

INCLINED BOTTOM BOUNDARY CONDITION FOR THE MILD-SLOPE EQUATION

T.H. Jung¹, M. Son², S. Son³ and H.S. Park⁴

ABSTRACT: A finite element method is one of the most effective methods to analyze hydrodynamic behaviors in the coastal zone because it can be applied to irregular and complex geometry. However, it is not easy to treat the boundary condition properly under the condition of vertically inclined boundary. In this study, a numerical method for treatment of inclined bottom boundary in the finite element method is introduced. The mild-slope equation is used as a governing equation. Comparison with an analytical solution shows the validity of the present method.

Keywords: Inclined bottom boundary, finite element method, coastal zone, mild-slope equation

INTRODUCTION

Numerous wave equations have been proposed in order to investigate the physical phenomena in the coastal region for the past several decades. One of the representative equations is the mild-slope equation first suggested by Berkhoff (1972). Although most of applications are limited to linear water waves, it is still very popular because it has a capability to simulate the refraction and diffraction at the same time, moreover, it is relatively easy to handle compared to other types of equations.

A numerical solution of the mild-slope equation is usually obtained by using the finite difference method or finite element method. Both methods have their own pros and cons according to the cases where they are applied. For complex geometry, such as coastlines, the finite element method is often used for the advantage of a numerical scheme (Tsay and Liu 1983; Part et al. 1994; Panchang et al. 2000; Bellotti et al. 2003).

In order to obtain an accurate result, both a well-established governing equation and a proper treatment of boundary condition are demanded. When waves propagate in an open boundary, the radiation boundary condition suggested by Sommerfeld is widely used at present. This technique is effective when waves propagate along one specific direction. However, under the condition of multi-directions, this method cannot treat the outgoing waves. As an alternative, the internal wave generation and sponge layer technique have been proposed (Larsen and Darcy 1983; Lee and Suh 1998).

The waves are generated in the computational domain and are attenuated as it passes the sponge layer. Therefore, when the sponge layers are placed at the boundary, it is not necessary to use the treatment of boundary condition because most of waves are dissipated before they reach the boundary. When the structure or beach is located in front of waves, some of them are reflected depending on the characteristics of the structure or the beach. Thus, these reflections should be treated suitably. When the structure or beach is perpendicular, it is easy to impose a partial or perfect reflection condition. However, it is not easy to deal with the boundary condition if they are not perpendicular. For this reason, we developed the boundary technique aimed at inclined bottom boundary. The water depth at the coastline is zero so that a shallow water region around the coastline always exist. In the shallow water region, the linear wave can be expressed in terms of the Bessel function if the bottom slope is constant (Dean 1964). Using these two conditions, we developed the technique for boundary treatment and validated it with the analytical solution.

THEORITICAL BACKGROUND

The computational domain is divided into three regions as shown in Fig. 1. Region 1 is a constant water depth region where the incident and reflected waves are superimposed. Region 2 is a varying water depth region to which the finite element method is applied. Finally, region 3 is the shallow water region. Under the

¹ Department of Civil and Environment Engineering, Hanbat National University, Yuseoung-gu, Daejeon, 305-719, SOUTH KOREA

² Department of Civil Engineering, Chungnam National University, Daehak-ro 99, Yusesong-gu, Daejeon, 305-764, SOUTH KOREA

³ Sonny Astani Department of Civil and Environmental Engineering, University of Southern California, 3620 S. Vermont Avenue, KAP 210, Los Angeles, California, 90089, US

⁴ Disaster Research Division, National Disaster Management Institute, 136 Mapo-daero Mapo-gu, Seoul, 121-719, SOUTH KOREA

assumption that Region 3 is small enough, a constant bottom slope is applied.

