Serb. Astron. J. № 177 (2008), 115 - 124 UDC 520–872 : 521–181
DOI: 10.2298/SAJ0877115M Professional paper

SPHERICAL TRIGONOMETRY OF THE PROJECTED BASELINE ANGLE

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(Received: March 11, 2008; Accepted: May 16, 2008)

SUMMARY: The basic vector geometry of a stellar interferometer with two telescopes is defined by the right triangle of (i) the baseline vector between the telescopes, of (ii) the delay vector which points to the star, and of (iii) the projected baseline vector in the plane of the wavefront of the stellar light. The plane of this triangle intersects the celestial sphere at the position of the star; the intersection is a circular line segment. The interferometric angular resolution is high (diffraction limited to the ratio of the wavelength over the projected baseline length) in the two directions along this line segment, and low (diffraction limited to the ratio of the wavelength over the telescope diameter) perpendicular to these. The position angle of these characteristic directions in the sky is calculated here, given either local horizontal coordinates, or celestial equatorial coordinates.

Key words. Methods: analytical – Techniques: interferometric – Reference systems

1. SCOPE

The paper describes a standard to define the plane that contains the baseline of a stellar interferometer and the star, and its intersection with the Celestial Sphere. This intersection is a great circle, and its section between the star and the point where the prolonged baseline meets the Celestial Sphere defines an oriented projected baseline. The incentive to use this projection onto the Celestial Sphere is that this ties the baseline to sky coordinates, which defines directions in equatorial coordinates independent of the observatory's location on the Earth. This is (i) the introduction of a polar coordinate system's polar angle in the $u-v$ coordinate system associated with the visibility tables of the Optical Interferometry Exchange Format (OIFITS) (Pauls et al. 2004, 2005), detailing the orthogonal directions which are well and poorly resolved by the interferometer, and (ii) a definition of celestial longitudes in a coordinate system where the pivot point has been relocated from the North Celestial Pole (NCP) to the star.

In overview, Section 2 introduces the standard set of variables in the familiar polar coordinate systems. Section 3 defines and computes the baseline position angle—which can be done purely in equatorial or horizontal coordinates or in a hybrid way if the parallactic angle is used to bridge these. Some comments on relating distances away from the optical axis (star) in the tangent plane to the optical path delay conclude the manuscript in Section 4.

2. SITE GEOMETRY

2.1. Telescope Positions

In a geocentric coordinate system, the Cartesian coordinates of the two telescopes of a stellar interferometer are related to the geographic longitude λ_i, geographic latitude ϕ_i and net Earth radius ρ (sum of the Earth radius and altitude above the sea level),

$$\mathbf{T}_i \equiv \rho \begin{pmatrix} \cos\lambda_i \cos\phi_i \\ \sin\lambda_i \cos\phi_i \\ \sin\phi_i \end{pmatrix}_g , \quad i = 1, 2. \tag{1}$$

We add a "g" to the geocentric coordinates to set them apart from other Cartesian coordinates that will be used further below. A great circle of radius $\rho \approx 6380$ km, centered at the Earth center, joins $\mathbf{T}_1$ and $\mathbf{T}_2$. The vector $\hat{\mathbf{J}} \equiv \frac{1}{\rho^2 \sin Z}\mathbf{T}_1 \times \mathbf{T}_2$ is perpendicular to this circle, where a baseline aperture angle Z is defined as the apparent size of baseline vector

$$\mathbf{b} = \mathbf{T_2} - \mathbf{T}_1 \tag{2}$$

as seen from the Earth center:

$$\mathbf{T}_1 \cdot \mathbf{T}_2 = |\mathbf{T}_1|\,|\mathbf{T}_2| \cos Z;$$
$$\cos Z = \cos\phi_1 \cos\phi_2 \cos(\lambda_2 - \lambda_1) + \sin\phi_1 \sin\phi_2.$$

$\hat{\mathbf{J}}$ is the axis of rotation when $\mathbf{T}_1$ is turned toward T_2. The nautical direction from $\mathbf{T}_1$ to $\mathbf{T}_2$ is computed as follows: define a tangent plane to the Earth at $\mathbf{T}_1$. Two orthonormal vectors that span this plane to the North and East are

$$\hat{\mathbf{N}}_1 \equiv \begin{pmatrix} -\cos\lambda_1 \sin\phi_1 \\ -\sin\lambda_1 \sin\phi_1 \\ \cos\phi_1 \end{pmatrix}_g \sim \partial\mathbf{T}_1/\partial\phi_1; \quad |\hat{\mathbf{N}}_1| = 1.$$

and

$$\hat{\mathbf{E}}_1 \equiv \begin{pmatrix} -\sin\lambda_1 \\ \cos\lambda_1 \\ 0 \end{pmatrix}_g \sim \partial\mathbf{T}_1/\partial\lambda_1; \quad |\hat{\mathbf{E}}_1| = 1.$$

The unit vector from $\mathbf{T}_1$ to $\mathbf{T}_2$ along the great circle is $\hat{\mathbf{J}} \times \frac{1}{\rho}\mathbf{T}_1$. The compass rose angle τ is computed by the partition $\hat{\mathbf{J}} \times \frac{1}{\rho}\mathbf{T}_1 = \cos\tau\hat{\mathbf{N}}_1 + \sin\tau\hat{\mathbf{E}}_1$ within the tangent plane. τ is zero if $\mathbf{T}_2$ is North of $\mathbf{T}_1$, and is $\pi/2$ if $\mathbf{T}_2$ is East of $\mathbf{T}_1$: Fig. 1.

