Minimal Prime Ideals of Ore over Commutative Dedekind Domain

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Abstract

Let $R = D[x; \sigma, \delta]$ be an Ore extension over a commutative Dedekind domain D, where σ is an automorphism on D. In the case $\delta = 0$ Marubayashi et. al. already investigated the class of minimal prime ideals in term of their contraction on the coefficient ring D. In this note we extend this result to a general case $\delta \neq 0$.

Keywords: minimal prime, Ore extension, derivation.

1 Introduction

This paper studies minimal prime ideals of an Ore extension over a commutative Dedekind domain. Ore extensions are widely used as the underlying rings of various linear systems investigated in the area Algebraic system theory. These systems may represent systems coming from mathematical physics, applied mathematics and engineering sciences which can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations, etc. If these systems are linear, they can be defined by means of matrices with entries in non-commutative algebras of functional operators such as the ring of differential operators, shift operators, time-delay operators, etc. An important class of such algebras is called Ore extensions (Ore Algebras).

The structure of prime ideals of various kind of Ore extensions have been investigated during the last few years. In [7], [8] primes of Ore extensions over commutative noetherian rings were considered. In [2], [3], and [11], prime ideals of Ore extensions of derivation type were described. These result recently were exploited in [9] to investigate properties of minimal prime rings of Ore extensions of derivation type in term of their contraction on the coefficient ring. In this note we extend this result to a general Ore extension of automorphism type .

2 Ore Extension

We recall some definitions, notations, and more or less well known facts concerning. A *(left) skew derivation* on a ring D is a pair (σ, δ) where σ is a ring endomorphism of D and δ is a (*left*) *σ-derivation* on D; that is, an additive map from D to itself such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in D$. For (σ, δ) any skew derivation on a ring D, we obtain

$$\delta(a^m) = \sum_{i=0}^{m-1} \sigma(a)^i \delta(a) a^{m-1-i}$$

for all $a \in D$ and $m = 1, 2, \cdots$. (See [4, Lemma 1.1])

Definition 1 *Let D be a ring with identity 1 and (σ, δ) be a (left) skew derivation on the ring D. The Ore Extension over D with respect to the skew derivation (σ, δ) is the ring consisting of all polynomials over D with an indeterminate x denoted by:*

$$D[x; \sigma, \delta] = \{ f(x) = a_n x^n + \cdots + a_0 \mid a_i \in D \}$$

satisfying the following equation, for all $a \in D$

$$xa = \sigma(a)x + \delta(a).$$

The notations $D[x;\sigma]$ and $D[x;\delta]$ stand for the particular Ore extensions where respectively $\delta = 0$ dan σ the identity map. For the case $\delta = 0$, Marubayashi et. al. [9] studied the factor rings of $D[x;\sigma]$ over minimal prime ideals where D is a commutative Dedekind domain. In order to extend their results to general cases, this paper investigates the class of minimal prime ideals in $D[x;\sigma,\delta]$.

The Ore extension $D[x;\sigma,\delta]$ is a free left D-module with basis $1, x, x^2, \cdots$ To abbreviate the assertion, the symbol R stands for the Ore extension $D[x;\sigma,\delta]$ constructed from a ring D and a skew derivation (σ,δ) on D. The *degree* of a nonzero element $f \in R$ is defined in the obvious fashion. Since the standard form for elements of R is with left-hand coefficients, the *leading coefficient* of f is f_n if

$$f = f_0 + f_1 x + \cdots + f_{n-1}x^{n-1} + f_n x^n$$

with all $f_i \in D$ and $f_n \neq 0$. If σ is an automorphism, f can also be written with right-hand coefficients, but then its x^n-coefficient is $\sigma^{-n}(f_n)$. While a general formula for $x^n a$ where $a \in D$ and $n \in \mathbb{N}$ is too involved to be of much use, an easy induction establishes that

$$x^n a = \delta^n(a) + a_x + \cdots + a_{n-1}x^{n-1} + \sigma^n(a)x^n$$

for some $a_1, \cdots, a_{n-1} \in D$.

In preparation for our analysis of the types of ideals occured when prime ideals of an Ore extension $D[x;\sigma,\delta]$ are contracted to the coefficient ring D, we consider σ-prime, δ-prime, and (σ,δ)-prime ideals of D.

