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ON MAXIMAL IDEAL OF SKEW POLYNOMIAL RINGS OVER A DEDEKIND DOMAIN

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Abstract

Let D be any ring with identity 1, σ be an endomorphism of D, and δ be a left σ-derivation. The skew polynomial ring over D in an indeterminate x, $R = D[x; \sigma, \delta]$ consists of polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_i \in D$ with standard coefficient-wise addition and multiplication rule $xa = \sigma(a)x + \delta(a)$ for all $a \in D$. This work investigates the maximal ideal of $D[x; \sigma]$, where D is a Dedekind domain and σ is an automorphism of D.

1. Introduction

This paper studies maximal ideals of a skew polynomial ring over a Dedekind domain. Skew polynomial rings are widely used as the underlying rings of various linear systems investigated in the area algebraic system

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theory. These systems may represent mathematical models coming from mathematical physics, applied mathematics and engineering sciences which can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations, etc. If these systems are linear, then they can be defined by means of matrices with entries in non-commutative algebras of functional operators such as the ring of differential operators, shift operators, time-delay operators, etc. An important class of such algebras is called *skew polynomial ring*.

The structure of ideals of various kind of skew polynomial rings have been investigated during the last few years. In [1], [2], [3], [6], and [7], prime ideals of skew polynomial ring automorphism type over Dedekind domain were considered. This paper investigates the maximal ideals of a skew polynomial ring over a Dedekind domain.

2. Definitions and Notations

We recall some definitions, notations, and more or less well known necessary facts. In the first part, we recall some facts about Dedekind domain. Some characteristics of Dedekind domain associated with its ideals presented in the following theorem.

Theorem 2.1 (Hungerford [4])**.** *The following conditions on an integral domain D are equivalent*:

1. *D is a Dedekind domain,*

2. *every proper ideal in D is uniquely a product of a finite number of prime ideals,*

3. *every nonzero ideal in D is invertible*.

By Theorem 2.1, every proper ideal in D is uniquely a product of a finite number of prime ideals. Based on this statement, obtained one type of relationship between the prime ideals, as stated in the following lemma.

Lemma 2.1 (Osserman [8])**.** *Let* $P, P_1, P_2, \ldots, P_n$ *be prime ideals of a Dedekind domain. If* $P \supseteq P_1P_2 \cdots P_n$, *then* $P = P_i$ *for some i.*

In the second part, we recall some definitions, notations and more or less well known necessary facts about skew polynomial ring.

Definition 2.1 (McConnel and Robson [7])**.** Let D be a ring with identity 1, σ be an endomorphism on the ring D, and δ be a σ-derivative on the ring D. The skew polynomial ring over D with respect to the skew derivation (σ, δ) is the ring consisting of all polynomials over D with an indeterminate x denoted by:

$$D[x; \sigma, \delta] = \{f(x) = a_n x^n + \cdots + a_0 \mid a_i \in D\}$$

satisfying the following equation, for all $a \in D,\ xa = \sigma(a)x + \delta(a)$.

The notations $D[x; \sigma]$ and $D[x; \delta]$ stand for the particular skew polynomial ring where respectively $\delta = 0$ and σ is the identity map. One important role in the investigation of the structure of a ring is the identification of its ideals. This paper investigated maximal ideals of the skew polynomial ring $D[x; \sigma]$. The investigation will be done by exploiting the knowledge of the maximal ideal of the ring D.

In preparation for our analysis of the type of ideals occurred when prime ideals of a skew polynomial ring $D[x; \sigma]$ are to the coefficient ring D, we consider σ-ideal, δ-ideal, (σ, δ)-ideal, σ-prime ideal, δ-prime, and (σ, δ)-prime ideals of D.

Definition 2.2 (Goodearl [3])**.** Let Σ be a set of maps from the ring D to itself. A Σ*-ideal* of D is any ideal I of D such that $\alpha(I) \subseteq I$ for all $\alpha \in \Sigma$. A Σ*-prime ideal* is any proper Σ-ideal I such that whenever J, K are Σ-ideal satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.

