)'$	$E_6(a_3)$		
$(5,3,2^4)$	A_3+A_2		
$(4^4)'$	A_4		
$(4^2,3,2^2,1)$	$D_4(a_1)+A_1$		
$(3,2^6,1)$	A_2+A_1		
$(2^8)'$	A_2		

Table 3. $A_1D_6 < E_7$.

$p > 2$		$p = 2$		
D_6-class of y	E_7-class of uy	D_6-class of y	E_7-class of y	E_7-class of uy
(11,1)	$E_7(a_3)$	c_6	$4A_1$	$4A_1$
(9,3)	$E_7(a_4)$	a_6'	$3A_1''$	$4A_1$
(7,5)	$E_7(a_5)$	a_6''	$3A_1'$	$3A_1'$
$(7,2^2,1)$	$D_5(a_1)$	c_4	$3A_1'$	$4A_1$
$(6^2)'$	$E_6(a_3)$	a_4	$2A_1$	$3A_1'$
$(5,3,2^2)$	A_3+A_2	c_2	$2A_1$	$3A_1''$
$(4^2,3,1)$	$D_4(a_1)+A_1$	a_2	A_1	$2A_1$
$(4^2,2^2)'$	$D_4(a_1)$	1	1	A_1
$(3,2^4,1)$	A_2+A_1			
$(2^6)'$	A_2			

TABLE 4. $A_7 < E_7$.

A_7-class	E_7-class
(8)	$E_6(a_3)$
$(6,2)$	$E_6(a_3)$
(4^2)	A_4
$(4,2^2)$	$D_4(a_1)$
(2^4)	$\begin{cases} A_2, & p>2 \\ (3A_1)', & p=2 \end{cases}$

TABLE 5. $C_4 < E_6,\ p > 2$.

C_4-class	E_6-class
(8)	$E_6(a_1)$
$(6,2)$	$E_6(a_3)$
$(6,1^2)$	A_5
(4^2)	A_4
$(4,2^2)$	$D_4(a_1)$
$(4,2,1^2)$	A_3+A_1
$(4,1^4)$	A_3
$(3^2,2)$	$2A_2+A_1$
$(3^2,1^2)$	$2A_2$
(2^4)	A_2
$(2^3,1^2)$	$3A_1$
$(2^2,1^4)$	$2A_1$
$(2,1^6)$	A_1

TABLE 6. $A_1C_3 < F_4,\ p > 2$.

C_3-class of y	F_4-class of uy
(6)	$F_4(a_2)$
$(4,2)$	$F_4(a_3)$
$(4,1^2)$	$C_3(a_1)$
(2^3)	A_2
$(2,1^4)$	$\widetilde{A}_1$

the notation λ and λ' to denote the two distinct $\bar{G}$-classes corresponding to λ. For example, in Table 2, a D_8-class labelled $(8^2)'$ corresponds (via the familiar Bala–Carter identification) to the pair $(L, P_{L'})$, where $L = A_7T_1$ is a Levi subgroup of D_8, $P_{L'}$ is a distinguished parabolic subgroup of $L' = A_7$ and L is not a Levi subgroup of E_8. This latter property distinguishes the D_8-class $(8^2)'$ from (8^2), and we adopt the same notation in Table 3. Convenient notation and tables of all unipotent classes in exceptional algebraic groups can be found in [**36**], and we use the notation therein. In addition, in Tables 3 and 6, u denotes a non-trivial unipotent element in A_1.

3. *Parabolic actions*

We continue with the notation of the previous section: G is an almost simple group with socle G_0, a simple group of exceptional Lie type over $\mathbb{F}_q$ with $q = p^a$ for a prime p; $\bar{G}$ is a simple exceptional algebraic group over the algebraic closure $\bar{\mathbb{F}}_q$ and σ is a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle G_0. In addition, H denotes a maximal parabolic subgroup of G and we write Ω for the set of right cosets of H in G. Observe that $H \cap \bar{G}_\sigma \leqslant \bar{P}_\sigma$, where $\bar{P}$ is a σ-stable parabolic subgroup of $\bar{G}$. In this section we prove Theorem 3.

3.1. *Fixed point ratios*

Here we explain how it is possible to calculate the exact value of $\widehat{Q}(G,c)$ for any $c \in \mathbb{N}$ (see (1.2)). The main reference here is [**38**, §§ 2,3].

(i) *Unipotent elements*. Let $x \in H \cap \bar{G}_\sigma$ be a unipotent element of order p and observe that $|C_\Omega(x)| = \chi(x)$, where $\chi = 1_{\bar{P}_\sigma}^{\bar{G}_\sigma}$ is the corresponding permutation character and

$$C_\Omega(x) = \{\omega \in \Omega :\ \omega x = \omega\}$$

is the fixed point set of x on Ω. Assume for now that $\bar{G}_\sigma$ is untwisted.

Let W denote the Weyl group of $\bar{G}$ and let $W_{\bar{P}}$ be the Weyl group of $\bar{P}$, so $W_{\bar{P}}$ is a standard parabolic subgroup of W. Write $\hat{W}$ for the set of (ordinary) irreducible characters of W. Then [**38**, Lemma 2.4] gives

$$\chi(x) = \sum_{\phi \in \hat{W}} n_\phi R_\phi(x), \tag{3.1}$$

where

$$n_\phi = \langle 1_{W_{\bar{P}}}^W, \phi\rangle = \langle 1_{W_{\bar{P}}}, \phi|_{W_{\bar{P}}}\rangle_{W_{\bar{P}}} = \frac{1}{|W_{\bar{P}}|}\sum_{w \in W_{\bar{P}}} \phi(w), \tag{3.2}$$

and the $R_\phi(x)$ are the so-called Foulkes functions of $\bar{G}_\sigma$. The integers n_ϕ are listed in [**38**, pp. 413–415] when $\bar{P}$ is a maximal parabolic subgroup of $\bar{G}$. The values of the n_ϕ in the remaining cases of interest are easily derived via (3.2). For example, if $\bar{G} = E_6$ and $\bar{P} = P_{1,6}$ then

$$\chi = R_{\phi_{1,0}} + 2R_{\phi_{6,1}} + 3R_{\phi_{20,2}} + R_{\phi_{15,5}} + R_{\phi_{30,3}} + 2R_{\phi_{64,4}} + R_{\phi_{24,6}}$$

with respect to the labelling in [**15**] of the irreducible characters of W. Therefore, it remains to determine the Foulkes functions of $\bar{G}_\sigma$. In fact, since each Foulkes function is a known linear combination of Green functions, it suffices to determine the Green functions of $\bar{G}_\sigma$.

In [**51**], Lusztig presents an algorithm to compute certain class functions associated to intersection cohomology complexes on the unipotent variety of $\bar{G}$. In later work, he proved that these functions are the desired Green functions of $\bar{G}_\sigma$ if p and q are sufficiently large (see [**52**, Theorem 1.14]), and this result was extended by Shoji to all values of p and q. Indeed, [**65**, Theorem 2.2] deals with the case where p is 'almost good' for $\bar{G}$, while the remaining cases are covered by [**65**, Theorem 7.4] and [**66**, Theorem 5.5].

The Green functions computed via Lusztig's algorithm are given as linear combinations of other functions, called characteristic functions of irreducible local systems on geometric unipotent classes. However, the values of these latter functions are in general known only up to a complex scalar of absolute value 1; the problem of determining these unknown scalars in full generality remains open.

If $\bar{G} = G_2$ then the scalar problem is easy to solve because the full character table of $G_2(q)$ is available in all characteristics; see [**17**, **22**, **23**]. Next suppose that $\bar{G} = F_4$, p is good for $\bar{G}$ and $x \in \bar{G}_\sigma$ is unipotent. In [**63**], Shoji specifies a unique so-called 'split' $\bar{G}_\sigma$-class in $(x^{\bar{G}})_\sigma$. This split class allows one to 'normalize' the aforementioned characteristic functions such that the relevant scalars appearing in the decomposition of the corresponding Green functions are all equal to 1 (for any value of q). These methods were extended to $\bar{G} = E_6$, E_7 and E_8 by Beynon and Spaltenstein [**2**], again under the hypothesis that p is good for $\bar{G}$. For more details on these calculations, we refer the reader to Shoji's survey article [**64**] on the computation of Green functions.

It follows that if p is good for $\bar{G}$ then it is possible to determine the aforementioned scalars and thus compute the precise Green (or Foulkes) functions of $\bar{G}_\sigma$. Indeed, using Lusztig's algorithm, Lübeck [**50**] has explicitly computed the Foulkes functions of $\bar{G}_\sigma$ when p is good for $\bar{G}$ (any p if $\bar{G} = G_2$). His results are presented in 2-dimensional arrays; rows indexed by the unipotent classes in $\bar{G}_\sigma$ and columns by the irreducible characters of W. The entries are polynomials in q with integer coefficients. In this way, using [**50**], we can compute the precise unipotent contribution to $\widehat{Q}(G,c)$ when p is good or $\bar{G} = G_2$.

Now assume that p is a bad prime for $\bar{G}$. In view of [**54**] and [**59**], the problem of scalars is solved if $(\bar{G},p) = (E_6,2)$, $(E_6,3)$ or $(F_4,2)$. (The methods employed in the unpublished diploma thesis of Porsch [**59**] are very similar to those in [**54**].) Here Lübeck [**50**] has computed the explicit Foulkes functions, and so the unipotent contribution to $\widehat{Q}(G,c)$ can be computed precisely in each of these cases.

Next set $A(x) = C_{\bar{G}}(x)/C_{\bar{G}}(x)^0$. If $|A(x)| = 1$ then $(x^G)_\sigma = x^{\bar{G}_\sigma}$ and x is split in the sense of Shoji [**64**] and Beynon–Spaltenstein [**2**]. As before, it is possible to normalize the characteristic functions such that the scalars involved are all equal to 1 (see [**2**, § 3] for a general discussion of split elements).

Now assume that $|A(x)| = 2$. Here $(x^{\bar{G}})_\sigma$ is a union of two $\bar{G}_\sigma$-classes, with representatives x and y say, precisely one of which is split. The relevant characteristic functions corresponding to the $\bar{G}_\sigma$-class of x are parametrized by the irreducible characters of the component group $A(x)$; the corresponding scalar for the trivial character is 1, and it is either 1 or -1 for the non-trivial character, depending on whether or not $x^{\bar{G}_\sigma}$ is split. If $|x^{\bar{G}_\sigma}| \neq |y^{\bar{G}_\sigma}|$ then we can determine if x is split, and thus the problem of scalars is solved in this case. Indeed, the class length of the split class in $(x^G)_\sigma$ can be computed as a by-product of Lusztig's algorithm, and so we can immediately determine if the given element x is split or not. On the other hand, if $|x^{\bar{G}_\sigma}| = |y^{\bar{G}_\sigma}|$ then for the purpose of computing $\widehat{Q}(G,c)$ we may as well assume that $x^{\bar{G}_\sigma}$ is the split class since the contribution to $\widehat{Q}(G,c)$ from the $\bar{G}_\sigma$-classes of x and y is the same if $x^{\bar{G}_\sigma}$ is split or not.

In this way, Lübeck [**50**] gives the explicit Foulkes functions $R_\phi(x)$ for all unipotent elements $x \in \bar{G}_\sigma$ with $|A(x)| \leqslant 2$, unless $|A(x)| = 2$ and $(x^{\bar{G}})_\sigma = x^{\bar{G}_\sigma} \cup y^{\bar{G}_\sigma}$, with $|x^{\bar{G}_\sigma}| = |y^{\bar{G}_\sigma}|$. In the latter situation, Lübeck has computed polynomials $f_\phi(q), g_\phi(q) \in \mathbb{Z}[q]$ such that $\{R_\phi(x), R_\phi(y)\} = \{f_\phi(q), g_\phi(q)\}$ for all $\phi \in \hat{W}$, where $R_\phi(x) = f_\phi(q)$ if and only if $x^{\bar{G}_\sigma}$ is split. As previously remarked, for the purpose of computing $\widehat{Q}(G,c)$, there is no harm in assuming that $x^{\bar{G}_\sigma}$ is split. It follows that we can calculate the precise contribution to $\widehat{Q}(G,c)$ from the set of unipotent elements $x \in G$ with $|A(x)| \leqslant 2$.

Finally, suppose that $\bar{G} = E_8$ or $G = E_7$, with p bad for $\bar{G}$, or $(\bar{G},p) = (F_4,3)$. Now, if $x \in \bar{G}_\sigma$ has order p and $|A(x)| > 2$ then we claim that $G = E_8$, $p = 5$ and x belongs to one of the $\bar{G}$-classes labelled $D_4(a_1)$ or $D_4(a_1) + A_1$. To see this, we first inspect the relevant tables in [**36**] to determine the unipotent $\bar{G}$-classes containing elements of order p. Here we use the fact that if $x \in \bar{G}$ has order p then there can be no Jordan blocks of size greater than p in the Jordan form of x on any $\bar{G}$-module. Finally we read off the $|A(x)|$ values from [**56**] (for $\bar{G} = E_8$ and E_7) and [**62**] (for $(\bar{G},p) = (F_4,3)$), and the claim follows.

Suppose that $G = E_8$, $p = 5$ and x is in $D_4(a_1)$ or $D_4(a_1) + A_1$. Here $A(x) \cong S_3$ and $(x^{\bar{G}})_\sigma$ is a union of precisely three distinct $\bar{G}_\sigma$-classes. In these cases one can check that the argument of Beynon–Spaltenstein, labelled Case III in [**2**, § 3], still applies when $p = 5$ (the only unipotent class in E_8 which behaves differently when $p = 5$, compared with $p > 5$, is the regular class). In particular, it is possible to determine the precise scalars involved and the corresponding explicit Foulkes functions are given in [**50**].

We conclude that it is possible to compute the precise unipotent contribution to $\widehat{Q}(G,c)$ whenever $\bar{G}_\sigma$ is untwisted.

Now assume that $\bar{G}_\sigma$ is twisted. For $\bar{G}_\sigma = {}^2E_6(q)$ we proceed as before: the precise values of the functions R_ϕ at unipotent elements of order p have been computed by Lübeck [**50**], while the numbers n_ϕ in (3.1) can be determined from the formula on [**38**, p. 416]. For the reader's convenience, we record the relevant decompositions of χ.

$P_{1,6}$	$R_{\phi_{1,0}} + R_{\phi_{15,5}} + R_{\phi_{20,2}} + R_{\phi_{24,6}} + R_{\phi_{30,3}}$
P_2	$R_{\phi_{1,0}} + R_{\phi_{6,1}} - R_{\phi_{15,4}} + R_{\phi_{20,2}} + R_{\phi_{30,3}}$
$P_{3,5}$	$R_{\phi_{1,0}} + R_{\phi_{10,9}} + R_{\phi_{15,5}} - R_{\phi_{15,4}} + R_{\phi_{20,2}} + 2R_{\phi_{24,6}} + 2R_{\phi_{30,3}} - R_{\phi_{60,8}} + R_{\phi_{80,7}} + R_{\phi_{60,11}} + R_{\phi_{81,10}}$
P_4	$R_{\phi_{1,0}} + R_{\phi_{10,9}} + R_{\phi_{6,1}} - 2R_{\phi_{15,4}} + R_{\phi_{20,2}} + R_{\phi_{24,6}} + 2R_{\phi_{30,3}} - R_{\phi_{60,8}} + R_{\phi_{80,7}} + R_{\phi_{60,5}} + R_{\phi_{81,6}}$

The remaining twisted groups are easy to deal with because the irreducible unipotent characters have been determined. We refer the reader to [**38**, p. 416] for further details and relevant references.

We conclude that the contribution to $\widehat{Q}(G,c)$ from unipotent elements can be calculated precisely, as claimed. Lübeck's tables of Foulkes functions [**50**] are currently unpublished and we thank him for making them available to us in GAP-readable form.

(ii) *Semisimple elements*. Next let $x \in H \cap \bar{G}_\sigma$ be a semisimple element of prime order and note that $|C_\Omega(x)| = \chi(x)$ as in (i). First assume that $\bar{G}_\sigma$ is untwisted. Let Φ be the root system of $\bar{G}$ with respect to a fixed maximal torus, let Π be a simple system of roots for $\bar{G}$ and write α_0 for the highest root of Φ with respect to Π. Then the possible centralizer types of semisimple elements in $\bar{G}_\sigma$ are parametrized by pairs $(J,[w])$, where J is a proper subset of $\Pi \cup \{\alpha_0\}$ (determined up to W-conjugacy), W_J is the subgroup of W generated by reflections in the roots in J, and $[w] = W_J w$ is a conjugacy class representative of $N_W(W_J)/W_J$.

An explicit formula for $\chi(x)$ is given in [**38**, Corollary 3.2]. With the aid of a computer, Lawther has used this formula to calculate $\chi(x)$ for all semisimple elements $x \in \bar{G}_\sigma$. The results are presented in tables [**37**]; rows are indexed by the pairs $(J,[w])$ and columns by the maximal parabolic subgroups. The entries in each table are polynomials in q with non-negative integer coefficients. Further, the polynomials are independent of the characteristic p. We are grateful to Lawther for making his unpublished tables available to us.

If $\bar{G}_\sigma = {}^2E_6(q)$ then Lawther's calculations apply, while the remaining cases are very easy because the irreducible unipotent characters of $\bar{G}_\sigma$ are known (see [**38**, p. 423] for further details).

(iii) *Field and graph-field automorphisms*. Let $x \in G$ be a field or graph-field automorphism of prime order r and write $\bar{G}_\sigma = G(q)$, $\bar{P}_\sigma = P(q)$ and $C_{\bar{G}_\sigma}(x) = G^\epsilon(q^{1/r})$. Then according to the proof of [**38**, Lemma 6.1] we have $x^{\bar{G}_\sigma} \cap \bar{P}_\sigma x = x^{\bar{P}_\sigma}$ and $C_{\bar{P}_\sigma}(x) = P^\epsilon(q^{1/r})$ is the corresponding parabolic subgroup of the group $C_{\bar{G}_\sigma}(x)$. In particular, we deduce that

$$\mathrm{fpr}(x) = \frac{|G^\epsilon(q^{1/r}) : P^\epsilon(q^{1/r})|}{|G(q) : P(q)|}.$$

(iv) *Graph automorphisms*. First assume that $\bar{G}_\sigma = E_6^\epsilon(q)$ and $x \in G$ is an involutory graph automorphism. If $p \neq 2$ then the precise value of $\mathrm{fpr}(x)$ can be determined from the proof of [**38**, Lemma 6.4]. Now assume that $p = 2$, so by [**1**, § 19] we have $C_{\bar{G}}(x) = F_4$ or $C_{\bar{G}}(x) = C_{F_4}(t)$, where $t \in F_4$ is a long root element. Now, if $C_{\bar{G}}(x) = F_4$ then $\mathrm{fpr}(x) \leqslant k_{\bar{P}}(q)^{-1}$, where the values of $k_{\bar{P}}(q)$ are given in [**38**, Proposition 2.6] and recorded in Table 7. As described in [**38**, p. 418], it is possible to compute $|C_\Omega(x)|$ precisely when $C_{\bar{G}}(x) = C_{F_4}(t)$. Here we thank Lawther for performing the necessary calculations which yield the relevant bounds listed in Table 7.

TABLE 7. *The values of $k_{\bar{P}}(q)$.*

$\bar{G}_\sigma$	$\bar{P}$	$C_{\bar{G}}(x) = F_4$	$C_{\bar{G}}(x) = C_{F_4}(t)$
$E_6(q)$	$P_{1,6}$	q^9	q^{13}
	P_2	q^9	q^{13}
	$P_{3,5}$	$\frac{1}{3}q^{11}$	$q^{15}(q-1)^2$
	P_4	q^9	$q^{15}(q-1)$
${}^2E_6(q)$	$P_{1,6}$	$q^8(q-1)$	$q^{12}(q-1)$
	P_2	q^6-q^3+1	$q^{10}(q-1)$
	$P_{3,5}$	$q^{10}(q-1)$	$q^{15}(q-1)^2$
	P_4	$q^6(q^2-1)(q-1)$	$q^{14}(q-1)^2$

Finally if $\bar{G}_\sigma = {}^3D_4(q)$ and x is a triality graph automorphism then precise fixed point ratios can be found in the proof of [**38**, Lemma 6.3]. We note that if $H = (\bar{P}_{1,3,4})_\sigma$ and $C_{G_0}(x) = G_2(q)$ then the proof of [**38**, Lemma 6.3] indicates that $\mathrm{fpr}(x)$ is independent of p, and hence

$$\mathrm{fpr}(x) = \frac{q^2+q+1}{q^8+q^4+1}$$

for all values of q.

3.2. *Proof of Theorem* 3

Recall that in order to establish the bound $b(G) \leqslant c$ it suffices to show that $\widehat{Q}(G,c) < 1$ (see (1.2)). As explained in Subsection 3.1, we can compute the exact value of $\widehat{Q}(G,c)$ for any $c \in \mathbb{N}$, so it is possible to determine the smallest integer c such that $\widehat{Q}(G,c) < 1$. In this way, with the exception of the case $G = E_6(2)$ with $H = P_1$ (or $H = P_6$), we obtain the upper bounds on $b(G)$ stated in Theorem 3. In the exceptional case we find that $\widehat{Q}(G,6) > 1$, and we use the computer package MAGMA to establish the bound $b(G) \leqslant 6$. We thank A. Hulpke for constructing the relevant permutation representation of degree 139503 which facilitates this calculation. (In fact, it is easy to check that $b(G) = 6$ in this example; see Remark 1.)

In practice, it is very laborious to calculate $\widehat{Q}(G,c)$ precisely; in general, we aim to derive an upper bound of the form $\widehat{Q}(G,c) < F(q)$ with the property that $F(q) < 1$ for all possible values of q. We illustrate our approach with a couple of specific examples. This is essentially careful book-keeping; the other cases are very similar and we omit the details.

PROPOSITION 3.1. *If $G_0 = E_8(q)$ and H is of type P_1 then $b(G) = 4$. Furthermore, the probability that a random 4-tuple in Ω forms a base for G tends to 1 as $|G| \to \infty$.*

Proof. First observe that $|\Omega| = f(q)$, where

$$f(q) = (q^{15}+1)(q^{12}+q^6+1)(q^{12}+1)(q^{10}+q^5+1)(q^{10}+1)(q^8+q^4+1)(q^7+1) \times (q^4+q^3+q^2+q+1),$$

so $|\Omega| > q^{78}$ and Proposition 2.4 yields $b(G) \geqslant 4$. To establish equality, it suffices to show that $\widehat{Q}(G,4) < 1$. We do this by estimating the contribution to $\widehat{Q}(G,4)$ from the various elements of prime order.

Let $x \in H$ be a unipotent element of order p. As described in Subsection 3.1, the Foulkes functions of $\bar{G}_\sigma$ are labelled by the irreducible characters of the corresponding Weyl group W, and [38, p. 414] gives

$$|C_\Omega(x)| = R_{\phi_{1,0}}(x) + R_{\phi_{8,1}}(x) + R_{\phi_{35,2}}(x) + R_{\phi_{560,5}}(x) + R_{\phi_{112,3}}(x) + R_{\phi_{84,4}}(x) + R_{\phi_{210,4}}(x) + R_{\phi_{50,8}}(x) + R_{\phi_{700,6}}(x) + R_{\phi_{400,7}}(x)$$

(see (3.1)). The polynomials $R_{\phi_{i,j}}(x)$ can be read off from [50] and $\mathrm{fpr}(x)$ quickly follows. In this way, we calculate that $\mathrm{fpr}(x) < q^{-61} = b_1$ if $\dim x^{\bar{G}} \geqslant 198$, while Proposition 2.2 implies that there are fewer than $q^{240} = a_1$ such elements. Similarly, if $166 \leqslant \dim x^{\bar{G}} \leqslant 196$ then $\mathrm{fpr}(x) < q^{-51} = b_2$ and by inspecting [56] we find that there are no more than $q^{198} = a_2$ of these elements in G. Now, if $146 \leqslant \dim x^{\bar{G}} \leqslant 164$ then $\mathrm{fpr}(x) < q^{-43} = b_3$ and there are fewer than $q^{166} = a_3$ such elements; similarly, the contribution to $\widehat{Q}(G, 4)$ from unipotent elements x with $128 \leqslant \dim x^{\bar{G}} \leqslant 144$ is less than $a_4 b_4^4$, where $a_4 = q^{137}$ and $b_4 = q^{-35}$. It remains to consider the $\bar{G}$-classes labelled A_2, $3A_1$, $2A_1$ and A_1. Using precise values for $|x^G|$ and $\mathrm{fpr}(x)$ it quickly follows that the combined contribution from these unipotent elements is less than $q^{-8} = c_1$.

