Proceedings of the 7th International Conference on Asian and Pacific Coasts
(APAC 2013) Bali, Indonesia, September 24-26, 2013

INTERNAL GENERATION OF WAVES WITH DAMPING USING EXTENDED BOUSSINESQ EQUATIONS

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ABSTRACT: It has been known that the waves generated internally propagate with the energy velocity (Lee and Suh 1998; Lee et al. 2001). Until now, this internal wave generation technique has been developed for waves without damping. In real sea, waves may experience energy dissipation when passing through porous media or in surf zone. In this study, we develop techniques of internal generation of waves with damping using the extended Boussinesq equations of Nwogu (1993). Using the Green's function method (Wei et al. 1999), we derive the Gaussian-shaped source functions. Through numerical experiments for linear and nonlinear waves, we find the source functions with damping generate waves accurately.

Keywords: Internal generation of waves, damping, green’s function method, extended boussinesq equations.

INTRODUCTION

Ocean waves are really important to human, especially, for the activities relate to the ocean such as ship navigation, harbor, and seashore protection, etc. To understand wave effects, people conduct field survey, physical experiments or numerical experiments. The former two things take much time and cost while the numerical experiment requires short time and economical cost.

For the tool of numerical experiment, Madsen and Sorensen (1992) and Nwogu (1993) derived the extended Boussinesq equations by adding some correction terms and using horizontal velocities at a certain level, respectively. Recently, Kim et al. (2009) extended the equations of Madsen and Sorensen by including both bottom curvature and squared bottom slope terms. These equations are able to simulate wave propagation from shallow to intermediate-depth waters.

In the numerical experiment, the technique of internal generation of waves has been used with sponge layers at outside boundaries in order to specify offshore boundary conditions. It has been known that the waves generated internally propagate with the energy velocity (Lee and Suh, 1998; Lee et al., 2001). Until now, this technique has been developed for waves without damping. In real sea, wave energy may be dissipated when passing through porous media or propagating in surf zone.

In this study, we develop techniques of internal generation of waves with mass absorption in the extended Boussinesq equations of Nwogu (1993). A source function is added to the continuity equation together with a damping term. Following Wei et al. (1999) we derive the mass source function. Using geometric optic approach we also get energy velocity as Lee and Suh (1998) and Lee et al. (2001) did for waves without damping. Then, we verify the developed theory by generating linear and nonlinear waves propagating over horizontal 1-dimensional domain.

DERIVATION OF SOURCE FUNCTION

The extended Boussinesq equations of Nwogu (1993) for waves with mass absorption may be described as

$$\frac{\partial \eta}{\partial t}+\nabla\cdot\left[(h+\eta)u\right]$$
$$+\nabla\cdot\left\{\left(\frac{z_\alpha^2}{2}-\frac{h^2}{6}\right)h\nabla(\nabla\cdot u)+\left(z_\alpha+\frac{h}{2}\right)h\nabla\left[\nabla\cdot(hu)\right]\right\}+D\eta=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+g\nabla\eta+z_\alpha\left\{\frac{z_\alpha}{2}\nabla\left(\nabla\cdot\frac{\partial u}{\partial t}\right)+\nabla\left[\nabla\cdot\left(h\frac{\partial u}{\partial t}\right)\right]\right\}=0 \tag{2}$$

In Eqs. (1), and (2), η is the water surface elevation, u is the horizontal velocity at a certain elevation z_α, h is the still water depth, ∇ is the horizontal gradient operator and D is the mass absorption rate (damping coefficient).

Neglecting nonlinear terms and considering horizontally one-dimensional domain on a constant water depth, a linearized form of Eq. (2) yields the relation between velocity potential ϕ and the surface elevation η as

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$$\eta = i\frac{\omega}{g}\frac{1}{1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2}\phi \tag{3}$$

where k_r and k_i are the real and imaginary parts of the complex wavenumber ($k = k_r + ik_i$), respectively, $i = \sqrt{-1}$, ω is the angular frequency and $\alpha = (z_\alpha / h)^2 / 2 + z_\alpha / h$. The potential velocity is determined by the relation of $u = \nabla\phi$.

