On Maximal Orders and Factor Rings of Ore Extension over a Commutative Dedekind Domain

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Abstract

Let $R = D[x; \sigma, \delta]$ be an Ore extension over a commutative Dedekind domain D, where σ is an automorphism on D. Chamarie [2] implicitly proved that R is a maximal order. In this paper we give an explicit and simpler proof. Then we use that result to study the prime factor ring of $D[x; \sigma, \delta]$ over prime ideals.

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1 Introduction

This paper studies maximal order and factor rings of an Ore extension over the prime ideals. Ore extensions are widely used as the underlying rings of various linear systems investigated in the area algebraic system theory. These systems may represent systems coming from mathematical physics, applied mathematics and engineering sciences which can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations, etc. If these systems are linear, they can be defined by means of matrices with entries in non-commutative algebras of functional operators such as the ring of differential operators, shift operators, time-delay operators, etc. An important class of such algebras is called Ore extensions (Ore Algebras).

Chamarie [2] implicitly proved that R is a maximal order. In this paper we give an explicit and simpler proof. Then we use that result to study the prime factor ring of $D[x;\sigma,\delta]$ over prime ideals.

2 Ore Extension as a Maximal Order

2.1 Definitions and Notations of Ore Extension

We recall some definitions, notations, and more or less well known facts concerning. A *(left) skew derivation* on a ring D is a pair (σ,δ) where σ is a ring endomorphism of D and δ is a *(left) σ-derivation* on D; that is, an additive map from D to itself such that $\delta(ab)=\sigma(a)\delta(b)+\delta(a)b$ for all $a,b\in D$. For (σ,δ) any skew derivation on a ring D, we obtain

$$\delta(a^m)=\sum_{i=0}^{m-1}\sigma(a)^i\delta(a)a^{m-1-i}$$

for all $a\in D$ and $m=1,2,\cdots$. (See [3, Lemma 1.1])

Definition 2.1 *Let D be a ring with identity 1 and (σ,δ) be a (left) skew derivation on the ring D. The Ore Extension over D with respect to the skew derivation (σ,δ) is the ring consisting of all polynomials over D with an indeterminate x denoted by:*

$$D[x;\sigma,\delta] \;=\; \{\, f(x) \;=\; a_nx^n \;+\; \cdots \;+\; a_0 \;\mid\; a_i \;\in\; D \,\}$$

satisfying the following equation, for all $a\in D$

$$xa \;=\; \sigma(a)x \;+\; \delta(a).$$

The notations $D[x;\sigma]$ stand for the particular Ore extensions where $\delta=0$ and $D[x;\delta]$ for σ the identity map. In this paper, we describe the Ore Extension $R=D[x;\sigma,\delta]$ where D is a commutative Dedekind domain and σ is an automorphism.

The Ore extension $D[x;\sigma,\delta]$ is a free left D-module with basis $1,x,x^2,\cdots$. To abbreviate the assertion, the symbol R stands for the Ore extension $D[x;\sigma,\delta]$ constructed from a ring D and a skew derivation (σ,δ) on D. The *degree* of a nonzero element $f\in R$ is defined in the obvious fashion. Since the standard form for elements of R is with left-hand coefficients, the *leading coefficient* of f is f_n if

$$f(x)=f_0+f_1x+\cdots+f_{n-1}x^{n-1}+f_nx^n$$

with all $f_i\in D$ and $f_n\neq 0$. If σ is an automorphism, f can also be written with right-hand coefficients, but then its x^n-coefficient is $\sigma^{-n}(f_n)$. While a general formula for x^na where $a\in D$ and $n\in\mathbb{N}$ is too involved to be of much use, an easy induction establishes that

$$x^na=\delta^n(a)+a_1x+\cdots+a_{n-1}x^{n-1}+\sigma^n(a)x^n$$

for some $a_1, \cdots, a_{n-1} \in D$.

In preparation for our analysis of the types of ideals occurred when prime ideals of an Ore extension $D[x;\sigma,\delta]$ are contracted to the coefficient ring D, we consider σ-prime, δ-prime, and (σ,δ)-prime ideals of D.

Definition 2.2 *Let Σ be a set of maps from the ring D to itself. A Σ-ideal of D is any ideal I of D such that $\alpha(I) \subseteq I$ for all $\alpha \in \Sigma$. A Σ-prime ideal is any proper Σ-ideal I such that whenever J, K are Σ-ideals satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.*

In the context of a ring D equipped with a skew derivation (σ,δ), we shall make use of the above definition in the cases $\Sigma = \{\sigma\}, \Sigma = \{\delta\}$ and $\Sigma = \{\sigma,\delta\}$; and simplify the prefix Σ to respectively σ, δ, or (σ,δ).