Definition 2 *Let Σ be a set of map from the ring D to itself. A Σ-ideal of D is any ideal I of D such that $\alpha(I) \subseteq I$ for all $\alpha \in \Sigma$. A Σ-prime ideal is any proper Σ-ideal I such that whenever J, K are Σ-ideals satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.*

In the context of a ring D equipped with a skew derivation (σ,δ), we shall make use of the above definition in the cases $\Sigma = \{\sigma\}, \Sigma = \{\delta\}$ or $\Sigma = \{\sigma,\delta\}$; and simplify the prefix Σ to respectively σ, δ, or (σ,δ). Concerning the contraction of prime ideals in an Ore Extension to its coefficient ring, Goodearl [4] obtained the following theorem which will be of use later.

Theorem 1 *Let $R = D[x; \sigma, \delta]$ where D is a commutative Dedekind domain and σ is an automorphism. If $\mathfrak{p}$ is any ideal of D which is (σ, δ)-prime, then $\mathfrak{p} = P \cap R$ for some prime ideal P of R and more specially $\mathfrak{p}R \in$ Spec(R) where Spec(R) denotes the set of all Prime ideal in R.*

3 Minimal Prime Ideals of Ore Extensions

Throughout this section, let D be a commutative Dedekind domain and $R = D[x; \sigma, \delta]$ be the Ore extension over D, for (σ, δ) is a skew derivation, $\sigma \neq 1$ is an automorphism of D and $\delta \neq 0$.

Marubayashi et. al. [9] already investigated the class of minimal prime ideals in R where $\delta = 0$. For the case $\delta \neq 0$ but it is inner, Goodearl [4] showed the existance of an isomorphism between $D[x; \sigma, \delta]$ and $D[y; \sigma]$ as described in the following. The σ-derivative δ is called inner if there exists an element $a \in D$ such that $\delta(b) = ab - \sigma(b)a$ for all $b \in D$.

Theorem 2 *Let $D[x; \sigma, \delta]$ be an Ore extension where $\sigma \neq 1$ and $\delta \neq 1$. If δ is an inner σ-derivation, i.e, there exists $a \in D$ such that $\delta(b) = ab - \sigma(b)a$ for all $b \in D$, then $D[x; \sigma, \delta]$ and $D[y; \sigma]$, where $y = x - a$, are isomorph.*

Hence, by combining Theorem 2 and the class of minimal prime ideals in Ore extensions R obtained in [9] we can derive the class of minimal prime ideals in Ore extension $R = D[x; \sigma, \delta]$ for σ and δ being respectively an automorphism and inner as the following. Notation $\mathrm{Spec}_0(R)$ stands for the set of all prime ideals in $R = D[x; \sigma, \delta]$ having zero intersection with D.

Theorem 3 *Let $R = D[x; \sigma, \delta]$ be an Ore extension.*

(1) *The set*

$\{\mathfrak{p}[x; \sigma, \delta],\ P \mid \mathfrak{p}$ *is a* $\sigma -$ *prime ideal of* D*and* $P \in \mathrm{Spec}_0(R)$ *with* $P \neq (0)\}$

consists of all minimal prime ideals of R.

(2) *Let $P \in \mathrm{Spec}(R)$ with $P \neq (0)$. Then P is invertible if and only if it is a minimal prime ideal of R.*

Now we shall investigate the class of minimal prime ideals for general $\delta \neq 0$. For this general case, we need the following lemma.

Lemma 1 *If $P = \mathfrak{p}[x; \sigma, \delta]$ is a minimal prime ideal of R where $\mathfrak{p}$ is a (σ, δ)-prime ideal of D, then $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D.*

Proof. Assume that $\mathfrak{p}$ is a (σ, δ)-prime ideal of D but is not minimal (σ, δ)-prime. Let $\mathfrak{q}$ be a (σ, δ)-prime ideal of D such that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\mathfrak{q} \neq (0)$. Then applying [2, Theorem 3.1] we have $\mathfrak{q}R \in \mathrm{Spec}(R)$. So, $\mathfrak{q}R \subsetneq \mathfrak{p}R = P$. This is a contradiction because P is a minimal prime ideal. □

Theorem 4 *Let P be a prime ideal of R and $P \cap D = \mathfrak{p} \neq (0)$. Then P is a minimal prime ideal of R if and only if either $P = \mathfrak{p}[x; \sigma, \delta]$ where $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D or (0) is the largest (σ, δ)-ideal of D in $\mathfrak{p}$.*

Proof. Let P be a minimal prime ideal of R and $P \cap D = \mathfrak{p} \neq (0)$. Since $\mathfrak{p} \neq (0)$, then there are two cases [4, Theorem 3.1]; namely, either $\mathfrak{p}$ is a (σ, δ)-prime ideal of D or $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$.