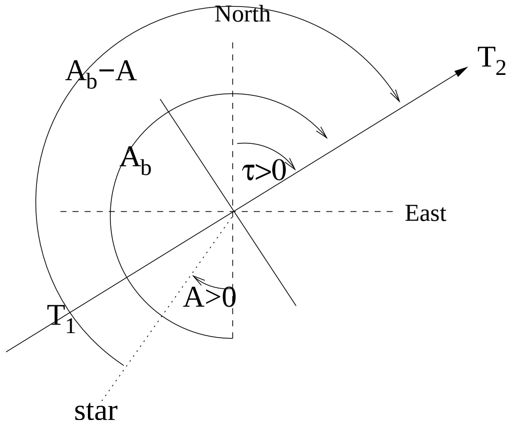

Fig. 1. *In the horizontal coordinate system, the baseline direction is characterized by the azimuth angle A_b. In the case shown, $0 < A < \pi < A_b$ and $D < 0$.*

Straight forward algebra establishes τ from

$$\cos\tau = \frac{\cos\phi_1 \sin\phi_2 - \sin\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2)}{\sin Z};$$
$$\sin\tau = \frac{\cos\phi_2 \sin(\lambda_2 - \lambda_1)}{\sin Z}.$$

There is a caveat of this spherical approximation: The angle τ does not transform exactly into $\tau \pm \pi$ if the roles of the two telescopes are swapped, because the great circle, which has been used to define the direction, is not a loxodrome. This asymmetry within the definition indicates that a more generic framework to represent the geometry is useful.

If we consider (i) geodetic rather than geocentric representations of geographic latitudes (NIMA 2000) or (ii) long baseline interferometers build into rugged landscapes, the five parameters above (ρ, λ_i, ϕ_i) are too constrained to handle the six degrees of freedom of two earth-fixed telescope positions. Nevertheless, a telescope array "platform" is helpful to define a zenith, a horizon, and associated coordinates like the zenith distance z or the star azimuth A. This leads to the OIFITS concept, an array center $\mathbf{C}$, for example

$$\mathbf{C} = \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix}_g ,$$

plus local telescope coordinates $\mathbf{t}_i$,

$$\mathbf{T}_i \equiv \mathbf{C} + \mathbf{t}_i, \quad i = 1, 2.$$

Geodetic longitude λ, geodetic latitude ϕ and site altitude H above the geoid are defined with the array center (Jones 2004, Vermeille 2004, Wood 1996, Zhang et al. 2005),

$$\begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix}_g = \begin{pmatrix} (N+H)\cos\phi\cos\lambda \\ (N+H)\cos\phi\sin\lambda \\ \left[N(1-e^2)+H\right]\sin\phi \end{pmatrix}_g ,$$

where e is the eccentricity of the Earth ellipsoid, and $N \equiv \rho_e/\sqrt{1-e^2\sin\phi}$ the distance from the array center to the Earth axis measured along the local vertical.

2.2. Sky coordinates

In a plane tangential to the geoid at $\mathbf{C}$ we define a star azimuth A, a zenith angle z, and a star elevation $a = \pi/2 - z$,

$$\mathbf{s} = \begin{pmatrix} -\cos A\sin z \\ \sin A \sin z \\ \cos z \end{pmatrix}_t = \begin{pmatrix} -\cos A\cos a \\ \sin A\cos a \\ \sin a \end{pmatrix}_t . \quad (3)$$

We label this coordinate system t as "topocentric," with the first component North, the second component West and the third component up. The value of A used in this script picks the convention that South means $A = 0$ and West means $A = +\pi/2$. This conventional definition of the horizontal means the star coordinates can be transformed to the equatorial parameters, hour angle $h = l - \alpha$, declination δ, and right ascension α (Lang 1998 eq. (5.45), Karttunen 1987 Eq. (2.13)),

$$\begin{aligned}
\cos a\sin A &= \cos\delta\sin h; &(4)\\
\cos a\cos A &= -\sin\delta\cos\phi + \cos\delta\cos h\sin\phi; &(5)\\
\sin a &= \sin\delta\sin\phi + \cos\delta\cos h\cos\phi; &(6)\\
\cos\delta\cos h &= \sin a\cos\phi + \cos a\cos A\sin\phi; &(7)\\
\sin\delta &= \sin a\sin\phi - \cos a\cos A\cos\phi. &(8)
\end{aligned}$$

These convert (3) into

$$\mathbf{s} = \begin{pmatrix} \cos\phi\sin\delta - \sin\phi\cos\delta\cos h \\ \cos\delta\sin h \\ \sin\phi\sin\delta + \cos\phi\cos\delta\cos h \end{pmatrix}_t . \quad (9)$$

A star at hour angle $h = 0$ is on the Meridian,

$$\mathbf{s} = \begin{pmatrix} \sin(\delta-\phi) \\ 0 \\ \cos(\delta-\phi) \end{pmatrix}_t , \quad (h=0),$$

north of the zenith at $A = \pi$ if $\delta - \phi > 0$, south at $A = 0$ if $\delta - \phi < 0$. A reference point on the Celestial Sphere is the North Celestial Pole (NCP), given by insertion of $\delta = \pi/2$ into (9),

$$\mathbf{s}_+ = \begin{pmatrix} \cos\phi \\ 0 \\ \sin\phi \end{pmatrix}_t . \quad (10)$$

The cosine of the angular distance between $\mathbf{s}$ and $\mathbf{s}_+$ is

$$\mathbf{s}\cdot\mathbf{s}_+ = \sin\delta. \quad (11)$$

2.3. Baseline: Generic Position

The baseline vector coordinates of the geocentric OIFITS system

$$\mathbf{b} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_g = \mathbf{T}_2 - \mathbf{T}_1$$