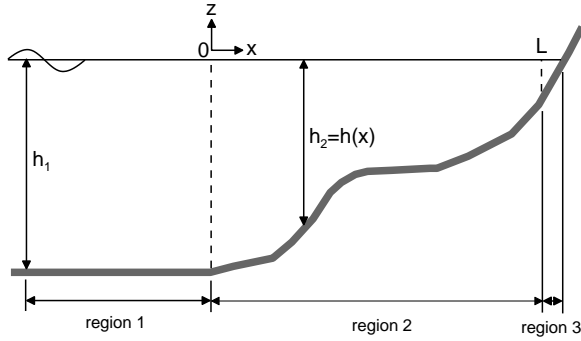


Fig. 1 Schematic definition of computational domain.

Governing Equation

Under the condition of incompressible and inviscid fluid and irrotational flow, the one-dimensional mild-slope equation for linear wave is given by:

$$\frac{d}{dx} \left(CC_g \frac{d\eta}{dx} \right) + k^2 CC_g \eta = 0 \quad (1)$$

where η is the free surface elevation, C is the phase velocity, C_g is the group velocity, and k is the wavenumber determined by Eq. (2):

$$\omega^2 = gk \tanh kh \quad (2)$$

where ω is the angular frequency, g gravitational acceleration, and h water depth.

Since the region 1 is the constant water depth, Eq. (1) is reduced to the second-order linear ordinary differential equation:

$$\frac{d^2\eta}{dx^2} + k^2\eta = 0 \quad (3)$$

The general solution of Eq. (3) is given by

$$\eta_1 = Ae^{ik_1x} + Be^{-ik_1x} \quad (4)$$

The subscript 1 of Eq. (4) represents the region. k_1 is the wavenumber corresponding to h_1 . If the incident wave amplitude is set to be 1, the unknown coefficient A and B become 1 and the reflection coefficient R , respectively.

Because Region 3 is the shallow water region, the phase and group velocities are approximated as:

$$C = C_g \approx \sqrt{gh} \quad (5)$$

Thus, Eq. (1) becomes:

$$g \frac{d}{dx} \left(h \frac{d\eta_3}{dx} \right) + \omega^2 \eta_3 = 0 \quad (6)$$

If the water depth in Region 3 varies linearly, Eq. (6) can be transformed into Eq. (7) by using the relation of $X = h(L) - h(L)(x-L)m$, where m is the bottom slope of Region 3.

$$X^2 \frac{d^2\eta_3}{dX^2} + \frac{d\eta_3}{dX} + s\eta_3 = 0 \quad (7)$$

$$s = \frac{\omega^2}{gm^2} \quad (8)$$

Use of an another variable substitution, $t^2 = 4Xs$, gives the following form of equation:

$$t^2 \frac{d^2\eta_3}{dt^2} + t \frac{d\eta_3}{dt} + t^2\eta_3 = 0 \quad (9)$$

As a result, the solution of Eq. (9) can be expressed by the Bessel function:

$$\eta_3 = CJ_0 \left(2\sqrt{X(x)s} \right) + DY_0 \left(2\sqrt{X(x)s} \right) \quad (10)$$

where J_0 is the Bessel function of the first kind and Y_0 is the Bessel function of the second kind. Since the second term of right-hand side diverges when the x approaches coastline, the coefficient D should be zero to avoid unrealistic results. Therefore, the free surface elevation in the region 3 can be expressed by:

$$\eta_3 = CJ_0 \left(2\sqrt{X(x)s} \right) \quad (11)$$

FEM Formulation

The governing equation represented by Eq. (1) should be transformed into finite element form in the region 2. If we set $\eta = \mathbf{N}^T \boldsymbol{\eta}_2^e$ and insert this into Eq. (1), the following integral for each element is obtained.

