INFORMATION
Volume 16, Number 11, pp.7837-7848

ISSN 1343-4500

Convolution and Correlation Based on Discrete Quaternion Fourier Transform

Mawardi Bahri[a], Ryuichi Ashino[b]* and Rémi Vaillancourt[c]

[a] Department of Mathematics, Hasanuddin University, Tamalanrea Makassar, Indonesia
e-mail: mawardibahri@gmail.com
[b] Division of Mathematical Sciences, Osaka Kyoiku University, Osaka 582-8582, Japan
e-mail: ashino@cc.osaka-kyoiku.ac.jp
[c] Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N
e-mail: remi@uottawa.ca

Abstract

In this paper we present the generalized convolution and correlation for the two-dimensional discrete quaternion Fourier transform (DQFT). We provide several new properties of the generalization. These results can be considered as the extensions of correlation and convolution properties of real and complex Fourier transform (FT) to the DQFT domain.
Keywords: quaternion convolution, quaternion correlation, and auto-correlation

AMS Subject Classification: 11R52, 42C40

1 Introduction

The real and complex Fourier transform (FT) is an important tool in signal processing. It transforms a signal in the time domain into frequency domain. The extension of the FT to quaternion algebra is called the *quaternion Fourier transform* (QFT). Recently the QFT has been widely used in signal and image processing (see, for example, [7, 8, 9, 15]). Many properties of the generalized transform are already known, such as translation, modulation, differentiation, and uncertainty principle (see [10, 11, 12, 1, 2]. The properties are extensions of the corresponding version of the FT with the some modifications. In [14, 15, 13], authors briefly introduced the quaternion convolution and correlation of the QFT. However, some properties of the relationships among the quaternion convolution, quaternion correlation and the DQFT have not been established.

In this paper, we first extend the definition of the convolution and correlation to quaternion algebra. We then discuss the relationships among the quaternion convolution, correlation and the DQFT. We also investigate some of their important properties. We finally define the quaternion autocorrelation and sate its relation with the DQFT.

*Corresponding author

2 Preliminaries

In this section we briefly review some basic ideas on quaternions and the DQFT. For a more complete discussion we refer the readers to [7].

2.1 Quaternion Algebra

The first concept of quaternions, which is a type of hypercomplex numbers, was formally introduced by Hamilton in 1843. The quaternion algebra, which is denoted by $\mathbb{H}$, is an associate non-commutative four-dimensional algebra.

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}. \tag{1}$$

The orthogonal imaginary units $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ follow the multiplication rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The scalar part of a quaternion q is denoted by $Sc(q) = q_0$ and a pure part of q is denoted by $Vec(q) = \mathbf{i}q_1 + \mathbf{j}q_1 + \mathbf{k}q_3$. The conjugate of a quaternion q is obtained by changing the signs of the pure quaternion, that is,

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3. \tag{2}$$

It is a linear anti-involution, that is, for every $p, q \in \mathbb{H}$, we have

$$\overline{\overline{p}} = p, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{pq}. \tag{3}$$

The norm of a quaternion q is defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2}, \tag{4}$$

and the inverse of q can be represented as

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

Hence, $\mathbb{H}$ is a normed division algebra. Any quaternion q can be represented as

$$q = |q|e^{\mu\theta}.$$

where $\theta = \arctan \dfrac{|Sc(q)|}{Vec(q)}$, $0 \le \theta \le \pi$ is the eigen angle (phase). A quaternion q is called a unit quaternion if $|q| = 1$. Euler's and De Moivre's formulas still hold in quaternion space, that is, for a pure unit quaternion μ the following holds:

$$\begin{aligned} e^{\mu\theta} &= \cos\theta + \mu \sin\theta \\ e^{\mu n\theta} &= (\cos\theta + \mu\sin\theta)^n = \cos n\theta + \mu \sin n\theta. \end{aligned}$$

For a fixed pure unit quaternion $\mathbf{q}$, every pure quaternion $\mathbf{p}$ can be resolved into its parallel component $\mathbf{p}_{\|}$ and perpendicular component $\mathbf{p}_{\perp}$ with respect to $\mathbf{q}$, that is,

$$\mathbf{p} = \mathbf{p}_{\|} + \mathbf{p}_{\perp}. \tag{5}$$

Here, $\mathbf{p}_{\|} = \frac{1}{2}(\mathbf{p} - \mathbf{qpq})$ for $\mathbf{p} \parallel \mathbf{q}$ and $\mathbf{p}_{\perp} = \frac{1}{2}(\mathbf{p} + \mathbf{qpq})$ for $\mathbf{p} \perp \mathbf{q}$. Therefore, we have the following result, which will be used in the next section.