Next let $x \in H$ be a semisimple element of prime order. Here we use Lawther's calculations [37], together with the information on semisimple conjugacy classes recorded in [25]. If $\dim x^{\bar{G}} \geqslant 216$ then [37] implies that $\mathrm{fpr}(x) < q^{-66} = b_5$ and, of course, there are fewer than $q^{248} = a_5$ such elements in G. Now, if $190 \leqslant \dim x^{\bar{G}} \leqslant 214$ then $\mathrm{fpr}(x) < q^{-59} = b_6$, and using [25] we calculate that there are no more than $q^{219} = a_6$ of these elements. Similarly, if $158 \leqslant \dim x^{\bar{G}} \leqslant 188$ then $\mathrm{fpr}(x) < q^{-50} = b_7$ and there are fewer than $q^{190} = a_7$ such elements. If $\dim x^{\bar{G}} < 158$ then $C_{\bar{G}}(x) = D_7T_1$, D_8, E_7T_1 or E_7A_1, and careful calculation reveals that the combined contribution here to $\widehat{Q}(G, 4)$ is less than $q^{-6} = c_2$.

Finally, suppose that $x \in H$ is a field automorphism of prime order r. Then $q = q_0^r$ and the proof of [38, Lemma 6.1] gives $\mathrm{fpr}(x) = f(q_0)/f(q) = h(r, q)$, where $|\Omega| = f(q)$ as above. Now

$$|x^G| < 2q^{248(1-r^{-1})} = g(r, q)$$

and if we set $j(r, q) = g(r, q)h(r, q)^4$ then the contribution to $\widehat{Q}(G, 4)$ from field automorphisms is less than

$$\sum_{r\in\pi} (r-1).j(r, q) < j(2, q) + 2j(3, q) + 4j(5, q) + \log_2 q.q^{248}h(7, q)^4 < q^{-10} = c_3,$$

where π is the set of distinct prime divisors of $\log_p q$. We conclude that $b(G) \leqslant 4$ since

$$\widehat{Q}(G, 4) < \sum_{i=1}^{7} a_i b_i^4 + \sum_{i=1}^{3} c_i = F(q) < q^{-1}$$

for all $q \geqslant 2$. The probabilistic statement follows at once because $F(q) \to 0$ as $q \to \infty$. □

Proposition 3.2. *If $G_0 = {}^2E_6(q)$ and H is of type P_2 then $b(G) \in \{4, 5\}$ and the probability that a random 5-tuple in Ω forms a base for G tends to 1 as $|G| \to \infty$.*

Proof. First observe that $|\Omega| = f(q)$, where

$$f(q) = (q^9 + 1)(q^6 + 1)(q^4 + 1)(q^2 + q + 1).$$

In view of Proposition 2.4, it suffices to show that $\widehat{Q}(G, 5) < 1$, with $\widehat{Q}(G, 5) \to 0$ as $q \to \infty$. We proceed as in the proof of the previous proposition. First let $x \in H$ be a unipotent element of order p. As remarked in Subsection 3.1, we have

$$|C_\Omega(x)| = R_{\phi_{1,0}}(x) + R_{\phi_{6,1}}(x) - R_{\phi_{15,4}}(x) + R_{\phi_{20,2}}(x) + R_{\phi_{30,3}}(x),$$

and thus $\mathrm{fpr}(x)$ can be calculated via [50]. If $\dim x^{\bar{G}} \geqslant 58$ then we find that $\mathrm{fpr}(x) < q^{-15} = b_1$, while there are fewer than $q^{72} = a_1$ such elements in G (see Proposition 2.2). Similarly, if $50 \leqslant$

$\dim x^{\bar{G}} \leqslant 56$ then $\mathrm{fpr}(x) < q^{-13} = b_2$ and G contains no more than $q^{56} = a_2$ of these elements (see [55]). The contribution to $\widehat{Q}(G,5)$ from unipotent elements $x \in H$ with $46 \leqslant \dim x^{\bar{G}} \leqslant 48$ is less than $a_3 b_3^5$, where $a_3 = 2q^{48}$ and $b_3 = q^{-11}$. Now, if $\dim x^{\bar{G}} < 46$ then x lies in one of the $\bar{G}$-classes labelled A_2, $3A_1$, $2A_1$ or A_1. Here a precise calculation reveals that the contribution from these elements is less than $c_1 = q^{-4}$. Arguing as in the proof of the previous proposition, using [37] and [24], the reader can check that the total contribution to $\widehat{Q}(G,5)$ from semisimple elements is less than $c_2 = 3/2q$.

Next suppose that $x \in H$ is a field automorphism of prime order r. Then r is odd, $q = q_0^r$, $|x^G| < 2q^{78(1-r^{-1})} = g(r,q)$ and $\mathrm{fpr}(x) = f(q^{1/r})/f(q) = h(r,q)$, where $|\Omega| = f(q)$ as before. If we set $j(r,q) = g(r,q)h(r,q)^5$ then the contribution to $\widehat{Q}(G,5)$ from field automorphisms is less than

$$\sum_{r\in\pi}(r-1).j(r,q) < 2j(3,q) + 4j(5,q) + 6j(7,q) + \log_2 q.q^{78}h(11,q)^5 < q^{-12} = c_3,$$

where π is the set of distinct odd primes which divide $\log_p q$. Finally, let $x \in H$ be an involutory graph automorphism. If $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2q^{26} = a_4$ and [38, Theorem 2] states that $\mathrm{fpr}(x) \leqslant (q^6 - q^3 + 1)^{-1} = b_4$. Similarly, if $C_{\bar{G}}(x) \neq F_4$ then $|x^G| < 2q^{42} = a_5$ and $\mathrm{fpr}(x) \leqslant q^{-10}(q-1)^{-1} = b_5$ (see Table 7 and the proof of [38, Lemma 6.4]). If $q \geqslant 3$ then we conclude that $b(G) \leqslant 5$ since

$$\widehat{Q}(G,5) < \sum_{i=1}^{5} a_i b_i^5 + \sum_{i=1}^{3} c_i < 2q^{-1}$$

for all $q \geqslant 3$. By direct calculation, it is easy to check that $\widehat{Q}(G,5) < 1$ when $q = 2$. □

4. *Non-parabolic actions*

In this section we prove Theorem 4 and this completes the proof of Theorem 1. We partition the proof into a number of subsections, according to the various possibilities for G_0. In each case, we first deal with the primitive actions of 'large' degree. More precisely, we establish Theorem 4 for actions with $|G_\omega| \leqslant q^{f(G_0)}$ for some fixed integer $f(G_0)$. For example, we set $f(E_8(q)) = 88$ and $f(E_7(q)) = 46$. By applying known facts about maximal subgroups, it is easy to determine a short list of possibilities for G_ω with $|G_\omega| > q^{f(G_0)}$; the non-parabolic subgroups which arise here are mainly subgroups of maximal rank, or subfield subgroups corresponding to a subfield of index two. We then consider each of these cases in turn.

We continue with our earlier notation. In particular, H is a maximal non-parabolic subgroup of G, and $b(G)$ denotes the smallest size of a base for G with respect to the natural action of G on the set Ω of right cosets of H in G.

Remark 4.1. In general, we show that $b(G) \leqslant c$ by defining a function F such that $\widehat{Q}(G,c) < F(q)$ for all sufficiently large q. In each case it is easy to check that $F(q) \to 0$ as $q \to \infty$ and this justifies the probabilistic statement in Theorem 4. We leave the reader to verify these asymptotic results.

4.1. $G_0 = E_8(q)$

Lemma 4.2. *If $|H| > q^{88}$ then H is of type $E_8(q^{1/2})$, $A_1(q)E_7(q)$ or $D_8(q)$.*

Proof. According to [41, Theorem 2], the possibilities for H are as follows:

(i) $H = N_G(\bar{M}_\sigma)$, where $\bar{M}$ is a σ-stable closed subgroup of $\bar{G}$ of positive dimension;

(ii) H is an exotic local subgroup (see [18, Table 1]);
(iii) $F^*(H) = A_5 \times A_6$;
(iv) H is of the same type as G over a subfield of $\mathbb{F}_q$ of prime index;
(v) H is almost simple, and not of type (i) or (iv).

Suppose that $|H| > q^{88}$. The subgroups of type (i) are determined in [40, 45], and the hypothesis on $|H|$ implies that H must be of type $A_1(q)E_7(q)$ or $D_8(q)$. Evidently, $E_8(q^{1/2})$ is the only possible subfield subgroup, H is not an exotic local subgroup by [18, Theorem 1(II)] and is clearly not of type (iii). Finally, suppose that H is almost simple, with socle H_0. If H_0 lies in Lie(p), where Lie(p) is the set of simple groups of Lie type in characteristic p, then the untwisted Lie rank of H_0 is at most 4 (see [46, Theorem 1.1]) and [47, Theorem 1.2] states that the subgroups which arise here have order less than $q^{56}.12\log_p q$. The possibilities with $H_0 \notin \mathrm{Lie}(p)$ are listed in [44, Tables 10.1–10.4] and by inspection it is easy to see that there are no examples with $|H| > q^{88}$. □

LEMMA 4.3. *If $|H| \leqslant q^{88}$ then $b(G) \leqslant 5$.*

Proof. It suffices to show that there exists a function $F(q)$ such that $\widehat{Q}(G,5) \leqslant F(q) < 1$ (see (1.2)). If $x \in G_0$ and $\dim x^{\bar{G}} \geqslant 112$ then $|x^G| > \frac{1}{2}q^{112} = b$ (see [25, 56]), and it is clear that this bound also holds if x is a field automorphism. Conversely, if $\dim x^{\bar{G}} < 112$ then x is unipotent and belongs to the $\bar{G}$-class A_1 or $2A_1$. There are fewer than $2q^{92} = c$ such elements in G and by [38, Theorem 2] we have $\mathrm{fpr}(x) \leqslant 2q^{-24} = d$. Applying Proposition 2.3 we conclude that

$$\widehat{Q}(G,5) < b(a/b)^5 + cd^5 = F(q),$$

where $a = q^{88}$. It is straightforward to check that $F(q) < 1$ for all $q \geqslant 2$. □

LEMMA 4.4. *If H is of type $A_1(q)E_7(q)$ then $b(G) \leqslant 5$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = A_1E_7$ is a σ-stable subgroup of $\bar{G}$. As before, it suffices to show that $\widehat{Q}(G,5) < 1$. Let $x \in H$ be a semisimple element of prime order. Then [38, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{|W(E_8):W(A_1E_7)|.2(q+1)^z}{q^{\delta(x)+z-8}(q-1)^8} = \frac{240(q+1)^z}{q^{\delta(x)+z-8}(q-1)^8}, \tag{4.1}$$

where $W(X)$ is the Weyl group of the reductive algebraic group X, $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$ and $z = \dim Z(\bar{D})$ for $\bar{D} = C_{\bar{G}}(x)$. If $\bar{D}$ has no E_7 or D_8 factor then [39, Theorem 2] gives $\delta(x) \geqslant 70$ and thus (4.1) implies that $\mathrm{fpr}(x) < q^{-51} = b_1$ if $z \leqslant 5$; the same bound holds if $z \geqslant 6$ since $|\Phi^+(\bar{D})| \leqslant 3$ and thus

$$\delta(x) = 2\left(|\Phi^+(\bar{G})| - |\Phi^+(\bar{M})| - |\Phi^+(\bar{D})| + |\Phi^+(\bar{D}\cap\bar{M})|\right) \geqslant 2(120-64-3) = 106,$$

where $|\Phi^+(X)|$ is the number of positive roots in the root system $\Phi(X)$ of X (see [39, § 5]). Of course, there are fewer than $q^{248} = a_1$ semisimple elements in G. If $\bar{D}$ does have an E_7 or D_8 factor then [38, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-37} = b_2$ and we calculate that there are less than $q^{130} = a_2$ such elements.

Next let $x \in H$ be a unipotent element of order p. According to [42, 2.1] we have

$$\mathscr{L}(E_8) \downarrow A_1E_7 = \mathscr{L}(A_1E_7) \oplus (V(\lambda_1) \otimes V(\lambda_7)), \tag{4.2}$$

where $\mathscr{L}(X)$ denotes the Lie algebra of the reductive algebraic group X, $V(\lambda_1)$ is the natural A_1-module and $V(\lambda_7)$ is the 56-dimensional irreducible E_7-module with highest weight λ_7

(we label weights as in Bourbaki [**5**]). Therefore we can determine the Jordan form of x on $\mathscr{L}(E_8)$ via [**36**, Tables 7, 8], and then identify the $\bar{G}$-class of x by inspecting [**36**, Table 9]. For example, suppose that $x = u_0u_1 \in A_1E_7$, where $u_0 \neq 1$ and u_1 has E_7-label $D_4(a_1) + A_1$. For convenience, let us assume that $p \geqslant 7$. Now, according to [**36**, Tables 7, 8], the Jordan form of u_1 on $\mathscr{L}(E_7)$ and $V(\lambda_7)$ is $[J_7^2, J_6^4, J_5^5, J_4^8, J_3^8, J_2^4, J_1^6]$, and $[J_6, J_5^4, J_4^2, J_3^4, J_2^5]$, respectively, where J_i denotes a standard Jordan block of size i. From (4.2) we deduce that the Jordan form of x on $\mathscr{L}(E_8)$ is

$$[J_7^2, J_6^4, J_5^5, J_4^8, J_3^8, J_2^4, J_1^6] \oplus [J_3] \oplus \big([J_2] \otimes [J_6, J_5^4, J_4^2, J_3^4, J_2^5]\big) = [J_7^3, J_6^8, J_5^8, J_4^{16}, J_3^{16}, J_2^8, J_1^{11}],$$

and inspecting [**36**, Table 9] we conclude that x lies in the $\bar{G}$-class labelled $A_3 + A_2$. Now, following the proof of [**38**, Lemma 4.5] we deduce that

$$\mathrm{fpr}(x) < \frac{\alpha.2(q+1).\beta}{q^{\delta(x)-7}(q-1)^8}, \tag{4.3}$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$, α is the number of distinct $\bar{M}$-classes in $x^{\bar{G}} \cap \bar{M}$ and $\beta = |C : C^0|$, where $C = C_{\bar{G}}(x)$ (note that $\dim Z(C^0/R_u(C^0)) \leqslant 1$, see [**56**], for example). If $\dim x^{\bar{G}} \geqslant 146$ then the prime order hypothesis implies that p is odd and we calculate that $\alpha \leqslant 3$ and $\delta(x) \geqslant 64$. Therefore (4.3) yields $\mathrm{fpr}(x) < q^{-54} = b_3$ since $\beta \leqslant 120$, and we note that there are fewer than $q^{240} = a_3$ of these elements (see Proposition 2.2). If $\dim x^{\bar{G}} \leqslant 112$ then [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-24} = b_4$ and there are less than $2q^{112} = a_4$ such elements. Similarly, if $p > 2$ and $112 < \dim x^{\bar{G}} < 146$ then (4.3) implies that $\mathrm{fpr}(x) < q^{-42} = b_5$ since $\alpha \leqslant 3$, $\beta \leqslant 2$ and $\delta(x) \geqslant 48$. Also, there are fewer than $2q^{136} = a_5$ of these elements. Finally, if $p = 2$ and $\dim x^{\bar{G}} > 112$ then x lies in the $\bar{G}$-class $4A_1$ and (4.3) yields $\mathrm{fpr}(x) < q^{-44}$ since $\alpha = 3$, $\beta = 1$ and $\delta(x) = 56$. In addition, there are no more than $2q^{128}$ of these elements.

Finally, suppose that $x \in G$ is a field automorphism of prime order r. Then $q = q_0^r$,

$$\mathrm{fpr}(x) \leqslant \frac{|A_1(q)E_7(q) : A_1(q^{1/r})E_7(q^{1/r})|}{|E_8(q) : E_8(q^{1/r})|} < 8q^{-112(1-1/r)} \leqslant 8q^{-56} = b_6$$

and we set $a_6 = \log_2 q.q^{248}$. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < \sum_{i=1}^{6} a_i b_i^5 < 1$ for all $q \geqslant 2$. □

LEMMA 4.5. *If H is of type $D_8(q)$ then $b(G) \leqslant 5$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = D_8$ is a σ-stable subgroup of $\bar{G}$. If $x \in H$ is semisimple then

$$\mathrm{fpr}(x) < \frac{|W(E_8) : W(D_8)|.2(q+1)^8}{q^{\delta(x)}(q-1)^8} = \frac{270(q+1)^8}{q^{\delta(x)}(q-1)^8},$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. Therefore, $\mathrm{fpr}(x) < q^{-59} = b_1$ if $C_{\bar{G}}(x)$ has no E_7 or D_8 factor since [**39**, Theorem 2] states that $\delta(x) \geqslant 80$. As in the proof of Lemma 4.4, the contribution to $\widehat{Q}(G,5)$ from the remaining semisimple elements is less than $a_2b_2^5$, where $a_2 = q^{130}$ and $b_2 = q^{-37}$.

Next suppose that $x \in H$ is a unipotent element of prime order p. As in the proof of Lemma 4.4, the contribution from unipotent elements $x \in G$ with $\dim x^{\bar{G}} \leqslant 112$ is less than $a_3b_3^5$, where $a_3 = 2q^{112}$ and $b_3 = 2q^{-24}$. Now assume that $\dim x^{\bar{G}} > 112$ and observe that (4.3) holds. First suppose that $p > 2$. By the familiar Bala–Carter theory (see [**15**, Theorem 5.9.6]; the extension to all good primes is due to Pommerening [**57**, **58**]) we can label the $\bar{M}$-class of x by a pair $(L, P_{L'})$, where L is a Levi subgroup of $\bar{M}$ and $P_{L'}$ is a distinguished parabolic subgroup of L'. If L is also a Levi subgroup of $\bar{G}$ then the $\bar{G}$-class of x has the same label and we can compute $\dim x^{\bar{M}}$ and $\dim x^{\bar{G}}$ via [**39**, Proposition 1.10] and [**15**, pp. 405–407], respectively. In the few cases where L is not a Levi of $\bar{G}$ we use [**35**] to determine the $\bar{G}$-class of x. The relevant results here are recorded in Section 2 (see Table 2). In this way, we deduce that $\delta(x) \geqslant 64$ and $\alpha \leqslant 3$ if $\dim x^{\bar{G}} \geqslant 146$, and thus (4.3) yields $\mathrm{fpr}(x) < q^{-54} = b_4$ since $\beta \leqslant 120$. Similarly, if $112 < \dim x^{\bar{G}} < 146$ then $\delta(x) \geqslant 64$ and $\alpha, \beta \leqslant 2$, so (4.3) gives $\mathrm{fpr}(x) < q^{-58}$ and we note that there are less than $2q^{136}$ such elements in G. Now, if $p = 2$ then the $\bar{G}$-class of each involution in $\bar{M}$ is determined in [**35**] and again we reproduce these results in Table 2. In particular, if $\dim x^{\bar{G}} > 112$ then x lies in the $\bar{G}$-class $4A_1$, so $|x^G| < 2q^{128} = a_5$ and (4.3) yields $\mathrm{fpr}(x) < q^{-52} = b_5$ since $\delta(x) = 64$, $\alpha = 3$ and $\beta = 1$.

Finally, suppose that $x \in G$ is a field automorphism of prime order r. Then $q = q_0^r$ and

$$\mathrm{fpr}(x) \leqslant \frac{|D_8(q) : D_8(q^{1/r})|}{|E_8(q) : E_8(q^{1/r})|} < 4q^{-128(1-1/r)} \leqslant 4q^{-64} = b_6.$$

We conclude that $\widehat{Q}(G,5) < \sum_{i=1}^{6} a_i b_i^5 = F(q)$, where $a_1 = q^{248}$, $a_4 = q^{240}$ and $a_6 = \log_2 q.q^{248}$. The reader can check that $F(q) < 1$ for all $q \geqslant 2$. □

PROPOSITION 4.6. *If $G_0 = E_8(q)$ and H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 5$.*

Proof. In view of Lemmas 4.2–4.5 we may assume that H is of type $E_8(q^{1/2})$. We claim that $b(G) \leqslant 4$. To see this, first let $x \in G$ be a semisimple element of prime order. Then $C_{\bar{G}}(x)$ is connected (since $\bar{G}$ is simply connected) and so a well-known corollary to the Lang–Steinberg Theorem [**68**, I, 2.7] implies that $x^{G_0} \cap H_0 = x^{H_0}$, where $H_0 = H \cap G_0 = E_8(q^{1/2})$. Therefore [**38**, Proposition 1.6] yields

$$\mathrm{fpr}(x) < \frac{2(q+1)^8}{q^{1/2\dim x^{\bar{G}}+4}(q^{1/2}-1)^8},$$

and thus $\mathrm{fpr}(x) < q^{-72} = b_1$ if $\dim x^{\bar{G}} \geqslant 156$. Similarly, if $\dim x^{\bar{G}} < 156$ then $\mathrm{fpr}(x) < q^{-51} = b_2$, and there are fewer than $3q^{128} = a_2$ such elements. Next let $x \in G$ be a unipotent element of order p. Then the class of x in both H_0 and G_0 is determined by the labelling of its class in $\bar{G}$ and we deduce that $x^{G_0} \cap H_0 = x^{H_0}$. First assume that $p > 2$. Then considering the centralizer orders $|C_{H_0}(x)|$ and $|C_{G_0}(x)|$ (see [**56**]) we calculate that $|(x^{\bar{G}})_\sigma| < 4q^{\dim x^{\bar{G}}}$ and $\mathrm{fpr}(x) < 8(q+1)q^{-(1/2)\dim x^{\bar{G}}-1}$, and hence the contribution to $\widehat{Q}(G,4)$ from unipotent elements of order p is less than

$$\sum 4q^{\dim x^{\bar{G}}}.(8(q+1)q^{-(1/2)\dim x^{\bar{G}}-1})^4 = 4.8^4(q+1)^4 \sum q^{-\dim x^{\bar{G}}-4} < q^{-53},$$

where we sum over a set of representatives for the distinct $\bar{G}$-classes of unipotent elements $x \in H$ of order p. Similarly, one can check that the contribution from unipotent elements is also less than q^{-53} when $p = 2$.

Finally, suppose that $x \in G$ is a field automorphism of prime order r. If r is odd then x induces a field automorphism on H_0 and therefore $\mathrm{fpr}(x) < 4q^{-248/3} = b_3$. On the other hand, if $r = 2$ then we may assume that x centralizes H_0, so

$$|x^G \cap H| = i_2(H_0) + 1 < 2q^{64}, \quad |x^G| < 2q^{124} = a_4$$

and thus $\mathrm{fpr}(x) < 4q^{-60} = b_4$. We conclude that $\widehat{Q}(G,4) < q^{-53} + \sum_{i=1}^{4} a_i b_i^4 < 1$, where $a_1 = q^{248}$ and $a_3 = \log_2 q.q^{248}$. □

4.2. $G_0 = E_7(q)$

LEMMA 4.7. *If $|H| > q^{46}$ then either H is of type $E_7(q^{1/2})$ or $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = T_1E_6.2$, A_1D_6, $A_7.2$ or A_1F_4.*

Proof. This is very similar to the proof of Lemma 4.2 and we omit the details. □

LEMMA 4.8. *If* $|H| \leqslant q^{46}$ *then* $b(G) \leqslant 6$.

