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ON ORBITS FOR A PARTICULAR CASE OF AXIAL SYMMETRY

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SUMMARY: A particular case of steady state and axial symmetry -the potential formula proposed by Miyamoto and Nagai - is studied. A number of orbits of a bound test particle is determined numerically, with both, the potential parameters and initial conditions, varied. Unlike special cases, such as nearly circular and nearly planar orbits, in the case of "truly spatial orbits" the time dependence of the coordinates becomes very complicated and a mathematical treatment including any known periodic functions is hardly possible. Bearing in mind that orbits studied in the present paper are determined by three elements, the authors propose the mean values over time of the squares of velocity components to characterize them.

Key words. Galaxy: kinematics and dynamics

1. INTRODUCTION

The question concerning the integrals of motion in the case of large or statistical stellar systems, like star clusters and galaxies, is among the fundamental ones in stellar dynamics. A study of such systems involves almost always the assumption of a steady state. As a consequence, for an arbitrary test particle (star) orbiting the system center the total mechanical energy will be constant, the fact known as the energy integral. In addition, the steady state, as an ideal state in time, is usually accompanied by a kind of symmetry, an ideal state in space. The two best known kinds of symmetry are the spherical and the axial ones. As is well known, in the former case the consequence is the conservation of the angular-momentum vector, whereas in the latter one only one component of this vector (that along the symmetry axis) is conserved. Indeed, this may be applied to real stellar systems as a first approximation: globular clusters and subsystems of galaxies, except discs, are examples for steady state and spherical symmetry, whereas the discs are examples for steady state and axial symmetry.

From the theoretical point of view the steady state has another important consequence - the total number of independent integrals of motion describing the motion of a test particle is five, not six. However, as easily seen, the mere validity of the two conservation laws (energy and angular momentum) in both variants (spherical and axial symmetries) does not yield five independent integrals of motion. It is usually said that the additional integrals of motion are generally non-isolating, unlike the energy and angular-momentum components, which are isolating.

In order to throw more light on the nature of generally non-isolating integrals, we have undertaken a study intended to result in a series of papers. In the first paper of this series (Ninković and Jovanović 2008), a particular case of steady state and spherical symmetry was considered where the attention was paid to the ratio of the sidereal period to the anomalistic one. The results clearly indicated a different nature of the fifth independent isolating integral of motion for that case (steady state and spherical symmetry), when it exists, appearing rather as a consequence of a resonance than following from a fundamental conservation law, like those concerning energy and angular momentum.

In the present paper the authors examine the more general case of steady state and axial symmetry. This case is rather well known in the literature (e.g. Binney and Tremaine 1987 - p. 114, Contopoulos 2002 - p. 434). Though from the point of view of the theory it is not yet clear if for this case in general there is another isolating integral of motion, independent of the two classical ones (energy and one component of angular momentum), it has attracted much attention. In particular, some formulae describing the gravitational potential within a stellar system and corresponding to the case of steady state and axial symmetry have been derived; well-known examples are the Hénon-Heiles formula (Hénon and Heiles 1964), the Kuzmin formula (Kuzmin 1956), the Miyamoto-Nagai (1975) formula, the so-called logarithmic potential (Richstone 1980), etc. It should be said that the Kuzmin formula corresponds to one of the pairs surface density-square of circular velocity ($n = 1$) found by Toomre (1963).

Any analytical presentation of the potential is very important. Not only because of the cases allowing an analytical solution of the equations of motion, but also for those where the only possibility is to look after a numerical solution. Today many algorithms for solving the equations of motion are available. The procedure is rather standardized; one applies the angular-momentum integral in order to reduce the problem to solving two equations only: in R (distance to the axis of symmetry) and in z (its modulus is the distance to the plane of symmetry). The energy integral is used as the means of controlling the accuracy of the numerical procedure.

As an example of such an approach, one can mention a paper by Ninković et al. (2002). In that paper orbits around the center of the Milky Way are calculated for a number of sets of initial conditions. The Milky Way was assumed to be in steady state with axial symmetry. Its potential was given analytically, combining terms of axial symmetry and the spherical one (special case of axial symmetry), which represented different subsystems of the Galaxy. The orbits obtained there were presented in projection onto the meridional plane and the shapes of these projections were studied. However, due to the constraints imposed on the distribution of dark matter, the same potential formulae cannot be applied beyond the limiting sphere of dark matter. For this reason the opinion of the present authors is that a better approach is to assume a one-component potential described by a single formula. The potential formula chosen here is that of Miyamoto and Nagai (1975).