The mass source function S is added to the right-hand side of the continuity equation (1). Then, we combine Eqs. (1) and (2) in favor of ϕ. Integrating the combined equation in space and using the relation given in Eq. (3) we get the following equation

$$\begin{aligned}&\left(\alpha+\frac{1}{3}\right)gh^3\frac{\partial^4\eta}{\partial x^4}-\alpha h^2\frac{\partial^4\eta}{\partial x^2\partial t^2}+gh\frac{\partial^2\eta}{\partial x^2}-\frac{\partial^2\eta}{\partial t^2}\\&+i\omega D\left[1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2\right]\eta\\&=i\omega\left[1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2\right]S\end{aligned} \tag{4}$$

For monochromatic waves, we may have the following expressions for water surface elevation η and source function S as

$$\eta = \tilde{\eta}(x)\exp(-i\omega t) \tag{5}$$

$$S = \tilde{s}(x)\exp(-i\omega t) \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (4) yields an ordinary differential equation for S with respect to x

$$C_1\frac{d^4\eta}{dx^4}+C_2\frac{d^2\eta}{dx^2}+C_3\eta = C_4\tilde{S} \tag{6}$$

where

$$C_1 = \left(\alpha+\frac{1}{3}\right)gh^3 \tag{7a}$$

$$C_2 = gh + \alpha\omega^2 h^2 \tag{7b}$$

$$C_3 = \omega^2\left\{1+i\frac{D}{\omega}\left[1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2\right]\right\} \tag{7c}$$

$$C_4 = i\omega\left[1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2\right] \tag{7d}$$

We can get a non-homogeneous solution for Eq. (6) by following Wei et al. (1999). The Gaussian-shaped source function is defined as

$$S_G = E\exp\left[-\beta(x-x_s)\right] \tag{8}$$

where E is the amplitude of the source term, β is a parameter associated with the width of the source function and x_s is the center point of the source region. Introducing the Green's function method to Eq. (6) to get the solution for the water surface elevation as

$$\begin{aligned}\eta(x)&=\frac{1}{2gh}\frac{1-\alpha(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2}{1+\alpha\frac{\omega^2 h}{g}-2\alpha_1(k_r h)^2\left(1+i\frac{k_i}{k_r}\right)^2}\\&\times\frac{\omega}{k_r\left(1+i\frac{k_i}{k_r}\right)}I_1 E\exp\left[\pm ik_r\left(1+i\frac{k_i}{k_r}\right)(x-x_s)\right]\end{aligned} \tag{9}$$

where $\alpha_1 = \alpha + 1/3$, I_1 is a parameter relating to the Gaussian-shaped function as

$$I_1 = \sqrt{\frac{\pi}{\beta}}\exp\left[-\frac{k_r^2}{4\beta}\left(1+i\frac{k_i}{k_r}\right)^2\right] \tag{10}$$

The target surface elevation is given by

$$\eta(x) = a_0\exp\left[\pm k_r\left(1+i\frac{k_i}{k_r}\right)(x-x_s)\right] \tag{11}$$

The amplitude of the source function can be determined by equating Eq. (9) to the target surface elevation given in Eq. (11). From that, the mass source function is determined as

$$\tilde{S}_G = 2\eta^I C_e\left(1+i\frac{k_i}{k_r}\right)\frac{\gamma}{I_1}\exp\left[-\beta(x-x_s)\right] \tag{12}$$

where $\eta^I = a_0\exp(-i\omega t)$ is the incident water surface elevation, C_e is the energy velocity, γ is a function of $k_r h$ and k_i / k_r. The energy velocity C_e is derived by applying geometric optic approach and given by

$$C_e = gh\frac{k_r}{\omega}\frac{1-2\alpha_1(k_r h)^2+\alpha\frac{\omega^2 h}{g}+6\alpha_1(k_r h)^2\left(\frac{k_i}{k_r}\right)^2}{1-\alpha(k_r h)^2+\alpha(k_r h)^2\left(\frac{k_i}{k_r}\right)^2} \tag{13}$$