Proof. If $x \in G_0$ has prime order and $\dim x^{\bar{G}} \geqslant 64$ then $|x^G| > \frac{1}{2}(q+1)^{-1}q^{65} = b$, and it is clear that this bound also holds if x is a field automorphism. By inspecting [**24**] and [**56**] we see that there are fewer than $3q^{55} = c$ elements $x \in G$ with $\dim x^{\bar{G}} < 64$, while [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-12} = d$. Applying Proposition 2.3 we conclude that $\widehat{Q}(G,6) < b(a/b)^6 + cd^6 < 1$, where $a = q^{46}$. □

LEMMA 4.9. *If* $H = N_G(\bar{M}_\sigma)$, *where* $\bar{M} = T_1E_6.2$, *then* $b(G) \leqslant 6$.

Proof. To begin with, let us assume that $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order. Then [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{|W(E_7):W(E_6).2|.2^2(q+1)^z.2}{q^{\delta(x)+z-6}(q-1)^6} = \frac{224(q+1)^z}{q^{\delta(x)+z-6}(q-1)^6}, \tag{4.4}$$

where $z = \dim Z(\bar{D}^0)$, $\bar{D} = C_{\bar{G}}(x)$ and $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. If $\bar{D}$ does not have an E_6, D_6 or A_7 factor then [**39**, Theorem 2] gives $\delta(x) \geqslant 34$ and thus (4.4) implies that $\mathrm{fpr}(x) < q^{-24} = b_1$ since $z \leqslant 7$. There are fewer than $q^{71} = a_2$ remaining semisimple elements $x \in G$ and [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant q^{-19} = b_2$.

Next let $x \in H$ be a unipotent element of order p, and assume for now that p is odd. Then $x \in \bar{M}^0$ and using [**36**] we can determine the $\bar{G}$-class of x by considering the restriction $V_{56} \downarrow E_6 = V_{27} \oplus (V_{27})^* \oplus 0^2$, where V_{56} and V_{27} denote the minimal modules for E_7 and E_6, respectively, $(V_{27})^*$ is the dual of V_{27} and 0 is the trivial 1-dimensional E_6-module. In this way we deduce that $x^{\bar{G}} \cap \bar{M} = x^{\bar{M}^0}$ and so the proof of [**38**, Lemma 4.5] yields

$$\mathrm{fpr}(x) < \frac{2^2(q+1)^7.6}{q^{\delta(x)+1}(q-1)^6}, \tag{4.5}$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. In addition, we calculate that $\delta(x) \geqslant 30$ if $\dim x^{\bar{G}} > 66$, and hence (4.5) gives $\mathrm{fpr}(x) < q^{-22} = b_3$ and Proposition 2.2 implies that there are fewer than $q^{126} = a_3$ such elements. If $\dim x^{\bar{G}} \leqslant 66$ then [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2q^{-12} = b_4$ and we note that there are less than $2q^{66} = a_4$ of these elements (see [**56**]).

Now assume that $p = 2$ and $x \in G_0$ is an involution. If $x \in \bar{M} - \bar{M}^0$ then x induces a graph automorphism on E_6; the proof of [**39**, Lemma 4.1] reveals that x lies in the $\bar{G}$-class $3A_1''$ if $C_{E_6}(x) = F_4$; otherwise x is in the class $4A_1$. If $x \in \bar{M}^0$ then the $\bar{G}$-class of x can be determined as before and the bounds $|x^G| < c_i$ and $\mathrm{fpr}(x) < d_i$ in Table 8 are easily verified. Here τ_1

TABLE 8. $T_1E_6.2 < E_7, p = 2$.

i	E_6-class of x	E_7-class of x	c_i	d_i
1	A_1	A_1	$2q^{34}$	$2q^{-12}$
2	$2A_1$	$2A_1$	$2q^{52}$	$6q^{-20}$
3	$3A_1$	$3A_1'$	$2q^{64}$	$4q^{-24}$
4	τ_1	$4A_1$	$2q^{54}$	$4q^{-28}$
5	τ_2	$3A_1''$	$2q^{70}$	$4q^{-28}$

is an F_4-type graph automorphism of E_6, while τ_2 represents the other E_6-class of graph automorphisms in $\mathrm{Aut}(E_6)$.

It follows that the contribution to $\widehat{Q}(G,6)$ from unipotent involutions is less than $\sum_{i=1}^{5} c_i d_i^6 < q^{-30}$. (Note that this bound is valid if $q=2$, while $\sum_{i=1}^{5} c_i d_i^6 < \sum_{i=3}^{4} a_i b_i^6$ for any q.) Finally, suppose that $x \in G$ is a field automorphism of prime order r, so $q = q_0^r$. If r is odd then

$$\mathrm{fpr}(x) \leqslant 2\left(\frac{q+1}{q^{1/r}+1}\right) \cdot \frac{|E_6^\epsilon(q) : E_6^\epsilon(q^{1/r})|}{|E_7(q) : E_7(q^{1/r})|} < 16q^{-54\left(1-\frac{1}{r}\right)} \leqslant 16q^{-36} = b_5,$$

while $|x^G| < 2q^{133/2} = a_6$ and [38, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-22} = b_6$ if $r=2$. We conclude that $\widehat{Q}(G,6) < \sum_{i=1}^{6} a_i b_i^6 < 1$ if $q \geqslant 3$, where $a_1 = q^{133}$ and $a_5 = \log_2 q.q^{133}$.

To complete the proof, let us assume that $q=2$. As previously noted, the contribution from involutions is less than 2^{-30}, so let $x \in G$ be an element of odd prime order. As before, set $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. First observe that there are fewer than $2^{69} = e_1$ elements $x \in G$ of odd prime order such that $\bar{D} = C_{\bar{G}}(x)$ has an E_6 or D_6 factor, and the proof of [38, Lemma 4.7] gives $\mathrm{fpr}(x) < 3.2^{-22} = f_1$. If $\bar{D}^0 = A_6T_1$, $D_5A_1T_1$ or $A_5A_1T_1$ then [39, Theorem 2] states that $\delta(x) \geqslant 34$, and therefore (4.4) yields $\mathrm{fpr}(x) < 2^{-19} = f_2$ since $z = \dim Z(\bar{D}^0) = 1$. In addition, we calculate that there are fewer than $2^{86} = e_2$ such elements (note that there are no semisimple elements $x \in G$ with $\bar{D}^0 = A_5A_1T_1$; see [24] for example).

Next we claim that $\mathrm{fpr}(x) < 2^{-24} = f_3$ if $z \geqslant 3$ and $\bar{D}^0 \neq D_4T_3$. This follows at once from (4.4) if $z \geqslant 4$ since the relation

$$\delta(x) = 2\left(|\Phi^+(\bar{G})| - |\Phi^+(\bar{M})| - |\Phi^+(\bar{D})| + |\Phi^+(\bar{D} \cap \bar{M})|\right) \tag{4.6}$$

(see [39, § 5]) implies that $\delta(x) \geqslant 2(63-36-6) = 42$ as $|\Phi^+(\bar{D})| \leqslant |\Phi^+(A_3)| = 6$. The case $z=3$ is entirely similar if $|\Phi^+(\bar{D})| \leqslant 7$. It remains to deal with the case $\bar{D}^0 = A_4T_3$. Now, if Ψ is a subsystem of the root system Φ and X is a type of root system, then we say that Ψ is X-*dense* in Φ if every subsystem of Φ of type X meets Ψ (see [39, § 5]). The A_2-dense subsystems of the simple root systems are listed in [39, Lemma 5.1]. Evidently, $\Phi(\bar{M})$ is A_2-dense in $\Phi(\bar{G})$, and thus $\Phi(\bar{D} \cap \bar{M})$ is A_2-dense in $\Phi(\bar{D})$. In particular, a further application of [39, Lemma 5.1] implies that $\Phi(\bar{D} \cap \bar{M}) = A_3$ or $\Phi(\bar{D} \cap \bar{M}) = A_2A_1$, so $\delta(x) \geqslant 2(63-36-10+4) = 42$ and the claim follows via (4.4).

Now, if $\bar{D}^0 = D_4T_3$ then arguing as above we deduce that $\Phi(\bar{D} \cap \bar{M}) = A_3$ or $\Phi(\bar{D} \cap \bar{M}) = A_1^4$, so $\delta(x) \geqslant 2(63-36-12+4) = 38$ and thus (4.4) implies that $\mathrm{fpr}(x) < 2^{-22} = f_4$. By inspecting [24] we calculate that G contains fewer than $2^{106} = e_4$ such elements. Next suppose that $z=0$ and that $\bar{D}$ has no E_6 or D_6 factor. Then the hypothesis $q=2$ implies that $\bar{D}^0 = A_5A_2$ (see [24]), and (4.4) implies that $\mathrm{fpr}(x) < 2^{-20} = f_5$ since [39, Theorem 2] gives $\delta(x) \geqslant 34$. Further, an easy calculation reveals that there are less than $2^{91} = e_5$ such elements in G.

Finally, suppose that $z=1$ or $z=2$. Excluding the cases considered above we see that $\bar{D}^0 = A_2^3T_1$, $A_2A_1^3T_2$, A_5T_2 or $D_4A_1T_2$. Using [24] we calculate that there are fewer than $2^{116} = e_6$ such elements in G and we claim that $\mathrm{fpr}(x) < 2^{-21} = f_6$. In view of (4.4), it suffices to show that $\delta(x) \geqslant 36$. This is clear in the first two cases since $|\Phi^+(\bar{D})| = 9$, and thus (4.6) implies that $\delta(x) \geqslant 2(63-36-9) = 36$. For the remaining possibilities we use the fact that $\Phi(\bar{D} \cap \bar{M})$ is A_2-dense in $\Phi(\bar{D})$. For example, if $\bar{D}^0 = A_5T_2$ then [39, Lemma 5.1] implies that $\Phi(\bar{D} \cap \bar{M}) = A_3A_1$ or $\Phi(\bar{D} \cap \bar{M}) = A_2^2$, and thus $|\Phi^+(\bar{D} \cap \bar{M})| \geqslant 6$ and (4.6) gives $\delta(x) \geqslant 2(63-36-15+6) = 36$. We conclude that

$$\widehat{Q}(G,6) < 2^{-30} + \sum_{i=1}^{6} e_i f_i^6 < 1 \quad \text{if } q=2,$$

where $e_3 = 2^{133}$. □

LEMMA 4.10. *If $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = A_1D_6$, $A_7.2$ or A_1F_4, then $b(G) \leqslant 6$.*

Proof. First consider the case $\bar{M} = A_1D_6$ and assume that $q \geqslant 3$ for now. If $x \in G$ is a semisimple element of odd prime order then [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{|W(E_7) : W(A_1D_6)|.2(q+1)^z.2}{q^{\delta(x)+z-7}(q-1)^7} = \frac{252(q+1)^z}{q^{\delta(x)+z-7}(q-1)^7}, \qquad (4.7)$$

where z and $\delta(x)$ are defined in the usual manner. If $\bar{D} = C_{\bar{G}}(x)$ does not have an E_6, D_6 or A_7 factor then [**39**, Theorem 2] gives $\delta(x) \geqslant 40$ and thus (4.7) implies that $\mathrm{fpr}(x) < q^{-30} = b_1$ since $z \leqslant 7$. As in the proof of Lemma 4.9, the contribution to $\widehat{Q}(G,6)$ from the remaining semisimple elements is less than $a_2b_2^6$, where $a_2 = q^{71}$ and $b_2 = q^{-19}$.

Now suppose that $x \in G$ is a unipotent element of order p, and first assume that $p > 2$. By Bala-Carter, the $\bar{M}$-class of x is labelled by a pair $(L, P_{L'})$, where L is a Levi subgroup of $\bar{M}$ and $P_{L'}$ is a distinguished parabolic subgroup of L'. If L is also a Levi subgroup of $\bar{G}$ then the $\bar{G}$-class of x has the same label, and thus $\dim x^{\bar{M}}$ and $\dim x^{\bar{G}}$ are easily determined. Now, if $x \in D_6 < \bar{M}$ then L is always a Levi subgroup of $\bar{G}$. However, if $x = uy$, where $y \in D_6$ and $1 \neq u \in A_1$, then there are a few cases for which L is not a Levi subgroup of $\bar{G}$. In each of these cases, the corresponding $\bar{G}$-class is determined in [**35**], and these results are listed in Section 2 (see Table 3). In this way, we deduce that $x^{\bar{G}} \cap \bar{M}$ is a union of at most three distinct $\bar{M}$-classes, and so the proof of [**38**, Lemma 4.5] yields

$$\mathrm{fpr}(x) < \frac{6(q+1)^7.6}{q^{\delta(x)}(q-1)^7},$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^G \cap \bar{M})$. Now, if $\dim x^{\bar{G}} > 66$ then our previous calculations imply that $\delta(x) \geqslant 34$ and thus $\mathrm{fpr}(x) < q^{-26} = b_3$. As in the proof of Lemma 4.9, the contribution from the other unipotent elements is less than $a_4b_4^6$, where $a_4 = 2q^{66}$ and $b_4 = 2q^{-12}$. Now assume that $p = 2$. There are 15 distinct conjugacy classes of involutions in $\bar{M}$, and the corresponding $\bar{G}$-classes are listed in Table 3, using results taken from [**35**]. It quickly follows that the unipotent involutions in G contribute less than q^{-44} to $\widehat{Q}(G,6)$. (This upper bound is still valid when $q = 2$.)

Finally, if $x \in G$ is a field automorphism of prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|A_1(q)D_6(q) : A_1(q^{1/r})D_6(q^{1/r})|}{|E_7(q) : E_7(q^{1/r})|} < 8q^{-64(1-1/r)} \leqslant 8q^{-32} = b_5,$$

and we conclude that $\widehat{Q}(G,6) < \sum_{i=1}^{5} a_ib_i^6 < 1$ if $q \geqslant 3$, where $a_1 = q^{133}$, $a_3 = q^{126}$ and $a_5 = \log_2 q.q^{133}$.

Now assume that $q = 2$. As before, the contribution from involutions is less than 2^{-44}. There are fewer than $2^{69} = c_1$ semisimple elements x in G such that $\bar{D} = C_{\bar{G}}(x)$ has an E_6 or D_6 factor; for such elements, [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2^{-12} = d_1$. We claim that $\mathrm{fpr}(x) < 2^{-23} = d_2$ if $\bar{D}$ has no E_6 or D_6 factor. First note that $\bar{D}^0 \neq A_7$ since $p = 2$, so [**39**, Theorem 2] implies that $\delta(x) \geqslant 40$ and thus (4.7) yields $\mathrm{fpr}(x) < 2^{-23}$ if $z \leqslant 3$. Now, if $z \geqslant 4$ then $|\Phi^+(\bar{D})| \leqslant 6$ and (4.7) gives $\mathrm{fpr}(x) < 2^{-32}$ since $\delta(x) \geqslant 52$ (see (4.6)). We conclude that $\widehat{Q}(G,6) < 2^{-44} + c_1d_1^6 + c_2d_2^6 < 1$, where $c_2 = 2^{133}$.

The case $\bar{M} = A_7.2$ is very similar and we omit the details. (Note that if $x \in A_7$ has order p and the A_7-class of x corresponds to the pair $(L, P_{L'})$, where L is a Levi subgroup of A_7 which is not a Levi of $\bar{G}$, then the $\bar{G}$-class of x is listed in Table 4; the relevant results originating in [**35**].)

Next we claim that $b(G) \leqslant 5$ if $\bar{M} = A_1F_4$. To see this, first let $x \in G$ be a semisimple element of prime order and write $\bar{D} = C_{\bar{G}}(x)$. If $\bar{D}$ does not have an E_6, D_6 or A_7 factor then $|x^G| > \frac{1}{3}q^{84} = f$, and we observe that $|H \cap \bar{G}_\sigma| < q^{55} = e$. There are fewer than $q^{71} = g_1$

remaining semisimple elements and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-22} = h_1$. Next let $x \in H$ be a unipotent element of order p. Now [**42**, Proposition 2.4] gives

$$\mathscr{L}(E_7) \downarrow A_1F_4 = \mathscr{L}(A_1F_4) \oplus (V(2\lambda_1) \otimes V(\lambda_4)),$$

and so we can determine the $\bar{G}$-class of x by inspecting the relevant tables in [**36**]. It turns out that $x^{\bar{G}} \cap \bar{M}$ is a union of at most two distinct $\bar{M}$-classes and therefore the proof of [**38**, Lemma 4.5] yields

$$\mathrm{fpr}(x) < \frac{2^2(q+1)^7.6}{q^{\delta(x)+2}(q-1)^5},$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. We can check that $\delta(x) \geqslant 36 + 10\delta_{2,p}$ if $\dim x^{\bar{G}} \geqslant 64$, and hence $\mathrm{fpr}(x) < q^{-28} = h_2$. There are fewer than $2q^{54} = g_3$ remaining unipotent elements and we note that [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-12} = h_3$. Finally, if $x \in G$ is a field automorphism of prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|A_1(q)F_4(q) : A_1(q^{1/r})F_4(q^{1/r})|}{|E_7(q) : E_7(q^{1/r})|} < 8q^{-78(1-\frac{1}{r})} \leqslant 8q^{-39} = h_4,$$

and applying Proposition 2.3 we deduce that $\widehat{Q}(G,5) < f(e/f)^5 + \sum_{i=1}^4 g_ih_i^5 < 1$, where $g_2 = q^{126}$ and $g_4 = \log_2 q.q^{133}$. □

Proposition 4.11. *If $G_0 = E_7(q)$ and H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 6$.*

Proof. In view of Lemmas 4.7–4.10, we may assume that H is of type $E_7(q^{1/2})$. Here it is easy to establish $b(G) \leqslant 4$ by arguing as in the proof of Proposition 4.6. We leave the details to the reader. □

4.3. $G_0 = E_6^{\epsilon}(q)$

We begin with two techinical lemmas on fixed point ratios for involutory graph automorphisms.

Lemma 4.12. *Suppose that $G = \mathrm{Aut}(E_6(2)) = E_6(2).2$, that H is a maximal subgroup of G with $|H| \leqslant 2^{32}$ and that $x \in G$ is an involutory graph automorphism. Then $\mathrm{fpr}(x) < 2^{-7}$.*

Proof. Let $G_0 = E_6(2)$. If $C_{G_0}(x) \neq F_4(2)$ then

$$|x^G| = |E_6(2) : C_{F_4(2)}(t)| = 2^{12}(2^4+1)(2^5-1)(2^9-1)(2^{12}-1) > 2^{42},$$

where $t \in F_4(2)$ is a long root element, and thus $\mathrm{fpr}(x) < 2^{-10}$ since $|x^G \cap H| \leqslant |H| \leqslant 2^{32}$. Now assume that $C_{G_0}(x) = F_4(2)$, so

$$|x^G| = |E_6(2) : F_4(2)| = 2^{12}(2^5-1)(2^9-1).$$

The maximal subgroups of G are determined in [**34**] and the possibilities for H are as follows:

$$\text{(i) } 3.(U_3(2) \times L_3(4)).D_{12}, \quad \text{(ii) } (L_3(2) \times L_3(2) \times L_3(2)) : D_{12}, \quad \text{(iii) } L_3(8) : 6,$$
$$\text{(iv) } L_3(2) : 2 \times G_2(2), \quad \text{(v) } 7^3 : 3^{1+2} : 2S_4.$$

For (v) we have $|H|/|x^G| < 2^{-7}$ and the claim follows at once. In the other cases we require more accurate calculations. First consider case (iii). Here $H \cap G_0 = L_3(8) : 3$ and thus $|x^G \cap H| \leqslant |L_3(8) : \Omega_3(8)| = 32704$ since x induces a graph automorphism on $L_3(8)$. This gives $\mathrm{fpr}(x) <$

2^{-10}. Similarly, in (iv) we calculate that $\mathrm{fpr}(x) < 2^{-12}$ since

$$|x^G \cap H| \leqslant |L_3(2){:}\Omega_3(2)|.(i_2(G_2(2))+1) = 8848,$$

while in (ii) we get $\mathrm{fpr}(x) < 2^{-10}$ since

$$|x^G \cap H| \leqslant |L_3(2) : \Omega_3(2)|^3 + 3|L_3(2)|.|L_3(2) : \Omega_3(2)| = 36064.$$

Finally, in (i) we have

$$|x^G \cap H| \leqslant |\mathrm{SU}_3(2) : \Omega_3(2)|\,(|\mathrm{SL}_3(4) : \Omega_3(4)| + |\mathrm{SL}_3(4) : \mathrm{SU}_3(2)| + |\mathrm{SL}_3(4) : \mathrm{SL}_3(2)|) = 59328,$$

and thus $\mathrm{fpr}(x) < 2^{-10}$. □

LEMMA 4.13. *Let G be an almost simple group with socle $G_0 = E_6(q)$, where $q \geqslant 3$, and let H be a maximal subgroup of G with $|H| \leqslant q^{32}$. Then $\mathrm{fpr}(x) < q^{-5}$ if $x \in G$ is an involutory graph automorphism and $C_{\bar{G}}(x) = F_4$.*

Proof. By [**41**, Theorem 2], the possibilities for H are as follows:

(i) $H = N_G(\bar{M}_\sigma)$, where $\bar{M}$ is a σ-stable closed subgroup of $\bar{G}$ of positive dimension;
(ii) H is an exotic local subgroup (see [**18**, Table 1]);
(iii) H is of type $E_6(q_0)$, where $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$ of odd prime index;
(iv) H is almost simple, and not of type (i) or (iii).

Now, if $|H| \leqslant q^{19}$ then $\mathrm{fpr}(x) < 6q^{-7} < q^{-5}$ since $|x^G| = |G_0 : F_4(q)| > \frac{1}{6}q^{26}$. Therefore we can assume that $|H| > q^{19}$. Arguing as in the proof of Lemma 4.2, we deduce that if H is a subgroup of type (i), (ii) or (iii) then either $H = N_G(\bar{M}_\sigma)$ with $\bar{M} \in \{T_2D_4.S_3, A_2^3.S_3, A_2G_2\}$, or H is of type $E_6(q_0)$ and $q = q_0^3$. In each case we can estimate $i_2(H)$ via [**38**, Proposition 1.3] and the desired result quickly follows. For example, suppose that $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = T_2D_4.S_3$. Then inspecting [**40**, Table 5.1] we deduce that

$$|x^G \cap H| \leqslant i_2(H) \leqslant 2(q+1)^2.\max\left(i_2(\mathrm{Aut}(\mathrm{P}\Omega_8^+(q))), i_2(\mathrm{Aut}(^3D_4(q)))\right) < 4(q+1)^3q^{15}, \quad (4.8)$$

and thus $\mathrm{fpr}(x) < 24(q+1)^3q^{-11} < q^{-5}$ for all $q \geqslant 3$. Similarly, if $H = N_G(\bar{M}_\sigma)$ and $\bar{M} = A_2^3.S_3$ then

$$|x^G \cap H| \leqslant i_2(H) \leqslant 4.(i_2(\mathrm{Aut}(L_3^{\epsilon'}(q))))^3 < 32(q+1)^3q^{12},$$

and we conclude that $\mathrm{fpr}(x) < 6.32(q+1)^3q^{-14} < q^{-5}$ as required. The other cases are very similar.