(defined to stretch from $\mathbf{T}_1$ to $\mathbf{T}_2$—the opposite sign is also in use (Pearson 1991)) could be converted with

$$\begin{aligned}
\mathbf{b} &= \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}_t = \begin{pmatrix} t_{2x} \\ t_{2y} \\ t_{2z} \end{pmatrix}_t - \begin{pmatrix} t_{1x} \\ t_{1y} \\ t_{1z} \end{pmatrix}_t \\
&= U_{tg}(\phi,\lambda)\cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_g
\end{aligned}$$

to the topocentric system via the rotation matrix

$$U_{tg}(\phi,\lambda) = \begin{pmatrix} -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \sin\lambda & -\cos\lambda & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{pmatrix}.$$

Just like the star direction (3), the unit vector $\hat{\mathbf{b}}$ along the baseline direction defines a baseline azimuth A_b and a baseline elevation a_b

$$\hat{\mathbf{b}} \equiv \frac{\mathbf{b}}{b} \equiv \begin{pmatrix} -\cos A_b\cos a_b \\ \sin A_b\cos a_b \\ \sin a_b \end{pmatrix}_t . \quad (12)$$

The standard definition of the delay vector $\mathbf{D}$ and projected baseline vector $\mathbf{P}$ is

$$\mathbf{b} = \mathbf{D} + \mathbf{P}; \quad \mathbf{D} \parallel \mathbf{s}; \quad \mathbf{P} \perp \mathbf{D}.$$

$\mathbf{P}$—and later its circle segment projected on the Celestial Sphere—inherits its direction from $\mathbf{b}$ such that heads and tails of the vectors are associated with the same portion of the wavefront: Fig. 2.

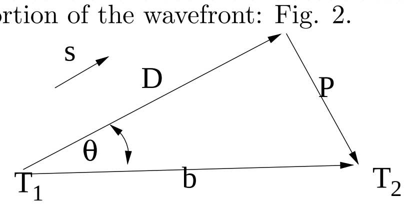

Fig. 2. *Sign conventions of the coplanar vectors* **b**, **D** *and* **P**.

We use the term "projected baseline" both ways: for the straight vector of length P, units meter, that connects the tails of $\mathbf{D}$ and $\mathbf{b}$, or the line segment of length θ, units radian, on the Celestial Sphere.

The dot product of (3) by (12) is

$$D = \mathbf{s}\cdot\mathbf{b} = b\cos\theta, \quad (0 \le \theta \le \pi), \quad (13)$$

where the angular distance θ between the baseline and star directions is introduced as

$$\cos\theta = \cos a_b \cos a \cos(A_b - A) + \sin a_b \sin a. \quad (14)$$

The star circles around the NCP in 24 hours, which changes the distance to the baseline in a centric periodic way (Appendix B).

3. PROJECTED BASELINE ANGLES

3.1. Definition

We define position angles of points on the Celestial Sphere as the bearing angle by which the NCP must be rotated around the star direction **s** (the axis) until the NCP and the point appear aligned in the same direction (along a celestial circle) from the star. The sign convention is *left-handed* placing the head of **s** at the center of the Celestial Sphere. This is equivalent to drawing a line between the star and the NCP, looking at it from inside the sphere, and defining position angles of points as the angles in polar coordinates in the mathematical sign convention, using this line as the abscissa and the star as the center. Another, fully equivalent definition uses a *right-handed* turn of the object around the star until it is North of the star. And finally, as an aid to memory, it is also the value of the nautical course while navigating a ship sailing the outer hull of the Celestial Sphere, which is currently poised at the star's coordinates.

This defines values modulo 2π; whether these are finally represented as numbers in the interval $[0, 2\pi)$ or in the interval $(-\pi, +\pi]$—as the `SLA_BEAR` and the `SLA_PA` routines of the SLA library (Wallace 2003) do—is largely a matter of taste.

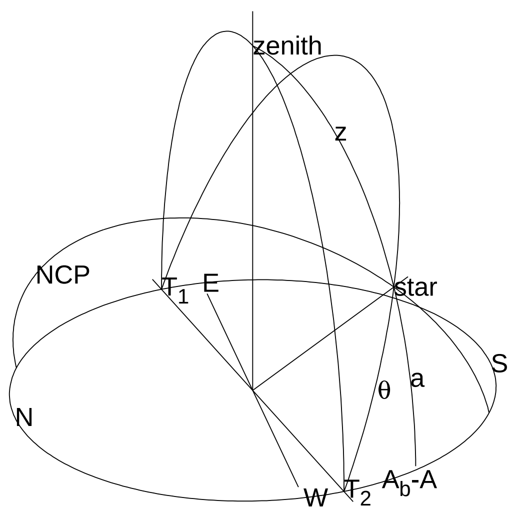

Fig. 3. *Celestial sphere, seen from the outside. The north direction through the object is given by the great circle passing through the celestial poles and the object. The angle θ is the vector $\mathbf{P} = \mathbf{b} - \mathbf{D}$ projected on the sphere.*

An overview of the relevant geometry is given in Fig. 3 which looks at the celestial sphere from outside.

In Fig. 3, the baseline has been infinitely extended straight outwards in both directions which define telescope coordinates T_1 and T_2, also where the baseline meets the Celestial Sphere. For an observer at the mid-point of the baseline, telescope 1 then is at azimuth τ, telescope 2 at azimuth $A_b = \tau + \pi$, see Fig. 1. Let the object be at azimuth A and elevation a.

The projection of the baseline occurs in a plane including the object and the baseline, thus defining the great circle labeled θ in Fig. 3. If θ is the angle on the sky between the object and the point on the horizon with azimuth A_b, then the length P of the projected baseline is given by

$$P = b \sin\theta,$$

where θ is calculated from the relation (14). With this auxiliary quantity, there are various ways of obtaining the position angle p_b of the projected baseline on the sky, some of which are detailed in Sections 3.2–3.4. The common theme is that the baseline orientation (A_b, a_b) in the local horizontal polar coordinate system is transferred to a rotated polar coordinate system with the star defining the new polar axis; the position angle is the difference of azimuths between $\mathbf{T}_2$ and the NCP in this rotated polar coordinate system.