2. DESCRIPTION OF THE POTENTIAL FORMULA CHOSEN HERE

The formula proposed by Miyamoto and Nagai (1975) which describes the gravitational potential within a stellar system (or subsystem) is

$$\Pi = \frac{G\mathcal{M}}{\sqrt{R^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}} \quad . \tag{1}$$

The designations mean: Π potential, G universal constant of gravitation, $\mathcal{M}$ total mass of the system, R distance to the main axis (axis of symmetry), z the coordinate along this axis, whereas a and b are two constants which have dimension of distance. Thus it is seen from formula (1) that the potential of a stellar system in this case has three independent parameters (three constants, $\mathcal{M}$, a and b, not universal).

Potential (1) originates (Poisson's equation) from a mass distribution given by

$$\rho = \frac{b^2\mathcal{M}\left[aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2\right]}{4\pi\left[R^2 + (a + \sqrt{z^2 + b^2})^2\right]^{5/2}(z^2 + b^2)^{3/2}}, \tag{2}$$

where ρ is the density. As easily seen from this expression, sufficiently close to the center ($R = 0, z = 0$) the density is almost constant. The volume, within which this mass distribution applies, is infinite; formula (1) shows that at very large both R and $|z|$ the potential becomes similar to that of a point mass. The behavior of the two constants having dimension of distance, a and b, determines the two limiting cases. If we have $a = 0$, then this density-potential pair (formulae (2) and (1)) becomes the case known as the Plummer or the Schuster one, in which the axial symmetry is reduced to its special case – the spherical one. On the other hand, if $b = 0$, the pair becomes Kuzmin's case (or Kuzmin-Toomre case) for which the system has a collapse so that the density dependence on z follows the Dirac delta function. Therefore, by varying the ratio a/b one varies the flattening of the system: from zero (no flattening) towards infinity (collapse into the plane of symmetry). As for these two extremes, the case corresponding to the spherical symmetry might be of interest. It is true that in both extremal cases the orbit of a test particle always remains in the same plane, but for the spherical symmetry the inclinations can have all possible values resulting in different pictures when projected onto the meridional plane (R, z plane). However, in the case of collapse all stars orbit the center in the same plane because, clearly, the system is self-consistent.

3. ORBITS

The orbit of a test particle is calculated by using the two well-known Lagrange equations: in the coordinates R and z. In the first one, we use the integral of angular momentum J_z ($J_z = R^2\dot{\theta}$, θ is the position angle in plane $z = 0$). In the case of potential (1) the two differential equations cannot be solved analytically. For this reason we apply numerical integrations based on the Runge-Kutta method. In our procedure the second-order differential equations are reduced to twice as many first-order equations. As the accuracy control of the algorithm we use the energy integral. The total energy of the test particle per unit mass (specific energy) is calculated at every step; this is the difference between the specific kinetic energy and the potential expressed in km ^{2} s $^{-2}$. The procedure used here enabled us to achieve required levels of accuracy and precision: accuracy goal 10^{-6}, and working precision 10^{-14}. (The precision defines the number of significant digits in the calculations. The accuracy concerns the number of reliable digits in the result.)

Though the values of the model parameters are not significant, we choose them in a way that our model (potential (1)) resembles a real galaxy or, more precisely, a subsystem of a real galaxy. The total mass ($\mathcal{M}$) had always the same value, namely 95 $\times 10^9$ solar masses, whereas for the two parameters having the dimension of distance, a and b, the values were varied, but their sum was kept constant - 4.73 kpc. According to the condition mentioned in the previous Section, the parameter a may attain zero, which is its minimum value. In the case of the other parameter, the minimum is arbitrarily taken to be 0.47 kpc. All these values, including that for the mass, have their origin in the parameter values for the Milky Way model proposed by one of the present authors (Ninković 1992). More particularly, they are referred to the disc component in that model. We can say that in this paper the galactic disc, as it was described in Ninković's (1992) paper, represents one of the two limits, where the other one is a sphere of the same total mass.

