ON SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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ABSTRACT
In this paper we define some generalized sequence spaces defined by a sequence of moduli. The results here are analogous to those by ASMA BEKTAS Cigdem (2006)[Journal of Zhejiang University Science A (2006), 7(12) 2093-2096].

Keywords: Difference sequence space , Sequence of moduli, Strongly almost convergent.

1. INTRODUCTION
Let ℓ0 be the set of all complex sequences and \( l_\infty, c \) and \( c_0 \) be the sets of all bounded, convergent and null sequences \( x = (x_k) \) with complex terms, respectively, normed by
\[
\| x \|_{\infty} = \sup_{k} |x_k|, \quad \text{where } k \in I \quad N = \{1, 2, \cdots\}.
\]
The idea of difference sequence space was introduced by Kizmaz (1981). In 1981, Kizmaz defined the sequence spaces:
\[
l_\infty(\Delta) = \{ x = (x_k) \in \ell^0 : (\Delta x_k) \in l_\infty \},
\]
\[
c(\Delta) = \{ x = (x_k) \in \ell^0 : (\Delta x_k) \in c \},
\]
and
\[
c_0(\Delta) = \{ x = (x_k) \in \ell^0 : (\Delta x_k) \in c_0 \},
\]
where \( \Delta x = (x_k - x_{k+1}) \). These are Banach spaces with the norm
\[
\| x \|_\Delta = \| x_1 \| + \| \Delta x \|_{\infty}.
\]
After then R. Colak and M. Et (1995) defined the sequence spaces:
\[
l_\infty(\Delta^m) = \{ x = (x_k) \in \ell^0 : (\Delta^m x_k) \in l_\infty \},
\]
\[
c(\Delta^m) = \{ x = (x_k) \in \ell^0 : (\Delta^m x_k) \in c \},
\]
and
\[
c_0(\Delta^m) = \{ x = (x_k) \in \ell^0 : (\Delta^m x_k) \in c_0 \},
\]
where \( m \in I \quad N \),
\[
\Delta^0 x = (x_k),
\]
\[
\Delta x = (x_k - x_{k+1}),
\]
\[
\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}),
\]
and so that
\[
\Delta^m x_k = \sum_{i=0}^{m} (-1)^{i} \left[ \begin{array}{c} m \\ i \end{array} \right] x_{k+i}.
\]
and show that these are Banach spaces with the norm
\[
\| x \|_\Delta = \sum_{i=1}^{m} |x_i| + \| \Delta^m x \|_{\infty}.
\]

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Definition 1.1. A function \( f : [0, \infty) \to [0, \infty) \) is called a modulus if
1. \( f(t) = 0 \) if and only if \( t = 0 \),
2. \( f(t + u) \leq f(t) + f(u) \) for all \( t, u \geq 0 \),
3. \( f \) is increasing, and
4. \( f \) is continuous from the right of \( 0 \).

Let \( X \) be a sequence space. Then the sequence space \( X(f) \) is defined as
\[
X(f) = \{ x = (x_k) \in \ell^0 : (f(|x_k|)) \in X \}
\]
for a modulus \( f \) (Maddox 1986 and Ruckle 1973).

Kolk (1993, 1994) gave an extension of \( X(f) \) by considering a sequence of moduli \( F = (f_k) \) i.e.
\[
X(F) = \{ x = (x_k) \in \ell^0 : (f_k(|x_k|)) \in X \}.
\]

A sequence \( x \in l_\infty \) is said to be almost convergent (Lorentz, 1984) if all Banach limits of \( x \) coincide. Lorentz (1984) proved that
\[
\hat{c} := \{ x = (x_k) \in \ell^0 : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ uniformly in } s \}.
\]

Maddox (1967; 1978) has defined \( x \) to be strongly almost convergent to \( L \) if
\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s \text{ for some } L > 0.
\]