3.2. In Horizontal Coordinates

We use a vector algebraic method to span a plane tangential to the celestial sphere; tangent point is the position **s** of the star. Within this orthographic zenithal projection (Calabretta and Greisen 2002) we consider the two directions from the origin, toward the NCP, $\mathbf{s}_+$, on one hand and toward T_2, $\hat{\mathbf{b}}$, on the other. The tangential plane is fixed by any two unit vectors perpendicular to the vector **s** of (9). Rather arbitrarily they are chosen along $\partial\mathbf{s}/\partial A$ and $\partial\mathbf{s}/\partial a$, explicitly

$$\mathbf{e}_A = \begin{pmatrix} \sin A \\ \cos A \\ 0 \end{pmatrix}_t, \quad (15)$$

$$\mathbf{e}_a = \begin{pmatrix} \cos A \sin a \\ -\sin A \sin a \\ \cos a \end{pmatrix}_t. \quad (16)$$

(The equivalent exercise with a different choice of axes follows in Section 3.4.) These are orthonormal,

$$\mathbf{s}\cdot\mathbf{e}_A = \mathbf{s}\cdot\mathbf{e}_a = \mathbf{e}_A\cdot\mathbf{e}_a = 0;$$
$$|\mathbf{e}_A| = |\mathbf{e}_a| = 1; \quad \mathbf{s}\times\mathbf{e}_a = \mathbf{e}_A.$$

Partitioning of the direction from **s** to the NCP defines three coefficients $c_{0,1,2}$,

$$\mathbf{s}_+ = c_0\mathbf{s} + c_1\mathbf{e}_A + c_2\mathbf{e}_a. \quad (17)$$

Building the square on both sides yields the familiar formula for the sum of squares of the projected cosines,

$$c_0^2 + c_1^2 + c_2^2 = 1. \tag{18}$$

Two dot products of (17) using (10) and (15)–(16) solve for two coefficients,

$$\begin{aligned} c_1 = \mathbf{s}_+ \cdot \mathbf{e}_A &= \cos\phi \sin A, \\ c_2 = \mathbf{s}_+ \cdot \mathbf{e}_a &= \cos\phi\cos A \sin a + \sin\phi\cos a. \end{aligned}$$

An angle φ between $\mathbf{e}_A$ and $\mathbf{s}_+$ is defined via Fig. 4,

$$\begin{aligned} \cos\varphi &= \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \\ \sin\varphi &= \frac{c_2}{\sqrt{c_1^2 + c_2^2}}. \end{aligned}$$

From (18), and since (11) equals c_0,

$$\sqrt{c_1^2 + c_2^2} = \cos\delta.$$

The equivalent splitting of $\hat{\mathbf{b}}$ into components within and perpendicular to the tangent plane is:

$$\hat{\mathbf{b}} = c_0'\mathbf{s} + c_1'\mathbf{e}_A + c_2'\mathbf{e}_a. \tag{19}$$

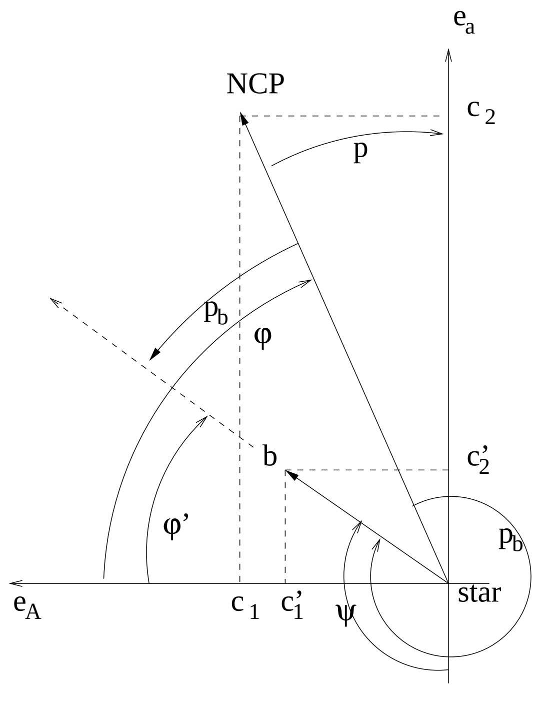

Fig. 4. *In the tangent plane spanned by the unit vectors (15) and (16), the components of* $\hat{\mathbf{b}}$ *and* $\mathbf{s}_+$ *are* c_1*,* c_1'*,* c_2 *and* c_2' *defined by (17) and (19). The view is from the outside onto the plane, so* $\mathbf{e}_A$*, the vector into the direction of increasing* A*, points to the left.*

Building the square on both sides yields

$$c_0'^2 + c_1'^2 + c_2'^2 = 1. \tag{20}$$

Dot products of (19) solve for the expansion coefficients, with (12) and (15)–(16):

$$\begin{aligned} c_1' &= \hat{\mathbf{b}} \cdot \mathbf{e}_A \\ &= -\cos A_b \cos a_b \sin A + \sin A_b \cos a_b \cos A \\ &= \cos a_b \sin(A_b - A), \qquad (21) \\ c_2' &= \hat{\mathbf{b}} \cdot \mathbf{e}_a \\ &= -\cos a_b \sin a \cos(A_b - A) + \sin a_b \cos a. \qquad (22) \end{aligned}$$

This defines an angle φ' in the polar coordinates of the tangent plane,

$$\begin{aligned} \cos\varphi' &= \frac{c_1'}{\sqrt{c_1'^2 + c_2'^2}}, \\ \sin\varphi' &= \frac{c_2'}{\sqrt{c_1'^2 + c_2'^2}}. \end{aligned}$$