In these circumstances a variety of initial conditions were specified. Every set of them yielded a new orbit. First orbits we were interested in are those characterized by the same values for the specific energy and specific angular momentum. For convenience, we examined them for the case of the same initial position or, more precisely, the same potential at the beginning of the orbit calculation. The fact that two different sets of initial conditions corresponding to the same values for the two classical integrals of motion (energy and angular momentum) result in two different orbits (say Binney and Tremaine 1987 p. 117) is well known. It is usual to call the sum of the potential and of the quantity $-\frac{1}{2}\frac{{J_z}^2}{R^2}$ effective potential (say, Binney and Tremaine 1987 - p. 115), but, perhaps, it might be better to use the term effective (specific) kinetic energy for the sum $\frac{1}{2}(\dot{R}^2 + \dot{z}^2)$. The reason is that then, for the conditions specified above, we study orbits with the same initial effective kinetic energy, but with different fractions of the two components (radial $\frac{1}{2}\dot{R}^2$ and vertical $\frac{1}{2}\dot{z}^2$).

As for the initial position of the test particle, for convenience, our choice is a point in the main plane, not too close to the center (due to axial symmetry the position angle in the plane has no influence). The value of its R coordinate is 8.5 kpc. This value is exactly equal to the distance of the Sun to the rotation axis of the Milky Way recommended by the IAU. The coincidence is not accidental, but with regard to the values of the model parameters (a and b) the initial position chosen in such a way is sufficiently far from the center. In this situation one can clearly distinguish three essential kinds of orbits. Those characterized by a value for the modulus of the specific angular momentum close to the product $Ru_c(R)$, where $u_c(R)$ is the circular speed, and by low values of moduli of the other two velocity components will be nearly circular; the test particle will be always close to a circle $R = R_m$ =const in the main plane. Such orbits are well described by the Lindblad formalism and they are usually referred to as epicyclic orbits (e.g. Contopoulos 2002 - p. 381). In this case the motions in the main plane with respect to the circle and along the z axis are sufficiently well represented as harmonic oscillations with constant amplitudes. The other two cases are not so simple. For this reason we calculate the dependence on time for all the six phase coordinates: R, θ, z, $\dot{R}$, $\dot{\theta}$ (or $\Theta = R\dot{\theta}$), $\dot{z}$ (cylindrical frame).

If the orbital eccentricity in the main plane, more precisely the interval in the R coordinate, is not too low, the motion becomes more complicated. This is shown in our calculation; the motion in R can be no longer represented as simple harmonic oscillations, but the extrema are clearly seen. On the other hand, in the case of the z coordinate not only that we have no simple harmonic oscillations, but the amplitude becomes variable, dependent on R. However, such a situation can be expected only as long as the test particle remains sufficiently close to the main plane. Orbits of that kind may be referred to as nearly planar orbits.

Finally, one has the most general case when the motion of a test particle is not limited either to the proximity of a circle in the main plane, or to that of the plane itself. Orbits of such kind are "true spatial orbits". In this case, the time dependence becomes very complicated and, unlike the earlier cases, we cannot find a good fit based on elementary functions in either of the two coordinates (R and z). This circumstance seems to be closely connected to the question of a third independent isolating integral of motion. In the case of the epicyclic orbits, as it is well known, the planar and vertical (along z) motions are approximately separated and three orbital elements are easily recognizable. They can be ascribed to the integral of angular momentum and to the splitting of the energy integral in two parts (planar and vertical). A similar situation takes place in the case of nearly planar orbits. The motion in the main plane is well characterized by two orbital elements, say the extremal values of R; in other words, it seems that there is a separate energy integral for the planar motion which combined with that of angular momentum yields two extremal values of R. As for the z coordinate, it can be characterized by one orbital element, its "own energy integral", but, as already said, the amplitude is dependent on R and the nature is not as simple as in the case of epicyclic orbits.

However, when the true spatial orbits are in question, quasi-integrals (or approximate integrals) cannot be so easily connected to the energy integral. For instance, if the test particle reaches the same pair of circles R =const, $|z|$=const several times, then the effective kinetic energy must always be the same, so the behavior of the components is of interest. On the basis of the integrals of energy and angular momentum we can expect each time different values of the components of the effective kinetic energy. However, this is not what we obtain. The values of the components of the effective kinetic energy on these pairs of circles are not constant. If this were the case we could easily relate each component of the effective kinetic energy to a function of the coordinates R and $|z|$ which resembles splitting of energy integral in two different parts. Some values of the components of effective kinetic energy are repeated at the moments of reaching such a pair of circles. Therefore, the situation as found here is something between the two extremal possibilities: always the same values of the components of effective kinetic energy and each time different values. In other words, we find no splitting of the energy integral, but also we do not have a chaotic situation (each time different values) which would be against the existence of any additional integral. In any case, all sets of initial conditions characterized by the same initial position and the same values for the energy and angular-momentum integrals result in different orbits which can be distinguished through the initial fraction in the effective kinetic energy of either of its components. It is clear that to define a single orbit one needs three elements.