To complete the proof, suppose that H is almost simple and is not of type (i) or (iii). Let H_0 denote the socle of H. The possibilities for H_0 are listed in [**44**, Tables 10.1–10.4] when H_0 is not in $\mathrm{Lie}(p)$, where $\mathrm{Lie}(p)$ is the set of finite simple groups of Lie type in characteristic p. Inspecting these tables we find that the only case with $|H| > q^{19}$ occurs when $H_0 = \mathrm{Fi}_{22}$ and $q = 4$. Here

$$|x^G \cap H| \leqslant i_2(\mathrm{Aut}(H_0)) = 79466751 < q^{14}$$

and thus $\mathrm{fpr}(x) < 6q^{-12}$. Now assume that $H_0 \in \mathrm{Lie}(p)$, with H_0 a simple group of Lie type over $\mathbb{F}_{q_0}$. According to [**43**], we may assume that the untwisted Lie rank of H_0 (that is, the rank of the ambient simple algebraic group corresponding to H_0) is at most 3 and that either $q_0 \leqslant 9$, $H_0 = L_3^{\epsilon'}(16)$, or $H_0 \in \{L_2(q_0), {}^2B_2(q_0), {}^2G_2(q_0)\}$ and $q_0 \leqslant (2, p-1).124$. In each case, the desired result follows from the obvious bound $|x^G \cap H| \leqslant i_2(H)$. For example, suppose that $H_0 = {}^2G_2(q_0)$, where $q_0 = 3^l$ and l is odd (note that $l \leqslant 5$ since we may assume that $q_0 \leqslant 248$). Now, if $l = 5$ then the hypothesis $|H| \leqslant q^{32}$ implies that $q \geqslant 9$, and applying [**38**, Proposition 1.3] we calculate that $i_2(H) < 2(q_0+1)q_0^3 < q^{11}$. Similarly, if $l < 5$ then $i_2(H) < 3^{13}$ and the desired conclusion quickly follows. If $H_0 = \mathrm{PSp}_6(q_0)$ then we may assume that

$q_0 \leqslant 9$ and that $q = 9$ if $q_0 = 9$ since $|H| > q^{32}$ if $(q_0, q) = (9, 3)$. Then [**38**, Proposition 1.3] gives $i_2(H) < 2(1 + q_0)q_0^{11} < q^{19}$ and the result follows. The other cases are just as easy. □

The proof of the next result follows that of Lemma 4.2.

LEMMA 4.14. *If $|H| > q^{32}$ then one of the following holds:*
(i) *$H = N_G(\bar{M}_\sigma)$, where $\bar{M} = T_1D_5$, A_1A_5, F_4, $T_2D_4.S_3$ or C_4 ($p \neq 2$);*
(ii) *$\epsilon = +$ and H is of type $E_6^\delta(q^{1/2})$;*
(iii) *$G_0 = {}^2E_6(2)$ and H has socle* Fi_{22}*;*
(iv) *$G = {}^2E_6(2).2$ and $H = \mathrm{SO}_7(3)$.*

LEMMA 4.15. *If $|H| \leqslant q^{32}$ then $b(G) \leqslant 6$.*

Proof. For now let us assume that $q \geqslant 3$. Suppose that $x \in \bar{G}_\sigma$ has prime order and note that $|x^G| > \frac{1}{2}q^{40} = b$ if $\dim x^{\bar{G}} \geqslant 40$ (see [**24**, **55**]). There are fewer than $2q^{32} = c_1$ semisimple elements x with $\dim x^{\bar{G}} < 40$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-12} = d_1$ since we are assuming that $q \geqslant 3$. If x is unipotent and $\dim x^{\bar{G}} < 40$ then x belongs to one of the $\bar{G}$-classes labelled A_1 and $2A_1$ (see [**39**, Table 2], for example). Now, if x is in the class A_1 then $|x^G| < 2q^{22} = c_2$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-6} = d_2$. Similarly, if x is in $2A_1$ then $|x^G| < 2q^{32} = c_3$, and we claim that $\mathrm{fpr}(x) \leqslant q^{-6} = d_3$. If H is not of maximal rank then this follows from [**38**, Theorem 2], so assume that $H = N_G(\bar{M}_\sigma)$, where $\bar{M}$ is a maximal closed σ-stable subgroup of $\bar{G}$ of maximal rank. According to [**45**, Table 10.3], the hypothesis $|H| \leqslant q^{32}$ implies that $\bar{M}^0 \in \{D_4T_2, A_2^3, T_6\}$. Now, if $\bar{M}^0 = D_4T_2$ then the proof of Lemma 4.18 gives $\mathrm{fpr}(x) < q^{-12}$ (see (4.14) below); in the other two cases it is clear that $|H \cap \bar{G}_\sigma| < \frac{2}{3}q^{26}$ and thus $\mathrm{fpr}(x) < q^{-6}$ since $|x^G| > \frac{2}{3}q^{32}$.

Next we observe that G contains fewer than $4q^{39} = c_4$ involutory field and graph-field automorphisms and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-12} = d_4$. If x is a field automorphism of odd prime order then $|x^G| > \frac{1}{6}q^{52} > b$. Now, if $x \in G$ is an involutory graph automorphism and $C_{\bar{G}}(x) \neq F_4$ then $|x^G| > \frac{1}{6}q^{42} > b$; there are fewer than $2q^{26} = c_5$ graph automorphisms x with $C_{\bar{G}}(x) = F_4$, and a combination of Lemma 4.13 and [**38**, Theorem 2] implies that $\mathrm{fpr}(x) < q^{-5} = d_5$ since we are assuming that $q \geqslant 3$. In view of Proposition 2.3 we conclude that $\widehat{Q}(G, 6) < b(a/b)^6 + \sum_{i=1}^{5} c_i d_i^6 < 1$ for all $q \geqslant 3$, where $a = q^{32}$.

To complete the proof let us assume that $q = 2$. If $x \in G$ is a semisimple element of prime order and $\dim x^{\bar{G}} \geqslant 42$ then $|x^G| > 2^{41}$. In addition, there are fewer than $2^{33} = g_1$ such elements with $\dim x^{\bar{G}} < 42$ and [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2^{-6} = h_1$. Next assume that x is a unipotent involution, so x lies in one of the classes A_1, $2A_1$ and $3A_1$. If x is in $3A_1$ then $|x^G| > 2^{40}$, while we have $|x^G| < 2^{22} = g_2$ and $\mathrm{fpr}(x) \leqslant 2^{-5} = h_2$ if x belongs to the class A_1 (see [**38**, Theorem 2]). If x is in $2A_1$ then $|x^G| < 2^{34} = g_3$, and we claim that $\mathrm{fpr}(x) \leqslant 2^{-6} = h_3$. This follows from [**38**, Theorem 2] if H is not of maximal rank, while the proof of Lemma 4.18 below yields $\mathrm{fpr}(x) < 6.2^{-20}$ if $H = N_G(\bar{M}_\sigma)$ with $\bar{M}^0 = D_4T_2$. If H is a different subgroup of maximal rank then the hypothesis $|H| \leqslant 2^{32}$ implies that

$$|x^G \cap H| \leqslant i_2(H \cap \bar{G}_\sigma) \leqslant i_2(L_3(2)^3.S_3) = (i_2(L_3(2)) + 1)^3 + 3|L_3(2)| = 11151$$

(see [**40**, Table 5.1]) and the claim follows since $|x^G| > 2^{31}$. Finally, if x is an involutory graph automorphism and $C_{\bar{G}}(x) \neq F_4$ then $|x^G| > \frac{1}{6}2^{42} = f$; if $C_{\bar{G}}(x) = F_4$ then a combination of Lemma 4.12 and [**38**, Theorem 2] implies that $\mathrm{fpr}(x) \leqslant (2^6 - 2^3 + 1)^{-1} = h_4$, while it is easy to see that there are fewer than $2^{27} = g_4$ such elements. Applying Proposition 2.3 we deduce that $\widehat{Q}(G, 6) < f(e/f)^6 + \sum_{i=1}^{4} g_i h_i^6 < 1$, where $e = 2^{32}$. □

Lemma 4.16. *If H is of type $A_1(q)A_5^{\epsilon}(q)$ then $b(G) \leqslant 5$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = A_1A_5$ is a σ-stable subgroup of $\bar{G}$. For now we will assume that $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order and set $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. Then [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{|W(E_6) : W(A_1A_5)|.2(q+1)^z.3}{q^{\delta(x)+z-6}(q-1)^6} = \frac{216(q+1)^z}{q^{\delta(x)+z-6}(q-1)^6}, \tag{4.9}$$

where $z = \dim Z(\bar{D}^0)$ and $\bar{D} = C_{\bar{G}}(x)$. Now, $\Phi(\bar{M})$ is A_2-dense in $\Phi(\bar{G})$, so $\Phi(\bar{D} \cap \bar{M})$ is A_2-dense in $\Phi(\bar{D})$ and using [**39**, Lemma 5.1] we calculate that $\delta(x) \geqslant 26$ if $\bar{D}$ has no D_5 or A_5 factor and $\bar{D}^0 \neq D_4T_2$ (see (4.6)). In this case, (4.9) yields $\mathrm{fpr}(x) < q^{-17} = b_1$ and clearly there are less than $q^{78} = a_1$ semisimple elements in G. If $\bar{D}^0 = D_4T_2$ then [**39**, Theorem 2] gives $\delta(x) \geqslant 24$, and hence (4.9) implies that $\mathrm{fpr}(x) < q^{-16} = b_2$ and we calculate that G contains fewer than $4q^{50} = a_2$ such elements. Similarly, if $\bar{D}$ has an A_5 factor then $\mathrm{fpr}(x) < q^{-12} = b_3$ since $\delta(x) \geqslant 20$ (see [**39**, Theorem 2]) and we note that there are less than $q^{45} = a_3$ of these elements. Finally, if $\bar{D}^0 = D_5T_1$ then $\mathrm{fpr}(x) < q^{-8} = b_4$ since $\delta(x) \geqslant 16$ and there are fewer than $q^{34} = a_4$ such elements.

Now suppose that $x \in G$ is a unipotent element of order p. By Bala-Carter, the $\bar{M}$-class of x corresponds to a pair $(L, P_{L'})$, where L is a Levi subgroup of $\bar{M}$ and $P_{L'}$ is a distinguished parabolic subgroup of L'. As before, if L is also a Levi subgroup of $\bar{G}$ then we find that the $\bar{G}$-class of x has the same label; this is indeed the case unless $L = A_1^4$, $A_3A_1^2$ or A_5A_1. In these cases we can determine the $\bar{G}$-class of x via [**36**, Table 5], by first calculating the Jordan form of x on the 27-dimensional module V_{27} for E_6. This is very straightforward since we have

$$V_{27} \downarrow A_1A_5 = (V(\lambda_1) \otimes V(\lambda_1)) \oplus (0 \otimes V(\lambda_4)).$$

It follows that we can calculate $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$ for all unipotent elements $x \in G$ of order p. First assume that $p > 2$. Then $x^{\bar{G}} \cap \bar{M}$ is a union of at most two distinct $\bar{M}$-classes and so the proof of [**38**, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{2^2(q+1)^2.6}{q^{\delta(x)-4}(q-1)^6} \tag{4.10}$$

since $\dim Z(C^0/R_u(C^0)) \leqslant 2$, where $C = C_{\bar{G}}(x)$ (see [**55**], for example). If $\dim x^{\bar{G}} \geqslant 40$ then $\delta(x) \geqslant 22$, so $\mathrm{fpr}(x) < q^{-16} = b_5$ and there are fewer than $q^{72} = a_5$ such elements in G (see Proposition 2.2). If $\dim x^{\bar{G}} < 40$ then x belongs to one of the classes A_1 or $2A_1$. If x is in A_1 then $|x^G| < 2q^{22} = a_6$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-6} = b_6$. Similarly, if $x \in 2A_1$ then $|x^G| < 2q^{32} = a_7$ and $\mathrm{fpr}(x) < q^{-9} = b_7$ since $\delta(x) = 16$. The case $p = 2$ is very similar. Here we calculate that $\mathrm{fpr}(x) < 4q^{-\delta(x)}$, and it is straightforward to check that unipotent involutions contribute less than $2q^{-27}$ (this upper bound is still valid when $q = 2$).

Next suppose that x is an involutory field or graph-field automorphism. There are fewer than $4q^{39} = a_8$ such elements and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-12} = b_8$. If x is a field automorphism of odd prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|A_1(q)A_5^{\epsilon}(q) : A_1(q^{1/r})A_5^{\epsilon}(q^{1/r})|}{|E_6^{\epsilon}(q) : E_6^{\epsilon}(q^{1/r})|} < 8q^{-40\left(1-\frac{1}{r}\right)} \leqslant 8q^{-80/3} = b_9,$$

and of course there are less than $\log_2 q.q^{78} = a_9$ such elements. Finally, suppose that $x \in G$ is an involutory graph automorphism. At the level of algebraic groups, the action of x on $\bar{M}$ induces an involutory graph automorphism on the A_5-factor; according to the proof of [**38**, Lemma 6.4] we have $C_{\bar{G}}(x) = F_4$ if and only if $C_{A_5}(x) = C_3$ and x centralizes the A_1 factor of $\bar{M}$. Therefore, if $C_{G_0}(x) = F_4(q)$, then we have

$$\mathrm{fpr}(x) = \frac{|\mathrm{SL}_6^{\epsilon}(q) : \mathrm{Sp}_6(q)|}{|E_6^{\epsilon}(q) : F_4(q)|} < 12q^{-12} = b_{10},$$

and there are less than $2q^{26} = a_{10}$ of these graph automorphisms. On the other hand, if p is odd and $C_{\bar{G}}(x) \neq F_4$ then $\mathrm{fpr}(x) < 24q^{-16} = b_{11}$ since $|x^G| > \frac{1}{6}q^{42}$ and

$$|x^G \cap H| \leqslant (i_2(\mathrm{SL}_2(q)) + 1).\left(\frac{|\mathrm{SL}_6^{\epsilon}(q)|}{|\mathrm{SO}_6^{+}(q)|} + \frac{|\mathrm{SL}_6^{\epsilon}(q)|}{|\mathrm{SO}_6^{-}(q)|} + 1\right) < 4q^{26}.$$

We can check that this bound is also valid when $p = 2$ and we note that there are fewer than $2q^{42} = a_{11}$ of these elements in G. In particular, we conclude that $\widehat{Q}(G,5) < \sum_{i=1}^{11} a_i b_i^5 < 1$ if $q \geqslant 3$.

Now assume that $q = 2$. Write $\widetilde{H} = H \cap \bar{G}_\sigma = \mathrm{SL}_6^{\epsilon}(2) \times \mathrm{SL}_2(2)$ and note that $|\widetilde{H}| < 2^{38}$. As before, the contribution to $\widehat{Q}(G,5)$ from involutions is less than $2^{-26} + a_{10}b_{10}^5 + a_{11}b_{11}^5 < 2^{-13}$, while Proposition 2.3 implies that the semisimple elements $x \in G$ with $|x^G| > 2^{48} = d$ contribute less than $d(c/d)^5$, where $c = 2^{38}$. Now let $x \in G$ be a semisimple element of odd prime order r such that $|x^G| \leqslant 2^{48}$, so $\bar{D}^0 = T_1D_5$, T_1A_5, T_2D_4 or $A_4A_1T_1$, where $\bar{D} = C_{\bar{G}}(x)$ (see Table 9 in Section 6). We claim that $\mathrm{fpr}(x) < 2^{-14} = f_1$ if $\bar{D}^0 \neq T_1D_5$. If $\bar{D}^0 = A_4A_1T_1$ or $\bar{D}^0 = T_1A_5$ then $r = 3$ (see Table 9) and the claim holds since $|x^G| > 2^{41}$ and

$$|x^G \cap H| \leqslant i_3(\widetilde{H}) = (i_3(\mathrm{SL}_6^{\epsilon}(2)) + 1).(i_3(\mathrm{SL}_2(2)) + 1) - 1 < 2^{26}.$$

Similarly, if $\bar{D}^0 = T_2D_4$ then $\mathrm{fpr}(x) < 2^{-14}$ since $|x^G| > 2^{45}$ and $|x^G \cap H| \leqslant i_7(\widetilde{H}) < 2^{31}$ since $r = 3$ or $r = 7$. In addition, we note that there are fewer than $2^{53} = e_1$ semisimple elements $x \in G$ with $\bar{D}^0 = T_1A_5$, T_2D_4 or $A_4A_1T_1$.

It remains to consider the case $\bar{D}^0 = T_1D_5$. Here $(\epsilon, r) = (-,3)$ and $|x^G| > 2^{31}$ (see Table 9 in Section 6). Now, it is easy to see that $|y^{\widetilde{H}}| < 2^{1+\dim y^{\bar{M}}}$ for any element $y \in \widetilde{H}$ of order 3, while there are precisely 19 distinct $\widetilde{H}$-classes of such elements. Arguing as in the proof of [**38**, Lemma 4.5], it follows that $\mathrm{fpr}(x) < 19.4.2^{-\delta(x)}$, where $\delta(x)$ is defined as before. By [**39**, Theorem 2] we have $\delta(x) \geqslant 16$, and hence $\mathrm{fpr}(x) < 2^{-10} = f_2$ and we note that there are fewer than $2^{33} = e_2$ such elements. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < 2^{-13} + d(c/d)^5 + \sum_{i=1}^{2} e_i f_i^5 < 1$. □

LEMMA 4.17. *If H is of type $D_5^{\epsilon}(q)T_1$ then $b(G) \leqslant 6$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = D_5T_1$ is a σ-stable subgroup of $\bar{G}$. To begin with, we will assume that $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order, so [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{|W(E_6):W(D_5)|.2(q+1)^z.3}{q^{\delta(x)+z-6}(q-1)^6} = \frac{162(q+1)^z}{q^{\delta(x)+z-6}(q-1)^6}, \tag{4.11}$$

where $z = \dim Z(\bar{D}^0)$, $\bar{D} = C_{\bar{G}}(x)$ and $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. Now, $\Phi(\bar{M})$ is A_2-dense in $\Phi(\bar{G})$, and so $\Phi(\bar{D} \cap \bar{M})$ is A_2-dense in $\Phi(\bar{D})$. By considering the possibilities for $\bar{D}$ and using [**39**, Lemma 5.1] we deduce that $\delta(x) \geqslant 24$ if $\dim x^{\bar{G}} \geqslant 60$, so (4.11) yields $\mathrm{fpr}(x) < q^{-15} = b_1$. There are fewer than $q^{64} = a_2$ semisimple elements x with $\dim x^{\bar{G}} < 60$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-12} = b_2$.

Next, let $x \in G$ be a unipotent element of order p. First assume that $p > 2$. By Bala-Carter, unipotent classes in $\bar{M}$ are parametrized by pairs $(L, P_{L'})$, where L is a Levi subgroup of $\bar{M}$ and $P_{L'}$ is a distinguished parabolic subgroup of L'. Evidently, every Levi subgroup of $\bar{M}$ is also a Levi of $\bar{G}$, and so the $\bar{G}$-class of x has the same label. In this way we deduce that either $x^{\bar{G}} \cap \bar{M} = x^{\bar{M}}$, or x belongs to one of the $\bar{G}$-classes $2A_1$ and A_3, and $x^{\bar{G}} \cap \bar{M}$ is a union of two distinct $\bar{M}$-classes. In particular, we see that (4.10) holds. Now, if $\dim x^{\bar{G}} \geqslant 50$ then $\delta(x) \geqslant 20$ and thus $\mathrm{fpr}(x) < q^{-14} = b_3$. If x is in one of the $\bar{G}$-classes labelled $A_2 + A_1$, A_2 or $3A_1$ then $\mathrm{fpr}(x) < q^{-10} = b_4$ since $\delta(x) \geqslant 16$. We also note that there are fewer than $3q^{46} = a_4$ such elements. If x lies in the class $2A_1$ then $|x^G| < 3q^{32} = a_5$ and $x^{\bar{G}} \cap \bar{M} = y^{\bar{M}} \cup z^{\bar{M}}$, where

y and z have respective Jordan forms $[J_3, I_7]$ and $[J_2^4, I_2]$ on the natural D_5-module. Therefore $\mathrm{fpr}(x) < q^{-9} = b_5$ since

$$|x^G \cap H| < 2(q-1)^{-1}(q^{17}+q^{21}),\ |x^G| > \tfrac{1}{2}(q+1)^{-1}q^{33}.$$

Similarly, if x is in the class A_1 then $|x^G| < 2q^{22} = a_6$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-6} = b_6$.

Now assume that x is unipotent and $p = 2$. Let V_{27} denote the 27-dimensional minimal module for $\bar{G}$. Then according to [**42**, Table 8.7] we have $V_{27} \downarrow D_5 = V(\lambda_1) \oplus V(\lambda_4) \oplus 0$, where $V(\lambda_4) = V_{16}$ is a 16-dimensional spin module for D_5 and 0 denotes the trivial 1-dimensional D_5-module. Since $V_{16} \downarrow D_4$ is a sum of two non-equivalent spin modules for D_4, it follows that

$$V_{27} \downarrow D_4 = V(\lambda_1) \oplus V(\lambda_3) \oplus V(\lambda_4) \oplus 0^3. \tag{4.12}$$

Now, every unipotent involution $x \in \bar{M}$ has a representative in a subgroup D_4, and therefore we can easily compute the Jordan form of x on V_{27} and then determine the $\bar{G}$-class of x via [**36**, Table 5]. In the notation of [**1**], we find that $a_2 \in A_1$; c_2 and a_4 are in $2A_1$, while c_4 is in the $\bar{G}$-class $3A_1$. It quickly follows that the contribution to $\widehat{Q}(G,6)$ from unipotent involutions is less than q^{-13}, and we note that this bound is valid for all $q \geqslant 2$. Furthermore, we observe that $q^{-13} < \sum_{i=3}^{6} a_i b_i^6$, where $a_3 = q^{72}$.

As in the proof of Lemma 4.16, the contribution from involutory field and graph-field automorphisms is less than $a_7 b_7^6$, where $a_7 = 4q^{39}$ and $b_7 = q^{-12}$. If x is a field automorphism of odd prime order r then x induces a field automorphism on the $D_5^{\epsilon}(q)$-factor and we deduce that $\mathrm{fpr}(x) < q^{-19} = b_8$ since

$$|x^G \cap H| \leqslant (q-\epsilon)\frac{D_5^{\epsilon}(q)}{D_5^{\epsilon}(q^{1/r})} < 2(q+1)q^{45(1-1/r)}, \quad |x^G| > \frac{1}{6}q^{78(1-1/r)}.$$

Finally, suppose that $x \in G$ is an involutory graph automorphism. If $C_{\bar{G}}(x) \neq F_4$ then $|x^G| < 2q^{42} = a_9$ and we calculate that $\mathrm{fpr}(x) < q^{-11} = b_9$ since $|x^G| > \frac{1}{6}q^{42}$ and [**38**, Proposition 1.3] implies that

$$|x^G \cap H| \leqslant (q+1).i_2(\mathrm{Aut}(\mathrm{P}\Omega_{10}^{\epsilon}(q))) < 2(q+1)^2 q^{24}.$$

Conversely, if $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2q^{26} = a_{10}$ and the proof of [**38**, Lemma 6.4] gives

$$\mathrm{fpr}(x) \leqslant (q+1)\frac{|D_5^{\epsilon}(q) : B_4(q)|}{|\bar{G}_{\sigma} : F_4(q)|} < 4(q+1)q^{-17} = b_{10}.$$

We conclude that $\widehat{Q}(G,6) < \sum_{i=1}^{10} a_i b_i^6 < 1$ if $q \geqslant 3$, where $a_1 = q^{78}$, $a_3 = q^{72}$ and $a_8 = \log_2 q.q^{78}$.