If $\mathfrak{p}$ is a (σ, δ)-prime ideal of D, then $\mathfrak{p}R \in \mathrm{Spec}(R)$, by [4, Theorem 3.1]. So, $\mathfrak{p}R = P$ because $\mathfrak{p}R \subseteq P$ and P is a minimal prime ideal. On the other hand $\mathfrak{p}R = \mathfrak{p}[x; \sigma, \delta]$. From here, we get $P = \mathfrak{p}[x; \sigma, \delta]$ and $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D, by Lemma 1.

Suppose $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$. Let $\mathfrak{m}$ be the largest (σ, δ)-ideal contained in $\mathfrak{p}$ and assume that $\mathfrak{m} \neq (0)$. Then by primeness of $\mathfrak{p}$ it can be shown that $\mathfrak{m}$ is a (σ, δ)-prime ideal of D. So, $\mathfrak{m}R$ is a prime ideal of R by [4, Proposition 3.3]. On the other hand, since $\sigma(\mathfrak{p}) \neq \mathfrak{p}$, we have $\mathfrak{m} \subsetneq \mathfrak{p}$. So, $\mathfrak{m}R \subsetneq \mathfrak{p}R \subseteq P$, i.e, P is not a minimal prime. This is a contradiction. So, (0) is the largest (σ, δ)-ideal of D in $\mathfrak{p}$.

Conversely, let $P = \mathfrak{p}[x; \sigma, \delta]$, where $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D. Then, according to [4, Theorem 3.3], $P = \mathfrak{p}[x; \sigma, \delta]$ is a prime ideal of R. Let Q be a prime ideal of R where $Q \subseteq P$. Set $\mathfrak{q} = Q \cap D$, then $\mathfrak{q} = Q \cap D \subseteq P \cap D = \mathfrak{p}$. Applying in [4, Theorem 3.1], we have two cases; namely, either $\mathfrak{q}$ is a (σ, δ)-prime ideal of D or $\mathfrak{q}$ is a prime ideal of D and $\sigma(\mathfrak{q}) \neq \mathfrak{q}$.

For the first case, suppose $\mathfrak{q}$ is a (σ, δ)-prime ideal of D. Then $\mathfrak{q} = \mathfrak{p}$ because $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D. So, $P = \mathfrak{p}[x; \sigma, \delta] = \mathfrak{q}[x; \sigma, \delta] \subseteq Q$. This implies $P = Q$. For the other case, if $\mathfrak{q}$ is a prime ideal of D, then $\mathfrak{q} = \mathfrak{p}$ because D is a Dedekind domain. So, $P = \mathfrak{p}[x; \sigma, \delta] = \mathfrak{q}[x; \sigma, \delta] \subseteq Q$. This implies $P = Q$. Thus P is a minimal prime ideal of R.

Let (0) be the largest (σ, δ)-ideal of D in $\mathfrak{p}$. Let Q be a prime nonzero ideal of R satisfying $Q \subseteq P$. Set $\mathfrak{q} = Q \cap D$, then $\mathfrak{q} = Q \cap D \subseteq P \cap D = \mathfrak{p}$. Similar to the above explanation, we have two cases; namely, either $\mathfrak{q}$ is a (σ, δ)-prime ideal of D or $\mathfrak{q}$ is a prime ideal of D and $\sigma(\mathfrak{q}) \neq \mathfrak{q}$ but the first case will not happen. If it is so, then, because of (0) being the largest (σ, δ)-ideal of D in $\mathfrak{p}$, $\mathfrak{q} = (0)$ implying a contradiction $Q \cap D = 0$ (see [5, Lemma 2.19]). Thus $\mathfrak{q}$ is a prime ideal of D with $\sigma(\mathfrak{q}) \neq \mathfrak{q}$. As a result, $\mathfrak{q} = \mathfrak{p}$. So $Q \cap D = P \cap D$, which, according to [5, Proposition 3.5], implies $Q = P$. Thus P is the minimal prime ideal of R. □

4 Concluding Remark

In this paper we identify all minimal prime ideals in an Ore extension over a Dedekind domain according to their contraction on the coefficient ring. Studying the results in Marubayashi et. al [9] it is expected that this identification can be used to study the structure of the corresponding factor rings which is currently under investigation.

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