It should be emphasized that an epicyclic orbit as the result of interchanging the values of the two components of the effective kinetic energy can hardly become a nearly planar one. The reason is that the condition of smallness for the effective kinetic energy is so strong that by varying the components within the same value of this quantity one always obtains harmonic oscillations in both R and z, but with different amplitudes. However, in the case of a nearly planar orbit the condition of smallness for the effective kinetic energy is not so strong. One only needs that the modulus of the z component is low. Therefore, when this component becomes dominant, then a nearly planar orbit can be transformed into a "true spatial orbit". Once again, this consideration concerns orbits characterized by the same values of energy and angular momentum and the same initial position. The only difference is in the fraction of either component in the total effective kinetic energy.

In order to examine the behavior of the obtained orbits in more detail we derive the dependence on time of the coordinates. We obtain an excellent analytical approximation with just a few harmonics (Figs. 1-3). Orbits which look like those in Fig. 3 are known as box orbits, as proposed by Contopoulos. Unfortunately, this is not the case with "true spatial orbits".

The conclusion concerning the necessary number of elements for defining a single orbit in the case of steady state and axial symmetry is rather well known (e.g. Hénon and Heiles 1964). Therefore, we try here to propose some quantities which characterize an orbit and can be used as some kinds of integrals of motion. In particular, we calculate for each orbit the mean values over time for the components of the effective kinetic energy. The time interval is that within which an orbit is integrated. The mean values are organized as components of a symmetric tensor: $(\dot{R}^2)_t$, $(\dot{R}\dot{z})_t$ and $(\dot{z}^2)_t$. In this case the two mean squares are of interest because the mixed mean value is practically zero. The mean squares are in general mutually different and if several orbits with the same total energy and angular momentum are compared, their ratios are also different. The mean squares of velocity components taken over time are generally different, i.e. this property does not concern epicyclic orbits and nearly planar planar ones where the time dependence can be represented analytically by applying trigonometric functions. Therefore, independent harmonic oscillations in each degree of freedom (R and z) cannot explain what we obtain here. This is seen from Figs. 4-6 in which a truly spatial orbit is presented. As noted in the captions to these figures, the time dependence of both R and z becomes very complicate compared to the analogous cases in Figs. 1-2. For this reason we cannot obtain a conservation of energy "per axis" $x^2 + \dot{x}^2$, where x is any of the coordinates. Again, any clear dependence of a velocity square on the coordinates would lead to a fixed value of the given velocity square always when the test particle passes through a given circle R =const, $|z|$ =const. However, in the cases of truly spatial orbits, like the one presented in Figs. 4-6, neither of the two velocity squares has always the same value when the test particle is on a given circle, but the situation with the velocity squares is not completely chaotic, either. Due to this, the mean values of the velocity squares (radial and vertical) for a given circle, i.e. involving all passages of the test particle are, in general, different. As a consequence, the mean values of the two velocity squares taken over the whole time interval covered by the integration are also, in general, different.

If we used the mean velocity squares over time as quasi-integrals of motion and applied the Jeans theorem to them, we would obtain different velocity dispersions in the R and z directions, for example for the position used here as the initial one. Of course, the same line of reasoning would be valid for any other position. It should be commented here that, namely, the well-known Schwarzschild triaxial distribution of random velocities at the galactocentric position of the Sun served as the first indication that the energy and angular-momentum integrals are not enough to explain this result of the stellar kinematics in the solar neighborhood. As it is well known, a lot of solutions have been proposed (e.g. Contopoulos 2002 - p. 435). Our attempt to explain the triaxility of the distribution of random velocities is similar to