To complete the proof, let us assume that $q = 2$. As above, the contribution to $\widehat{Q}(G,6)$ from involutions is less than $2^{-13} + a_9 b_9^6 + a_{10} b_{10}^6 < 2^{-12}$ so suppose that $x \in G$ is a semisimple element of odd prime order r, and hence $x^G \cap H \subseteq \widetilde{H}$, where $\widetilde{H} = \Omega_{10}^{\epsilon}(2) \times (2-\epsilon)$. We claim that $\mathrm{fpr}(x) < 2^{-17} = d_1$ if $\dim x^{\bar{G}} > 48$. This is trivial if $\dim x^{\bar{G}} > 60$ since $|x^G| > 2^{64}$ (see Table 9) and $|\widetilde{H}| < 2^{47}$. If $48 < \dim x^{\bar{G}} \leqslant 60$ then $\bar{D}^0 = A_2^3$, A_3T_3 or $A_2^2T_2$, where $\bar{D} = C_{\bar{G}}(x)$. If $\bar{D}^0 = A_2^3$ then $r = 3$ and thus $\mathrm{fpr}(x) < 2^{-19}$ since $i_3(\widetilde{H}) < 3.2^{31}$ and $|x^G| > 2^{52}$. Similarly, if $\bar{D}^0 = A_3T_3$ or $\bar{D}^0 = A_2^2T_2$ then $r = 5$ or $r = 7$, respectively, and the claim follows since $|x^G| > 2^{58}$ and $i_r(\widetilde{H}) < 2^{37}$. This justifies the claim.

Now assume that $\dim x^{\bar{G}} \leqslant 48$, so $\bar{D}^0 = T_1D_5$, T_1A_5 or T_2D_4 (see Table 9). If $\bar{D}^0 = T_1D_5$ then $\epsilon = -$, $|x^G| < 2^{32} = c_2$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2^{-6} = d_2$. If $\bar{D}^0 = T_1A_5$ then $r = 3$, $|x^G| < 2^{42} = c_3$ and we have $\mathrm{fpr}(x) < 2^{-8} = d_3$ since $|x^G| > 2^{41}$ and $i_3(\widetilde{H}) < 3.2^{31}$. Finally, suppose that $\bar{D}^0 = T_2D_4$, so $|x^G| < 2^{48} = c_4$. If $\epsilon = -$ then $r = 3$ and thus $\mathrm{fpr}(x) < 2^{-12}$ since $i_3(\widetilde{H}) < 3.2^{31}$ and $|x^G| > 2^{45}$. On the other hand, if $\epsilon = +$ then $r = 3$ or $r = 7$ and thus $\mathrm{fpr}(x) < 2^{-10} = d_4$ since $|x^G \cap H| \leqslant i_7(\widetilde{H}) < 2^{37}$ and $|x^G| > 2^{47}$. We conclude that $b(G) \leqslant 6$ since $\widehat{Q}(G,6) < 2^{-12} + \sum_{i=1}^{4} c_i d_i^6 < 1$, where $c_1 = 2^{78}$. □

LEMMA 4.18. *If $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = T_2D_4.S_3$ is a σ-stable subgroup of $\bar{G}$, then $b(G) \leqslant 6$.*

Proof. We start with the case $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order and define $\delta(x)$ as before. Then [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{(|W(E_6) : W(D_4).6| + 3)\,.12(q+1)^z.3}{q^{\delta(x)+z-6}(q-1)^6} = \frac{1728(q+1)^z}{q^{\delta(x)+z-6}(q-1)^6}, \tag{4.13}$$

where $z = \dim Z(\bar{D}^0)$ and $\bar{D} = C_{\bar{G}}(x)$. If $\bar{D}$ has no D_5 or A_5 factor then $\delta(x) \geqslant 26$ by [**39**, Theorem 2], and hence (4.13) gives $\mathrm{fpr}(x) < q^{-14} = b_1$. In fact, the same bound holds if $\bar{D}$ has an A_5 factor since $z \leqslant 1$ and $\delta(x) \geqslant 24$. There are fewer than $2q^{34} = a_2$ elements $x \in G$ with $\bar{D}^0 = T_1D_5$ and [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2q^{-12} = b_2$.

Now suppose that $x \in G$ is a unipotent element of order p. First assume that $p > 2$. If $x \in \bar{M}^0$ then we can determine the Jordan form of x on V_{27} via (4.12) and then identify the $\bar{G}$-class of x by inspecting [**36**, Table 5]. If $p = 3$ and $x \in \bar{M} - \bar{M}^0$ then x induces a triality graph automorphism on D_4. Now x permutes the D_4-modules $V(\lambda_1)$, $V(\lambda_3)$ and $V(\lambda_4)$ and therefore (4.12) implies that x has Jordan form $[J_3^9]$ on V_{27}, and hence [**36**, Table 5] indicates that x belongs to either $2A_2$ or $2A_2 + A_1$. If $C_{D_4}(x) = G_2$ then it is clear that x belongs to $2A_2$ since $|C_H(x)|$ divides $|C_G(x)|$. It quickly follows that if $x \in \bar{M}$ has order p then $x^{\bar{G}} \cap \bar{M}$ is a union of at most three distinct $\bar{M}$-classes and thus [**38**, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{36(q+1)^2.6}{q^{\delta(x)-4}(q-1)^6}, \tag{4.14}$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. Now, if $\dim x^{\bar{G}} \geqslant 40$ then we can check that $\delta(x) \geqslant 24$, and thus (4.14) gives $\mathrm{fpr}(x) < q^{-16} = b_3$. If x belongs to the $\bar{G}$-class $2A_1$ then $|x^G| < 2q^{32} = a_4$ and (4.14) implies that $\mathrm{fpr}(x) < q^{-12} = b_4$ since $\delta(x) = 20$. Finally, if x is a long root element then $|x^G| < 2q^{22} = a_5$ and we have $\mathrm{fpr}(x) \leqslant 2q^{-6} = b_5$ by [**38**, Theorem 2].

Now assume that $p = 2$ and that $x \in G$ is a unipotent involution. If $x \in \bar{M} - \bar{M}^0$ then x acts as an involutory graph automorphism on D_4; in the notation of [**1**], x is D_4-conjugate to b_1 or b_3. Now $x = b_l$ swaps the D_4-modules $V(\lambda_3)$ and $V(\lambda_4)$ and acts on $V(\lambda_1)$ with Jordan form $[J_2^l, I_{8-2l}]$, so (4.12) implies that x has Jordan form $[J_2^{9+l}, I_{9-2l}]$ on V_{27}. Inspecting [**36**, Table 5], we conclude that b_1 lies in the $\bar{G}$-class $2A_1$, while b_3 is in $3A_1$. Now, according to [**40**, Table 5.1] we have $x^G \cap H \subseteq \widetilde{H}$, where $\widetilde{H} = O_8^+(q)$ or $\widetilde{H} = {}^3D_4(q)$. First assume that $\widetilde{H} = {}^3D_4(q)$. There are two classes of involutions in $\widetilde{H}$, labelled A_1 and $3A_1$ in [**67**], and it is easy to see that the corresponding classes in $\bar{G}$ have the same labels. For example, if x lies in the $\widetilde{H}$-class $3A_1$ then x is a c_4-involution in the overgroup $\Omega_8^+(q^3)$ and thus (4.12) implies that x has Jordan form $[J_2^{12}, I_3]$ on V_{27}, so x lies in the $\bar{G}$-class $3A_1$ (see [**36**, Table 5]). In this case, the contribution to $\widehat{Q}(G,6)$ from unipotent involutions is less than

$$2q^{22}.(2q^{-12})^6 + 2q^{40}.(4q^{-24})^6 < q^{-42}.$$

If $\widetilde{H} = O_8^+(q)$ then there are precisely six distinct classes of involutions, with representatives labelled b_1, a_2, c_2, b_3, a_4 and c_4 in [**1**]. We can check that $b_1, a_2 \in A_1$; $c_2, a_4 \in 2A_1$ and $b_3, c_4 \in 3A_1$. It quickly follows that the contribution here is less than q^{-32}. In addition, we note that $q^{-32} < \sum_{i=3}^{5} b_i(a_i/b_i)^6$ for all $q \geqslant 3$, where $a_3 = q^{72}$.

Next suppose that $x \in G$ is a field or graph-field automorphism of prime order r. Then $q = q_0^r$ and the proof of [**38**, Lemma 6.1] gives

$$\mathrm{fpr}(x) \leqslant \frac{6(q+1)^2q^{28}}{(q^{1/r}-1)^6q^{24/r}|x^{G_0}|} < \frac{36(q+1)^2q^{28}}{(q^{1/r}-1)^6q^{24/r}q^{78(1-1/r)}} < q^{12-48(1-1/r)}. \tag{4.15}$$

In particular, if $r = 2$ then $\mathrm{fpr}(x) < q^{-12} = b_6$, and we note that G contains fewer than $4q^{39} = a_6$ such elements. If $r \geqslant 3$ then (4.15) gives $\mathrm{fpr}(x) < q^{-20} = b_7$. Now, if $x \in G$ is an involutory graph automorphism and $C_{\bar{G}}(x) \neq F_4$ then $|x^G| < 2q^{42} = a_8$ and $\mathrm{fpr}(x) < q^{-17} = b_8$ since $|x^G| > \frac{1}{6}q^{42}$ and (4.8) holds. Similarly, if $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2q^{26} = a_9$ and (4.8) implies that $\mathrm{fpr}(x) < q^{-5} = b_9$ since we are assuming that $q \geqslant 3$. We conclude that $b(G) \leqslant 6$ if $q \geqslant 3$ since $\widehat{Q}(G,6) < \sum_{i=1}^{9} a_i b_i^6 < 1$, where $a_1 = q^{78}$, $a_3 = q^{72}$ and $a_7 = \log_2 q.q^{78}$.

Now let us assume that $q = 2$. As above, the contribution to $\widehat{Q}(G,6)$ from unipotent involutions and non-F_4-type graph automorphisms is less than $2^{-32} + a_9 b_9^6 < 2^{-31}$. Now, if x is a graph automorphism with $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2^{27} = c_1$, and we claim that $\mathrm{fpr}(x) \leqslant (2^6 - 2^3 + 1)^{-1} = d_1$. This follows from [**38**, Theorem 2] if $\epsilon = -$. On the other hand, if $\epsilon = +$ then we may assume that $H \leqslant (D_{14} \times {}^3D_4(2)).3 = J$ (see [**34**]) and the claim holds since $|x^G \cap H| \leqslant i_2(J) = 556927$ and $|x^G| = 2^{12}(2^5-1)(2^9-1)$. Next let $x \in G$ be a semisimple element of odd prime order r with $\bar{D} = C_{\bar{G}}(x)$, and note that $|H \cap \bar{G}_\sigma| \leqslant 3^2|\Omega_8^+(2)|.6 < 2^{34} = e$ (see [**40**, Table 5.1]). By Proposition 2.3, such elements x with $|x^G| > 2^{41} = f$ contribute less than $f(e/f)^6 = 2^{-1}$. If $|x^G| \leqslant 2^{41}$ then $(r,\epsilon) = (3,-)$ and $\bar{D}^0 = T_1D_5$ (see Table 9). Moreover, there are fewer than $2^{33} = c_2$ such elements and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2^{-6} = d_2$. This implies that $\widehat{Q}(G,6) < 2^{-31} + \sum_{i=1}^{2} c_i d_i^6 + f(e/f)^6 < 1$ as required. □

Lemma 4.19. *If H is of type $F_4(q)$ then $b(G) \leqslant 6$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = F_4$ is a σ-stable subgroup of $\bar{G}$. For now we will assume that $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order and note that $|x^G| > \frac{1}{2}(q+1)^{-4}q^{70} = b$ if $\dim x^{\bar{G}} \geqslant 66$. Now there are fewer than $q^{68} = c_1$ semisimple elements x with $\dim x^{\bar{G}} < 66$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-12} = d_1$. Next suppose that x is a unipotent element of order p and assume for now that p is odd. Inspecting [**36**, Table A] we deduce that $x^{\bar{G}} \cap \bar{M} = x^{\bar{M}}$, and so the proof of [**38**, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{2(q+1)^2.|C:C^0|}{q^{\delta(x)-4}(q-1)^6}, \tag{4.16}$$

where $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$ and $C = C_{\bar{G}}(x)$. We note that $|C:C^0| \leqslant 6$ (see [**55**], for example). If $\dim x^{\bar{G}} \geqslant 54$ then $\delta(x) \geqslant 18$ (see [**36**, Table A]) and thus $\mathrm{fpr}(x) < q^{-13} = d_2$. There are fewer than $2q^{52} = c_3$ unipotent elements $x \in G$ such that $48 \leqslant \dim x^{\bar{G}} < 54$ and (4.16) yields $\mathrm{fpr}(x) < q^{-14} = d_3$ since $\delta(x) \geqslant 16$. Similarly, there are less than $2q^{42} = c_4$ such elements x with $40 \leqslant \dim x^{\bar{G}} < 48$ and this time (4.16) gives $\mathrm{fpr}(x) < q^{-8} = d_4$ since $\delta(x) \geqslant 12$ and $|C:C^0| \leqslant 2$. There are no more than $3q^{32} = c_5$ remaining unipotent elements and [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant q^{-6} = d_5$. Now assume that $p = 2$, so x lies in one of the $\bar{G}$-classes A_1 and $2A_1$ (see [**36**, Table A]). Applying [**38**, Theorem 2] we deduce that the contribution to $\widehat{Q}(G,6)$ from unipotent involutions is less than $3q^{32}.q^{-6.6} = 3q^{-4}$.

Next suppose that $x \in G$ is a field or graph-field automorphism of prime order r. As in the proof of Lemma 4.15, the contribution to $\widehat{Q}(G,6)$ from involutory field and graph-field automorphisms is less than $c_6 d_6^6$, where $c_6 = 4q^{39}$ and $d_6 = q^{-12}$. On the other hand, if r is odd then

$$\mathrm{fpr}(x) \leqslant \frac{|F_4(q):F_4(q^{1/r})|}{|E_6^\epsilon(q):E_6^\delta(q^{1/r})|} < 12q^{-26(1-1/r)} < 12q^{-17} = d_7,$$

and of course there are fewer than $\log_2 q.q^{78} = c_7$ such elements. Finally, suppose that $x \in G$ is an involutory graph automorphism. If $C_{\bar{G}}(x) \neq F_4$ then $|x^G| < 2q^{42} = c_8$ and applying [**38**, Proposition 1.3] we deduce that $|x^G \cap H| \leqslant i_2(\mathrm{Aut}(F_4(q))) < 2(q+1)q^{27}$ and thus $\mathrm{fpr}(x) < q^{-11} = d_8$ since $|x^G| > \frac{1}{6}q^{42}$. If $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2q^{26} = c_9$ and the proof of [**38**, Lemma 5.4] implies that

$$\mathrm{fpr}(x) \leqslant \frac{|F_4(q):B_4(q)|}{|E_6^\epsilon(q):F_4(q)|} < 12q^{-10} = d_9$$

(note that this bound is valid for all p). Applying Proposition 2.3 we conclude that

$$\widehat{Q}(G,6) < b(a/b)^6 + \sum_{i=1}^{9} d_i(c_i/d_i)^6 < 1,$$

where $a = q^{52}$ and $c_2 = q^{72}$.

Finally, suppose that $q = 2$ and note that $G \leqslant G_0.2$. Using MAGMA we can compute precise fixed point ratios for all elements $x \in G_0$, while $\mathrm{fpr}(x)$ is given in the proof of [**38**, Lemma 5.4] when x is an involutory graph automorphism. It follows that $b(G) \leqslant 4$ since $\widehat{Q}(G,4) < 1$. (By Proposition 2.4, this implies that $b(G) = 4$ if $G = E_6^{\epsilon}(2).2$.) □

LEMMA 4.20. *If H is of type $C_4(q)$ then $b(G) \leqslant 5$.*

Proof. Here p is odd and $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = C_4$ is a σ-stable subgroup of $\bar{G}$. If $x \in G$ is semisimple and $\dim x^{\bar{G}} \geqslant 48$ then $|x^G| > \frac{1}{2}(q+1)^{-2}q^{50} = b$; there are fewer than $q^{46} = c_1$ remaining semisimple elements and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-12} = d_1$. Now assume that $x \in H$ is a unipotent element of order p. Then the $\bar{G}$-class of x is determined in [**35**] (see Table 5 in Section 2) and we deduce that (4.16) holds since $x^{\bar{G}} \cap \bar{M} = x^{\bar{M}}$. Now, if $\dim x^{\bar{G}} \geqslant 40$ then $\delta(x) \geqslant 22$ and thus (4.16) yields $\mathrm{fpr}(x) < q^{-17} = d_2$; there are less than $3q^{32} = c_3$ remaining unipotent elements and (4.16) gives $\mathrm{fpr}(x) < q^{-9} = d_3$ since $\delta(x) \geqslant 14$.

Now if x is a field or graph-field automorphism of prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|\mathrm{Sp}_8(q) : \mathrm{Sp}_8(q^{1/r})|}{|E_6^{\epsilon}(q) : E_6^{\delta}(q^{1/r})|} < 12q^{-42(1-1/r)} \leqslant 12q^{-21} = d_4.$$

Finally, suppose that $x \in G$ is an involutory graph automorphism. If $C_{\bar{G}}(x) = C_4$ then $|x^G| < 2q^{42} = c_5$ and we may assume that x centralizes $\bar{M}$. Therefore [**38**, Proposition 1.3] implies that

$$|x^G \cap H| \leqslant i_2(\mathrm{Aut}(\mathrm{PSp}_8(q))) < 2(q+1)q^{19}$$

and thus $\mathrm{fpr}(x) < q^{-19} = d_5$ since $|x^G| > \frac{1}{6}q^{42}$. Conversely, if $C_{\bar{G}}(x) = F_4$ then $|x^G| < 2q^{26} = c_6$ and the proof of [**38**, Lemma 5.4] gives

$$\mathrm{fpr}(x) \leqslant \frac{|\mathrm{Sp}_8(q) : \mathrm{Sp}_2(q)\mathrm{Sp}_6(q)|}{|E_6^{\epsilon}(q) : F_4(q)|} < 12q^{-14} = d_6.$$

Applying Proposition 2.3 we conclude that $\widehat{Q}(G,5) < b(a/b)^5 + \sum_{i=1}^{6} c_i d_i^5 < 1$, where $a = q^{36}$, $c_2 = q^{72}$ and $c_4 = 2\log_2 q.q^{78}$. □

PROPOSITION 4.21. *If H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 6$.*

Proof. In view of Lemmas 4.15–4.20 we may assume that H is one of the cases (ii)–(iv) in the statement of Lemma 4.14. Now if H has socle Fi_{22} then using MAGMA we can check that $\widehat{Q}(G,3) < 1$ and thus $b(G) = 3$ since $\log|G|/\log|\Omega| > 2$ (see Proposition 2.4). In a similar fashion, we deduce that $b(G) = 2$ if $G = {}^2E_6(2).2$ and $H = \mathrm{SO}_7(3)$.

Now assume that $\epsilon = +$ and H is of type $E_6^{\delta}(q^{1/2})$. Then $H_0 = H \cap G_0 = C_{G_0}(\tau)$, where τ is an involutory field automorphism of G_0 if $\delta = +$, and τ is a graph-field automorphism of G_0 if $\delta = -$. We claim that $b(G) \leqslant 5$. To see this, first let $x \in G$ be a semisimple element of prime order. Then $x^{G_0} \cap H_0$ is a union of at most $(3, q-1)$ distinct H_0-classes and thus

$$\mathrm{fpr}(x) < \frac{6(q+1)^6}{q^{\frac{1}{2}\dim x^{\bar{G}}+3}(q^{1/2}-1)^6}.$$

In particular, if $\dim x^{\bar{G}} \geqslant 48$ then $\mathrm{fpr}(x) < q^{-18} = b_1$. There are fewer than $q^{46} = a_2$ remaining semisimple elements and [38, Theorem 2] states that $\mathrm{fpr}(x) \leqslant q^{-12} = b_2$. Next let $x \in G$ be a unipotent element of order p. If $p > 2$ then $\mathrm{fpr}(x) < 8(q+1)^2 q^{-(1/2)\dim x^{\bar{G}}-2}$ and so the contribution to $\widehat{Q}(G,5)$ from unipotent elements is less than

$$\sum 4q^{\dim x^{\bar{G}}}.\left(8(q+1)^2 q^{-\frac{1}{2}\dim x^{\bar{G}}-2}\right)^5 < q^{-22} = c,$$

where we sum over a set of representatives for the distinct $\bar{G}$-classes of unipotent elements $x \in H$ of order p. Similarly, if $p = 2$ then $x^{G_0} \cap H_0 = x^{H_0}$ and we quickly deduce that the contribution from unipotent involutions is less than q^{-27}.

Next let $x \in G$ be a field or graph-field automorphism of prime order r. If r is odd then x induces a field automorphism on H_0 and thus $\mathrm{fpr}(x) < 12q^{-26} = b_3$. As before, the contribution to $\widehat{Q}(G,5)$ from involutory field and graph-field automorphisms is less than $a_4 b_4^5$, where $a_4 = 4q^{39}$ and $b_4 = q^{-12}$. Now, if $x \in G$ is an involutory graph automorphism then x induces a graph automorphism on H_0 such that the centralizers $C_{H_0}(x)$ and $C_{G_0}(x)$ are of the same type. It follows that $\mathrm{fpr}(x) < 12q^{-39+(1/2)\dim x^{\bar{G}}}$, so we have $|x^G| < 2q^{26} = a_5$ and $\mathrm{fpr}(x) < 12q^{-13} = b_5$ if $C_{\bar{G}}(x) = F_4$, otherwise $|x^G| < 2q^{42} = a_6$ and $\mathrm{fpr}(x) < 12q^{-21} = b_6$. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < c + \sum_{i=1}^{6} a_i b_i^5 < 1$, where $a_1 = q^{78}$ and $a_3 = \log_2 q.q^{78}$. □

4.4. $G_0 = F_4(q)$

The conjugacy classes of G are determined in [60] for even q and in [62] for odd q. If q is odd then there are precisely two classes of semisimple involutions, with representatives labelled t_1 and t_2 in [62, Table 9], where $C_{\bar{G}}(t_1) = A_1C_3$ and $C_{\bar{G}}(t_2) = B_4$. If $p = 2$ then there are exactly four classes of unipotent involutions, with representatives labelled x_1, x_2, x_3 and x_4 in [60, 2.1]; these correspond to the four $\bar{G}$-classes labelled A_1, $\widetilde{A}_1$, $\widetilde{A}_1^{(2)}$ and $A_1 + \widetilde{A}_1$ in [39, Table 2].

Lemma 4.22. *If* $|H| \leqslant q^{22}$ *then* $b(G) \leqslant 6$.

Proof. First assume that $q \geqslant 3$. If $x \in \bar{G}_\sigma$ has prime order and $\dim x^{\bar{G}} \geqslant 28$ then $|x^G| > q^{28} = b$ (see [60] and [62]). If $\dim x^{\bar{G}} < 28$ then [38, Theorem 2] gives

$$\mathrm{fpr}(x) \leqslant (q^4 - q^2 + 1)^{-1} = d_1$$

and we note that there are fewer than $2q^{22} = c_1$ such elements. If $x \in G$ is an involutory field or graph-field automorphism then $|x^G| < 2q^{26} = c_2$ and [38, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-6} = d_2$. (Note that G cannot simultaneously contain automorphisms of both types.) Finally, if x is a field automorphism of odd prime order then $|x^G| > b$, and applying Proposition 2.3 we conclude that $b(G) \leqslant 6$ since $\widehat{Q}(G,6) < b(a/b)^6 + \sum_{i=1}^{2} c_i d_i^6 < 1$, where $a = q^{22}$.