Without damping ($k_i = 0$), C_e given in Eq. (13) is the same as C_e given by Lee et al. (2001). The function γ is given by

$$\gamma = \left[1-\alpha_1 (k_r h)^2 \left(1+i\frac{k_i}{k_r}\right)^2\right]\left[1-\alpha (k_r h)^2 + \alpha (k_r h)^2 \left(\frac{k_i}{k_r}\right)^2\right]$$
$$\times \left[\begin{array}{l} \dfrac{1}{1-\alpha (k_r h)^2 \left(1+i\dfrac{k_i}{k_r}\right)^2 + i\dfrac{D}{\omega}} \\ -\dfrac{4\alpha_1 (k_r h)^2 \left(\dfrac{k_i}{k_r}\right)^2 + 4i\alpha_1 (k_r h)^2 \left(\dfrac{k_i}{k_r}\right)}{1-2\alpha_1 (k_r h)^2 + \alpha\dfrac{\omega^2 h}{g} + 6\alpha_1 (k_r h)^2 \left(\dfrac{k_i}{k_r}\right)^2} \\ \times \dfrac{1}{1-\alpha (k_r h)^2 \left(1+i\dfrac{k_i}{k_r}\right)^2 + i\dfrac{D}{\omega}} \end{array}\right] \tag{14}$$

Using the geometric optics approach we also find the dispersion relation for the extended Boussinesq equations of Nwogu (1993) for waves with damping as

$$C^2 = \left(\frac{\omega}{k_r}\right)^2 = \frac{gh}{1-\alpha (k_r h)^2 + \alpha (k_r h)^2 \left(\dfrac{k_i}{k_r}\right)^2}$$
$$\times \left\{\begin{array}{l} 1-\alpha_1 (k_r h)^2 - \left(\dfrac{k_i}{k_r}\right)^2 \left[1-6\alpha_1 (k_r h)^2\right] \\ -\alpha_1 (k_r h)^2 \left(\dfrac{k_i}{k_r}\right)^4 \end{array}\right\} \tag{15}$$

It should be noted that, without damping, Eq. (15) returns to the dispersion relation of the extended Bousinesq equation of Nwogu obtained by Lee et al. (2001).

The ratio of the damping coefficient and the angular frequency can be expressed as

$$\frac{D}{\omega} = 2\frac{g}{\omega^2 h}(k_r h)^2 \times \left\{\frac{k_i}{k_r}\left[1-2\alpha_1 (k_r h)^2 + \alpha\frac{\omega^2 h}{g}\right] + 2\alpha_1 (k_r h)^2 \left(\frac{k_i}{k_r}\right)^3\right\} \tag{16}$$

NUMERICAL EXPERIMENTS

Linear Waves

We conduct the numerical experiments to verify the developed theory by applying FUNWAVE 1D model to generate 1D linear waves over a constant water depth domain as shown in Fig. 1

Eqs. (1) and (2) can be rewritten as

$$\frac{\partial \eta}{\partial t} = E(\eta, u) \tag{17}$$

$$\frac{\partial U(u)}{\partial t} = F(\eta) \tag{18}$$

where

$$E(\eta, u) = -h\frac{\partial u}{\partial x} - \alpha_1 h^3 \frac{\partial^3 u}{\partial x^3} - D\eta \tag{19}$$

$$U(u) = u + \alpha h^2 \frac{\partial^2 u}{\partial x^2} \tag{20}$$

$$F(\eta) = -g\frac{\partial \eta}{\partial x} \tag{21}$$

We can express Eq. (20) in a discretized form as

$$U(u) = u_i^n + \alpha (h_i^n)^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{22}$$

We use the Adams-Bashforth-Moulton method to discretize Eqs. (17) and (18) in time. First, the third-order Adams-Bashforth predictor scheme is applied as

$$\eta_i^{n+1} = \eta_i^n + \frac{\Delta t}{12}\left(23E_i^n - 16E_i^{n-1} + 5E_i^{n-2}\right) \tag{23}$$

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{12}\left(23F_i^n - 16F_i^{n-1} + 5F_i^{n-2}\right) \tag{24}$$

Then, the fourth-order Adams-Moulton corrector scheme is applied as

$$\eta_i^{n+1} = \eta_i^n + \frac{\Delta t}{24}\left(9E_i^{n+1} + 19E_i^n - 5E_i^{n-1} + E_i^{n-2}\right) \tag{25}$$

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{24}\left(9F_i^{n+1} + 19F_i^n - 5F_i^{n-1} + F_i^{n-2}\right) \tag{26}$$