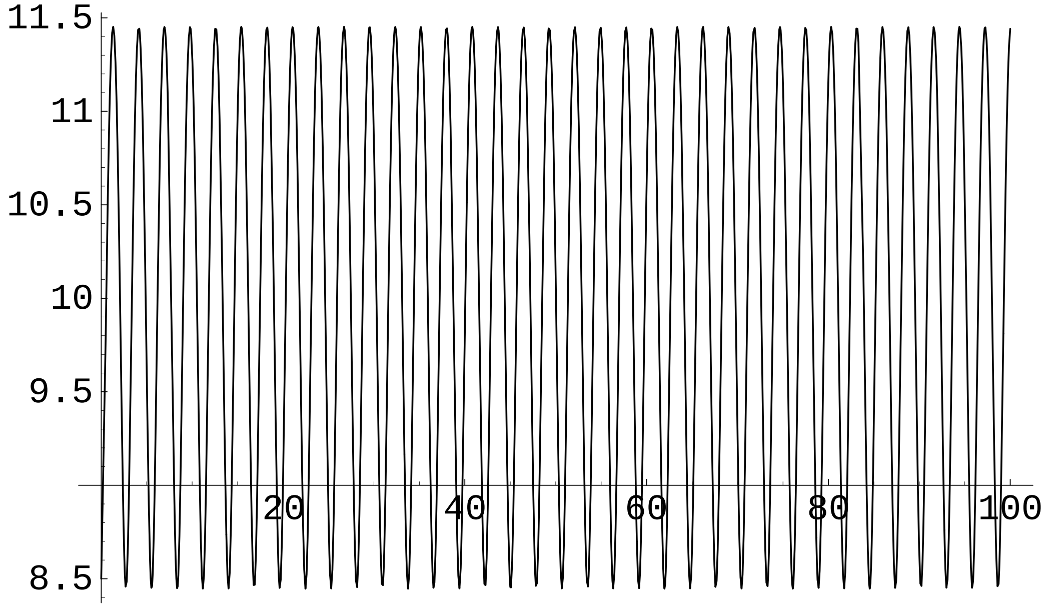

Fig. 1. *Dependence of distance R to z axis on time (time unit $= 10^8$ yr) for an orbit defined by: $a = 2.26$ kpc $b = 2.47$ kpc (model parameters) and $R = 8.5$ kpc, $z = 0$ kpc, $\dot{R} = 10$ kms^{-1}, $\dot{z} = 5$ kms^{-1}, $\Theta = 200$ kms^{-1} (initial conditions). This dependence is fitted by* $R(t) = 10.0631 + 1.48934\cos(2.22697t - 2.916945) + 0.112048\cos(4.45392t - 2, 69157) + 0.01256\cos(6.68092t - 2.46872) + 0.00166\cos(8.90804t - 2.25064) + 0.00024\times \cos(11.1360t - 2.08511) + 0.00008\cos(4.77324t + 135152) + \dots$.

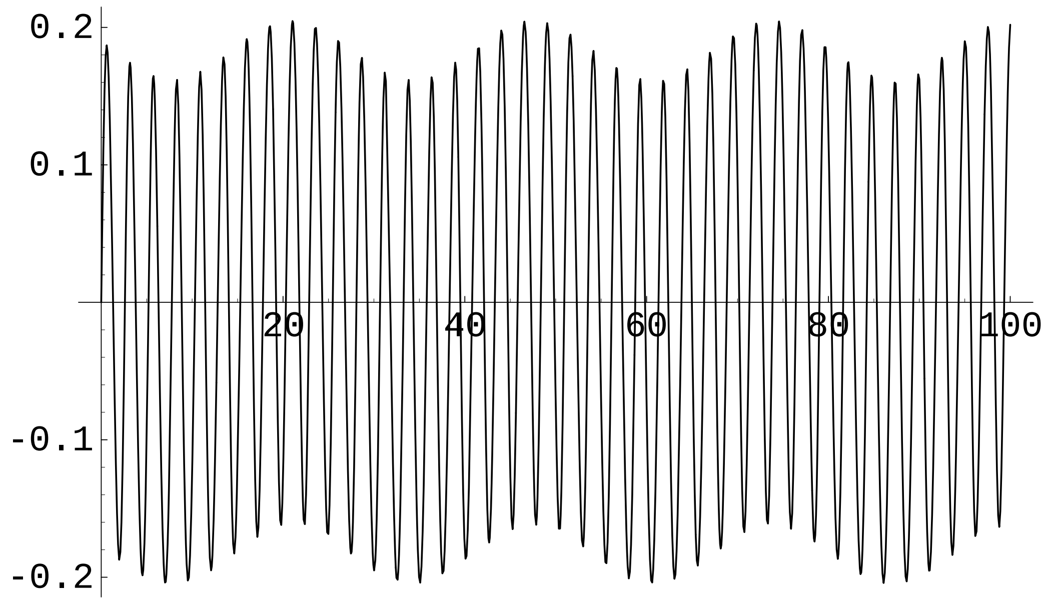

Fig. 2. *Dependence of z coordinate on time (time unit $= 10^8$ yr), model parameters and initial conditions as in Fig. 1. This dependence is fitted by* $z(t) = 0.18275\cos(2.46497t - 1.64078) + 0.03412\cos(0.23797t + 1.27748) + 0.01277\cos(4.69191t - 1.41489) + 0.00127\cos(6.91892t - 1.19329) + 0.00049\cos(2.00564t - 1.88559) + 0.00015\cos(9.14486t - 0.89823) + \dots$.