Since c_0' equals $\hat{\mathbf{b}} \cdot \mathbf{s} = \cos\theta$ in (14),

$$\sqrt{c_1'^2 + c_2'^2} = \sin\theta.$$

The baseline position angle is the difference between the two angles, as to redefine the reference direction from $\mathbf{e}_A$ to the direction of the NCP (see Fig. 4):

$$p_b = \varphi' - \varphi \pmod{2\pi}.$$

Sine and cosine of this imply

$$\begin{aligned} \sin p_b &= \sin\varphi'\cos\varphi - \cos\varphi'\sin\varphi \\ &= \frac{c_1 c_2' - c_1' c_2}{\cos\delta\sin\theta}; \qquad (23) \\ \cos p_b &= \cos\varphi'\cos\varphi + \sin\varphi'\sin\varphi \\ &= \frac{c_1 c_1' + c_2 c_2'}{\cos\delta\sin\theta}, \qquad (24) \end{aligned}$$

where

$$\begin{aligned} c_1 c_2' - c_1' c_2 &= \cos\phi(\sin A \sin a_b \cos a \\ &\quad - \cos a_b \sin a \sin A_b) \\ &\quad - \sin\phi\cos a \cos a_b \sin(A_b - A), \qquad (25) \end{aligned}$$

and

$$\begin{aligned} c_1 c_1' + c_2 c_2' &= \cos\phi(\cos A \cos a \cos\theta \\ &\quad - \cos a_b \cos A_b) \\ &\quad + c_2' \sin\phi \cos a. \qquad (26) \end{aligned}$$

The computational strategy is to build

$$\tan p_b = \frac{c_1 c_2' - c_1' c_2}{c_1 c_1' + c_2 c_2'}. \tag{27}$$

The (positive) values of $\cos\delta$ and $\sin\theta$ in the denominators of (23) and (24) need not to be calculated. The branch ambiguity of the arctan is typically handled by use of the `atan2()` functionality in the libraries of higher programming languages.

3.3. Via Parallactic Angle

If the position angle of the zenith p

$$\tan p = \frac{\cos\phi\sin A}{\cos\phi\cos A\sin a+\sin\phi\cos a} = \frac{\sin h}{\cos\delta\tan\phi-\sin\delta\cos h}$$

is known from any other source, a quicker approach to the calculation of p_b employs an auxiliary angle ψ (Ch. Leinert 2006, priv. commun.)

$$p_b \equiv p+\pi+\psi \pmod{2\pi}. \quad (28)$$

The calculation of ψ is delegated to the calculation of its sines and cosines

$$\cos\psi = -\cos p_b\cos p - \sin p_b \sin p;$$
$$\sin\psi = -\sin p_b\cos p + \cos p_b\sin p.$$

In the right hand sides we insert (23)–(24)

$$\sin p = \frac{\cos\phi\sin A}{\cos\delta}; \quad (29)$$
$$\cos p = \frac{\cos\phi\cos A\sin a+\sin\phi\cos a}{\cos\delta}, \quad (30)$$

and after some standard manipulations

$$\sin\psi = \frac{\cos a_b\sin(A_b-A)}{\sin\theta}; \quad (31)$$
$$\cos\psi = \frac{\sin a\cos\theta-\sin a_b}{\sin\theta\cos a} = \frac{\cos a_b\sin a\cos(A_b-A)-\sin a_b\cos a}{\sin\theta}. \quad (32)$$

[these are c_1' and $-c_2'$ of (21) and (22) divided by $\sin\theta$.] In Fig. 4, ψ can therefore be identified as the angle between $-\mathbf{e}_a$ and $\hat{\mathbf{b}}$, and this is consistent with (28) which decomposes the rotation into the angle p, a rotation by the angle π (which would join the end of p with the start of ψ in Fig. 4), and finally a rotation by ψ.

In summary, the disadvantage of this approach is that p must be known by other means, and the advantage is that computation of ψ via the arctan of (31) and (32) only requires c_1' and c_2', but not c_1 or c_2.

3.4. In Equatorial Coordinates

Section 3.2 provides p_b, given star coordinates (A,a). Subsequently we derive the same value in terms of δ and h. The straightforward approach is the substitution of (4)–(8) in (25) and (26). A less exhaustive alternative works as follows (R. Köhler 2006, priv. commun.): the axes $\mathbf{e}_A$ and $\mathbf{e}_a$ in the tangential plane are replaced by two different orthonormal directions, better adapted to δ and h, namely the directions along $\partial\mathbf{s}/\partial h$ and $\partial\mathbf{s}/\partial\delta$:

$$\mathbf{e}_h = \begin{pmatrix}\sin\phi\sin h\\ \cos h\\ -\cos\phi\sin h\end{pmatrix}_t;$$
$$\mathbf{e}_\delta = \begin{pmatrix}\cos\phi\cos\delta+\sin\phi\sin\delta\cos h\\ -\sin\delta\sin h\\ \sin\phi\cos\delta-\cos\phi\sin\delta\cos h\end{pmatrix}_t.$$

The further calculation follows the scheme of Section 3.2: The axes are orthonormal,

$$\mathbf{s}\cdot\mathbf{e}_h = \mathbf{s}\cdot\mathbf{e}_\delta = \mathbf{e}_h\cdot\mathbf{e}_\delta = 0;$$
$$|\mathbf{e}_h| = |\mathbf{e}_\delta| = 1; \quad \mathbf{s}\times\mathbf{e}_\delta = \mathbf{e}_h.$$

The decomposition of $\mathbf{s}_+$ in this system defines three expansion coefficients $d_{0,1,2}$,

$$\mathbf{s}_+ = d_0\mathbf{s}+d_1\mathbf{e}_\delta+d_2\mathbf{e}_h.$$

Multiplying this equation in turn with $\mathbf{e}_h$ and $\mathbf{e}_\delta$ yields

$$d_2 = 0; \quad d_1 = \mathbf{s}_+\cdot\mathbf{e}_\delta = \cos\delta. \quad (33)$$