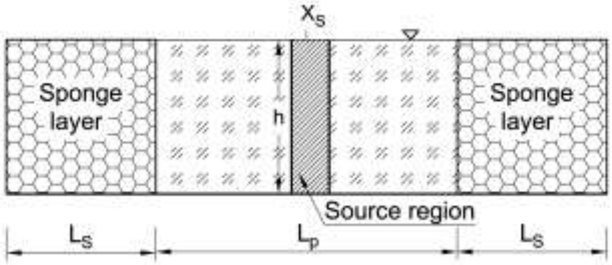


Fig. 1 Computational domain to generate horizontally one-dimensional waves

The computation domain which covers about 14 wavelengths includes an inner domain and two sponge layers at both ends of the computation domain. The source region is placed at the center of the domain. Waves with 6 second period are generated in shallow water ($kh = 0.083\pi$). The grid space is chosen as $\Delta x = 0.05m$ which gives about 288 grid points in one wavelength. The time step is chosen as $\Delta t = 0.01\text{sec}$ to guarantee a stable solution.

Fig. 2 shows the variation of the wave number ratio k_i / k_r with respect to the dimensionless damping coefficient D/ω. As the damping effect increases, the wave number ratio k_i / k_r up to 0.9.

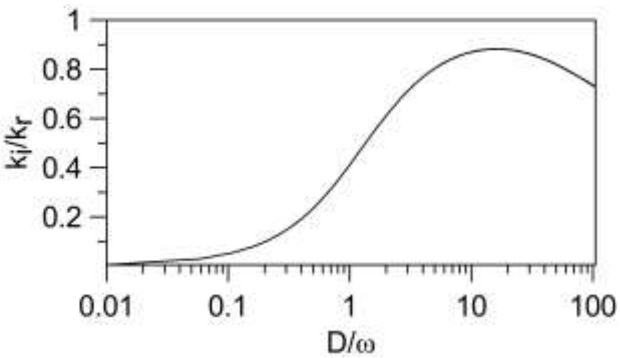


Fig. 2 Wave number ratio k_i / k_r vs. dimensionless damping coefficient D/ω

Fig. 3 shows that, when damping is small (i.e., D/ω is smaller than 0.3 or k_i / k_r is less than 0.2), the phase velocity which is defined in Eq. (15) is very close to the energy velocity with damping defined in Eq. (13). However, when damping becomes larger, the energy velocity with damping is greater than the phase velocity while the energy velocity without damping is smaller than phase velocity. In Fig. 3, C_{e0} is energy velocity without damping.

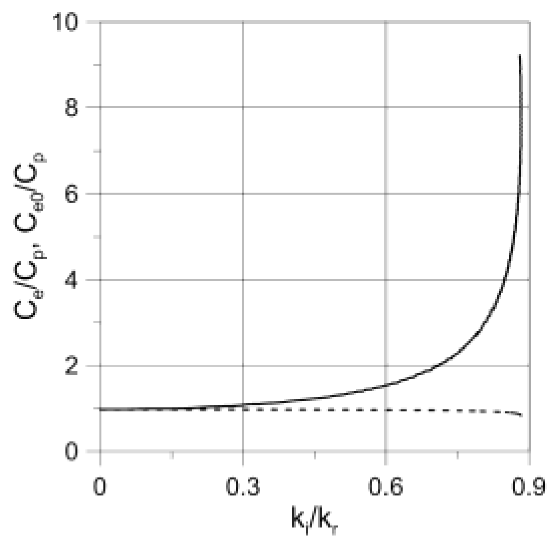


Fig. 3 Ratio of energy velocity to phase velocity vs. wave number ratio k_i / k_r. Line definition: solid line = C_e / C_p ; dashed line = C_{e0} / C_p

Figs. 4(a), (b), (c) compare numerical solutions of the surface elevation against the exact solutions with values of damping coefficient D/ω = 0.01, 0.1, 0.5, respectively. In all the cases, the numerical wave amplitudes are very close to the exact solutions (a_{ex}) which is defined as