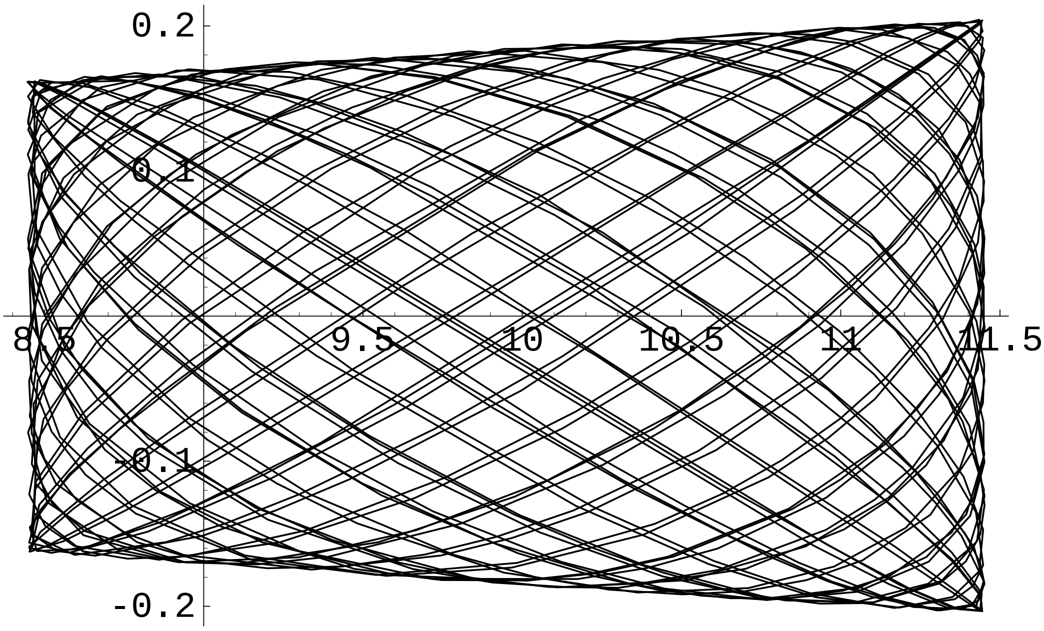

Fig. 3. *Dependence $z(R)$ for the same model parameters and initial conditions as in Figs. 1-2.*