The direction $\hat{\mathbf{b}}$ to T_2 is projected into the same tangent plane, defining three expansion coefficients $d'_{0,1,2}$,

$$\hat{\mathbf{b}} = d_0'\mathbf{s}+d_1'\mathbf{e}_\delta+d_2'\mathbf{e}_h. \quad (34)$$

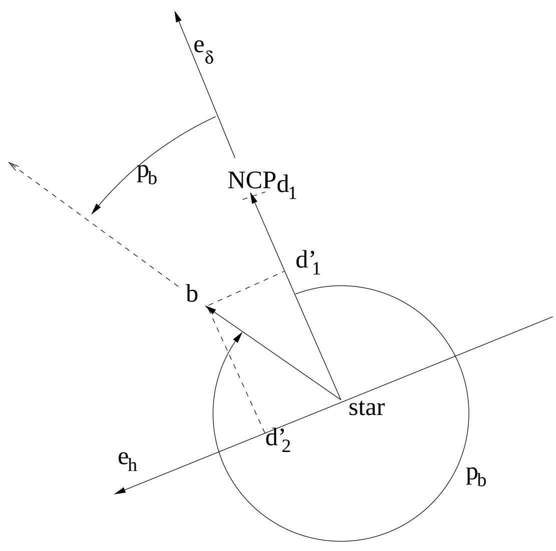

Fig. 5. *Fig. 4 after switching from the $(\mathbf{e}_A,\mathbf{e}_a)$ axes to $(\mathbf{e}_h,\mathbf{e}_\delta)$. Axis components of $\mathbf{s}_+$ and $\hat{\mathbf{b}}$ are indicated.*

Building the dot product of this equation with $\mathbf{e}_h$ using (12) yields

$$d_2' = -\cos A_b\cos a_b\sin\phi\sin h+\sin A_b\cos a_b\cos h - \sin a_b\cos\phi\sin h. \quad (35)$$

Building the dot product of (34) with $\mathbf{e}_\delta$ yields

$$\begin{aligned} d_1' &= -\cos A_b \cos a_b \cos\phi\cos\delta \\ &\quad -\cos A_b\cos a_b\sin\phi\sin\delta\cos h \\ &\quad -\sin A_b\cos a_b\sin\delta\sin h + \sin a_b\sin\phi\cos\delta \\ &\quad -\sin a_b\cos\phi\sin\delta\cos h. \end{aligned} \tag{36}$$

Some simplification in this formula is obtained by trading the hour angle h for the angle θ, which might be more readily available since θ is measured via the delay: In the second term we use (5) to substitute

$$\sin\phi\cos h \to \frac{\cos\phi\sin\delta + \cos A\cos a}{\cos\delta}.$$

In the third term we use (4) to substitute

$$\sin h \to \frac{\sin A\cos a}{\cos\delta},$$

and in the last term we use (6) to substitute

$$\cos\phi\cos h \to \frac{\sin a - \sin\delta\sin\phi}{\cos\delta}.$$

Further standard trigonometric identities and (14) yield

$$d_1' = \frac{\sin a_b\sin\phi - \cos a_b\cos A_b\cos\phi - \sin\delta\cos\theta}{\cos\delta}. \tag{37}$$

In the numerator we recognize a baseline declination δ_b,

$$\begin{aligned} \mathbf{s}_+\cdot\hat{\mathbf{b}} &\equiv \sin\delta_b \\ &= \sin a_b\sin\phi - \cos a_b\cos A_b\cos\phi. \end{aligned} \tag{38}$$

p_b is the angle that rotates the projected vector (d_2, d_1) within the tangent plane into the direction (d_2', d_1') (Fig. 5),

$$\begin{aligned} \cos p_b &= \frac{d_1'}{\sin\theta}, \\ \sin p_b &= -\frac{d_2'}{\sin\theta}. \end{aligned}$$

Again, $\sin\theta$ does not need actually to be calculated, but only

$$\tan p_b = \frac{-d_2'}{d_1'}, \tag{39}$$

and again, selection of the correct branch of the arctan is easy with `atan2()` functions if the negative sign is kept attached to d_2'.

The use of (37) is optional: d_2' and d_1' are completely defined in terms of the baseline direction (A_b, a_b), the geographic latitude ϕ and the star coordinates (h,δ) via (35) and (36). With these, one can immediately proceed to (39).

OIFITS (Pauls) defines no associated angle in the plane perpendicular to the direction of the phase center.

The segment of the baseline in the tangent plane primarily defines an orientation in the $u-v$ plane. However, modal decomposition of the amplitudes in products of radial and angular basis functions (akin to Zernike polynomials) means that the Fourier transform to the $x-y$ plane preserves the angular basis functions; see the calculation for the 3D case, the Spherical Harmonics. In this sense, the angle p_b is also "applicable" in the $x-y$ plane.

4. DIFFERENTIALS IN THE FIELD-OF-VIEW

4.1. Sign Convention

The provisions of the previous section, namely

- the OIFITS sign convention of the baseline vector, Eq. (2) and Fig. 2,
- the conventional formula (13)

fix the sign of the delay D as follows: if the wavefront hits T_2 prior to T_1, the values of D and $\cos\theta$ are positive; if it hits T_1 prior to T_2, both values are negative.

The transitional case $D = 0$ ($\theta = \pi/2$) occurs if the star passes through the "baseline meridian" plane perpendicular to the baseline; the two lines to the Northwest and Southeast in Fig. 1 show (projections of) these directions. The baseline in most optical stellar interferometers is approximately horizontal, $a_b \approx 0$. From (13) we see that this case is approximately equivalent to $\cos(A_b - A) \approx 0$. Since $\cos a > 0$, the sign of D coincides with the sign of $\cos(A_b - A)$. This is probably the fastest way to obtain the sign of the Optical Path Difference from FITS (Hanisch et al. 2001) header keywords; the advantage of this recipe is that the formula is independent of which azimuth convention is actually in use.