Now assume that $q = 2$. As above, the combined contribution to $\widehat{Q}(G,6)$ from graph-field automorphisms and elements $x \in G$ with $|x^G| > 2^{28}$ is less than $b(a/b)^6 + c_2 d_2^6 < 2^{-5}$, so assume that $x \in G_0$ and $|x^G| \leqslant 2^{28}$. Then x is an involution which belongs to one of the $\bar{G}$-classes labelled A_1, $\widetilde{A}_1$ and $\widetilde{A}_1^{(2)}$ in [39, Table 2]. Together, there are fewer than $3.2^{16} = e_1$ elements in the G-classes A_1 and $\widetilde{A}_1$ (see [60]) and [38, Theorem 2] states that $\mathrm{fpr}(x) \leqslant (2^4 - 2^2 + 1)^{-1} = f_1$. Now there are less than $2^{23} = e_2$ elements in the class $\widetilde{A}_1^{(2)}$ and we claim that $\mathrm{fpr}(x) \leqslant 2^{-4} = f_2$. This is trivial if $|H| \leqslant 2^{18}$ since $|x^G| > 2^{22}$, and it follows from [38, Theorem 2] if H is not a subgroup of maximal rank. According to [40], if H is a maximal rank subgroup and $2^{18} < |H| \leqslant 2^{22}$ then $H \cap G_0 = \mathrm{Sp}_4(2) \wr S_2$ or $\mathrm{Sp}_4(4).2$, and hence $i_2(H \cap G_0) < 2^{18}$ and the claim follows. For instance,

$$i_2(\mathrm{Sp}_4(2) \wr S_2) = (i_2(\mathrm{Sp}_4(2)) + 1)^2 - 1 + |\mathrm{Sp}_4(2)| = 6495 < 2^{18}.$$

We conclude that $\widehat{Q}(G,6) < 2^{-5} + \sum_{i=1}^{2} e_i f_i^6 < 1$. □

Lemma 4.23. *If $|H| > q^{22}$ then one of the following holds:*
(i) *$H = N_G(\bar{M}_\sigma)$, where $\bar{M}^0 = B_4$, D_4, A_1C_3 or C_4 ($p=2$);*
(ii) *H is of type $F_4(q^{1/2})$ or ${}^2F_4(q)$;*
(iii) *$q=2$ and H has socle $L_4(3)$.*

Lemma 4.24. *If $G_0 = F_4(2)$ and H has socle $H_0 = L_4(3)$ then $b(G) \leqslant 6$.*

Proof. Let $x \in G$ be a semisimple element of prime order. If $\dim x^{\bar{G}} \geqslant 36$ then $|x^G| > 2^{36}$ and thus $\mathrm{fpr}(x) < |\mathrm{Aut}(H_0)|.2^{-36} < 2^{-11} = b_1$. There are less than $2^{31} = a_2$ semisimple elements $x \in G$ with $\dim x^{\bar{G}} < 36$, while [**38**, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2^{-6} = b_2$. Next let $x \in G$ be a unipotent involution. As in the proof of Lemma 4.22, if $x \in A_1$ or $x \in \widetilde{A}_1$ then [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2^{-4} = b_3$ and we note that there are fewer than $3.2^{16} = a_3$ such elements. The remaining class of involutions contains fewer than $2^{30} = a_4$ elements and we have $\mathrm{fpr}(x) < 2^{-7} = b_4$ since $i_2(\mathrm{Aut}(H_0)) = 27639$ and $|x^G| > 2^{22}$. Finally, if x is an involutory graph-field automorphism then $|x^G| < 2^{27} = a_5$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2^{-6} = b_5$. We conclude that $b(G) \leqslant 6$ since $\widehat{Q}(G,6) < \sum_{i=1}^{5} a_i b_i^6 < 1$, where $a_1 = 2^{52}$. □

Lemma 4.25. *If H is of type $B_4(q)$ then $b(G) \leqslant 6$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = B_4$ is a σ-stable subgroup of $\bar{G}$ and $H \cap G_0 = H_0 = B_4(q)$. If $q=2$ then generators for H and G are given in the Web Atlas [**72**] and an easy Magma calculation yields $b(G) = 4$. Now assume that $q \geqslant 3$. Let $x \in G$ be a semisimple element of prime order. Then [**38**, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{|W(F_4):W(B_4)|.2(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4} = \frac{6(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4}, \tag{4.17}$$

where $z = \dim Z(\bar{D})$, $\bar{D} = C_{\bar{G}}(x)$ and

$$\begin{aligned}\delta(x) := \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M}) &= 2\left(|\Phi^+(\bar{G})| - |\Phi^+(\bar{M})| - |\Phi^+(\bar{D})| + |\Phi^+(\bar{D} \cap \bar{M})|\right)\\ &= 16 - 2\left(|\Phi^+(\bar{D})| - |\Phi^+(\bar{D} \cap \bar{M})|\right).\end{aligned}$$

If $z > 2$ then $|\Phi^+(\bar{D})| \leqslant 1$, so $\delta(x) \geqslant 14$ and (4.17) implies that $\mathrm{fpr}(x) < q^{-9} = b_1$. We also observe that $\delta(x) \geqslant 14$ (and thus $\mathrm{fpr}(x) < b_1$) if $\bar{D}' = A_3$, $A_2\widetilde{A}_1$, $A_1^2\widetilde{A}_1$, A_2, A_1^2 or $A_1\widetilde{A}_1$ because $\Phi(\bar{D} \cap \bar{M})$ contains all the long roots of $\Phi(\bar{D})$. Next suppose that $\bar{D}' = A_2\widetilde{A}_2$, $A_1\widetilde{A}_2$, $\widetilde{A}_2$ or B_2. Inspecting [**60**] and [**62**] we calculate that there are fewer than $q^{46} = a_2$ such elements and thus (4.17) gives $\mathrm{fpr}(x) < 2q^{-9} = b_2$ since $z \leqslant 2$ and $\delta(x) \geqslant 12$ by [**39**, Theorem 2]. Similarly, there are fewer than $2q^{31} = a_3$ semisimple elements in G with $\bar{D}' = B_3$ or $\bar{D}' = C_3$ and we claim that $\mathrm{fpr}(x) < q^{-6} = b_3$. If $\bar{D}' = B_3$ then [**39**, Theorem 2] gives $\delta(x) \geqslant 10$ and the claim follows since the proof of [**38**, Lemma 4.5] yields $\mathrm{fpr}(x) < 3(q+1).(q-1)^{-4}q^{-7}$ because $|x^G| > (q+1)^{-1}q^{31}$. Now, if $\bar{D}' = C_3$ then $\Phi(\bar{D} \cap \bar{M}) = A_1C_2$ since $\Phi(\bar{D} \cap \bar{M})$ is A_2-dense in $\Phi(\bar{D})$ and must contain all the long roots of $\Phi(\bar{D})$ (see [**39**, Lemma 5.1]). In particular, we have

$$|C_{H_0}(x)| = |\mathrm{SO}_5(q)||\mathrm{GL}_2^{\epsilon}(q)| > (q-1)^2 q^{12},$$

and arguing as in the proof of [**38**, Lemma 4.5] we deduce that

$$\mathrm{fpr}(x) < 3(q+1).(q-1)^{-2}q^{-7} \leqslant q^{-6}$$

since $|x^G| > (q+1)^{-1}q^{31}$ and $\delta(x) = 8$. This justifies the claim. For semisimple elements, it remains to consider involutions. Now there are fewer than $2q^{16} = a_4$ involutions $x \in G$ with $\bar{D} = B_4$, while [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-5} = b_4$. Similarly, there are less than $2q^{28} = a_5$ involutions x with $\bar{D} = A_1C_3$ and the proof of [**38**, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < 3(q-1)^{-2}q^{-6} < q^{-6} = b_5$$

since $|x^G| > q^{28}$ and $|C_{H_0}(x)| > (q-1)^2 q^{\dim C_{\bar{M}}(x)-2}$.

Next suppose that $x \in G$ has order p and assume for now that p is odd. If the $\bar{M}$-class of x is labelled by the pair $(L, P_{L'})$ and the Levi subgroup $L < \bar{M}$ is also a Levi subgroup of $\bar{G}$ then the $\bar{G}$-class of x inherits the same label. In the few remaining cases we use the fact that $V_{26} \downarrow B_4 = V(\lambda_1) \oplus V(\lambda_4) \oplus 0$ to calculate the Jordan form of x on the 26-dimensional $\bar{G}$-module V_{26} and we can then identify the $\bar{G}$-class of x by inspecting [**36**, Table 3] (note that the Jordan form of x on $V(\lambda_4)$ is listed in [**8**, Table 5] if $\dim x^{\bar{M}} \geqslant 24$, we refer the reader to the proof of [**6**, Lemma 2.8]). In this way, using [**62**, Tables 4–6], we deduce that $\mathrm{fpr}(x) < 3q^{-10} = d_1$ if $\dim x^{\bar{G}} \geqslant 34$. For example, if x lies in the $\bar{G}$-class labelled B_2 then

$$\mathrm{fpr}(x) \leqslant \frac{2(|B_4(q) : q^9A_1(q)| + |B_4(q) : q^7A_1(q)|)}{|F_4(q) : q^{10}A_1(q^2)|} = \frac{2(q^2+1)^2(q^4-1)}{q^6(q^{12}-1)} < 3q^{-10}.$$

Similarly, if $\dim x^{\bar{G}} < 34$ then we derive the bounds $|(x^{\bar{G}})_\sigma| < c_i$ and $\mathrm{fpr}(x) < d_i$ (listed in Table 10):

TABLE 10. $B_4 < F_4, p > 2$.

i	$\bar{G}$-class of x	c_i	d_i
2	A_1	$2q^{16}$	$(q^4-q^2+1)^{-1}$
3	$\widetilde{A}_1$	$2q^{22}$	$3q^{-6}$
4	$A_1+\widetilde{A}_1$	$2q^{28}$	q^{-8}
5	A_2	q^{30}	$3q^{-8}$

We conclude that if $p > 2$ then the contribution to $\widehat{Q}(G,6)$ from unipotent elements is less than $\sum_{i=1}^{5} c_i d_i^6 < 2q^{-6}$, where $c_1 = q^{48}$ (see Proposition 2.2). Now assume that $p = 2$. As described in [**1**], there are six distinct classes of involutions in B_4; the corresponding $\bar{G}$-classes are listed in the proof of [**39**, Lemma 4.6] and thus $\delta(x) := \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$ is easily determined. From [**60**, Lemma 2.1] we deduce that

$$q^{\dim x^{\bar{G}}} < |x^G| < 2q^{\dim x^{\bar{G}}}$$

and thus $\mathrm{fpr}(x) < 2q^{-\delta(x)}$ since $|x^G \cap H| < 2q^{\dim x^{\bar{M}}}$ (see [**10**, Proposition 3.22], for example). In this way we calculate that unipotent involutions contribute less than $2q^{-3} = c$.

As in the proof of Lemma 4.22, the contribution to $\widehat{Q}(G,6)$ from involutory field and graph-field automorphisms is less than $a_6b_6^6$, where $a_6 = 2q^{26}$ and $b_6 = q^{-6}$. If $x \in G$ is a field automorphism of odd prime order r then

$$\mathrm{fpr}(x) = \frac{|B_4(q) : B_4(q^{1/r})|}{|F_4(q) : F_4(q^{1/r})|} < 4q^{-16(1-1/r)} \leqslant 4q^{-32/3} = b_7,$$

and we conclude that $b(G) \leqslant 6$ since $\widehat{Q}(G,6) < c + \sum_{i=1}^{7} a_ib_i^6 < 1$, where $a_1 = q^{52}$ and $a_7 = \log_2 q.q^{52}$. □

LEMMA 4.26. *If H is of type $D_4(q)$ or ${}^3D_4(q)$ then $b(G) \leqslant 6$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = D_4.S_3$ is a σ-stable closed subgroup of $\bar{G}$ and H has socle $H_0 = \mathrm{P}\Omega_8^+(q)$ or $H_0 = {}^3D_4(q)$ (see [**40**, Table 5.1]). We note that if $p = 2$ then the maximality of H implies that G does not contain an involutory graph-field automorphism. The case $q = 2$ can be handled using MAGMA: we calculate that $\widehat{Q}(G,4) < 1$ and thus $b(G) \leqslant 4$. (In fact, if $H = D_4(2).S_3$ then $\widehat{Q}(G,3) < 1$ and thus Proposition 2.4 implies that $b(G) = 3$ in this particular case.) For the remainder we will assume that $q \geqslant 3$.

Let $x \in G$ be a semisimple element of prime order. Then [**38**, Lemma 4.5] gives

$$\mathrm{fpr}(x) < \frac{(|W(F_4) : W(D_4).6| + 3).12(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4} = \frac{48(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4}, \tag{4.18}$$

where z and $\delta(x)$ are defined as before. If $\bar{D} = C_{\bar{G}}(x)$ does not have a B_4, C_3 or B_3 factor then (4.18) yields $\mathrm{fpr}(x) < q^{-9} = b_1$ since [**39**, Theorem 2] states that $\delta(x) \geqslant 16$. Similarly, we deduce that $\mathrm{fpr}(x) < q^{-6} = b_2$ if $\bar{D}$ has a C_3 or B_3 factor and we note that there are fewer than $2q^{31} = a_2$ such elements in G. Finally, if $\bar{D} = B_4$ then $|x^G| < 2q^{16} = a_3$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant 2q^{-5} = b_3$.

Next let $x \in G$ be a unipotent element of order p and first assume that $p > 2$. Now, if $p = 3$ and $x \in \bar{M} - \bar{M}^0$ then x induces a triality graph automorphism on D_4 and we can determine the $\bar{G}$-class of x by considering the restriction

$$V_{26} \downarrow D_4 = V(\lambda_1) \oplus V(\lambda_3) \oplus V(\lambda_4) \oplus 0^2. \tag{4.19}$$

Indeed, we see that x has Jordan form $[J_3^8, J_2]$ on V_{26} because x permutes the 8-dimensional modules $V(\lambda_1)$, $V(\lambda_3)$ and $V(\lambda_4)$, while interchanging the two trivial modules. Then [**36**, Table 3] indicates that x lies in either $\widetilde{A}_2$ or $\widetilde{A}_2 + A_1$. By considering centralizer orders, it is easy to see that $x \in \widetilde{A}_2$ if $C_{D_4}(x) = G_2$, otherwise $x \in \widetilde{A}_2 + A_1$ (see [**62**, Table 6]). In the same way we can determine the $\bar{G}$-class of each unipotent element $x \in \bar{M}^0$.

Now, there are fewer than $3q^{22} = a_4$ unipotent elements $x \in G$ with $\dim x^{\bar{G}} < 28$ and we calculate that $\mathrm{fpr}(x) < 2q^{-6} = b_4$. Similarly, if $\dim x^{\bar{G}} \geqslant 28$ then $\mathrm{fpr}(x) < 8q^{-12} = b_5$. Now assume that $p = 2$ and that $x \in G$ is a unipotent involution. If $x \in \bar{M} - \bar{M}^0$ then x induces an involutory graph automorphism on D_4, so in the notation of [**1**], x is either a b_1 or a b_3 involution. If $x = b_l$, where $l = 1$ or $l = 3$, then (4.19) implies that the Jordan form of x on V_{26} has precisely $9 + l$ Jordan 2-blocks and thus [**36**, Table 3] reveals that x lies in the $\bar{G}$-class $\widetilde{A}_1$ if $l = 1$, otherwise x is in the class $A_1 + \widetilde{A}_1$. The $\bar{G}$-class of each involution in D_4 can be determined in a similar fashion. For any p, the reader can check that the total contribution to $\widehat{Q}(G,6)$ from unipotent elements is less than $a_4 b_4^6 + a_5 b_5^6$, where $a_5 = q^{48}$.

Finally, suppose that $x \in G$ is a field automorphism of prime order r. As in the proof of Lemma 4.22, the contribution to $\widehat{Q}(G,6)$ from involutory field automorphisms is less than $a_6 b_6^6$, where $a_6 = 2q^{26}$ and $b_6 = q^{-6}$. If $r = 3$ then $|x^G \cap H| \leqslant i_3(\mathrm{Aut}(H_0)) < 3q^{16}$ (see [**38**, Proposition 1.3]) and thus $\mathrm{fpr}(x) < 6q^{-56/3} = b_7$ since $|x^G| > \frac{1}{2}q^{104/3}$. We also observe that there are fewer than $4q^{104/3} = a_7$ such elements. Finally, if $r \geqslant 5$ and $H_0 = D_4(q)$ then

$$\mathrm{fpr}(x) \leqslant 6\frac{|D_4(q) : D_4(q^{1/r})|}{|F_4(q) : F_4(q^{1/r})|} < 24q^{-24(1-1/r)} \leqslant 24q^{-96/5} = b_8,$$

and it is easy to see that the same bound $\mathrm{fpr}(x) < b_8$ holds if $H_0 = {}^3D_4(q)$. We conclude that $b(G) \leqslant 6$ since $\widehat{Q}(G,6) < \sum_{i=1}^{8} a_i b_i^6 < 1$, where $a_1 = q^{52}$, $a_5 = q^{48}$ and $a_8 = \log_2 q.q^{52}$. □

LEMMA 4.27. *If H is of type $A_1(q)C_3(q)$ then $b(G) \leqslant 5$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = A_1C_3$ is a σ-stable subgroup of $\bar{G}$. According to [**40**, Table 5.1] we may assume that q is odd. If $x \in G$ is a semisimple element of prime order then [38, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{|W(F_4):W(A_1C_3)|.2(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4} = \frac{192(q+1)^z}{q^{\delta(x)+z-4}(q-1)^4}, \quad (4.20)$$

where z and $\delta(x)$ are defined in the usual way. Now, if $\bar{D} = C_{\bar{G}}(x)$ does not have a B_4, C_3 or B_3 factor then (4.20) yields $\mathrm{fpr}(x) < \frac{3}{2}q^{-11} = b_1$ since $\delta(x) \geqslant 18$ by [39, Theorem 2]. Combined, there are fewer than $2q^{31} = a_2$ semisimple elements x such that $\bar{D}$ has a C_3 or B_3 factor, and (4.20) gives $\mathrm{fpr}(x) < q^{-7} = b_2$ since $z \leqslant 1$ and $\delta(x) \geqslant 14$. Finally, if $\bar{D} = B_4$ then $|x^G| < 2q^{16} = a_3$ and [38, Theorem 2] states that $\mathrm{fpr}(x) \leqslant 2q^{-5} = b_3$.

Now assume that $x = u_1u_2 \in \bar{M}$ is a unipotent element of order p, where $u_1 \in A_1$ and $u_2 \in C_3$. Since p is odd, the $\bar{M}$-class of x is labelled by a pair $(L, P_{L'})$, where L is a Levi subgroup of $\bar{M}$ and $P_{L'}$ is a distinguished parabolic subgroup of L'. Now, if L is also a Levi subgroup of $\bar{G}$ then the $\bar{G}$-class of x has the same label. This always holds if $u_1 = 1$, but there are a few cases where it fails when $u_1 = u$ is non-trivial. In all cases the $\bar{G}$-class of x is given in [35], and the relevant results can be found in Section 2 (see Table 6). In this way we deduce that $x^{\bar{G}} \cap \bar{M}$ is a union of at most two distinct $\bar{M}$-classes for any $x \in \bar{M}$ of order p. Therefore the proof of [38, Lemma 4.5] implies that

$$\mathrm{fpr}(x) < \frac{2.2(q+1)^y|C:C^0|}{q^{\delta(x)+y-4}(q-1)^4} \leqslant \frac{96}{q^{\delta(x)-4}(q-1)^4}, \quad (4.21)$$

where $C = C_{\bar{G}}(x)$, $y = \dim Z(C^0/R_u(C^0)) = 0$ (see [62]) and $\delta(x) = \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M})$. Now, if $\dim x^{\bar{G}} \geqslant 28$ then $\delta(x) \geqslant 16$ and thus (4.21) yields $\mathrm{fpr}(x) \leqslant 2q^{-11} = b_4$. Similarly, if x belongs to the $\bar{G}$-class $\widetilde{A}_1$ then $|x^G| < 2q^{22} = a_5$ and $\mathrm{fpr}(x) < 2q^{-7} = b_5$, while we have $|x^G| < 2q^{16} = a_6$ and $\mathrm{fpr}(x) < 2q^{-5} = b_6$ if x is in A_1.

As observed in the proof of Lemma 4.22, the contribution to $\widehat{Q}(G,5)$ from involutory field automorphisms is less than $a_7b_7^5$, where $a_7 = 2q^{26}$ and $b_7 = q^{-6}$. If x is a field automorphism of odd prime order r then

$$\mathrm{fpr}(x) = \frac{|A_1(q)C_3(q):A_1(q^{1/r})C_3(q^{1/r})|}{|F_4(q):F_4(q^{1/r})|} < 8q^{-28(1-1/r)} \leqslant 8q^{-56/3} = b_8,$$

and we conclude that $\widehat{Q}(G,5) < \sum_{i=1}^{8} a_ib_i^5 < 1$, where $a_1 = q^{52}$, $a_4 = q^{48}$ and $a_8 = \log_2 q.q^{52}$. □

PROPOSITION 4.28. *If H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 6$.*

Proof. By the previous results, may assume that H is of type $F_4(q^{1/2})$ or ${}^2F_4(q)$. If $q = 2$ then $G = F_4(2)$, $H = {}^2F_4(2)'.2$ and a MAGMA calculation yields $\widehat{Q}(G,3) < 1$, and hence $b(G) \leqslant 3$. For the remainder, we will assume that $q \geqslant 3$. We claim that $b(G) \leqslant 5$.

We will assume that $H_0 = H \cap G_0 = {}^2F_4(q)$ since a very similar argument applies when H is of type $F_4(q^{1/2})$. Here $q = 2^{2m+1}$ for some $m \geqslant 1$ and we note that $H_0 = C_{G_0}(\tau)$ for an involutory graph-field automorphism τ of G_0. Let $x \in H$ be a semisimple element of prime order, and observe that $x^{G_0} \cap H_0 = x^{H_0}$ since $\bar{D} = C_{\bar{G}}(x)$ is connected. Since τ swaps long and short roots, $\bar{D}$ must contain an equal number of long and short roots, so $|x^G| > q^{36} = b$ because $\bar{D} = A_2\widetilde{A}_2$, B_2T_2, $A_1\widetilde{A}_1T_2$ or T_4. Similarly, if $x \in H$ is a unipotent involution then x belongs to one of the $\bar{G}$-classes labelled $\widetilde{A}_1^{(2)}$ and $A_1 + \widetilde{A}_1$. According to [61], if $p = 2$ then H_0 contains precisely two classes of involutions, represented by t_2 and t_2', where $|C_{H_0}(t_2)| = q^{10}(q^2-1)$ and $|C_{H_0}(t_2')| = q^{12}(q^2+1)(q-1)$. Further, Lagrange's theorem implies that $t_2 \in A_1 + \widetilde{A}_1$ and $t_2' \in \widetilde{A}_1^{(2)}$, so $\mathrm{fpr}(x) < q^{-11} = d_1$ and we note that G contains fewer than $2q^{22} = c_1$ unipotent involutions. If x is a field automorphism of prime order r then r must be odd and

$$\mathrm{fpr}(x) \leqslant \frac{|{}^2F_4(q):{}^2F_4(q^{1/r})|}{|F_4(q):F_4(q^{1/r})|} < 4q^{-26(1-1/r)} \leqslant 4q^{-\frac{52}{3}} = d_2.$$

Finally, if $x \in G$ is an involutory graph-field automorphism then $|x^G| < 2q^{26} = c_3$, and we may assume that x centralizes H_0. Therefore $|x^G \cap H| = i_2(H_0)+1 < 2q^{14}$ and thus $\mathrm{fpr}(x) < 2q^{-12} = d_3$. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < b(a/b)^5 + \sum_{i=1}^{3} c_i d_i^5 < 1$, where $a = q^{26}$ and $c_2 = \log_2 q.q^{52}$. □

4.5. $G_0 = G_2(q)'$

The maximal subgroups of G are determined in [**19**] for even q, and in [**31**] for odd q. In addition, detailed information on the conjugacy classes in G can be found in [**16**] when $p \geqslant 5$, and in [**21**] for $p < 5$. The following lemma is an easy consequence of [**19**, **31**].