Let \( p = (p_k) \) be a sequence of strictly positive real numbers. Nanda (1984) defined
\[
[\hat{c}, p] := \{ x = (x_k) \in \ell^0 : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^p = 0, \text{ uniformly in } s \text{ for some } L > 0 \},
\]
\[
[\hat{c}, p]_0 := \{ x = (x_k) \in \ell^0 : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^p = 0, \text{ uniformly in } s \},
\]
\[
[\hat{c}, p]_\infty := \{ x = (x_k) \in \ell^0 : \sup_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^p = 0, \text{ uniformly in } s \}.
\]

2. MAIN RESULTS

Let \( F = (f_k) \) be a sequence of moduli, \( u = (u_k) \) be any sequence such that \( u_k \neq 0 \) for all \( k \) and \( p = (p_k) \) be any sequence space of strictly positive real numbers then we define the following sequence spaces :
\[
[\hat{c}, F, p](\Delta^m) := \{ x = (x_k) \in \ell^0 : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} f_k(|u_k \Delta^m x_{k+s} - L|)^p = 0, \text{ uniformly in } s \text{ for some } L > 0 \},
\]
\[
[\hat{c}, F, p]_0(\Delta^m) := \{ x = (x_k) \in \ell^0 : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} f_k(|u_k \Delta^m x_{k+s}|)^p = 0, \text{ uniformly in } s \},
\]
\[
[\hat{c}, F, p]_\infty(\Delta^m) := \{ x = (x_k) \in \ell^0 : \sup_{n} \frac{1}{n} \sum_{k=1}^{n} f_k(|u_k \Delta^m x_{k+s}|)^p < \infty, \text{ uniformly in } s \}.
\]

If \( f_k(x) = x \) for every \( k \), then \([\hat{c}, F, p](\Delta^m) = [\hat{c}, p] \), \([\hat{c}, F, p]_0(\Delta^m) = [\hat{c}, p]_0(\Delta^m) \) and \([\hat{c}, F, p]_\infty(\Delta^m) = [\hat{c}, p]_\infty(\Delta^m) \). We denote \([\hat{c}, F, p](\Delta^m) \), \([\hat{c}, F, p]_0(\Delta^m) \) and \([\hat{c}, F, p]_\infty(\Delta^m) \) by \([\hat{c}, F](\Delta^m) \), \([\hat{c}, F]_0(\Delta^m) \) and \([\hat{c}, F]_\infty(\Delta^m) \), when \( p_k = 1 \) for all \( k \).
**Theorem 2.1.** For a sequence $F = (f_k)$ of moduli, the following statements are equivalent:

1. $[\hat{c}, p]_\infty(\Delta^m) \subseteq [\hat{c}, F, p]_\infty(\Delta^m)$,
2. $[\hat{c}, p]_0(\Delta^m) \subseteq [\hat{c}, F, p]_0(\Delta^m)$,
3. $\sup_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k(t)|^p_k < \infty \quad (t > 0)$.

**Proof.**

(i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let $[\hat{c}, p]_0(\Delta^m) \subseteq [\hat{c}, F, p]_0(\Delta^m)$. Suppose that (iii) is not true. Then for some $t$

$$\sup_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k(t)|^p_k = \infty,$$

and there exists a sequence $(n_i)$ of positive integers such that

$$\frac{1}{n_i} \sum_{k=1}^{n_i} f_k(\frac{1}{i}) > i \quad for \quad i = 1, 2, \ldots (1)$$

Now we define $x = \{x_k\}$ by

$$x_k = \begin{cases} \frac{1}{i} & , \quad if \quad 1 \leq k \leq n_i, \quad i = 1, 2, \ldots, \\ 0 & , \quad (k > n_i). \end{cases}$$

Then $x \in [\hat{c}, p]_0(\Delta^m)$ but by Eqn (1), $x \notin [\hat{c}, F, p]_\infty(\Delta^m)$ which contradicts (ii).

Hence (iii) is true.