Proposition 2.1. *If* $\mathbf{p}$ *and* $\mathbf{q}$ *are two pure quaternions, then*

- $\mathbf{p}$ *and* $\mathbf{q}$ *are parallel* ($\mathbf{p} \parallel \mathbf{q}$) *if and only if* $\mathbf{pq} = \mathbf{qp}$,
- $\mathbf{p}$ *and* $\mathbf{q}$ *are perpendicular* ($\mathbf{p} \perp \mathbf{q}$) *if and only if* $\mathbf{pq} = -\mathbf{qp}$.

2.2 Discrete QFT

Denote $\mathbb{Z}_{M\times N} = \{(m,n) \in \mathbb{Z}^2 \mid 0 \le m \le M-1,\ 0 \le n \le N-1\}$. Hereinafter, we will simply denote a finite sequence $\{f(m,n)\}_{0\le m\le M-1,\ 0\le n\le N-1} \in \mathbb{H}^{M\times N}$ of the quaternion numbers by $f(m,n)$, which can be regarded as a quaternion function over the discrete domain $\mathbb{Z}_{M\times N}$.

Definition 2.1. *There are two standard extensions of $f(m,n)$ to functions on $\mathbb{Z}^2$.*

(i) (Zero extension) *By putting $f(m,n) = 0$ for $(m,n) \in \mathbb{Z}^2 \setminus \mathbb{Z}_{M\times N}$, we can extend $f(m,n) \in \mathbb{H}^{M\times N}$ to a function on $\mathbb{Z}^2$, which is called the zero extension of $f(m,n)$ and denoted by the same $f(m,n)$ when there is no confusion.*

(ii) (Periodic extension) *By extending $f(m,n)$ periodically, we can extend $f(m,n) \in \mathbb{H}^{M\times N}$ to a function on $\mathbb{Z}^2$, which is called the periodic extension of $f(m,n)$ and denoted by the same $f(m,n)$ when there is no confusion.*

Similar to the two-dimensional quaternion Fourier transform (QFT), we define the two-dimensional discrete quaternion Fourier transform (DQFT) and the inverse discrete quaternion Fourier transform (IDQFT). Due to the non-commutative property of the quaternion multiplication, there are at least three different definitions of the DQFT. In this paper we choose the DQFT proposed by Ell and Sangwine in [9]. A more general formulation is given by the following definition.

Definition 2.2. *Let $f(m,n) \in \mathbb{H}^{M\times N}$. The DQFT $F^q\{f\} \in \mathbb{H}^{M\times N}$ of $f(m,n)$ is defined by*

$$F^q\{f\}(u,v) = F^q_f(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}, \tag{6}$$

where μ is any pure unit quaternion such that $\mu^2 = -1$.

Remark 2.1. *To calculate the two-dimensional discrete quaternion Fourier transform by iteration of the one-dimensional discrete quaternion Fourier transform, the definition of the two-dimensional discrete quaternion Fourier transform should be*

$$F^q\{f\}(u,v) = \sum_{m=0}^{M-1} e^{-\mu 2\pi\frac{mu}{M}} \sum_{n=0}^{N-1} f(m,n)\, e^{-\mu 2\pi\frac{nv}{N}}, \tag{7}$$

which is not equivalent to (6) because of the non-commutativity of the quaternion multiplication.

Theorem 2.1. *The IDQFT can be expressed as*

$$F_f^{-q}[F_f^q](m,n) = f(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F_f^q(u,v) e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}. \tag{8}$$

Some basic properties of the DQFT such as the linearity, shift, modulation, conjugation and Plancherel's formula can be found in [4].

3 Convolution for DQFT

Convolution plays an important role in signal processing, such as edge detection, sharpening and smoothing in image processing. It is well known that the classical convolution in the discrete Fourier domain can be represented as the product of the discrete Fourier transforms individually (see [8]). In this section, we shall show that this fact is a special case of the quaternion convolution in the discrete quaternion Fourier domain. Let us begin with the definition of the three standard convolutions of two quaternion sequences.