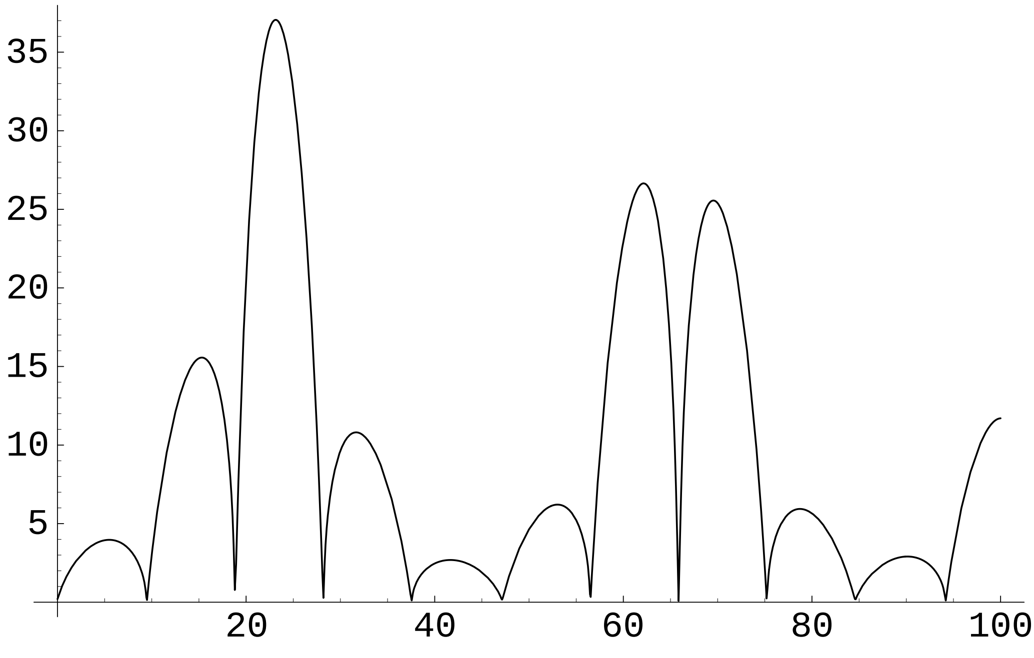

Fig. 4. *Dependence of distance R to z axis on time (time unit = 10^8 yr) for an orbit determined by: $a = 4.26$ kpc $b = 0.47$ kpc (model parameters) and $R = 0.2$ kpc, $z = 0$ kpc, $\dot{R} = 15$ kms^{-1}, $\dot{z} = 391.2$ kms^{-1}, $\Theta = 20$ kms^{-1} (initial conditions). This dependence cannot be approximated by a sum of harmonics as accurately as in the previous case (Fig. 1).*

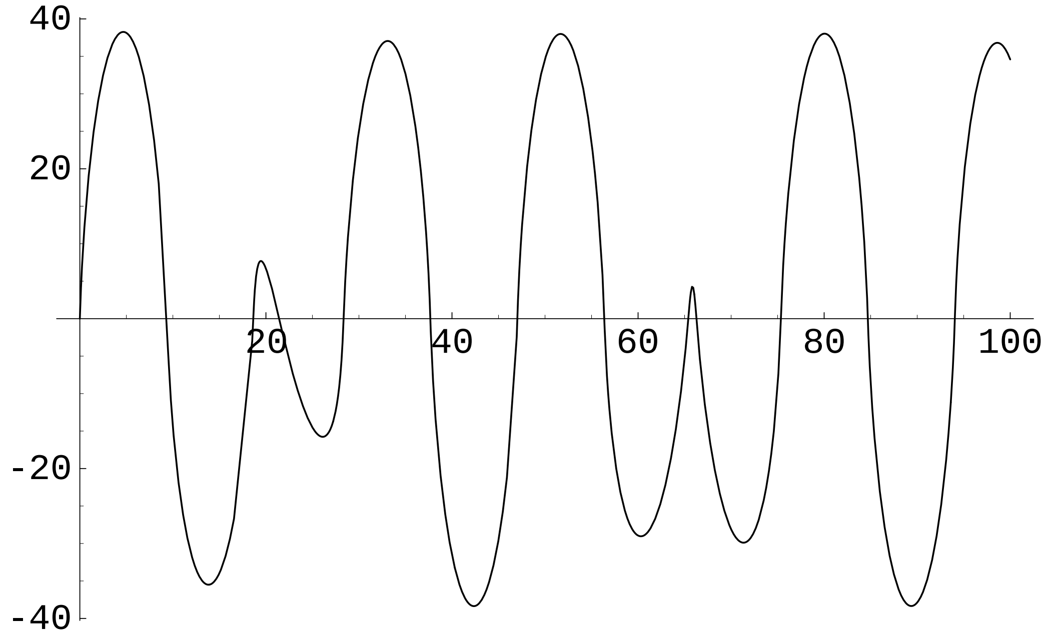

Fig. 5. *Dependence of z coordinate on time (time unit $= 10^8$ yr), model parameters and initial conditions as in Fig. 4. Although some kind of regularity is evident, the harmonic representation of this dependence is significantly less accurate than the one shown in Fig. 2.*