$$a_{ex} = \exp(-k_i x) \tag{27}$$

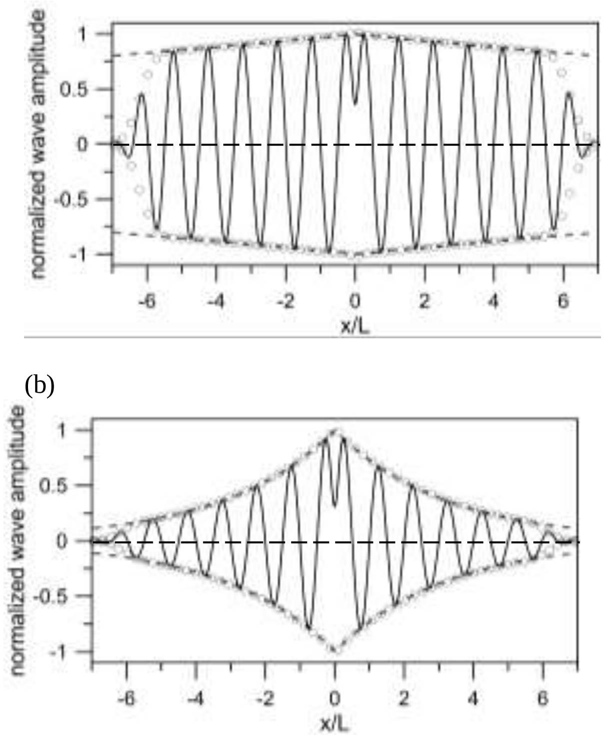
For the case of small damping coefficient as in Fig. 4(a), wave amplitudes are attenuated about 20% until at the end of the computation domain. However, with large damping as in Fig. 4(c), wave amplitudes decay down to almost zero just after about 2 wavelengths.

Cnoidal Waves

We also generate cnoidal waves with condition as linear waves. The water depth is $h = 0.6$ m, wave period is $T = 6$ s and the incident wave height is $H = 0.18$ m which gives Ursell number of U_r = 2.18.

Fig. 5(a) shows the surface elevation with no damping which has flat trough and steep crest. The numerical solution is close to the exact. Fig. 5(b) shows the surface elevation with damping ($D = 0.05\omega$) which is attenuating from the wave generation point.

(a)



(c)

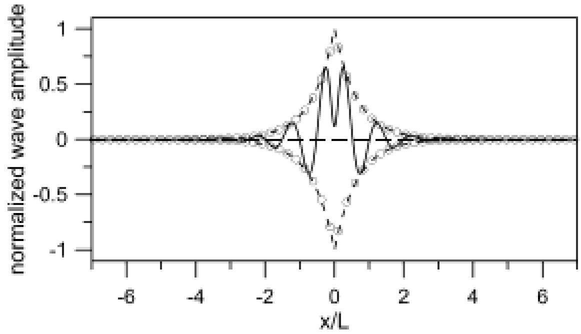


Fig. 4 Normalized water surface elevations and amplitudes of monochromatic waves with damping. Line definition: solid line = numerical solution of water surface elevation; circle = numerical solution of wave amplitude; dashed line = exact solution of wave amplitude. (a) $D = 0.01\omega$, (b) $D = 0.1\omega$, (c) $D = 0.5\omega$

(a)

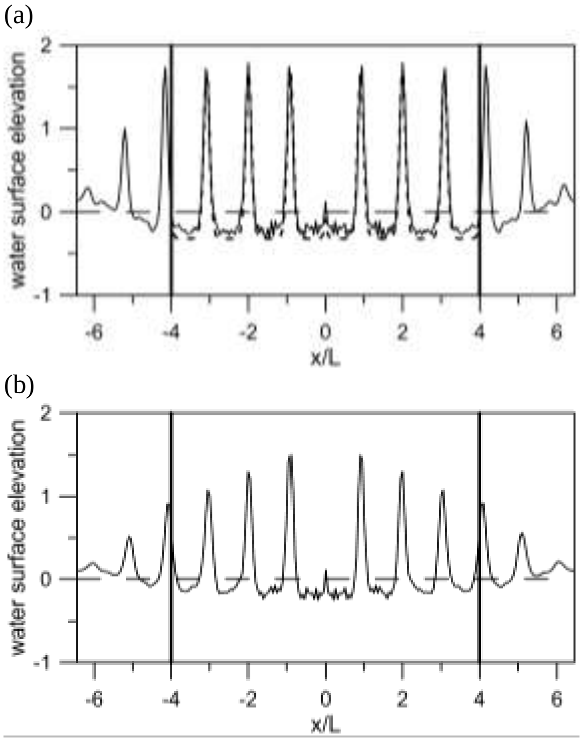


Fig. 5 Normalized water surface of cnoidal waves. Line definition: solid line = numerical solution; dashed line = exact solution; solid vertical line = starting point of sponge layer. (a) $D = 0$, (b) $D = 0.05\omega$

ACKNOWLEDGEMENTS

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2043775).

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