Definition 3.1.

(i) *Let* $f(m_1,n_1) \in \mathbb{H}^{M_1\times N_1}$ *and* $g(m_2,n_2) \in \mathbb{H}^{M_2\times N_2}$. *Put* $M_3 = \max\{M_1+M_2-1, M_1, M_2\}$ *and* $N_3 = \max\{N_1+N_2-1, N_1, N_2\}$. *The discrete convolution* $h = f * g \in \mathbb{H}^{M_3\times N_3}$ *of two quaternion sequences is defined by*

$$\begin{aligned} h(m,n) &= (f*g)(m,n) \\ &= \sum_{m'=0}^{M_1-1}\sum_{n'=0}^{N_1-1} f(m',n')g(m-m',n-n'), \end{aligned} \tag{9}$$

which is so-called 'full' convolution.

(ii) *Let* $f \in \mathbb{H}^{M_1\times N_1}$ *and* $g \in \mathbb{H}^{M_2\times N_2}$. *Assume that* $f,\ g \in \mathbb{H}^{\mathbb{Z}^2}$ *are their zero extensions. Then, the discrete convolution* $h \in \mathbb{H}^{\mathbb{Z}^2}$ *is defined by*

$$h(m,n) = \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} f(m',n')g(m-m',n-n'). \tag{10}$$

(iii) *Let* $f(m,n),\ g(m,n) \in \mathbb{H}^{M\times N}$. *Assume that* $f,\ g \in \mathbb{H}^{\mathbb{Z}^2}$ *are their periodic extensions with period* (M,N). *Then, the cyclic convolution* $h \in \mathbb{H}^{\mathbb{Z}^2}$ *is defined by*

$$h(m,n) = \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} f(m',n')g(m-m',n-n'), \tag{11}$$

which is a periodic function with the same period (M,N).

To illustrate another type of convolution so-called '*same*' convolution, let us give a simple example in Example 1 below. The convolution in Example 1 produces a quaternion discrete convolution h of the same size as f by taking the proper same size block from (10).

Example 1. *Consider the quaternion sequences* $f \in \mathbb{H}^{5\times 5}$ *and* $g \in \mathbb{H}^{3\times 3}$ *as follows*

$$f = \begin{bmatrix} \mathbf{i}+2\mathbf{j} & 2\mathbf{i}-\mathbf{k} & \mathbf{i} & \mathbf{k} & \mathbf{i}-\mathbf{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{j}+\mathbf{k} & \mathbf{i} & 2\mathbf{j}-\mathbf{k} \\ \mathbf{j}+\mathbf{k} & \mathbf{i} & \mathbf{i}+\mathbf{j} & 2\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} \\ \mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{i} \\ \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j} & \mathbf{j} \end{bmatrix},$$