The baseline length b follows immediately from the Euclidean distance of the two telescopes entries for column `STAXYZ` of the OIFITS table `OI_ARRAY` (Pauls et al. 2005).

4.2. External Path Delay

The total differential of (14) relates a direction $(\Delta A, \Delta a)$ away from the star to a change in the angular distance between the star and the baseline,

$$\begin{aligned} -\sin\theta\Delta\theta &= \Delta a\big[-\cos a_b\sin a\cos(A_b - A) \\ &\qquad + \sin a_b\cos a\big] \\ &\quad + \Delta A\cos a\cos a_b\sin(A_b - A). \end{aligned}$$

(31) and (32) turn this into

$$\sin\theta\Delta\theta = \sin\theta(\cos\psi\Delta a - \sin\psi\cos a\Delta A), \tag{40}$$

consistent with the decomposition in Fig. 4. The two directions of constant delay are characterized by

$\Delta\theta = 0$. Equating the right hand side of the previous equation with zero, this means

$$\frac{\Delta a}{\cos a \Delta A} = \tan\psi; \quad (\Delta\theta = \Delta D = 0). \qquad (41)$$

For the direction of maximum change in ΔD, which runs perpendicular to (41),

$$\frac{\Delta a}{\cos a\, \Delta A} = -\frac{1}{\tan\psi}, \quad (p_b : \max \Delta D)$$

which leads to

$$\begin{aligned}\Delta D &= -b\sin\theta\Delta\theta = b\sin\theta\frac{\cos a\,\Delta A}{\sin\psi}\\ &= -b\sin\theta\frac{\Delta a}{\cos\psi}, \ (\max \Delta D).\end{aligned}$$

Example: a field of view of $\Delta\theta = \pm 1$" at a baseline of $b = 100$ m at a position $\theta = 45°$ scans $\Delta D = \pm 340$ μm. The details of the optics determine how far this contributes to the instrumental visibility (loss).

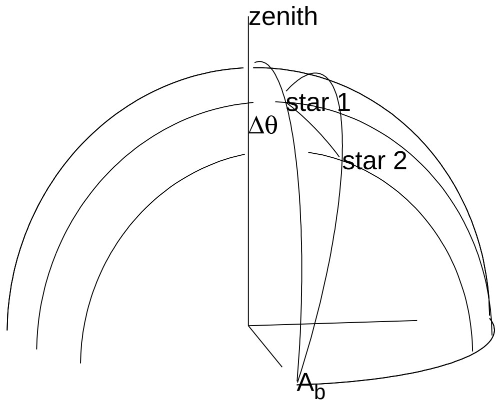

Fig. 6. *Example of three concentric circles on the Celestial Sphere, centered at the baseline, which unite star directions of $D = const$. Two star positions are connected by the short diagonal dash. A quarter of each of the two projected baselines, which intersect at (A_b, a_b) near the horizon, is also shown.*

Fig. 6 sketches this geometry: the change ΔD while re-pointing from one star by $(\Delta A, \Delta a)$ to another star along some segment of the Celestial Sphere can be split into a change associated with the radial direction toward/away from T_2 along a great circle, plus no change moving along a circle of radius $\cos\theta$ centered on the baseline azimuth.

We may repeat the calculation in the equatorial system and split $\Delta\theta$ into $\Delta\delta$ and Δh,

$$\begin{aligned}\Delta D &= -b\sin\theta\Delta\theta\\ &= b(\Delta\delta \mathbf{e}_\delta \cdot \hat{\mathbf{b}} + \Delta h \cos\delta \mathbf{e}_h \cdot \hat{\mathbf{b}})\\ &= b(\Delta\delta\, d_1' + \Delta h \cos\delta\, d_2')\\ &= b\sin\theta(\Delta\delta\cos p_b - \Delta h\cos\delta\sin p_b). \qquad (42)\end{aligned}$$

This translates Calabretta's (Calabretta and Greisen 2002) Appendix C to our variables. When $\Delta\theta = 0$, we have the directions of zero change in ΔD, perpendicular to the projected baseline,

$$\frac{\Delta\delta}{\cos\delta\Delta h} = \tan p_b, \quad (\Delta\theta = \Delta D = 0). \qquad (43)$$

The direction of maximum change is

$$\frac{\Delta\delta}{\cos\delta\Delta h} = -\frac{1}{\tan p_b}, \quad (p_b : \max \Delta D),$$

and along this gradient

$$\Delta D = -b\sin\theta\frac{\cos\delta\Delta h}{\sin p_b} = b\sin\theta\frac{\Delta\delta}{\cos p_b}, \ (\max \Delta D).$$

5. SUMMARY

To standardize the nomenclature, we propose to choose the North Celestial Pole as the "reference" direction (direction of position angle zero) and a handedness to define the sign of the position angles (mathematically positive if looking at the Celestial Sphere from the inside).

The set of position angles discussed here includes

- the direction of the zenith, the "parallactic" angle,
- the direction toward the point where the baseline pinpoints the Celestial Sphere, the "projected baseline angle,"
- position angles of "secondary" stars in differential astrometry.

Computation of the angle proceeds via (27) if the object coordinates are given in the local altitude-azimuth system, or via (39) if they are given in the equatorial system.

The mathematics involved is applicable to observatories at the Northern and the Southern hemisphere: positions (A, a) or (δ, h) are mapped onto a 2D zenithal coordinate system. The position angles play the role of the longitude (the North Celestial Pole the role of Greenwich or Aries). A distance between the object and the star—in the range from 0 to π along great circles through the Celestial Pole in an `ARC` projection—might take the role of the polar distance—although an orthographic `SIN` projection had been used for the calculations in this script.