(iii) $\Rightarrow$ (i):

Let (iii) is true and $x \in [\hat{c}, p]_\infty(\Delta^m)$. If we suppose that $x \notin [\hat{c}, F, p]_\infty(\Delta^m)$, then

$$\sup_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k([u_k \Delta^m x_{k+s}])|^p_k = \infty(2).$$

If we take $t = |u_k \Delta^m x_{k+s}|$ for each $k$ and fixed $s$, then by Eqn(2)

$$\sup_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k(t)|^p_k = \infty,$$

which contradicts (iii). Hence $[\hat{c}, p]_\infty(\Delta^m) \subseteq [\hat{c}, F, p]_\infty(\Delta^m)$.

**Theorem 2.2.** Let $1 \leq p_k \leq \sup_{k} p_k < \infty$. For a sequence of moduli $F = (f_k)$ the following statements are equivalent:

1. $[\hat{c}, F, p]_0(\Delta^m) \subseteq [\hat{c}, p]_0(\Delta^m)$,
2. $[\hat{c}, F, p]_0(\Delta^m) \subseteq [\hat{c}, p]_\infty(\Delta^m)$,
3. $\inf_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k(t)|^p_k > 0 \quad (t > 0)$.

**Proof.**

(i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let $[\hat{c}, F, p]_0(\Delta^m) \subseteq [\hat{c}, p]_\infty(\Delta^m)$. Suppose that (iii) does not hold. Then

$$\inf_{n} \frac{1}{n} \sum_{k=1}^{n} |f_k(t)|^p_k = 0 \quad (t > 0), (3)$$

and there exists a sequence $(n_i)$ of positive integers such that
\[
\frac{1}{n_i} \sum_{k=1}^{n_i} |f_k(i)|^{p_k} < \frac{1}{i}, \text{ for } i = 1, 2, \ldots.
\]

Now define the sequence \( x = \{x_k\} \) by

\[
x_k = \begin{cases} 
i, & \text{if } 1 \leq k \leq n_i, \text{ for } i = 1, 2, \ldots, \\ 0, & k > n_i.
\end{cases}
\]

By Eqn(3), \( x \in [\hat{c}, F, p]_0(\Delta^m_u) \) but \( x \not\in [\hat{c}, p]_\infty(\Delta^m_u) \), which contradicts (ii).

Hence (iii) is true.

(iii) \( \Rightarrow \) (i):

Let (iii) is true and \( x \in [\hat{c}, F, p]_0(\Delta^m_u) \) i.e.

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} \left| f_k \left( u_k \Delta^m x_{k+s} \right) \right|^{p_k} = 0, \text{ uniformly in } s \}. (4)
\]

Suppose that \( x \not\in [\hat{c}, p]_0(\Delta^m_u) \). Then for some number \( \varepsilon_0 > 0 \) and positive integer \( n_0 \) we have \( \left| u_k \Delta^m x_{k+s} \right| \geq \varepsilon_0 \) for some \( s \geq s' \) and \( 1 \leq k \leq n_0 \). Therefore

\[
[f_k(\varepsilon_0)]^{p_k} \leq \left| f_k \left( u_k \Delta^m x_{k+s} \right) \right|^{p_k}
\]

and hence \( \lim_{n} \sum_{k=1}^{n} \left| f_k(\varepsilon_0) \right|^{p_k} = 0 \), which contradicts (iii). Thus \( [\hat{c}, F, p]_0(\Delta^m_u) \subseteq [\hat{c}, p]_0(\Delta^m_u) \).

**Theorem 2.3.**

Let \( 1 \leq p_k \leq \sup \ p_k < \infty \). The inclusion \([\hat{c}, F, p]_\infty(\Delta^m_u) \subseteq [\hat{c}, p]_0(\Delta^m_u)\) holds if and only if

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} \left| f_k(t) \right|^{p_k} = \infty \text{ for } t > 0. (5)
\]

**Proof.** Let \([\hat{c}, F, p]_\infty(\Delta^m_u) \subseteq [\hat{c}, p]_0(\Delta^m_u)\). Suppose Eqn(5) does not hold. Then there exists a number \( t_0 > 0 \) and a sequence \( \{n_i\} \) of positive integers such that

\[
\frac{1}{n_i} \sum_{k=1}^{n_i} \left| f_k(t_0) \right|^{p_k} \leq M < \infty, \text{ i = 1, 2, \ldots}(6)
\]