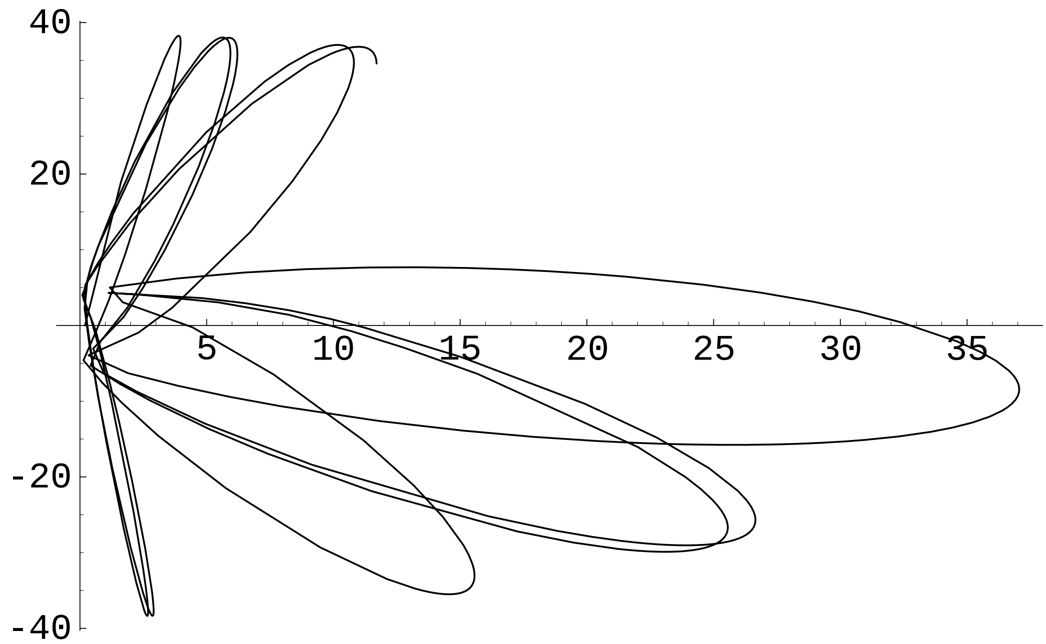

Fig. 6. *Dependence $z(R)$ for the same model parameters and initial conditions as in Figs. 4-5.*

that of Barbanis who proposed an additional integral, a function of $\dot{z}^2$ (Contopoulos 2002, p. 436).

It is also important to examine the problem with another initial position, i.e. different initial values of the coordinates for which the initial values of the velocity components are varied. The positions in the main plane are preferred for clear reasons. Due to the properties of the potential formula (1) assumed here, any choice where the distance to the axis of symmetry is shifted outwards, will bring nothing new. The most interesting case is if a position sufficiently near the center is chosen. Then, since the density corresponding to potential (1) is approximately constant near the center, the second partial derivatives in both R and z taken at the center will be finite, the first ones for reasons of symmetry will be zero, so that potential (1) near the center can be approximated by a polynomial containing the squares in R and z only. In other words, its approximation in the central parts is a potential function separating the variables, known to admit splitting of energy integral. The obtained orbits confirm this because the boundaries in z are two symmetric planes and those in R two coaxial cylinders. We can say that this is quite a simple case.

4. CONCLUSION

In the present paper a number of orbits of a test particle moving in a force field characterized by a steady state and axial symmetry is integrated. The particular potential chosen for this purpose is that of Miyamoto and Nagai (formula (1)). This potential is more realistic than those used by other authors (e.g. Hénon and Heiles 1964, Richstone 1980) and offers a satisfactory approximation in describing the mass distribution within subsystems of galaxies (e.g. Ninković 1992). Among its parameters the total mass is kept constant, whereas the other two, having distance dimension, are varied but provided that their sum is constant. The variation of the parameter values does not lead to essential differences. On the other hand, some special cases, such as epicyclic orbits, nearly planar orbits and orbits in the central parts, are confirmed as relatively simple ones. They are all characterized by some kind of splitting of the energy integral in two parts referring to the motions in R and z, respectively. Our confirmation concerns the dependence on time of the coordinates where we find rather simple periodical functions. However, the situation with the so-called real spatial orbits is different. Though they are described with three independent orbital elements, the question concerning the third integral and its relation to the energy integral remains unclear which is confirmed by the very complicated time dependences of the coordinates. Also, the same initial position, the same energy and angular momentum yield different orbits depending on the initial fraction in the effective kinetic energy ($\frac{1}{2}(\dot{R}^2 + \dot{z}^2)$) of either of its components. Without regard to this circumstance, the present authors are inclined to propose to characterize an orbit, in addition to its specific angular momentum (component along z), by the mean values of $\dot{R}^2$ and $\dot{z}^2$, over time as a possible replacement of its energy and possibly another integral. In this way the present authors try to explain why, for example, in the solar neighborhood the mean value of the square in the z random-velocity component is different from that in R, no matter what sample of local stars is examined.

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ПУТАЊЕ ЗА КОНКРЕТАН СЛУЧАЈ ОБРТНЕ СИМЕТРИЈЕ

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Предмет проучавања је један конкретан случај стационарног стања и обртне симетрије - формула за потенцијал коју су предложили Мијамото и Нагаи. Израчунат је низ орбита везане пробне материјалне тачке при чему се мењају и параметри потенцијала и почетни услови. За разлику од неких посебних случајева, као што су скоро кружне и скоро раванске орбите, у случају "чисто просторних" орбита зависност координата од времена је веома сложена па је математичка обрада уз помоћ периодичних фунција једва могућа. Имајући у виду да су орбите проучаване у овом раду одређене са три елемента, аутори предлажу да вредности квадрата компонената брзина усредњене по времену служе као карактеристике ових орбита.