In the context of a ring D equipped with a skew derivation (σ, δ), we shall make use of the above definition in the cases $\Sigma = \{\sigma\}, \Sigma = \{\delta\}$ and $\Sigma = \{\sigma, \delta\}$; and simplify the prefix Σ to, respectively, $\sigma, \delta,$ or (σ, δ).

According to the above definition, we can conclude that if I is a prime ideal and also σ-ideal, then I is a σ-prime ideal. The relations between prime ideal with σ-ideal are given in the following two lemmas.

Lemma 2.2 (Goodearl [3]). *Let σ be an automorphism on R and I be a σ-ideal of R. If R is a Noetherian ring, then $\sigma(I) = I$.*

Lemma 2.3 (Goodearl [3]). *Let σ be an automorphism on Noetherian ring R and I be a σ-ideal of R. Then I is a σ-prime ideal if and only if there exists a prime ideal P consisting I and positif integer n such that $\sigma^{n+1}(P) = P$ and $I = P \cap \sigma(P) \cap \cdots \cap \sigma^n(P)$.*

3. The Main Results

Lemma 3.1. *Let σ be an automorphism on Dedekind domain D and $\mathfrak{p}$ be an ideal which is not a prime ideal but σ-ideal of D. Then there exists a prime ideal $\mathfrak{m}$ of D such that $\mathfrak{m} \supset \mathfrak{p}$ and $\mathfrak{m} = \mathfrak{m}^2 + \mathfrak{p}$.*

Proof. According to the conditions on the lemma and using Lemma 2.3, then there exists prime ideal $\mathfrak{m}$ consisting $\mathfrak{p}$ and positif integer n such that $\sigma^{n+1}(\mathfrak{m}) = \mathfrak{m}$ and $\mathfrak{p} = \mathfrak{m} \cap \sigma(\mathfrak{m}) \cap \cdots \cap \sigma^n(\mathfrak{m})$. This leads to

$$\mathfrak{m} \supseteq \mathfrak{m}^2 + \mathfrak{p} \supseteq \mathfrak{p} \supseteq \mathfrak{m}\alpha(\mathfrak{m}) \cdots \sigma^n(\mathfrak{m}).$$

Moreover, from Lemma 2.1 we know that the set of prime ideals consisting $\mathfrak{p}$ is $\{\mathfrak{m}, \sigma(\mathfrak{m}), ..., \sigma^n(\mathfrak{m})\}$. Assume that $\mathfrak{m} \supsetneqq \mathfrak{m}^2 + \mathfrak{p}$, then using Theorem 2.1, we get

$$\mathfrak{m} \supsetneqq \mathfrak{m}^2 + \mathfrak{p} = \mathfrak{m}\sigma^{i_1}(\mathfrak{m}) \cdots \sigma^{i_k}(\mathfrak{m})$$

for some $i_1, ..., i_k \in \{1, ..., n\}$. Moreover,

$$\mathfrak{m}^2 \subseteq \mathfrak{m}^2 + \mathfrak{p} = \mathfrak{m}\sigma^{i_1}(\mathfrak{m}) \cdots \sigma^{i_k}(\mathfrak{m})$$

$$\mathfrak{m} \subseteq \sigma^{i_1}(\mathfrak{m}) \cdots \sigma^{i_k}(\mathfrak{m}) \subseteq \sigma^{i_1}(\mathfrak{m}).$$

Since $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m} = \sigma^{i_1}(\mathfrak{m})$. This contradicts with $\mathfrak{p} = \mathfrak{m} \cap \sigma(\mathfrak{m}) \cap \cdots \cap \sigma^n(\mathfrak{m})$ and $\mathfrak{p} \neq \mathfrak{m}$.

Theorem 3.1. *Let* $R = D[x;\, \sigma]$, *where D is a Dedekind domain and* σ *is an automorphism. Let P be a minimal prime ideal of R, where* $P = \mathfrak{p}[x;\, \sigma]$ *and* $\mathfrak{p}$ *is a* σ*-prime but not prime ideal of D. If* $\mathfrak{m}$ *is a maximal ideal consisting* $\mathfrak{p}$ *where* $\sigma(\mathfrak{m}) \neq \mathfrak{m}$, *then* $M = \mathfrak{m} + xR$ *is maximal ideal of R and* $M = M^2 + P$.