2.2 Ore Extension as a Maximal Order

Let $R = D[x;\sigma,\delta]$ be an Ore extension, where σ is an automorphism and δ is a $\sigma-$derivative on D. By [6, Theorem 2.1.14 and 2.1.15], R has a right quotient division ring, denoted by $Q(R)$ or Q for short . So, R is right order in Q, i.e, for all $q \in Q, q = a(x)b(x)^{-1}$ for some $a(x), b(x) \in R$. In this section we will show that R is a maximal order. We start with some easy lemmas.

Lemma 2.1 *Let I be an ideal of R and $a(x), b(x) \in R$. Then*
(i). $Ia(x)b(x)^{-1} \subseteq I \Longleftrightarrow Ia(x) \subseteq Ib(x)$.
(ii). $Ia(x) \subseteq Ib(x) \Longrightarrow der(a(x)) \geq der(b(x))$

Proof.
We get them by simple calculation. ■

Lemma 2.2 *Let I be an ideal of R. Set $T = \{d \in D \mid d$ is a leading coefficient of $f(x)$ for some $f(x) \in I$, where $f(x) \neq 0\} \cup \{0\}$. Then T is an ideal of D and $\sigma(T) = T$.*

Proof.
It is easy to see that T is an ideal of D and $\sigma(T) \subseteq T$. Using the facts that σ is an automorphism and T is an ideal in Dedekind domain D, we get $\sigma(T) = T$. ■

As T is an ideal in Dedekind domain D, T can be generated by two elements, say s_1 and s_2. Since $s_1, s_2 \in T$ then there are two polynomials $p_1(x), p_2(x) \in I$ such that s_1, s_2 are leading coefficients of $p_1(x), p_2(x)$, respectively, where $\text{der}(p_1(x)) = \text{der}(p_2(x)) = t$ for some natural number t. Now set $S = \{d \in D \mid d$ is a leading coefficient of a polynomial $d_1p_1(x) + d_2p_2(x)$, where $d_1s_1 + d_2s_2 \neq 0$, for some $d_1, d_2 \in D\} \cup \{0\}$. It is easy to see that $T = S$. Using S and t we prove the following lemma.

Lemma 2.3 *Let I be an ideal of R and $a(x), b(x) \in R$ where $a(x) = a_mx^m + \cdots + a_0$ and $b(x) = b_lx^l + \cdots + b_0$. . If $Ia(x) \subseteq Ib(x)$ then $a_m = c\sigma^{m-l}(b_l)$ for some $c \in D$.*

Proof.
The proof is done in two steps. In the first step, it will be shown that $Sa_m \subseteq S\sigma^{m-l}(b_l)$. In the second step, it will be shown that $a_m = c\sigma^{m-l}(b_l)$ for some $c \in D$.
Proof step I.
Let $v \in S$, then there exists $w \in S$ such that $\sigma^{-t}(w) = v$. The existence of w is guarantee by automorphism of σ and $\sigma(S) = S$. Since $w \in S$, there is a polynomial, say $q(x) = wx^t + \cdots \in I$. By definition of S, $q(x)$ has degrre t. Using relation, $Ia(x) \subseteq Ib(x)$, we get

$$[wx^t + \cdots]a(x) = w\sigma^t(a_m)x^{t+m} + \cdots \in Ia(x) \subseteq Ib(x).$$

So,

$$w\sigma^t(a_m)x^{t+m} + \cdots = \big[qx^{t+m-l} + \cdots\big]b(x) = q\sigma^{t+m-l}(b_l)x^{t+m} + \cdots .$$

for some $qx^{t+m-l} + \cdots \in I$ where $q \in T = S$. From the last equation, we get

$$\begin{aligned} w\sigma^t(a_m) &= q\sigma^{t+m-l}(b_l) \\ \sigma^{-t}(w)a_m &= \sigma^{-t}(q)\sigma^{m-l}(b_l) \\ va_m &= \sigma^{-t}(q)\sigma^{m-l}(b_l). \end{aligned}$$

Since $\sigma^{-t}(q) \in S$, it means that

$$Sa_m \subseteq S\sigma^{m-l}(b_l) \text{ or } Sa_m[\sigma^{m-l}(b_l)]^{-1} \subseteq S \text{ or } a_m[\sigma^{m-l}(b_l)] \in D.$$

Proof step II.
Since S is an ideal in Dedekind domain D which is a maximal order, then from the last relation, we get

$$S\Big((a_m)[\sigma^{m-l}(b_l)]^{-1}\Big) \subseteq S \quad \text{or} \quad a_m[\sigma^{m-l}(b_l)]^{-1} \in D.$$