$$-\int_{\Omega^e} \mathbf{N} \left[\frac{d}{dx} \left(CC_g \frac{d\mathbf{N}^T}{dx} \right) \boldsymbol{\eta}_2^e \right] d\Omega^e - k^2 CC_g \int_{\Omega^e} \mathbf{N} \mathbf{N}^T d\Omega^e \boldsymbol{\eta}_2^e = 0 \quad (12)$$

where, Ω^e is an element, \mathbf{N} is a shape function, \mathbf{T} is a transpose of matrix, and $\boldsymbol{\eta}_2^e$ is a nodal value vector of a free surface elevation. Conducting integration by part on the first integral of the left-hand side and assembling

over the entire elements give the following system matrix equation:

$$\sum_{\epsilon} \left(CC_g \int_{\Omega^{\epsilon}} \frac{\partial \mathbf{N}}{\partial x} \frac{\partial \mathbf{N}^T}{\partial x} d\Omega^{\epsilon} - k^2 CC_g \int_{\Omega^{\epsilon}} \mathbf{N} \mathbf{N}^T dA \right) \boldsymbol{\eta}_2^{\epsilon} - CC_g \mathbf{N} \frac{\partial \boldsymbol{\eta}_2^{\epsilon}}{\partial x} \Big|_{x=L} + CC_g \mathbf{N} \frac{\partial \boldsymbol{\eta}_2^{\epsilon}}{\partial x} \Big|_{x=0} = 0 \quad (13)$$

To complete the problem, the following matching conditions are used.

$$\eta_1 = \eta_2, \quad \frac{\partial \eta_1}{\partial x} = \frac{\partial \eta_2}{\partial x} \quad \text{at } x=0 \quad (14)$$

$$\eta_2 = \eta_3, \quad \frac{\partial \eta_2}{\partial x} = \frac{\partial \eta_3}{\partial x} \quad \text{at } x=L \quad (15)$$

In combination with Eqs. (4) and (11), Eqs. (14) and (15) become

$$\frac{\partial \eta_2}{\partial x} \Big|_{x=0} = (-ik_1 \eta_2 + 2ik_1) \Big|_{x=0} \approx \mathbf{N} \frac{\partial \boldsymbol{\eta}_2^{\epsilon}}{\partial x} \Big|_{x=0} \quad (16)$$

$$\frac{\partial \eta_2}{\partial x} \Big|_{x=L} = \frac{J'_0(2\sqrt{h(x)s})}{J_0(2\sqrt{h(x)s})} \eta_2(x) \Big|_{x=L} \approx \mathbf{N} \frac{\partial \boldsymbol{\eta}_2^{\epsilon}}{\partial x} \Big|_{x=L} \quad (17)$$

Substituting Eqs. (16) and (17) into Eq. (13) gives final set of equation.

$$\sum_{\epsilon} \left(\int_{\Omega^{\epsilon}} CC_g \frac{\partial \mathbf{N}}{\partial x} \frac{\partial \mathbf{N}^T}{\partial x} d\Omega^{\epsilon} \right) \boldsymbol{\eta}_2^{\epsilon} - \frac{J'_0(2\sqrt{h(x)s})}{J_0(2\sqrt{h(x)s})} CC_g \mathbf{N} \boldsymbol{\eta}_2^{\epsilon} \Big|_{x=L} - ik_1 CC_g \mathbf{N} \boldsymbol{\eta}_2^{\epsilon} \Big|_{x=0} = -2ik_1 CC_g \Big|_{x=0} \quad (18)$$

VALIDATION

To validate the present method, we compared the calculation result with an analytical solution for constant slope as shown in Fig. 2.

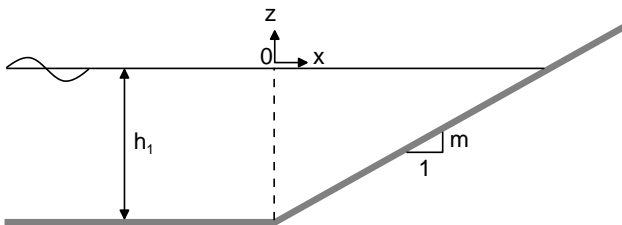


Fig. 2. Computational bathymetry of constant bottom slope.