$$g = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

According to Definition 3.1, we obtain

$$
\begin{aligned}
h &= f * g \\
&= \begin{bmatrix} \mathbf{i}+2\mathbf{j} & 2\mathbf{i}-\mathbf{k} & \mathbf{i} & \mathbf{k} & \mathbf{i}-\mathbf{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{j}+\mathbf{k} & \mathbf{i} & 2\mathbf{j}-\mathbf{k} \\ \mathbf{j}+\mathbf{k} & \mathbf{i} & \mathbf{i}+\mathbf{j} & 2\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} \\ \mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{i} \\ \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j} & \mathbf{j} \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{i}+2\mathbf{j} & 2\mathbf{i}-\mathbf{k} & \mathbf{i} & \mathbf{k} & \mathbf{i}-\mathbf{k} & 0 & 0 \\ 0 & 0 & \mathbf{i} & \mathbf{j} & \mathbf{j}+\mathbf{k} & \mathbf{i} & 2\mathbf{j}-\mathbf{k} & 0 & 0 \\ 0 & 0 & \mathbf{j}+\mathbf{k} & \mathbf{i} & \mathbf{i}+\mathbf{j} & 2\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} & 0 & 0 \\ 0 & 0 & \mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j} & \mathbf{j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\mathbf{i}+\mathbf{j}-\mathbf{k} & 3\mathbf{i}+3\mathbf{j}+\mathbf{k} & 3\mathbf{i}+\mathbf{j} & 2\mathbf{i}+3\mathbf{j}-\mathbf{k} & \mathbf{i}+\mathbf{k} \\ 3\mathbf{i}+\mathbf{j}-\mathbf{k} & 4\mathbf{i}+5\mathbf{j}+2\mathbf{k} & 4\mathbf{i}+3\mathbf{j}+\mathbf{k} & 4\mathbf{i}+5\mathbf{j} & \mathbf{i}+2\mathbf{j}+2\mathbf{k} \\ 2\mathbf{i}+2\mathbf{j}+\mathbf{k} & 2\mathbf{i}+4\mathbf{j}+3\mathbf{k} & 4\mathbf{i}+4\mathbf{j}+3\mathbf{k} & 3\mathbf{i}+6\mathbf{j}+\mathbf{k} & 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \\ 2\mathbf{i}+\mathbf{j}+2\mathbf{k} & 3\mathbf{i}+4\mathbf{j}+4\mathbf{k} & 4\mathbf{i}+4\mathbf{j}+4\mathbf{k} & 4\mathbf{i}+4\mathbf{j}+2\mathbf{k} & 2\mathbf{i}+3\mathbf{j}+2\mathbf{k} \\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} & 3\mathbf{i}+2\mathbf{j}+3\mathbf{k} & 2\mathbf{i}+2\mathbf{j}+\mathbf{k} & 2\mathbf{i}+2\mathbf{j}+\mathbf{k} \end{bmatrix},
\end{aligned}
$$

which is of the same size as f.

The following Theorem 3.1 is of fundamental importance and describes the connection between discrete quaternion convolution of two quaternion valued functions and DQFT.

Theorem 3.1. *Let $f,\ g \in \mathbb{H}^{M\times N}$. The DQFT of the cyclic convolution f of two quaternion sequences $f,\ g \in \mathbb{H}^{M\times N}$ is given by*

$$F_h^q(u,v) = F_g^q(u,v)F_{f_0}^q(u,v) + \mathbf{i}F_g^q(u,v)F_{f_1}^q(u,v) + \mathbf{j}F_g^q(u,v)F_{f_2}^q(u,v) + \mathbf{k}F_g^q(u,v)F_{f_3}^q(u,v).$$

Proof. It follows from the definition of discrete quaternion convolution (9) that

$$
\begin{aligned}
F_h^q(u,v) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} (f*g)(m,n) \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)g(m-m_0,n-n_0)\right] e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} g(m',n')\right] e^{-\mu 2\pi\left(\frac{(m'+m_0)u}{M}+\frac{(n'+n_0)v}{N}\right)} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} g(m',n')e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right] e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m_0,n_0)F_g^q(u,v)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}.
\end{aligned}
$$

For the second equality, we used the change of variables $m' = m - m_0$, $n' = n - n_0$. For the last equality, we applied (6). We subsequently write $f(m_0, n_0) = f_0(m_0, n_0) + \mathbf{i}f_1(m_0, n_0) + \mathbf{j}f_2(m_0, n_0) + \mathbf{k}f_3(m_0, n_0)$, then the above expression leads to

$$\begin{aligned}
F_g^q(u,v) &= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} (f_0(m_0,n_0) + \mathbf{i}f_1(m_0,n_0) + \mathbf{j}f_2(m_0,n_0) + \mathbf{k}f_3(m_0,n_0)) \\
&\quad \times F_g^q(u,v) e^{-\mu 2\pi(\frac{m_0 u}{M} + \frac{n_0 v}{N})} \\
&= F_g^q(u,v) \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_0(m_0,n_0) e^{-\mu 2\pi(\frac{m_0 u}{M} + \frac{n_0 v}{N})} \\
&\quad + \mathbf{i}F_g^q(u,v) \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_1(m_0,n_0) e^{-\mu 2\pi(\frac{m_0 u}{M} + \frac{n_0 v}{N})} \\
&\quad + \mathbf{j}F_g^q(u,v) \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_2(m_0,n_0) e^{-\mu 2\pi(\frac{m_0 u}{M} + \frac{n_0 v}{N})} \\
&\quad + \mathbf{k}F_g^q(u,v) \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_3(m_0,n_0) e^{-\mu 2\pi(\frac{m_0 u}{M} + \frac{n_0 v}{N})} \\
&= F_g^q(u,v)F_{f_0}^q(u,v) + \mathbf{i}F_g^q(u,v)F_{f_1}^q(u,v) + \mathbf{j}F_g^q(u,v)F_{f_2}^q(u,v) + \mathbf{k}F_g^q(u,v)F_{f_3}^q(u,v).
\end{aligned}$$