This implies, $a_m = c\sigma^{m-l}(b_l)$ for some $c \in D$. ∎

Lemma 2.4 *Let I be an ideal in R and $a(x), b(x) \in R$. Then*

$$I\big[a(x)b(x)^{-1}\big] \subseteq I \quad \Longrightarrow \quad a(x)b(x)^{-1} \in R.$$

Proof.
Let $a(x) = a_m x^m + \cdots + a_0$ and $b(x) = b_l x^l + \cdots + b_0$. The proof is done by induction on $m-l$. First let $m-l = 0$. By Lemma 2.3, $a_m = c\sigma^{m-l}(b_l)$ for some $c \in D$. We can find a polynomial $p(x) \in R$ where $\text{der}(p(x)) < m = l$, such that $a(x) = \big(cx^{m-l}\big)b(x) + p(x)$. Moreover,

$$a(x)b(x)^{-1} = \Big[\big(cx^{m-l}\big)b(x) + p(x)\Big]b(x)^{-1} = cx^{m-l} + p(x)b(x)^{-1}.$$

Since $I\big[a(x)b(x)^{-1}\big] \subseteq I$, then $I\big[cx^{m-l} + p(x)b(x)^{-1}\big] \subseteq I$. Therefore, $Ip(x)b(x)^{-1} \subseteq I$. Since $\text{der}(p(x)) < l$, then by Lemma 2.1, we conclude that $p(x) = 0$. This leads to,

$$a(x)b(x)^{-1} = cx^{m-l} \in R.$$

Now, let the statement,

$$I\big[a(x)b(x)^{-1}\big] \subseteq I \implies a(x)b(x)^{-1} \in R,$$

be true for $0 \le m - l \le k$. Next, we will prove that it is true for $m - l = k + 1$. From above we have, $Ip(x)b(x)^{-1} \subseteq I$, where $\mathrm{der}(p(x)) < m$. Since $\mathrm{der}(p(x)) < m$, then $\mathrm{der}(p(x)) - l \le k$. So, using induction hypothesis, we conclude that $p(x)b(x)^{-1} \in R$. Finally,

$$a(x)b(x)^{-1} = cx^{m-l} + p(x)b(x)^{-1} \in R.$$

■

Lemma 2.5 *Let I be an ideal in R and $a(x), b(x) \in R$. Then*

$$\big[a(x)b(x)^{-1}\big]I \subseteq I \implies a(x)b(x)^{-1} \in R.$$

Proof.
Since Q is a quotient ring of Ore extension $R = D[x;\sigma,\delta]$, where D is a commutative Dedekind domain, then Q is two side quotient ring of R, by [4, p.6]. It means, for all $q(x) \in Q(R)$, $q(x) = b(x)^{-1}a(x)$ for some $a(x), b(x) \in R$. Therefore, to prove this lemma, it is enough to prove the following.

$$\big[b(x)^{-1}a(x)\big]I \subseteq I \implies b(x)^{-1}a(x) \in R.$$

Using the same technique as the proof of Lemma 2.4, the proof follows. ■

For the theorem below, we need the following notations. Let I is an ideal of R,

$$O_r(I) = \{q \in Q(R) \mid Iq \subseteq I\} \text{ and } O_l(I) = \{q \in Q(R) \mid qI \subseteq I\}.$$

Theorem 2.1 *Let $R = D[x;\sigma,\delta]$ be an Ore extension, then $R = D[x;\sigma,\delta]$ is a maximal Order.*

Proof. Using Lemma 2.4 and Lemma 2.5, we get, respectively, $O_r(I) = R$ and $O_l(I) = R$ for all ideals I of R. So, $R = D[x;\sigma,\delta]$ is a maximal order, by [6, Theorem 5.1.4]. ■.

3 Factor Rings of Ore Extension

Throughout this section, let D be a commutative Dedekind domain and σ be an automorphism of D. Let $R = D[x;\sigma,\delta]$, an Ore extension over D.

In this section, we study the structure of the prime factor ring R/P for any prime ideal P of R, which is one of the ways to investigate the structure of rings. This investigation will be described into three subsections. In the first and the second subsection, we study the structure of the prime factor ring R/P where P is a minimal prime ideal of R while in the last subsection P is not a minimal prime ideal of R.

3.1 Factor Ring as a Maximal Order

Let P be a minimal prime of R. In this subsection we will show that factor ring R/P is a maximal order.