LEMMA 4.29. *If $q \geqslant 7$ and $|H \cap G_0| > q^6$ then H is of type $G_2(q^{1/2})$, ${}^2G_2(q)$ or $\mathrm{SL}_3^{\epsilon}(q)$.*

LEMMA 4.30. *If $|H \cap G_0| \leqslant q^6$ then $b(G) \leqslant 5$.*

Proof. If $q \leqslant 5$ then the lemma is easily checked using MAGMA (see Tables 11 and 12 in Section 6), so we will assume that $q \geqslant 7$. Let $x \in G_0$ be an element of prime order. If $\dim x^{\bar{G}} \geqslant 8$ then [**16**] and [**21**] imply that $|x^G| \geqslant (q^2-1)(q^6-1) = b_1$. There are fewer than $3q^6 = c_1$ elements $x \in G_0$ of prime order with $\dim x^{\bar{G}} < 8$ and [**38**, Theorem 2] gives that

$$\mathrm{fpr}(x) \leqslant (q^2-q+1)^{-1} = d_1.$$

Similarly, if x is an involutory field or graph-field automorphism then $|x^G| < 2q^7 = c_2$, and again we have $\mathrm{fpr}(x) \leqslant (q^2-q+1)^{-1} = d_2$. (Note that G cannot contain both involutory field and graph-field automorphisms.) Finally, if x is a field automorphism of odd prime order then $|x^G| > \frac{1}{2}q^{28/3} = b_2$, and applying Proposition 2.3 we conclude that

$$\widehat{Q}(G,5) < \sum_{i=1}^{2} b_i(a_i/b_i)^5 + \sum_{i=1}^{2} c_i d_i^5 < 1, \quad \text{where } a_1 = q^6 \text{ and } a_2 = \log_2 q.q^6. \qquad \square$$

LEMMA 4.31. *If H is of type $\mathrm{SL}_3^{\epsilon}(q)$ then $b(G) \leqslant 5$.*

Proof. Here $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = A_2.2$ is a σ-stable subgroup of $\bar{G}$. Using MAGMA we calculate that $b(G) = 3$ when $q \leqslant 5$ (see Tables 11 and 12), so we will assume that $q \geqslant 7$. Note that the maximality of H in G implies that G does not contain a graph-field automorphism when $p = 3$ (see [**31**]).

Let $x \in G$ be a semisimple element of odd prime order, so $x^{\bar{G}} \cap \bar{M} \subseteq \bar{M}^0$. Evidently, $\Phi(\bar{M})$ is the set of long roots in the root system of G_2, and hence $\Phi(\bar{D} \cap \bar{M})$ consists of the long roots in $\Phi(\bar{D})$, where $\bar{D} = C_{\bar{G}}(x)$. Therefore (4.6) implies that $\delta(x) := \dim x^{\bar{G}} - \dim(x^{\bar{G}} \cap \bar{M}) \geqslant 4$ and thus [**38**, Lemma 4.5] yields

$$\mathrm{fpr}(x) < \frac{|W(G_2):W(A_2).2|.4(q+1)^2}{q^{\delta(x)}(q-1)^2} \leqslant \frac{64}{9}q^{-4} = b_1.$$

If p is odd then G_0 contains precisely $q^4(q^4+q^2+1) = a_2$ involutions and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant (q^2-q+1)^{-1} = b_2$. Next let $x \in H$ be a unipotent element of order p. To determine the $\bar{G}$-class of x we first calculate the Jordan form of x on the 7-dimensional module for G_2. This is easy since $V_7 \downarrow A_2 = V_3 \oplus (V_3)^* \oplus 0$, and we can then identify the $\bar{G}$-class of x by inspecting [36, Table 1]. In particular, if $p = 2$ and $x \in \bar{M}^0$ then x is in the $\bar{G}$-class A_1, whence $|x^G| < q^6 = a_3$ and $\mathrm{fpr}(x) < 2q^{-2} = b_3$. Similarly, if $p = 2$ and $x \in \bar{M} - \bar{M}^0$ then x is in $\widetilde{A}_1$, so $|x^G| < q^8 = a_4$ and $\mathrm{fpr}(x) < 2q^{-3} = b_4$. Now if $p > 2$ then each regular unipotent element in A_2 lies in the $\bar{G}$-class $G_2(a_1)$ (since x has Jordan form $[J_3^2, I_1]$ on V_7), while the non-regular unipotent elements belong to the $\bar{G}$-class A_1. It is easy to check that the contribution to $\widehat{Q}(G,5)$ from unipotent elements is less than $a_3b_3^5 + a_4b_4^5$ for any q.

Finally, let x be a field automorphism of prime order r. If $r = 2$ then $|x^G| < 2q^7 = a_5$, and [38, Theorem 2] gives $\mathrm{fpr}(x) \leqslant (q^2 - q + 1)^{-1} = b_5$, whereas

$$\mathrm{fpr}(x) \leqslant 2\frac{|\mathrm{SL}_3^\epsilon(q) : \mathrm{SL}_3^\epsilon(q^{1/r})|}{|G_2(q) : G_2(q^{1/r})|} < 8q^{-6(1-1/r)} \leqslant 8q^{-4} = b_6$$

if r is odd. We conclude that $\widehat{Q}(G,5) < \sum_{i=1}^6 a_ib_i^5 < 1$, where $a_1 = q^{14}$ and $a_6 = \log_2 q.q^{14}$. □

PROPOSITION 4.32. *If H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 5$.*

Proof. We may assume that H is of type $G_2(q^{1/2})$ or ${}^2G_2(q)$. For brevity, we only give details for H of type ${}^2G_2(q)$ since the other case is very similar. Here $H_0 = H \cap G_0 = C_{G_0}(\tau)$, where τ is an involutory graph-field automorphism of G_0 and $q = 3^{2m+1}$ for some integer $m \geqslant 0$. If $m = 0$ then $H_0 \cong L_2(8).3$ and $b(G) \leqslant 3$ (see Table 11) so we will assume that $m \geqslant 1$. Let $x \in H$ be a semisimple element of prime order r and note that $x^{G_0} \cap H_0 = x^{H_0}$ since $C_{\bar{G}}(x)$ is connected. If $r > 2$ then $C_{\bar{G}}(x) = T_2$ is the only possibility since τ swaps long and short roots, whence $|x^G| > \frac{1}{2}(q+1)^{-2}q^{14} = b$. If $r = 2$ then $|x^G| < 2q^8 = c_1$ and $\mathrm{fpr}(x) < q^{-4} = d_1$ since both H_0 and G_0 contain a unique class of involutions (see [71]).

Next suppose that $x \in H$ is a unipotent element of order 3. Since $H_0 = C_{G_0}(\tau)$ and τ swaps long and short roots, it follows that x lies in one of the $\bar{G}$-classes labelled $\widetilde{A}_1^{(3)}$ and $G_2(a_1)$. As described in [71], there are three classes of elements of order 3 in H_0, with representatives t_i where $|C_{H_0}(t_1)| = |C_{H_0}(t_2)| = 2q^2$ and $|C_{H_0}(t_3)| = q^3$. By Lagrange's theorem, we have $t_1, t_2 \in G_2(a_1)$ and $t_3 \in \widetilde{A}_1^{(3)}$. In particular, if x is in the $\bar{G}$-class $G_2(a_1)$ then $|x^G| < q^{10} = c_2$ and $\mathrm{fpr}(x) < 4q^{-5} = d_2$, otherwise we have $|x^G| < q^8 = c_3$ and $\mathrm{fpr}(x) < 2q^{-4} = d_3$. If x is a field automorphism of prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|{}^2G_2(q) : {}^2G_2(q^{1/r})|}{|G_2(q) : G_2(q^{1/r})|} < 4q^{-7(1-1/r)} \leqslant 4q^{-14/3} = d_4.$$

Finally, if $x \in G$ is an involutory graph-field automorphism then $|x^G| < 2q^7 = c_5$, and we may assume that x centralizes H_0. Therefore $\mathrm{fpr}(x) < 2q^{-3} = d_5$ since $|x^G \cap H| = i_2(H_0) + 1 < 2q^4$ and $|x^G| > q^7$. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < b(a/b)^5 + \sum_{i=1}^5 c_id_i^5 < 1$, where $a = q^7$ and $c_4 = \log_3 q.q^{14}$. □

4.6. $G_0 = {}^2F_4(q)'$

Here $q = 2^{2m+1}$ for an integer $m \geqslant 0$. We refer the reader to Table 12 for the precise values of $b(G)$ when $q = 2$. For the remainder of this section we will assume that $q \geqslant 8$. The conjugacy classes in G_0 are described by Shinoda in [61]. In particular, we note that G has two classes of involutions and a unique class of elements or order 3, with respective representatives t_2, t_2' and t_3, where

$$(q-1)q^{13} < |t_2^G| < q^{14}, \quad (q-1)q^{10} < |t_2'^G| < q^{11}, \quad (q-1)q^{17} < |t_3^G| < q^{18}.$$

Furthermore, if $x \in G_0$ has order at least 5 then $|x^G| > \frac{1}{3}q^{20}$.

The maximal subgroups of G are determined in [53] and Lemma 4.33 follows.

Lemma 4.33. *If $q \geqslant 8$ and $|H| > q^9$ then H is of type ${}^2B_2(q) \wr S_2$ or $B_2(q).2$.*

Lemma 4.34. *If $|H| \leqslant q^9$ then $b(G) \leqslant 3$.*

Proof. As previously remarked, we may assume that $q \geqslant 8$. Suppose that $x \in G_0$ has prime order r and note that $|x^G| > \frac{1}{3}q^{20} = b$ if $r \geqslant 5$ (see [**61**]). If $r = 3$ then $|x^G| < q^{18} = c_1$ and $\mathrm{fpr}(x) < 2q^{-9} = d_1$ since $|x^G \cap H| < |H|$ and $|x^G| > \frac{1}{2}q^{18}$. Similarly, if $r = 2$ and x is G-conjugate to t_2 (see above) then $|x^G| < q^{14} = c_2$ and $\mathrm{fpr}(x) < (q-1)^{-1}q^{-4} = d_2$ since $|x^G \cap H| < |H|$ and $|x^G| > (q-1)q^{13}$. If x is conjugate to t_2' then $|x^G| < q^{11} = c_3$ and [**38**, Theorem 2] gives $\mathrm{fpr}(x) \leqslant q^{-4} = d_3$. Finally, suppose that $x \in G$ is a field automorphism of prime order r. If $r \geqslant 5$ then $|x^G| > \frac{1}{2}q^{104/5} > b$. On the other hand, if $r = 3$ then $|x^G \cap H| < |H|$ and $|x^G| > \frac{1}{2}q^{52/3}$, so $\mathrm{fpr}(x) < 2q^{-25/3} = d_4$ and we note that G contains fewer than $4q^{52/3} = c_4$ such elements. Applying Proposition 2.3 we conclude that $b(G) \leqslant 3$ since $\widehat{Q}(G,3) < b(a/b)^3 + \sum_{i=1}^{3} c_i d_i^3 < 1$, where $a = q^9$. □

Proposition 4.35. *If H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 3$.*

Proof. In view of Lemmas 4.33 and 4.34 we may assume that H is of type ${}^2B_2(q) \wr S_2$ or $B_2(q).2$. As before, we may also assume that $q \geqslant 8$. Write $H_0 = H \cap G_0$ and let $x \in H_0$ be an element of prime order r. If $r \geqslant 5$ then $\mathrm{fpr}(x) < 6q^{-10} = b_1$ since $|x^G| > \frac{1}{3}q^{20}$ and $|H_0| < 2q^{10}$. Similarly, if $r = 3$ then $|x^G| < q^{18} = a_2$ and $\mathrm{fpr}(x) < 2(q-1)^{-1}q^{-7} = b_2$. Now, if $r = 2$ and x is conjugate to t_2 then $|x^G| < q^{14} = a_3$,

$$i_2(H_0) \leqslant i_2(B_2(q).2) = (q^2-1)\left[2(q^2+1) + q^4 - 1 + q^2(q+1)\right] < 2q^6,$$

and it follows that $\mathrm{fpr}(x) < 2(q-1)^{-1}q^{-7} = b_3$. Similarly, if x is conjugate to t_2' then $|x^G| < q^{11} = a_4$ and $\mathrm{fpr}(x) < 2(q-1)^{-1}q^{-4} = b_4$. Finally, if $x \in G$ is a field automorphism of prime order r then

$$\mathrm{fpr}(x) \leqslant \frac{|{}^2B_2(q)^2 : {}^2B_2(q^{1/r})^2|}{|{}^2F_4(q) : {}^2F_4(q^{1/r})|} < 8q^{-16(1-1/r)}.$$

In particular, if $r = 3$ then $\mathrm{fpr}(x) < 8q^{-32/3} = b_5$, and we note that G contains fewer than $4q^{52/3} = a_5$ such elements. If $r \geqslant 5$ then $\mathrm{fpr}(x) < 8q^{-64/5} = b_6$ and we conclude that $b(G) \leqslant 3$ since $\widehat{Q}(G,3) < \sum_{i=1}^{6} a_i b_i^3 < 1$, where $a_1 = q^{26}$ and $a_6 = \log_2 q.q^{26}$. □

4.7. $G_0 = {}^2G_2(q)'$

Here $q = 3^{2m+1}$, where m is a non-negative integer. We may assume that $m \geqslant 1$ since ${}^2G_2(3)' \cong \mathrm{SL}_2(8)$. Further, we refer the reader to Table 11 for the precise $b(G)$ values when $m = 1$, and so in fact we will assume that $m \geqslant 2$. The maximal subgroups of G are determined in [**31**] and detailed information on the conjugacy classes of G_0 can be found in [**71**]. In particular, we note that $C_{G_0}(x) = 2 \times L_2(q)$ if $x \in G_0$ is an involution and that any two involutions are conjugate. In addition, there are precisely three conjugacy classes containing elements of order 3; the G_0-centralizers of class representatives are of size $2q^2$, $2q^2$ and q^3. The possibilities for $|C_{G_0}(x)|$ when $x \in G_0$ is a semisimple element of odd order are as shown in Table 13.

Lemma 4.36. *If H is of type $2 \times L_2(q)$ then $b(G) \leqslant 3$.*

Proof. Here $H = C_G(z)$ and $H \cap G_0 = 2 \times L_2(q)$, where z is an involution. If $q = 3^3$ then $b(G) = 2$ (see Table 11) so we can assume that $q \geqslant 3^5$. Let $x \in H_0$ be an element of prime

TABLE 13. *Semisimple elements of odd order.*

$\lvert C_{G_0}(x)\rvert$	Number of G_0-classes
$q-1$	$\frac{1}{2}(q-3)$
$q+1$	$\frac{1}{6}(q-3)$
$q\pm\sqrt{3q}+1$	$\frac{1}{6}(q\pm\sqrt{3q})$

order r. From the proof of [**38**, Lemma 6.2] we see that the combined contribution to $\widehat{Q}(G,3)$ from elements of order 2 and 3 in G_0 is less than $a_1b_1^3+a_2b_2^3$, where

$$a_1=q^2(q^2-q+1), b_1=q^{-2}, a_2=q(q^3+1)(q-1) \quad \text{and} \quad b_2=2q^{-1}(q^2-q+1)^{-1}.$$

Now assume that $r\geqslant 5$, so Lagrange implies that $|C_{G_0}(x)|=q-\delta$ for some $\delta=\pm1$. If $\delta=1$ then

$$|x^{G_0}\cap H|\leqslant \tfrac{1}{2}(q-2).|\mathrm{GL}_2(q):\mathrm{GL}_1(q)^2|=\tfrac{1}{2}q(q-2)(q+1)=c_1, \quad |x^G|=q^3(q^3+1)=d_1,$$

and there are precisely $\frac{1}{2}(q-3)=n_1$ distinct G_0-classes of this type (see Table 13). Similarly, if $\delta=-1$ then x belongs to one of $\frac{1}{6}(q-3)=n_2$ distinct G_0-classes and we have

$$|x^{G_0}\cap H|\leqslant \frac{1}{2}q.|\mathrm{GL}_2(q):\mathrm{GL}_1(q^2)|=\frac{1}{2}q^2(q-1)=c_2, \quad |x^G|=q^3(q^2-q+1)(q-1)=d_2.$$

Finally, if x is a field automorphism of prime order r then

$$|x^{G_0}\cap H|<2q^{3(1-r^{-1})} \quad \text{and} \quad |x^G|>\frac{1}{2}q^{7(1-r^{-1})}=f(r,q),$$

so $\mathrm{fpr}(x)<4q^{-4(1-r^{-1})}=g(r,q)$. In particular, if we set $h(r,q)=f(r,q)g(r,q)^3$ then the contribution to $\widehat{Q}(G,3)$ from field automorphisms is less than

$$\sum_{r\in\pi}(r-1).h(r,q)<2h(3,q)+\log_3 q.q^7g(5,q)^3,$$

where π is the set of distinct prime divisors of $\log_3 q$. We conclude that

$$\widehat{Q}(G,3)<\sum_{i=1}^{2}a_ib_i^3+\sum_{i=1}^{2}n_id_i(c_i/d_i)^3+2h(3,q)+\log_3 q.q^7g(5,q)^3=F(q),$$

and the reader can check that $F(q)<1$ for all $q\geqslant 3^5$. □

LEMMA 4.37. *If H is the normalizer of a torus then $b(G)=2$.*

Proof. As before, we may assume that $q\geqslant 3^5$. According to [**31**] we have

$$|H|\leqslant \log_3 q.6(q+\sqrt{3q}+1)=a$$

and we note that $|x^G|\geqslant (q^3+1)(q-1)=b$ for all $x\in G$ (minimal if $x\in G_0$ has order 3 and $|C_{G_0}(x)|=q^3$). We conclude that $b(G)=2$ since Proposition 2.3 implies that $\widehat{Q}(G,2)<b(a/b)^2<1$ for all $q\geqslant 3^5$. □

PROPOSITION 4.38. *If H is a maximal non-parabolic subgroup of G then $b(G)\leqslant 3$.*

Proof. According to [31] we may assume that H is a subfield subgroup of type ${}^2G_2(q_0)$, where $q = q_0^k$ and k is an odd prime. We claim that $b(G) = 2$. First assume that $k \geqslant 5$. Then

$$H_0 = H \cap G_0 = {}^2G_2(q_0),$$

so $|H| < \log_3 q.q^{7/5} = a$, and the claim follows as in the proof of Lemma 4.37 since

$$\widehat{Q}(G,2) < b(a/b)^2 < 1, \quad \text{where } b = (q^3+1)(q-1).$$

Now assume that $k = 3$. If $q = 3^3$ then a MAGMA calculation yields $b(G) = 2$ (see Table 11), so we may assume that $q \geqslant 3^9$. Let $x \in H_0$ be an element of prime order r. If $r = 2$ then $|x^G \cap H| = q^{2/3}(q^{2/3} - q^{1/3} + 1) = a_1$ and $|x^G| = q^2(q^2 - q + 1) = b_1$, while the contribution to $\widehat{Q}(G,2)$ from unipotent elements of order 3 is precisely $\sum_{i=2}^{3} b_i(a_i/b_i)^2$, where

$$a_2 = (q+1)(q^{1/3}-1), \;\; b_2 = (q^3+1)(q-1), \;\; a_3 = q^{1/3}(q+1)(q^{1/3}-1), \;\; b_3 = q(q^3+1)(q-1).$$

Now assume that $r \geqslant 5$. Then $x^{G_0} \cap H_0 = x^{H_0}$ since $C_{\bar{G}}(x)$ is connected (see the proof of [38, Lemma 5.7]), and we observe that either $|C_{G_0}(x)| = q+1$, or $|C_{H_0}(x)| = q_0 - 1$ and $|C_{G_0}(x)| = q-1$. It follows that the contribution here is at most $\sum_{i=4}^{5} n_i b_i(a_i/b_i)^2$, where $n_4 = \frac{1}{2}(q-3)$, $n_5 = \frac{1}{6}(q-3)$ and

$$a_4 = q(q+1), \;\; b_4 = q^3(q^3+1), \;\; a_5 = q(q^{1/3}-1)(q^{2/3}-q^{1/3}+1), \;\; b_5 = q(q^3+1)(q-1).$$

Finally, suppose that $x \in G$ is a field automorphism of prime order r. If $r = 3$ then we may assume that x centralizes H_0, whence

$$|x^G \cap H| = i_3(H_0) + 1 = (q+1)(q^{2/3}-1) + 1 = a_6, \quad |x^G| > \frac{1}{2}q^{7/3} = b_6$$

and we set $n_6 = 2$. If $r \geqslant 5$ then $|x^G| > \frac{1}{2}q^{28/5} = d$ and we note that $|H| < \log_3 q.q^{7/3} = c$. Applying Proposition 2.3 we conclude that $\widehat{Q}(G,2) < \sum_{i=1}^{6} n_i b_i(a_i/b_i)^2 + d(c/d)^2 = F(q)$, where $n_i = 1$ for $i < 4$. The reader can check that $F(q) < 1$ for all $q \geqslant 3^9$. □

4.8. $G_0 = {}^2B_2(q)$

In this case, we have $q = 2^{2m+1}$ for an integer $m \geqslant 1$. We refer the reader to Table 11 for precise results when $m \leqslant 2$, so we can assume that $m \geqslant 3$. The conjugacy classes and maximal subgroups of G_0 are determined in [70]. In particular, if $x \in G$ is an involution then $|C_{G_0}(x)| = q^2$ and any two involutions are G_0-conjugate. The possibilities for $|C_{G_0}(x)|$ when x is semisimple are listed in Table 14. In addition, we remind the reader that G_0 does not contain any elements of order 3.

TABLE 14. *Semisimple elements.*

$\lvert C_{G_0}(x)\rvert$	Number of G_0-classes
$q-1$	$\frac{1}{2}(q-2)$
$q \pm \sqrt{2q}+1$	$\frac{1}{4}(q \pm \sqrt{2q})$

According to [70], a maximal non-parabolic subgroup of G is either a subfield subgroup or the normalizer of a maximal torus.