For the last equality, we used Definition 2.2. This completes the proof. □

Corollary 3.1. *Let $f,\ g \in \mathbb{H}^{M\times N}$. If the DQFT of g is a real sequence, then Theorem 3.1 takes the form*

$$F_h^q(u,v) = F_g^q(u,v)F_f^q(u,v). \tag{12}$$

We see that the above result has the same form as the two-dimensional discrete Fourier transform (see[4]).

The following result provides the alternative form of the above theorem.

Theorem 3.2. *Let $f,\ \mathbf{g} \in \mathbb{H}^{M\times N}$. Assume that $\mathbf{g}(m,n)$ is a pure quaternion sequence, that is,*

$$\mathbf{g}(m,n) = \mathbf{i}g_1(m,n) + \mathbf{j}g_2(m,n) + \mathbf{k}g_3(m,n).$$

Then, Theorem 3.1 reduces to

$$F_h^q(u,v) = F_f^q(u,v)F_{g_\parallel}^q(u,v) + F_f^q(-u,-v)F_{g_\perp}^q(u,v).$$

Proof. Because $\mathbf{g}$ is a pure quaternion sequence, according to Proposition 2.1, we may decompose $\mathbf{g}$ with respect to the axis μ into $\mathbf{g}_\parallel + \mathbf{g}_\perp$. Furthermore, we get

$$\begin{aligned}
F_h^q(u,v) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\mathbf{g}(m-m_0,n-n_0)\right] e^{-\mu 2\pi(\frac{mu}{M} + \frac{nv}{N})} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\big(\mathbf{g}_\parallel(m-m_0,n-n_0) \\
&\quad + \mathbf{g}_\perp(m-m_0,n-n_0)\big)\, e^{-\mu 2\pi\left(\frac{(m'+m_0)u}{M} + \frac{(n'+n_0)v}{N}\right)}
\end{aligned}$$

$$= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} f(m_0,n_0)\left(\mathbf{g}_{\|}(m',n')+\mathbf{g}_{\perp}(m',n')\right)$$
$$\times\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)} e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}$$
$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\mathbf{g}_{\|}(m',n')\,e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}$$
$$+\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)e^{\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\mathbf{g}_{\perp}(m',n')\,e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}.$$

Here, in the fourth equality, we applied the fact that

$$e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\mathbf{g}_{\|}(m',n') = \mathbf{g}_{\|}(m',n')e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)},$$
$$e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\mathbf{g}_{\perp}(m',n') = \mathbf{g}_{\perp}(m',n')e^{\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}.$$

This proves the theorem according to Definition 2.2. □

4 Correlation for DQFT

The quaternion cross-correlation (correlation) and quaternion convolution are in fact closely related where the quaternion correlation can be regarded as the conjugate of quaternion convolution. This section investigates the relationships between the quaternion correlation and the DQFT. For this purpose we first define the correlation of the two quaternion sequences as follows.

Definition 4.1. *Let $f,\ g \in \mathbb{H}^{M\times N}$. The cyclic correlation $f \star g$ of quaternion sequences f and g is defined by*

$$Cr(m,n) = (f\star g)(m,n) = \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m,n)\overline{g(m_0-m,n_0-n)}. \tag{13}$$

In particular, $f\star f$ is called the cyclic quaternion auto-correlation.

Recently, some general properties of continuous quaternion correlation have been investigated in [5, 3].

We are now ready to collect some important results in the following theorems.