Now we define the sequence \( x = (x_k) \) by

\[
x_k = \begin{cases} \ t_0, & \text{if } 1 \leq k \leq n_i, \text{ for } i = 1, 2, \ldots, \\ 0, & k > n_i.
\end{cases}
\]

Thus by Eqn (6), \( x \in [\hat{c}, F, p]_\infty(\Delta^m_u) \) but \( x \not\in [\hat{c}, p]_0(\Delta^m_u) \). So that Eqn (5) must hold.

Conversely let Eqn (5) hold. If \( x \in [\hat{c}, F, p]_0(\Delta^m_u) \), then for each \( s \) and \( n \)

\[
\frac{1}{n} \sum_{k=1}^{n} \left| f_k \left( u_k \Delta^m x_k \right) \right|^{p_k} \leq M < \infty. (7)
\]

Suppose that \( x \not\in [\hat{c}, p]_0(\Delta^m_u) \). Then for some number \( \varepsilon_0 > 0 \) and positive integer \( s_0 \) and index \( n_0 \) we have \( \left| u_k \Delta^m x_{k+s} \right| \geq \varepsilon_0 \) for \( s \geq s_0 \). Therefore

\[
[f_k(\varepsilon_0)]^{p_k} \leq \left| f_k \left( u_k \Delta^m x_{k+s} \right) \right|^{p_k},
\]

and hence for each \( k \) and \( s \) we get
\[
\frac{1}{n} \sum_{k=1}^{n} [f_k(e_0)]^{pk} \leq M < \infty,
\]
for some \( M > 0 \), by Eqn (7) which contradicts Eqn (5). Hence
\[
[\hat{c}, F, p]_{\infty}(\Delta^m_n) \subseteq [\hat{c}, F, p]_0(\Delta^m_n)
\]

**Theorem 2.4.**

Let \( 1 \leq p_k \leq \sup_k p_k < \infty \). The inclusion \([\hat{c}, p]_{\infty}(\Delta^m_n) \subseteq [\hat{c}, F, p]_0(\Delta^m_n)\), holds if and only if
\[
\lim \frac{1}{n} \sum_{k=1}^{n} [f_k(t_0)]^{pk} = 0 \quad \text{for} \quad t > 0. \tag{8}
\]

**Proof.** Suppose that \([\hat{c}, p]_{\infty}(\Delta^m_n) \subseteq [\hat{c}, F, p]_0(\Delta^m_n)\), but \(8\) does not hold. Then for some \( t_0 > 0 \)
\[
\lim \frac{1}{n} \sum_{k=1}^{n} [f_k(t_0)]^{pk} = L \neq 0. \tag{9}
\]

Define the sequence \( x = (x_k) \) by
\[
x_k = t_0^{-m} \sum_{v=0}^{k-v} (-1)^m \begin{pmatrix} m+k-v-1 \\ k-v \end{pmatrix}
\]
for \( k = 1, 2, \cdots \). Then \( x \notin c_0(F, p, \Delta^m_n) \), by Eqn (6). Hence Eqn (5) must hold.

Conversely let \( x \in [\hat{c}, p]_{\infty}(\Delta^m_n) \), and suppose that Eqn \(8\) holds. Then for every \( k \) and \( s \)
\[
|u_k \Delta^m x_{k+s}| \leq M < \infty.
\]
Therefore
\[
[f_k(|u_k \Delta^m x_{k+s}|)]^{pk} \leq [f_k(M)]^{pk},
\]
and
\[
\lim \frac{1}{n} \sum_{k=1}^{n} [f_k(|u_k \Delta^m x_{k+s}|)]^{pk} \leq \lim \frac{1}{n} \sum_{k=1}^{n} [f_k(M)]^{pk} = 0, \quad \text{by Eqn}(8).
\]

Hence \( x \in [\hat{c}, F, p]_0(\Delta^m_n) \).

3. REFERENCES