Acknowledgements – This work is supported by the NWO VICI grant 639.043.201.

REFERENCES

Calabretta, M. R., Greisen, E. W.: 2002, *Astron. Astrophys.*, **395**, 1077.
Hanisch, R. J., Farris, A., Greisen, E. W., Pence, W. D., Schlesinger, B. M., Teuben, P. J., Thompson, R. W., Warnock III, A.: 2001, *Astron. Astrophys.*, **376**, 359.
Jones, G. C.: 2004, *J. Geod.*, **76**, 437.
Karttunen, H., Kröger, P., Oja, H., Poutanen, M., Donner, K. J. (eds.): 1987, Fundamental Astronomy, Springer, Berlin, Heidelberg.
Lang, K. R.: 1998, Astrophysical Formulae, vol. II of Astronomy and Astrophysics Library, (3rd edn.), Springer, Berlin, Heidelberg. E: In the second line of (5.45), $\sin A$ should read $\cos A$.
National Imagery and Mapping Agency (NIMA): 2000, Department of defense world geodetic system 1984, Tech. Rep. TR8350.2.
Pauls, T. A., Young, J. S., Cotton, W. D., Monnier, J. D.: 2004, In: *New Frontiers in Stellar Interferometry*, Traub, W. A. (ed.), *Proc. SPIE* **5491**, Int. Soc. Optical Engineering, 1231.
Pauls, T. A., Young, J. S., Cotton, W. D., Monnier, J. D.: 2005, *Publ. Astron. Soc. Pacific*, **117**, 1255.
Pearson, T. J.: 1991, Introduction to the Caltech VLBI programs, Tech. rep., Astronomy Department, California Institute of Technology. `http://www.astro.caltech.edu/~tjp/citvlb/`
Vermeille, H.: 2004, *J. Geod.*, **76**, 451.
Wallace, P. T.: 2003, SLALIB—Positional Astronomy Library 2.4–12 Programmer's Manual.
Wood, T. E.: 1996, *SIAM Review*, **38**, 637.
Zhang, C.-D., Tsu, H. T., Wu, X. P., Li, S. S., Wang, Q. B., Chai, H. Z., Du, L.: 2005, *J. Geod.*, **79**, 413.

APENDICES

A.1. Diurnal motion

The diurnal motion of the external path difference is described by separating terms proportional to $\sin h$ and proportional to $\cos h$. We rewrite (13) in equatorial coordinates by multiplying (9) with (12), and collect terms with the aid of (38):

$$D = \mathbf{s} \cdot \mathbf{b} = b[\sin\delta \sin\delta_b + \cos\delta\cos\delta_b \cos(h - h_b)].$$

A time-independent offset $b \sin\delta \sin\delta_b$ and an amplitude $b\cos\delta\cos\delta_b$, defined by products of the polar and equatorial components of star and baseline, respectively, constitute the regular motion of the delay. The phase lag h_b is determined by the arctan of

$$\begin{aligned} \cos\delta_b \sin h_b &= \sin A_b \cos a_b; \\ \cos\delta_b \cos h_b &= \cos A_b \cos a_b \sin\phi + \sin a_b \cos\phi, \end{aligned}$$

and plays the role of an interferometric hour angle.

A.2. Closure Relations

An array T_1, T_2 and T_3 of telescopes forms a baseline triangle

$$\mathbf{b}_{12} + \mathbf{b}_{23} + \mathbf{b}_{31} = 0.$$

If all point to a common direction $\mathbf{s}$, some phase closure sum rules result in:

$$\begin{aligned} \mathbf{D}_{12} + \mathbf{D}_{23} + \mathbf{D}_{31} = 0; \\ D_{12} + D_{23} + D_{31} = 0; \\ b_{12}\cos\theta_{12} + b_{23}\cos\theta_{23} + b_{31}\cos\theta_{31} = 0. \end{aligned}$$

The triangle of projected baselines in the tangent plane is

$$\mathbf{P}_{12} + \mathbf{P}_{23} + \mathbf{P}_{31} = 0.$$

These vectors can be represented as numbers in a complex plane by introduction of three moduli P and three orientation angles p_b:

$$\begin{aligned} &P_{12}e^{ip_{b12}} + P_{23}e^{ip_{b23}} + P_{31}e^{ip_{b31}} \\ &= b_{12}\sin\theta_{12}e^{ip_{b12}} + b_{23}\sin\theta_{23}e^{ip_{b23}} + b_{31}\sin\theta_{31}e^{ip_{b31}} \\ &= 0. \end{aligned}$$

These equations remain correct if transformed by complex conjugation or multiplied by a complex phase factor; therefore these closure relations of the projected baseline angles are correct for (i) both senses of defining their orientation, and (ii) any reference direction of the zero angle.

СФЕРНА ТРИГОНОМЕТРИЈА ПРОЈЕКЦИЈЕ УГЛА ОСНОВЕ

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УДК 520–872 : 521–181
Стручни чланак

Основна геометрија звезданог интерферометра кога чине два телескопа одређена је правоуглим троуглом који граде (i) основа, одосно вектор који спаја два телескопа, (ii) вектор "кашњења" у правцу ка звезди, и (iii) пројекција вектора основе интерферометра на раван таласног фронта упадног зрачења. Раван троугла пресеца небеску сферу по кругу који пролази кроз звезду; сам пресек представља сегмент кружне линије. Раздвојна моћ интерферометра је велика у два правца дуж овог линијског сегмента (ограничена дифракцијом на однос таласне дужине и дужине пројектоване основе интерферометра), а мала нормално на ове правце (ограничена дифракцијом на однос таласне дужине и пречника телескопа). У овом чланку израчунати су позициони углови ових карактеристичних праваца у хоризонтским и небеским екваторским координатама.