Theorem 4.1. *Let $f,\ g\in\mathbb{H}^{M\times N}$. The DQFT of the cyclic correlation $f\star g$ is given by*

$$F^q_{Cr} = \left(F^q_{g_0}-F^q_{\mathbf{g}}\right)F^q_{f_0} + \mathbf{i}\left(F^q_{g_0}-F^q_{\mathbf{g}}\right)F^q_{f_1} + \mathbf{j}\left(F^q_{g_0}-F^q_{\mathbf{g}}\right)F^q_{f_2} + \mathbf{k}\left(F^q_{g_0}-F^q_{\mathbf{g}}\right)F^q_{f_3}.$$

Proof. Using (6) and (13), we immediately get

$$F^q_{Cr}(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\overline{g(m_0-m,n_0-n)}\right]e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$
$$= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\overline{g(m',n')}\right]e^{-\mu 2\pi\left(\frac{(m_0-m')u}{M}+\frac{(n_0-n')v}{N}\right)}$$

$$= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\overline{g(m',n')}e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right]e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}.$$

Here, the second equality uses the change of variables $m' = m_0 - m$, and $n' = n_0 - n$. Notice also that $\overline{g(m',n')} = g_0(m',n') - \mathbf{g}(m',n')$. We thus arrive at

$$\begin{aligned} F^q_{Cr}(u,v) &= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}(g_0(m',n') - \mathbf{i}g_1(m',n') - \mathbf{j}g_2(m',n')\right.\\ &\qquad \left. -\mathbf{k}g_3(m',n'))e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right]e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)[F^q_{g_0}(-u,-v) - F^q_{\mathbf{g}}(-u,-v)]e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}. \end{aligned}$$

Since $f_0(m_0,n_0) = f_0(m_0,n_0) + \mathbf{i}f_1(m_0,n_0) + \mathbf{j}f_2(m_0,n_0) + \mathbf{k}f_3(m_0,n_0)$, we have

$$\begin{aligned} F^q_{Cr}(u,v) &= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}(f_0(m_0,n_0) + \mathbf{i}f_1(m_0,n_0) + \mathbf{j}f_2(m_0,n_0) + \mathbf{k}f_3(m_0,n_0))\\ &\qquad \times F^q_{g_0}(-u,-v)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &\quad - \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}(f_0(m_0,n_0) + \mathbf{i}f_1(m_0,n_0) + \mathbf{j}f_2(m_0,n_0) + \mathbf{k}f_3(m_0,n_0))\\ &\qquad \times F^q_{\mathbf{g}}(-u,-v)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &= \left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{g_0}(-u,-v)f_0(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right.\\ &\qquad + \mathbf{i}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{g_0}(-u,-v)f_1(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &\qquad + \mathbf{j}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{g_0}(-u,-v)f_2(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &\qquad \left. + \mathbf{k}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{g_0}(-u,-v)f_3(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right]\\ &\quad - \left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{\mathbf{g}}(-u,-v)f_0(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right.\\ &\qquad + \mathbf{i}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{\mathbf{g}}(-u,-v)f_1(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &\qquad + \mathbf{j}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{\mathbf{g}}(-u,-v)f_2(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\ &\qquad \left. + \mathbf{k}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} F^q_{\mathbf{g}}(-u,-v)f_3(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right] \end{aligned}$$

$$
\begin{aligned}
= &\left[F_{g_0}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_0(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right.\\
&+\mathbf{i}F_{g_0}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_1(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\
&+\mathbf{j}F_{g_0}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_2(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\
&\left.+\mathbf{k}F_{g_0}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_3(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right]\\
&-\left[F_{\mathbf{g}}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_0(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right.\\
&+\mathbf{i}F_{\mathbf{g}}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_1(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\
&+\mathbf{j}F_{\mathbf{g}}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_2(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\\
&\left.+\mathbf{k}F_{\mathbf{g}}^q(-u,-v)\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f_3(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\right].
\end{aligned}
$$