Proof. Let N be an ideal of R and $M \subsetneq N$. Then there exists $a(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in N$ but $a(x) \notin M$. Since

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x \in M \subseteq N,$$

then $0 \neq a_0 \in N \cap D$ and $a_0 \notin \mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal, $N \cap D = D$. This implies $N = R$.

To show that $M = M^2 + P$, it is enough to show that $M \subseteq M^2 + P$, because $M \supseteq M^2 + P$. Let $f(x) \in M = \mathfrak{m} + xR$. Then we can write $f(x)$ in the form as follows.

$$f(x) = a + x[g_n x^n + g_{n-1}x^{n-1} + \cdots + g_0]$$

$$= a + \sigma(g_n)x^{n+1} + \sigma(g_{n-1})x^n + \cdots + \sigma(g_0)x,$$

where $a \in \mathfrak{m}$ and $g_n x^n + g_{n-1}x^{n-1} + \cdots + g_0 \in R$.

On the other hand,

$$M^2 = (\mathfrak{m} + xR)(\mathfrak{m} + xR) = \mathfrak{m}^2 + xR\mathfrak{m} + \mathfrak{m}xR + xRxR.$$

Let

$$u_i x^{i+1} \in xR\mathfrak{m} \text{ and } v_i x^{i+1} \in \mathfrak{m}xR \text{ for } i = 1, \ldots, n.$$

Then we can choose $w_i x^{i+1} \in xRxR$ such that

$$\sigma(g_i)x^{i+1} = u_i x^{i+1} + v_i x^{i+1} + w_i x^{i+1}.$$

Therefore $\sigma(g_i)x^{i+1} \in M^2$ for $i = 1, ..., n$. The next, we will show that $\sigma(g_0)x \in M^2$.

We divide the proof into two cases, namely: $\sigma(g_0) \in \mathfrak{m}$ and $\sigma(g_0) \notin \mathfrak{m}$.

If $\sigma(g_0) \in \mathfrak{m}$, then $\sigma(g_0)x \in \mathfrak{m}xR \subseteq M^2$. On the other side, if $\sigma(g_0) \notin \mathfrak{m}$, then choose $b \in \mathfrak{m}$ such that $\sigma(b) \notin \mathfrak{m}$. We can choose such b because $\sigma(\mathfrak{m}) \neq \mathfrak{m}$. Now, we have the following conditions:

- $\sigma(g_0) \notin \mathfrak{m}$,
- $\sigma(b) \notin \mathfrak{m}$,
- $D/\mathfrak{m}$ is a field.

So, we can choose $\sigma(c) \in D\backslash\mathfrak{m}$ such that $\sigma(g_0) = \sigma(c)\sigma(b) + l$ for some $l \in \mathfrak{m}$, this implies

$$\sigma(g_0)x = \sigma(c)\sigma(b)x + lx = xcb + lx \in xR\mathfrak{m} + \mathfrak{m}xR \subseteq M^2.$$

Furthermore, using identity $\mathfrak{m} = \mathfrak{m}^2 + \mathfrak{p}$ in Lemma 3.1, we get $a \in M^2 + P$. Therefore, we have $f(x) \in M^2 + P$. This proves that $M \subseteq M^2 + P$.

4. Conclusion

Let $\mathfrak{p}$ be a σ-ideal but not a prime ideal of a Dedekind domain D. Then we can choose a prime ideal $\mathfrak{m}$ of D such that $\mathfrak{m} \supset \mathfrak{p}$ and $\mathfrak{m} = \mathfrak{m}^2 + \mathfrak{p}$. Furthermore the prime ideal $\mathfrak{m}$ can be extended to be a maximal ideal of skew polynomial ring $R = D[x; \sigma]$. In this case, we get that $M = \mathfrak{m} + xR$ is a maximal ideal of $R = D[x; \sigma]$.

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