Theorem 3.1 *If P is a prime ideal of R, then R/P is a maximal order*

Proof.
In the first step, we will show that

$$O_r(\tilde{I}) = O_l(\tilde{I}) = R/P, \text{ for all ideal } \tilde{I} \text{ of } R/P.$$

Let $\tilde{q} \in O_r(\tilde{I})$. It means $\tilde{q} \in \widetilde{Q}$ and $\tilde{I}\tilde{q} \subseteq \tilde{I}$. This implies $Iq \subseteq I$, where I is the ideal of R and $q \in Q$. Since R is maximal order then $O_r(I) = O_l(I) = R$ by [6, Theorem 5.1.4] . So, $qI \subseteq I$. This implies $\tilde{q}\tilde{I} \subseteq \tilde{I}$. So,$O_r(\tilde{I}) \subseteq O_l(\tilde{I})$. With similar technique, it easy to show that $O_l(\tilde{I}) \subseteq O_r(\tilde{I})$ and $O_r(\tilde{I}) = R/P$.
Now we have $O_r(\tilde{I}) = O_l(\tilde{I}) = R/P$, for all ideal $\tilde{I}$ ofR/P. So, using again [6, Theorem 5.1.4] we get R/P is a maximal order. ■

3.2 Dedekind Factor Ring of Ore Extension

In this subsection, we study the structure of the prime factor ring R/P where P is a minimal prime ideal of R.

Theorem 3.2 *Let P be a minimal prime ideal of R with $P = \mathfrak{p}[x; \sigma, \delta]$, where $\mathfrak{p}$ is a (σ, δ)-prime ideal of D. Then R/P is a Dedekind domain if and only if $\mathfrak{p} \in \operatorname{Spec}(D)$.*

Proof.
$\Leftarrow$
Since $\mathfrak{p}$ is a (σ, δ)-prime ideal of D, then according to [3, page 330]

$$\big(R/\mathfrak{p}R\big) \cong \big(D/\mathfrak{p}\big)[x; \sigma, \delta].$$

Moreover, $\mathfrak{p}R \in \operatorname{Spec}(R)$ by [3, Theorem 3.1]. So, $P = \mathfrak{p}R$ since $\mathfrak{p}R \subseteq P$ and P is minimal prime. It means, we are in the situation $\big(R/P\big) \cong \big(D/\mathfrak{p}\big)[x; \sigma, \delta]$. On the other hand, if $\mathfrak{p} \in \operatorname{Spec}(D)$ then $D/\mathfrak{p}$ is a field. So, $\big(D/\mathfrak{p}\big)[x; \sigma, \delta]$ is a principal ideal domain by [1, Theorem 1.3.2]. This implies $\big(D/\mathfrak{p}\big)[x; \sigma, \delta]$ is a Dedekind domain and so, R/P is a Dedekind domain.
$\Rightarrow$
Let R/P be a Dedekind domain. Then it is clear that $D/\mathfrak{p} \subseteq R/P$ also Dedekind domain. Hence $\mathfrak{p} \in \operatorname{Spec}(D)$. ■

3.3 Field Factor Rings of Ore Extension

In this subsection, we study the structure of the prime factor ring R/P where P is a prime ideal of R but not a minimal prime. For the case P is not a minimal prime ideal and $P \cap D = \mathfrak{p}$ is not a (σ, δ)-ideal of D, we will show that R/P is a field.

With the situations above, by [3, Theorem 3.1], we get the following:

i. $\mathfrak{p}$ is a prime ideal of D

ii. $\sigma(\mathfrak{p}) \neq \mathfrak{p}$

iii. P is the unique prime ideal of R where $P \cap D = \mathfrak{p}$

iv. R/P is a commutative domain.

From the above conditions, we get the following theorem.

Theorem 3.3 *R/P is a field.*

Proof.
Let S/P be an ideal of R/P, then S/P is contained in a maximal ideal, say M/P. Since R/P a commutative domain then M/P also a prime ideal. Hence M a prime ideal of R such that $M \cap D \supseteq P \cap D = \mathfrak{p}$. Since $\mathfrak{p}$ is a maximal ideal then $M \cap D = \mathfrak{p}$. Using [3, Theorem 3.1], we get $M = P$. So $(M/P) = (0)$ and $S/P = (0)$. ■

4 Concluding Remark

In this paper we study the factor rings of $D[x;\sigma,\delta]$ over the prime ideals P, where $P \cap D = \mathfrak{p} \neq 0$. Studying these results, it is expected that this identification can be used to study the structure of the corresponding factor rings of $D[x;\sigma,\delta]$ over the minimal prime ideals P, where $P \cap D = 0$, which is currently under investigation.

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