We finally obtain

$$
\begin{aligned}
F_{Cr}^q(-u,-v) &= F_{g_0}^q(-u,-v)F_{f_0}^q(u,v)+\mathbf{i}F_{g_0}^q(-u,-v)F_{f_1}^q(u,v)\\
&\quad+\mathbf{j}F_{g_0}^q(-u,-v)F_{f_2}^q(u,v)+\mathbf{k}F_{g_0}^q(-u,-v)F_{f_3}^q(u,v)\\
&\quad-[F_{\mathbf{g}}^q(-u,-v)F_{f_0}^q(u,v)+\mathbf{i}F_{\mathbf{g}}^q(-u,-v)F_{f_1}^q(u,v)\\
&\quad+\mathbf{j}F_{\mathbf{g}}^q(-u,-v)F_{f_2}^q(u,v)+\mathbf{k}F_{\mathbf{g}}^q(-u,-v)F_{f_3}^q(u,v)]\\
&= \left(F_{g_0}^q(-u,-v)-F_{\mathbf{g}}^q(u,v)\right)F_{f_0}^q(u,v)\\
&\quad+\mathbf{i}\left(F_{g_0}^q(-u,-v)-F_{\mathbf{g}}^q(-u,-v)\right)F_{f_1}^q(u,v)\\
&\quad+\mathbf{j}\left(F_{g_0}^q(-u,-v)-F_{\mathbf{g}}^q(-u,-v)\right)F_{f_2}^q(u,v)\\
&\quad+\mathbf{k}\left(F_{g_0}^q(-u,-v)-F_{\mathbf{g}}^q(-u,-v)\right)F_{f_3}^q(u,v),
\end{aligned}
$$

which completes the proof. □

As a special case of Theorem 4.1, we obtain the following important result.

Theorem 4.2. *Let* f, $\mathbf{g}\in\mathbb{H}^{M\times N}$. *Assume that* $\mathbf{g}$ *is a pure quaternion sequence. Then, we have*

$$F_{Cr}^q(u,v) = -F_f^q(u,v)F_{\mathbf{g}_{\|}}^q(-u,-v)-F_f^q(-u,-v)F_{\mathbf{g}_{\perp}}^q(-u,-v).$$

Proof. By Definition 2.2, we obtain

$$F_{Cr}^q(u,v)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\overline{\mathbf{g}(m-m_0,n-n_0)}\right]e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\overline{\mathbf{g}(m',n')}\right] e^{-\mu 2\pi\left(\frac{(m_0-m)u}{M}+\frac{(n-n_0)v}{N}\right)}$$
$$= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\overline{\mathbf{g}(m',n')e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}}\right] e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}$$
$$= \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}\mathbf{g}(m',n')e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right] e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}$$
$$= -\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}(\mathbf{g}_{\|}(m',n')+\mathbf{g}_{\perp}(m',n'))e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right]$$
$$\times\ e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}$$
$$= -\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}(\mathbf{g}_{\|}(m',n')e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right]$$
$$-\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1}(\mathbf{g}_{\perp}(m',n')e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right]$$
$$= -F_f^q(u,v)F_{\mathbf{g}_{\|}}^q(-u,-v) - F_f^q(-u,-v)F_{\mathbf{g}_{\perp}}^q(-u,-v).$$

This completes the proof. □

Theorem 4.3. *Let $f \in \mathbb{H}^{M\times N}$. Assume that*

$$e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}F_f^q(u,v) = F_f^q(u,v)e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}. \tag{14}$$

Then, the DQFT of the auto-correlation f is given by

$$|F_f^q|^2 = F_{f\star f}^q = F^q\{f\star f\}.$$

Proof. Using the norm of a quaternion (4), we easily get

$$\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F_f^q(u,v)|^2 e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$
$$= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F_f^q(u,v)\overline{F_f^q(u,v)}e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$
$$= \frac{1}{MN}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(m_0,n_0)e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\overline{F_f^q(u,v)}e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$
$$= \frac{1}{MN}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(m_0,n_0)\overline{e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}F_f^q(u,v)e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}}.$$

By (14), the above expression can be rewritten as

$$\begin{aligned}
&\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F_f^q(u,v)|^2 e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}\\
&=\frac{1}{MN}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(m_0,n_0)F_f^q(u,v)e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}\overline{F_f^q(u,v)e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}e^{\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}}\\
&=\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1}\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\overline{F_f^q(u,v)e^{\mu 2\pi\left(\frac{(m_0-m)u}{M}+\frac{(n-n_0)v}{N}\right)}}\\
&=\sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m_0,n_0)\overline{f(m_0-m,n_0-n)}.
\end{aligned}$$

Here the last equality follows from (8). Therefore, we finally obtain

$$F_f^{-q}|F_f^q(u,v)|^2(m_0,n_0)=(f\star f)(m_0,n_0).$$

Or, equivalently,

$$|F_f^q(u,v)|^2=F^q\{f\star f\}(u,v)=F_{f\star f}^q(u,v),$$

which completes the proof. □

5 Conclusion

We yield the DQFT of convolution and correlation of two quaternion sequences and prove that it holds not only for the pure quaternion sequences but also full quaternion sequences. We also show the relationship between the auto-correlation and the DQFT and find that the DQFT of the auto-correlation is the modulus (or magnitude) of the DQFT of a quaternion sequences.