The analytical solution is obtained by Eqs. (4), (11), and (14) (see Dean, 1964, for details).

$$\eta_{analytic} = \begin{cases} e^{ik_1 x} + \frac{ik_1 J_0(2\sqrt{X(0)s}) - J'_0(2\sqrt{X(0)s})}{ik_1 J_0(2\sqrt{X(0)s}) + J'_0(2\sqrt{X(0)s})} e^{-ik_1 x}, & x \leq 0 \\ \frac{2ik_1 \cdot J_0(2\sqrt{X(x)s})}{ik_1 J_0(2\sqrt{X(0)s}) + J'_0(2\sqrt{X(0)s})}, & x > 0 \end{cases} \quad (19)$$

The relative water depth is set to $kh_1 = 0.05\pi$ because the analytical solution is valid only when the shallow water region. The constant water depth of h_1 is 5.0 m.

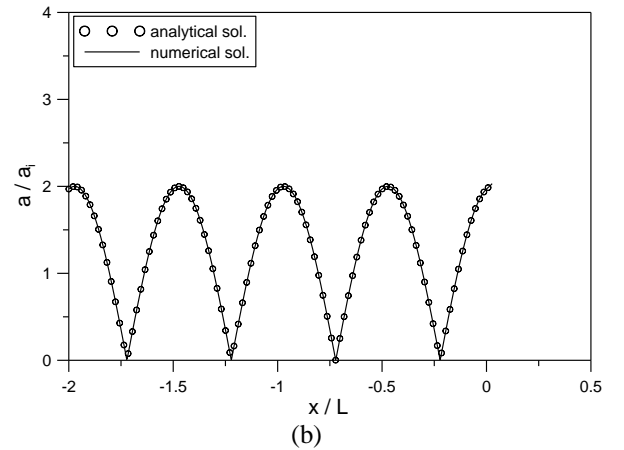
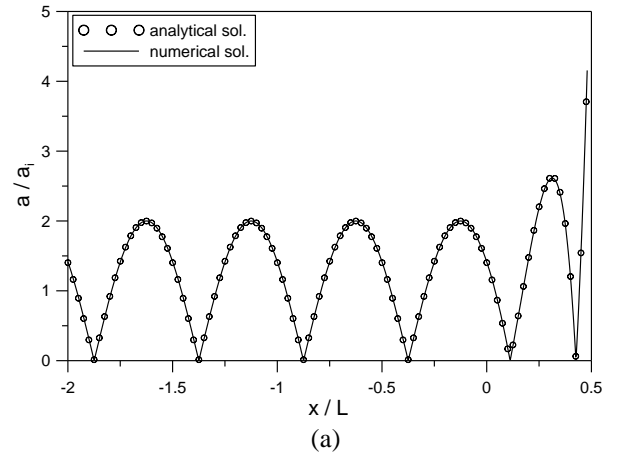


Fig. 3. Comparison between numerical and analytical solutions for $kh_1 = 0.05\pi$: (a) $m = 0.05$; (b) $m = 0.9$.

Fig. 3 shows the wave amplitude of the numerical solution and the analytical solution under the condition of different bottom slopes. The x - and y - axes are normalized by the wavelength and amplitude of incident

waves, respectively. Fig. 3(a) and Fig. 3(b) show the results for mild-slope ($m=0.05$) and steep slope ($m=0.9$) respectively.

To obtain the numerical solution, the computational domain should be divided into three regions as mentioned above. The outer boundary where Region 2 and Region 3 overlap is fixed at the end of slope. The inner boundary where Region 1 and Region 2 are overlapped is not fixed. In our calculation, the relative water depth in the entire domain is less than 0.1π . Thus, the location of inner boundary does not affect the result. At the point of $kh=0.01\pi$ of inner boundary, the calculation is conducted. However, if there is a region whose relative water depth is greater than 0.1π , the location of inner boundary should be determined with care.

CONCLUSION

In this study, a new boundary treatment method for a sloping bottom boundary is proposed. If the coastline or vertically inclined structure is located in the computation domain, it is not easy to deal with these boundaries because the water depth at this point becomes always zero. To overcome this limitation, a small region surrounding the coastline is separated from the computational domain and, then, is calculated individually. For this, two assumptions are used. One is that the separated small region is under shallow water region. Another assumption is that the bottom slope of small region is always constant. When the region is small enough, these two assumptions are considered to be appropriate. The proposed approach has been validated by comparing with an analytical solution.

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