LEMMA 4.39. *If H is the normalizer of a maximal torus then $b(G) = 2$.*

Proof. By [**70**, § 15] we have

$$|H \cap G_0| \leqslant 4(q + \sqrt{2q} + 1) = a_1$$

and we note that $|x^G| \geqslant (q^2+1)(q-1) = b_1$ for all $x \in G_0$ of prime order (minimal if x is an involution). Now assume that x is a field automorphism of prime order r. If $r \geqslant 5$ then $|x^G| > \frac{1}{2}q^4 = b_2$ and we have $|H| < \log_2 q.4(q + \sqrt{2q} + 1) = a_2$. On the other hand, the contribution to $\widehat{Q}(G,2)$ from field automorphisms of order 3 is at most $n_3 b_3 (a_3/b_3)^2$, where $n_3 = 2$, $a_3 = a_1$, $b_3 = g(q)/g(q^{1/3})$ and $g(t) = t^2(t^2+1)(t-1)$. We conclude that $b(G) = 2$ since

$$\widehat{Q}(G,2) < \sum_{i=1}^{3} n_i b_i (a_i/b_i)^2 < 1, \quad \text{where } n_1 = n_2 = 1. \qquad \square$$

Proposition 4.40. *If H is a maximal non-parabolic subgroup of G then $b(G) = 2$.*

Proof. We may assume that H is a subfield subgroup of type ${}^2B_2(q_0)$, where $q = q_0^k$ for a prime k which divides $\log_2 q$. If $k \geqslant 5$ then $|H| < \log_2 q.q = a$, $|x^G| \geqslant (q^2+1)(q-1) = b$ and thus $b(G) = 2$ since $\widehat{Q}(G,2) < b(a/b)^2 < 1$. Now assume that $k = 3$. Let $x \in H \cap G_0$ be an element of prime order r. If $r = 2$ then

$$|x^G \cap H| = (q^{2/3}+1)(q^{1/3}-1) = a_1 \quad \text{and} \quad |x^G| = (q^2+1)(q-1) = b_1.$$

Next suppose that $r > 2$ and observe that $x^{G_0} \cap H = x^{H_0}$ since $C_{\bar{G}}(x)$ is connected. By Lagrange we see that $|C_{G_0}(x)| = q-1$ if $|C_{H_0}(x)| = q_0 - 1$, while $|C_{G_0}(x)| = q - \epsilon\sqrt{2q} + 1$ if $|C_{H_0}(x)| = q_0 + \epsilon\sqrt{2q_0} + 1$. In particular, the contribution to $\widehat{Q}(G,2)$ from these elements is at most $\sum_{i=2}^{4} n_i b_i (a_i/b_i)^2$, where $n_2 = \frac{1}{2}(q-2)$, $n_3 = \frac{1}{4}(q - \sqrt{2q})$, $n_4 = \frac{1}{4}(q + \sqrt{2q})$ and

$$a_2 = q^{2/3}(q^{2/3}+1), \;\; a_3 = q^{2/3}(q^{1/3} + \sqrt{2}q^{1/6} + 1)(q^{1/3}-1),$$
$$a_4 = q^{2/3}(q^{1/3} - \sqrt{2}q^{1/6} + 1)(q^{1/3}-1), \;\; b_2 = q^2(q^2+1),$$
$$b_3 = q^2(q - \sqrt{2q} + 1)(q-1), \;\; b_4 = q^2(q + \sqrt{2q} + 1)(q-1).$$

Finally, let us assume that x is a field automorphism of prime order r. If $r = 3$ then we may assume that x centralizes H_0, whence $|x^G \cap H| = 1 = a_5$ and $|x^G| = g(q)/g(q^{1/3}) = b_5$, where $g(t) = t^2(t^2+1)(t-1)$. If $r \geqslant 5$ then $|x^G| \geqslant g(q)/g(q^{1/5}) = d$, and we note that $|H| \leqslant \log_2 q.g(q^{1/3}) = c$. Set $\alpha = 1$ if $\log_2 q$ is divisible by 15, otherwise $\alpha = 0$. Then applying Proposition 2.3 we conclude that $\widehat{Q}(G,2) \leqslant \sum_{i=1}^{5} n_i b_i (a_i/b_i)^2 + \alpha d(c/d)^2 < 1$, where $n_1 = 1$ and $n_5 = 2$. $\square$

4.9. $G_0 = {}^3D_4(q)$

The maximal subgroups of G are determined in [**32**], while the G-conjugacy classes are described in [**20**, **67**]. If q is odd and $x \in G_0$ is an involution then $|C_{G_0}(x)| = q^8(q^8+q^4+1)$ and any two involutions are G_0-conjugate. If q is even then there are two classes of unipotent involutions, labelled A_1 and $3A_1$ in [**67**]. We note that $\dim x^{\bar{G}} \geqslant 18$ for all semisimple elements $x \in G_0$ of odd order (see [**20**, Table 4.4]).

Lemma 4.41. *If $|H| \leqslant q^{12}$ then $b(G) \leqslant 5$.*

Proof. The case $q = 2$ can be handled using Magma (see Table 12) so assume that $q \geqslant 3$. Let $x \in H$ be an element of prime order. If $|x^G| \leqslant q^{16} = b$ then x is either a long root element, an involutory field automorphism, or a G_2-type triality graph automorphism. Further, [**38**, Theorem 1] gives $\mathrm{fpr}(x) \leqslant (q^4 - q^2 + 1)^{-1} = d$ and we note that there are fewer than $4q^{14} = c$

of these elements in G. In view of Proposition 2.3 we conclude that $\widehat{Q}(G,5) < b(a/b)^5 + cd^5 < 1$, where $a = q^{12}$. □

Lemma 4.42. *If H is of type $G_2(q)$ then $b(G) \leqslant 5$.*

Proof. If $q = 2$ then a Magma calculation yields $b(G) = 3$, so assume that $q \geqslant 3$. Write $H_0 = H \cap G_0 = G_2(q)$ and note that $|H_0| < q^{14} = a$. Let $x \in H_0$ be an element of prime order r. If $|x^G| \leqslant \frac{1}{4}q^{18} = b$ then either x is a semisimple involution, or $r = p$ and x lies in one of the $\bar{G}$-classes labelled A_1 and $3A_1$. In particular, there are precisely $(2q^8-1)(q^8+q^4+1) = c_1$ such elements and [**38**, Theorem 1] gives $\mathrm{fpr}(x) \leqslant (q^4-q^2+1)^{-1} = d_1$. Next let $x \in G$ be a field automorphism of prime order r and observe that

$$\mathrm{fpr}(x) = \frac{|G_2(q) : G_2(q^{1/r})|}{|{}^3D_4(q) : {}^3D_4(q^{1/r})|} < 4q^{-14(1-1/r)}.$$

In particular, if $r = 2$ then

$$|x^G| < 2q^{14} = c_2 \quad \text{and} \quad \mathrm{fpr}(x) < 4q^{-7} = d_2,$$

while $\mathrm{fpr}(x) < 4q^{-56/5} = d_3$ if $r \geqslant 5$. Finally suppose that $x \in G$ is a triality graph automorphism. If $C_{\bar{G}}(x) \neq G_2$ then $\mathrm{fpr}(x) < 2q^{-6} = d_4$ since $|x^G| > \frac{1}{2}q^{20}$, while G contains fewer than $4q^{20} = c_4$ such elements. On the other hand, if $C_{\bar{G}}(x) = G_2$ then the proof of [**38**, Lemma 6.3] gives $|x^G \cap H| \leqslant q^3(q^3+1)+1$, so $\mathrm{fpr}(x) < 2q^{-8} = d_5$, and we note that there are no more than $4q^{14} = c_5$ of these automorphisms in G.

We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < b(a/b)^5 + \sum_{i=1}^{5} c_i d_i^5 < 1$, where $c_3 = \log_2 q.q^{28}$. □

Proposition 4.43. *If H is a maximal non-parabolic subgroup of G then $b(G) \leqslant 5$.*

Proof. In view of [**32**] and Lemmas 4.41 and 4.42 we may assume that

$$H_0 = H \cap G_0 = {}^3D_4(q^{1/2}).$$

Let $x \in H$ be a semisimple element of odd prime order and observe that $x^{G_0} \cap H_0 = x^{H_0}$ since $C_{\bar{G}}(x)$ is connected. Then

$$\mathrm{fpr}(x) < 4q^{-(1/2)\dim x^{\bar{G}}} \leqslant 4q^{-9} = b_1$$

since $\dim x^{\bar{G}} \geqslant 18$ (see [**20**, Table 4.4]). If q is odd then both H_0 and G_0 contain a unique class of involutions and thus $|x^G| < 2q^{16} = a_2$ and $\mathrm{fpr}(x) < 2q^{-8} = b_2$. Next let $x \in H$ be a unipotent element of order p. Then $x^{G_0} \cap H_0 = x^{H_0}$ since the class of x in both H_0 and G_0 is determined by the labelling of the class of x in $\bar{G}$. In particular, if x belongs to the class labelled A_1 then $|x^G| < q^{10} = a_3$ and $\mathrm{fpr}(x) < 2q^{-5} = b_3$, otherwise $\mathrm{fpr}(x) < 4q^{-8} = b_4$.

Next suppose that $x \in G$ is a field automorphism of prime order r. If $r \geqslant 5$ then x induces a field automorphism on H_0 and thus $\mathrm{fpr}(x) < 4q^{-56/5} = b_5$; if $r = 2$ then $|x^G| < 2q^{14} = a_6$ and we may assume that x centralizes H_0, so $\mathrm{fpr}(x) < 4q^{-6} = b_6$ since $|x^G \cap H| = i_2(H_0)+1 < 2q^8$. Finally, let $x \in G$ be a triality graph automorphism. Then x induces a triality automorphism on H_0 and we note that the centralizers $C_{H_0}(x)$ and $C_{G_0}(x)$ are of the same type. It follows that $\mathrm{fpr}(x) < 4q^{-7} = b_7$ if $C_{\bar{G}}(x) = G_2$, otherwise $\mathrm{fpr}(x) < 4q^{-10} = b_8$. We conclude that $b(G) \leqslant 5$ since $\widehat{Q}(G,5) < \sum_{i=1}^{7} a_i b_i^5 < 1$, where $a_1 = q^{28}$, $a_4 = q^{24}$, $a_5 = \log_2 q.q^{28}$, $a_7 = 4q^{14}$ and $a_8 = 4q^{20}$. □

This completes the proof of Theorem 4.

5. *Proof of Theorem* 2

Let G be a finite almost simple group, and let Ω be a faithful primitive non-standard G-set. Recall that the strong form of the Cameron–Kantor Conjecture asserts that there exists an absolute constant c' such that the probability that a random c'-tuple in Ω forms a base for G tends to 1 as the order of G tends to infinity. Although this conjecture has now been established (see [**14**, **26**, **48**]), it is strictly an existence result and until this paper, no explicit value for c' was known. In view of Theorem 3, it follows that $c' \geqslant 5$. In this section we prove that the result holds with a constant $c' = 6$. It would be interesting to know whether $c' = 5$ is in fact sufficient (cf. Remark 1).

As explained in the Introduction, we may assume that G is a classical group over $\mathbb{F}_q$, with socle G_0 and natural module of dimension $n \leqslant 15$. As before, it is convenient to write $Q(G,6)$ for the probability that a random 6-tuple in Ω is not a base for G. Then in order to prove the theorem we need to show that $Q(G,6)$ tends to zero as q tends to infinity.

First suppose that $8 \leqslant n \leqslant 15$ and assume (as we may) that q is large. For $t \in \mathbb{R}$ set

$$\eta^{\tilde{G}}(t) = \sum_{C \in \mathcal{C}(\tilde{G})} |C|^{-t},$$

where $\mathcal{C}(\tilde{G})$ is the set of conjugacy classes in $\tilde{G} := G \cap \mathrm{Inndiag}(G_0)$. Then proceeding as in the proof of [**49**, Theorem 1.11], using the bound on fixed point ratios in [**9**, Theorem 1], we deduce that

$$Q(G,6) < \eta^{\tilde{G}}\left(\frac{1}{4}\right) - 1 + o(1),$$

where $o(1)$ is a term which tends to zero as q tends to infinity. Let $\bar{G}$ be the corresponding simple algebraic group and write h for the Coxeter number of $\bar{G}$. Then the hypothesis $n \geqslant 8$ implies that $h \geqslant 6$, and hence $\eta^{\tilde{G}}(1/4) \to 1$ as $q \to \infty$ by [**49**, Theorem 1.10(i)]. We conclude that $Q(G,6) \to 0$ as $q \to \infty$.

Next assume that $n = 7$ and q is large. Then [**9**, Theorem 1] gives $\mathrm{fpr}(x) < |x^G|^{-31/126}$ for all $x \in G$ of prime order. Therefore $Q(G,6) < \eta^{\tilde{G}}(10/21) - 1 + o(1)$ and once again the desired result follows via [**49**, Theorem 1.10(i)] since $h \geqslant 6$. Similarly, when $n = 6$ we quickly reduce to the case $G_0 = \mathrm{PSL}_6^{\epsilon}(q)$, with H of type $\mathrm{Sp}_6(q)$. Here we argue as in the proof of [**7**, Lemma 3.5]. More precisely, we use the proof of [**10**, Proposition 8.1] to show that $Q(G,6) \leqslant \widehat{Q}(G,6) \leqslant F(q)$ (see (1.2)) for a function F such that $F(q) \to 0$ as $q \to \infty$. We leave the details to the reader.

Finally, let us assume that $n \leqslant 5$. If $n = 4$ or $n = 5$ then the fact that Ω is non-standard implies that $\mathrm{fpr}(x) < |x^G|^{-1/2+1/n}$ for all $x \in G$ of prime order (see [**9**, Theorem 1] and Remark 5.1). Therefore, $Q(G,6) < \eta^{\tilde{G}}(1/2) - 1 + o(1)$ and the proof is complete. To deal with the remaining cases $n \in \{2,3\}$ we argue as in [**7**, Proposition 4.1], using (1.2) and fixed point ratio bounds. Here [**7**, Table 3] provides a convenient list of the cases which need to be considered; in each case it is easy to derive a bound $\widehat{Q}(G,6) \leqslant F(q)$ with $F(q) \to 0$ as $q \to \infty$.

This completes the proof of Theorem 2.

Remark 5.1. For classical groups, the notion of a non-standard action in the statement of Theorem 2 differs slightly from the notion of a non-subspace action adopted in [**9**]. Here we follow [**7**, Definition 1.1]. For example, if $G_0 = \mathrm{P}\Omega_8^+(q)$ and H is an irreducible almost simple subgroup with socle $\Omega_7(q)$ then the corresponding action of G is non-subspace in the sense of [**9**, Definition 1]. However, this action is equivalent via a triality automorphism to the action of G on the set of 1-dimensional non-singular subspaces of the natural G_0-module, so in accordance with [**7**, Definition 1.1] we say that the original action is standard. A list of these standard, non-subspace actions can be found in [**7**, Table 1].

6. *The tables*

In this final section we record some miscellaneous results which are relevant to the proof of Theorem 1. First, in Table 9, we provide some useful information on semisimple elements of prime order in the groups $E_6(2)$, ${}^2E_6(2).3$ and $F_4(2)$. Here the relevant character tables are available in the GAP Character Table Library and we use a combination of [**55**] and [**60**] to determine the structure of the centralizers in $\bar{G}$. In the second column we list all the G-classes which contain semisimple elements of prime order.

Next, in Tables 11 and 12, we present the precise base size results referred to in Proposition 1. Here we list $b(G)$ for each faithful primitive action of an almost simple group G with socle G_0, where

$$G_0 \in \{{}^2B_2(8),\ {}^2B_2(32),\ {}^2G_2(27),\ G_2(3), G_2(4), G_2(5),\ {}^3D_4(2),\ {}^2F_4(2)'\}.$$

To obtain these results we use the computer package Magma. Here we provide a brief sketch of the methods involved.

Suppose that $G = G_0$. First, with the aid of the Web Atlas [**72**], we construct G as a permutation group on two generators, a and b say. Now, generators for each maximal

Table 9. *Elements of odd prime order in $E_6(2)$, ${}^2E_6(2).3$ and $F_4(2)$.*

G	x	$C_{\bar{G}}(x)^0$	$\lvert x^G\rvert$	$\lvert x^G\rvert >$
$E_6(2)$	$3a$	T_1A_5	$2^{21}.3.5.7.13.17.73$	2^{41}
	$3b$	T_2D_4	$2^{24}.3.5.7^2.13.31.73$	2^{47}
	$3c$	A_2^3	$2^{27}.5.7^2.13.17.31.73$	2^{53}
	$5a$	A_3T_3	$2^{30}.3^2.7^3.13.17.31.73$	2^{60}
	$7a, 7b$	T_2D_4	$2^{24}.3^2.5^2.17.31.73$	2^{47}
	$7c$	$T_2A_2^2$	$2^{30}.3^4.5^2.13.17.31.73$	2^{59}
	$7d$	A_2T_4	$2^{33}.3^5.5^2.13.17.31.73$	2^{64}
	$13a$	T_6	$2^{36}.3^6.5^2.7^2.17.31.73$	2^{70}
	$17a, 17b$	T_6	$2^{36}.3^5.5^2.7^3.13.31.73$	2^{71}
	$31a-f$	A_1T_5	$2^{35}.3^5.5^2.7^3.13.17.73$	2^{69}
	$73a-h$	T_6	$2^{36}.3^6.5^2.7^3.13.17.31$	2^{71}
${}^2E_6(2).3$	$3a$	T_1A_5	$2^{21}.3^2.5.7.13.17.19$	2^{41}
	$3b$	T_2D_4	$2^{24}.3^2.7.11.13.17.19$	2^{45}
	$3c$	A_2^3	$2^{27}.5^2.7^2.11.13.17.19$	2^{52}
	$3d, 3e$	T_1D_5	$2^{16}.3^3.7.13.19$	2^{31}
	$3f, 3g$	T_2D_4	$2^{24}.3^4.5^2.11.17.19$	2^{46}
	$3h, 3i$	$A_4A_1T_1$	$2^{25}.3^3.5.13.17.19$	2^{44}
	$3j, 3k$	A_2^3	$2^{27}.3^4.5^2.7.11.13.17$	2^{52}
	$5a$	A_3T_3	$2^{30}.3^7.7.11.13.17.19$	2^{59}
	$7a$	$T_2A_2^2$	$2^{30}.3^7.5.11.13.17.19$	2^{58}
	$7b$	T_4A_2	$2^{33}.3^8.5^2.11.13.17.19$	2^{65}
	$11a, 11b$	A_1T_5	$2^{35}.3^8.5^2.7^2.13.17.19$	2^{69}
	$13a$	T_6	$2^{36}.3^9.5^2.7^2.11.17.19$	2^{72}
	$17a, 17b$	T_6	$2^{36}.3^9.5^2.7^2.11.13.19$	2^{71}
	$19a, 19b$	T_6	$2^{36}.3^9.5^2.7^2.11.13.17$	2^{71}
$F_4(2)$	$3a$	C_3T_1	$2^{15}.3.5.7.13.17$	2^{29}
	$3b$	B_3T_1	$2^{15}.3.5.7.13.17$	2^{29}
	$3c$	$A_2\widetilde{A}_2$	$2^{18}.5^2.7^2.13.17$	2^{36}
	$5a$	B_2T_2	$2^{20}.3^4.7^2.13.17$	2^{39}
	$7a$	A_2T_2	$2^{21}.3^5.5^2.13.17$	2^{41}
	$7b$	$\widetilde{A}_2T_2$	$2^{21}.3^5.5^2.13.17$	2^{41}
	$13a$	T_4	$2^{24}.3^6.5^2.7^2.17$	2^{47}
	$17a, 17b$	T_4	$2^{24}.3^6.5^2.7^2.13$	2^{47}

subgroup of G are also presented in the Web Atlas as words in a and b, and hence we can construct H as a subgroup of G. In order to show that $b(G) = c$ we use random search to find $c-1$ elements $x_2, \ldots, x_c$ in G such that $\bigcap_{i=1}^{c} H^{x_i} = 1$, where $x_1 = 1$. Of course, this only implies that $b(G) \leqslant c$, but with three exceptions the desired conclusion $b(G) = c$ follows from Proposition 2.4. The exceptions are the cases

$$(G,H) \in \{(G_2(3), 2^3.L_3(2)),\ (G_2(4), U_3(3):2),\ ({}^3D_4(2), 2^{1+8}:L_2(8))\}.$$

Here the previous approach yields $b(G) \leqslant 3$, but $\log|G|/\log|\Omega| < 2$ so Proposition 2.4 does not imply equality. To settle these cases we use the MAGMA command `CosetAction` to explicitly construct G as a permutation group on the cosets of H. It is then easy to calculate the size of each two-point stabilizer in G and check that $b(G) > 2$.

Now assume that $G \neq G_0$. As before, we can construct G as a permutation group and then obtain G_0 as the socle of G. In general, generators for the maximal subgroups of G are not listed in the Web Atlas, and so we need to work a little harder to construct H. First we use the `Classes` command to obtain a representative of each conjugacy class in G_0. Using these representatives, it is easy to find the so-called standard generators for G_0 by random search.

Let H be a maximal subgroup of G and suppose that $(G,H) \notin \mathscr{A}$, where

$$\mathscr{A} = \{(G_2(3):2, 3^2.[3^4]:D_8),\ ({}^2F_4(2), 3^{1+2}:SD_{16}),\ ({}^2F_4(2), 13:12)\}.$$

TABLE 11. *Some precise base size results,* I.

G	H	$b(G)$	G	H	$b(G)$
${}^2B_2(8)$	$2^{3+3}:7$	3	$G_2(3)$	$U_3(3):2$	3
	$13:4$	2		$(3^2 \times 3^{1+2}):2S_4$	3
	$5:4$	2		$L_3(3):2$	3
	D_{14}	2		$L_2(8):3$	2
				$2^3.L_3(2)$	3
${}^2B_2(8):3$	$2^{3+3}:7:3$	3		$L_2(13)$	2
	$13:12$	2		$2^{1+4}:3^2:2$	2
	$5:4\times 3$	2			
	$7:6$	2	$G_2(3):2$	$3^2.[3^4]:D_8$	3
				$L_2(8):3\times 2$	3
${}^2B_2(32)$	$2^{5+5}:31$	3		$2^3.L_3(2):2$	3
	$41:4$	2		$L_2(13):2$	3
	$25:4$	2		$2^{1+4}:(S_3\times S_3)$	2
	D_{62}	2			
			$G_2(4)$	J_2	4
${}^2B_2(32):5$	$2^{5+5}:31:5$	3		$2^{2+8}:(A_5\times 3)$	3
	$41:20$	2		$2^{4+6}:(A_5\times 3)$	3
	$25:20$	2		$U_3(4):2$	3
	$31:10$	2		$3.L_3(4):2$	3
				$U_3(3):2$	3
${}^2G_2(27)$	$3^{3+3+3}:26$	3		$A_5\times A_5$	2
	$2\times L_2(27)$	2		$L_2(13)$	2
	$3\times L_2(8)$	2			
	$37:6$	2	$G_2(4):2$	$J_2:2$	4
	$(2^2\times D_{14}):3$	2		$2^{2+8}:(A_5\times 3):2$	3
	$19:6$	2		$2^{4+6}:(A_5\times 3):2$	3
				$U_3(4):4$	3
${}^2G_2(27):3$	$3^{3+3+3}:26:3$	3		$3.L_3(4).2.2$	3
	$2\times L_2(27):3$	2		$U_3(3):2\times 2$	3
	$3\times L_2(8):3$	2		$(A_5\times A_5):2$	2
	$37:18$	2		$L_2(13):2$	2
	$A_4\times 7:6$	2			
	$19:18$	2			

TABLE 12. *Some precise base size results*, II.

G	H	$b(G)$	G	H	$b(G)$
$G_2(5)$	$5^{1+4}:GL_2(5)$	3	${}^2F_4(2)'$	$L_3(3):2$	3
	$5^{2+1+2}:GL_2(5)$	3		$2.[2^8].5.4$	3
	$3.U_3(5):2$	3		$L_2(25)$	3
	$L_3(5):2$	3		$2^2.[2^8].S_3$	3
	$2.(A_5\times A_5).2$	2		$A_6.2^2$	2
	$U_3(3):2$	2		$5^2:4A_4$	2
	$2^3.L_3(2)$	2			
			${}^2F_4(2)$	$2.[2^9].5.4$	3
${}^3D_4(2)$	$2^{1+8}:L_2(8)$	4		$L_2(25).2_3$	3
	$[2^{11}]:(7\times S_3)$	3		$2^2.[2^9].S_3$	3
	$U_3(3):2$	3		$5^2:4S_4$	2
	$S_3\times L_2(8)$	2		$3^{1+2}:SD_{16}$	2
	$(7\times L_2(7)):2$	2		$13:12$	2
	$3^{1+2}.2S_4$	2			
	$7^2:2A_4$	2			
	$3^2:2A_4$	2			
	$13:4$	2			
${}^3D_4(2):3$	$2^{1+8}:L_2(8):3$	4			
	$[2^{11}]:(7:3\times S_3)$	3			
	$3\times U_3(3):2$	3			
	$S_3\times L_2(8):3$	2			
	$(7:3\times L_2(7)):2$	2			
	$3^{1+2}.2S_4.3$	2			
	$7^2:(2A_4\times 3)$	2			
	$3^2:2A_4\times 3$	2			
	$13:12$	2			

Then $H = N_G(H_0)$ for some maximal subgroup H_0 of G_0. As previously remarked, generators for H_0 are given in the Web Atlas in terms of the standard generators for G_0, and hence we can easily construct H as a subgroup of G and compute $b(G)$ as before. Finally, the cases in $\mathscr{A}$ are easy to deal with because $H = N_G(S)$, where S is a Sylow 3-subgroup of G in the first two cases, while S is a Sylow 13-subgroup of G in the latter case.

NOTATION. In Table 9 we use the notation of the GAP Character Table Library for labelling conjugacy classes; in particular, classes labelled ra, rb, etc. contain elements of order r. In Tables 11 and 12 we write $[n]$ for an unspecified group of order n.

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