Acknowledgments

The first author is partially supported by Hibah Penelitian Kompetisi Internal tahun 2013 (No. 110/UN4-.42/LK.26/SP-UH/2013) from the Hasanuddin University, Indonesia. The second author is partially supported by JSPS.KAKENHI (C)22540130, (C)25400202 of Japan and the third author is partially supported by NSERC of Canada.

References

[1] Bahri, M., Hitzer, E., Hayashi, A. & Ashino, R., *An uncertainty principle for quaternion Fourier transform*, Comput. Math. Appl., 56(9) pp. 2398–2410, 2008.

[2] Bahri, M., Ashino, R. & Vaillancourt, R., *Two-dimensional quaternion wavelet transform*, Appl. Math. Comput., 218(1), pp. 10–21, 2011.

[3] Bahri, M., Ashino, R. & Vaillancourt, R., *Two-dimensional quaternion Fourier transform of type II and quaternion wavelet transform*, IEEE Conference Publications, 2012 International Conference on Wavelet Analysis and Pattern Recognition (ICWAPR), Xian, 15-17 July, pp. 359–364, 2012.

[4] Bahri, M. & Surahman, *Discrete quaternion Fourier transform and properties*, Int. Journal of Math. Analysis, 7(25), pp.1207–1215, 2013.

[5] Bahri, M. & Ashino, R., *Correlation theorems for type II quaternion Fourier transform*, IEEE Conference Publications, 2013 International Conference on Wavelet Analysis and Pattern Recognition (ICWAPR), Tianjin, 14-17 July, pp136–141, 2013.

[6] Bracewell, R., The Fourier Transform and Its Applications, McGraw-Hill, 24–25, 2000.

[7] Bülow, T., *Hypercomplex spectral signal representations for the processing and analysis of images*, Ph.D. thesis, University of Kiel, Germany, 1999.

[8] Ell, T.A., *Quaternion-Fourier transformations for analysis of two-dimensional linear time-invariant partial differential systems*, Proceeding of the 32$^{\text{nd}}$ IEEE Conference on Decision and Control, vol. 2, San Antonio, 15-17 Dec., 1830–1841, 1993.

[9] Ell, T.A. & Sangwine, S.J., *Hypercomplex Fourier transforms of color images*, IEEE Trans. Signal Process. 16(1), pp. 22–35, 2007.

[10] Hitzer, E., *Quaternion Fourier transform on quaternion fields and generalizations*, Adv. Appl. Clifford Algebr., 17(3), pp. 497–517, 2007.

[11] Hitzer, E., *Directional uncertainty principle for quaternion Fourier transform*, Adv. Appl. Clifford Algebr., 20(2), pp. 271–284, 2010.

[12] Hitzer, E., & Mawardi, B., *Clifford Fourier transform on multivector fields and uncertainty principles for dimensions* $n = 2 (\bmod 4)$ *and* $n = 3 (\bmod 4)$, Adv. Appl. Clifford Algebr., 18(3-4), pp. 715–736, 2008.

[13] Moxey, C.E., Sangwine, S.J. & Ell, T.A., *Hypercomplex correlation techniques for vector images*, IEEE Trans. Signal Processing. 51(7), pp. 1941–1953, 2003.

[14] Pei, S.C., Ding, J.J. & Chang, J.H., *Efficient implementation for quaternion Fourier transform, convolution, and correlation by 2-D complex FFT*, IEEE Trans. Signal Process., 49(11), pp. 2783–2797, 2001.

[15] Zhu, C., Shen, Y. & Wang, Q., *New fast algorithm for hypercomplex decomposition and hypercomplex cross-correlation*, Journal of Systems Engineering and Electronics, 21(3), pp. 514–519, 2010.